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Article

On Network Designs with Coding Error Detection and Correction Application

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Abstract: The detection of error and its correction is an important area of mathematics that is vastly constructed in all communication systems. Furthermore, combinatorial design theory has several applications like detecting or correcting errors in communication systems. Network (graph) designs (GDs) are introduced as a generalization of the symmetric balanced incomplete block designs (BIBDs) that are utilized directly in the above mentioned application. The networks (graphs) have been represented by vectors whose entries are the labels of the vertices related to the lengths of edges linked to it. Here, a general method is proposed and applied to construct new networks designs. This method of networks representation has simplified the method of constructing the network designs. In this paper, a novel representation of networks is introduced and used as a technique of constructing the group generated network designs of the complete bipartite networks and certain circulants. A technique of constructing the group generated network designs of the circulants is given with group generated graph designs (GDs) of certain circulants. In addition, the GDs are transformed into an incidence matrices, the rows and the columns of these matrices can be both viewed as a binary nonlinear code. A novel coding error detection and correction application is proposed and examined.

Keywords: Network decomposition; network designs; network edge covering; circulant graphs

1 Introduction

Graph (Network) designs are introduced as a generalization of symmetric balanced incomplete block designs (BIBDs) (see, e.g., [1,2]) which are decompositions of complete graphs (networks) to subgraphs (subnetworks) satisfying certain conditions (see [1]). There are several research papers on the subject of graph decompositions; for more details see [3]. Through the paper we use the word (graph) to mean (network).

As defined in [1], a symmetric graph design, or SGD, with parameters $(n, G, \lambda; F)$, where n, λ are positive integers and G and F are graphs with n vertices, is a set $\{G_1, \ldots, G_n\}$ of spanning subgraphs of the complete graph K_n such that

(a) $G_i \cong G$ for $i = 1, \ldots, n$;
(b) any edge of K_n is contained in exactly λ subgraphs G_i, and
(c) $G_i \cap G_j \cong F$ for $i, j = 1, \ldots, n$, $i \neq j$.

In [4], Dalibor Fronček and Alex Rosa determined all graphs F and all orders for which there exists an $(n, G, \lambda; F)$-SGD where $G \cong F_{\frac{n-1}{2},3}$, the friendship graph on n vertices.

In this paper, a generalization of symmetric BIBDs is investigated and we introduce a new graph representation that will help in constructing new graph designs (GDs).

Definition 1.1 *Let H be a r-regular Cayley graph of order n and B be a non-empty set of spanning subgraphs of H. A $(H, B, \lambda; F)$-GD (Graph Design, GD) is a collection $\Sigma = \{G_0, G_1, \ldots, G_s\}$ of spanning subgraphs of H such that*

(1) *all graphs G in B have the same size $e = |E(G)| \leq r$,*
(2) *any graph of Σ is isomorphic to one graph of B,*
(3) *every edge of H belongs to exactly λ elements of Σ,*
(4) *for any two different subgraphs G_i and G_j of Σ, we have $G_i \cap G_j \cong F$.*

If $H \cong K_{n,n}$, $\lambda = 2$, $|G| < n$ and $|F| = 1$ or 0, then the $(K_{n,n}, \{G\}, 2; F)$-GD is equivalent to the sub-orthogonal double covers (SODCs) of the complete bipartite graph by G. SODC's have been studied by many authors (for SODCs of $K_{n,n}$ by G, see [5–7] and for SODCs of K_n by G, see, [8–10]. The $(H, \{G\}, 2; K_2)$-GD with $|G| = r$ is equivalent to the orthogonal double covers (ODCs) of Cayley graphs which have been studied in [11]. Also, the $(K_{n,n}, \{G\}, 2; K_2)$-GD with $|G| = n$ is equivalent to ODCs of $K_{n,n}$ by G that have been investigated by many authors (see, e.g., [12–15]). Studying the case when $H \cong K_{n,n}$, $\lambda > 2$, $|G| = n$ and $F \cong K_2$ is equivalent to studying the mutually orthogonal graph squares which have been studied by many authors (see, e.g., [7,16–19]) and for more details see the survey [20]. Since SODCs and ODCs can be considered as graph designs, its construction tools can be used to construct new graph designs as will be done in this work.

Here, all graphs are assumed to be finite, simple and with non-empty edge set. We use the usual notations: $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ for the group of all residual classes modulo n, $\varnothing$ for the empty set, $K_{n,n}$ for the complete bipartite graphs, K_n for the complete graph, P_{n+1} for the path graph with n edges, S_n for the star of size n, E_n the empty graph of order n, the circulant graph $H = Circ(\mathbb{Z}_n, A)$ is defined by $V(H) = \mathbb{Z}_n$ and $E(H) = \{(i, i+l) : i \in \mathbb{Z}_n, l \in A\}$, see [21].

In our current study, we concentrate on the case when $H \cong K_{n,n}$ or $Circ(\mathbb{Z}_n, A)$ and $F \cong K_2$ or $F \cong E_n$. Note that, if $|G| < n$, then $|\Sigma| > n$.

From now on, all addition and subtraction shall be done modulo n.

The vertices of $K_{n,n}$ shall be labeled by the elements of $\mathbb{Z}_n \times \mathbb{Z}_2$. Namely, for $(v, i) \in \mathbb{Z}_n \times \mathbb{Z}_2$ we shall write v_i for the corresponding vertex and define $\{u_i, v_j\} \in E(K_{n,n})$ if and only if $i \neq j$, for all $u, v \in \mathbb{Z}_n$ and $i, j \in \mathbb{Z}_2$. To avoid ambiguity, the edge $\{u_0, v_1\}$ shall be written as (u, v).

All designs can be represented by a corresponding incidence matrix [22]. Following the method produced in [23], the incidence matrices can be used in coding error detection and corrections. Here, the suggested codes are not linear codes.

The arrangement of our paper is as follows: In Section 2, a new representation of graphs is introduced. In Section 3, a technique of constructing the group generated graph designs of $K_{n,n}$ is studied. In Section 4, detection of error and its correction is suggested as an application of the codes generated by the constructed graph designs. In Section 5, we construct new group generated graph designs of $K_{n,n}$. In Section 6, a technique of constructing the group generated graph designs of the circulants is given with group generated graph designs of certain circulants. The conclusion shall be in Section 7.

2 New Representation of Graphs

In this section, we introduce a new representation of graphs following the method that has been introduced in [13]. In [13], the graphs have been represented by a vector whose entries are the labels of the vertices related to the lengths of edges linked to it. This method of graph representation has simplified the method of constructing the graph designs. Here, a general method is proposed and applied to construct new graph designs.

Let G be a spanning subgraph of H and let $\alpha \in \mathbb{Z}_n$. Then the graph G with

$$E(G+\alpha) = \{(u+\alpha, v+\alpha) : (u, v) \in E(G)\}$$

is called the α-translate of G. The length of an edge $e = (u, v) \in E(G)$ is defined by $l(e) = v - u$.

For any subgraph G of $K_{n,n}$, let $(G) = \{y - x : (x, y) \in E(G)\}$

be the multiset containing the length of every edge in G. For any two subgraphs G_1 and G_2 of H, let

$$D(G_1, G_2) = \{u - x : (x, y) \in E(G_1), (u, v) \in E(G_2), y - x = v - u\}$$

be the multiset containing the distance of every pair of equal length edges in G_1 and G_2. Note that the distance set $D(G, G)$ means the set of distances between the different edges in G which have the same lengths. For any collection of graph $\Omega = \{G_i : 0 \le i \le k-1\}$, we define rd-matrix as a $k \times k$ matrix whose entries are $rd_\Omega(i,j) = D(G_i, G_j)$ for $0 \le i \le j \le k-1$.

Let G be a graph of order n and its vertices are the elements of $\mathbb{Z}_n$, G can be represented by a map $\psi(G)$ from $\mathbb{Z}_n$ to its power set (i.e., $\psi(G) : \mathbb{Z}_n \longrightarrow P(\mathbb{Z}_n)$) where for all $i \in \mathbb{Z}_n$, $\psi_i(G) = A_i \subseteq \mathbb{Z}_n$ such that for all $a \in A_i$, the edge $(a, a+i) \in E(G)$. $\psi(G)$ can be written in the form of n-tuple where

$$\psi_i(G) = \{u \in \mathbb{Z}_n : (u, u+i) \in E(G)\}$$

for all $0 \le i \le n-1$ (a vector whose *ith* entry is a set of vertices, from $\mathbb{Z}_n$, incident to the edges with length equal i). Then the following are clear.

$$\psi(K_{n,n}) = (\mathbb{Z}_n, \mathbb{Z}_n, \mathbb{Z}_n, \ldots, \mathbb{Z}_n), \psi(K_n) = (\varnothing, \mathbb{Z}_n, \mathbb{Z}_n, \ldots, \mathbb{Z}_n)$$

and for all $i \in \mathbb{Z}_n$, $|\psi_i(S_n)| = 1$ and $\psi_i(E_n) = \varnothing$.

Let $H = Circ(\mathbb{Z}_n,\ A = A^+ \cup -A^+)$ where $A^+ = A \cap \{1, 2, \ldots, \lfloor n/2 \rfloor\}$. Then

$$\psi_i(H) = \psi_{-i}(H) = \begin{cases} \mathbb{Z}_n & \text{for all } i \in A, \\ \varnothing & \text{otherwise.} \end{cases}$$

Let G and H be two spanning subgraphs of $K_{n,n}$, G and H are said to be orthogonal if they share at most one edge (i.e., $|E(G) \cap E(H)| \le 1$), see [8,10] or [24]. Then the collection Ω is mutually orthogonal if and only if all cells of rd_Ω matrix are sets.

For $H = r$-regular $Circ(\mathbb{Z}_n, A)$, the existence of $(H, B, \lambda; K_2)$-GD immediately implies the following two necessary conditions that is recorded as

Lemma 2.1 *Let* $\Sigma = \{G_0, G_1, \ldots, G_s\}$ *be a* $(H, B, \lambda; K_2)$*-GD and* e *is the size of any element of* B*. Then*

$$\begin{cases} \lambda nr/2 = 0\, mod\{e\} \\ s \geq \lambda n. \end{cases}$$

Proof. From Definition 1.1 of the $(H, B, \lambda; K_2)$-GD, we have

$$|E(G_1)| + |E(G_2)| + \cdots + |E(G_s)| = \lambda\, |E(H)| = \lambda nr/2.$$

Since all elements of Σ are isomorphic to one element of B and all elements of B have the same size e, this implies that $\lambda nr/2 = se$. Also, we have $e \leq n$ by Definition 1.1, which imply that $s \geq \lambda n$.

3 Group Generated Graph Designs of $K_{n,n}$

Definition 3.1 *Let* $\Omega = \{G_0, G_1, \ldots, G_k)$ *be a collection of spanning subgraphs of* $K_{n,n}$*. We call* Ω *a* $(K_{n,n}, B, \lambda; K_2)$*-GD generator if it satisfies the following conditions:*

(1) *Every element of* $\mathbb{Z}_n$ *appears exactly* λ *times in the sum of the multisets*

$L(G_i), \quad i = 0, 1, 2, \ldots, k-1.$

(2) *For all pairs i,j with* $0 \leq i \leq j \leq k-1$*, the cells of the* rd_Ω *matrix are sets, that is* $D(G_i, G_j)$ *are all sets.*

The elements of the generator Ω are called $(K_{n,n}, B, \lambda; K_2)$-GD *pre-starters* graphs.

Theorem 3.2 *Let* $\Omega = \{G_0, G_1, \ldots, G_k)$ *be a* $(K_{n,n}, B, \lambda; K_2)$*-GD generator. Then for all* $0 \leq i \leq k-1$*, the collection of all the translates of* $G_i + \alpha$ *for all* $\alpha \in \mathbb{Z}_n$*, forms a* $(K_{n,n}, B, \lambda; K_2)$*-GD by* B*.*

Proof. It is clear that the collection of all translates covers every edge of $K_{n,n}$ exactly λ times. Now, It is to show that the collection of all translates are mutually orthogonal, that is any two graphs of the collection of all translates share at most one edge. Consider two translates $G_i + \alpha$ and $G_j + \beta$ where $\alpha, \beta \in \mathbb{Z}_n$ and assume that they share two edges $e_1 = (x, y)$ with length $l_1 = y - x$ and $e_2 = (u, v)$ with length $l_2 = v - u$. Then the two edges $(x - \alpha, y - \alpha)$, $(u - \alpha, v - \alpha) \in G_i$ with lengths l_1, l_2 respectively and $(x - \beta, y - \beta)$, $(u - \beta, v - \beta) \in G_j$ with lengths l_1, l_2 respectively. Then the distance between the two edges with length l_1 in G_i and G_j is $\alpha - \beta$, and also the distance between the two edges with length l_2 in G_i and G_j is $\alpha - \beta$ and then $D(G_i, G_j)$ is not a set. This is a contradiction of the second condition in the Definition 3.1 of the $(K_{n,n}, B, \lambda; K_2)$-*GD* generator. Consequently, all subgraphs in the collection of all translates of *GD*-generator are mutually orthogonal, that is a $(K_{n,n}, B, \lambda; K_2)$-GD.

Lemma 3.3 *Let* $\Omega = \{G_0, G_1, \ldots, G_{k-1})$ *be a* $(K_{n,n}, B, \lambda; K_2)$*-GD generator, then*

(*i*) *the number of pre-starters in* Ω *is* $k = \lambda n/e$.
(*ii*) *For all* $0 \leq i \leq k-1$*, if* $d \in D(G_i, G_i)$ *then* $-d \notin D(G_i, G_i)$*,*
(*iii*) *For all* $0 \leq i \leq k-1$*, if* n *is even then* $n/2 \notin D(G_i, G_i)$*.*

Proof. (i) $\Sigma = \{G_0, G_1, \ldots, G_s) = (K_{n,n}, B, \lambda; K_2)$-GD. Since $s = kn$ then $kne = \lambda n^2$ and hence $k = \lambda n/e$.

(ii) Let $D(G_i, G_i)$ contains $\pm d$. then G_i contains four edges each pair of them has the same length l_1 and l_2, that is $(x, x+l_1), (x+d, x+d+l_1)$, $(u, u+l_2), (u-d, u-d+l_2) \in G_i$.

Then $G_i + d$ contains $(x+d, x+d+l_1)$, $(x+2d, x+2d+l_1)$, $(u+d, u+d+l_2)$, $(u, u+l_2)$ which imply that $|G_i \cap G_i + d| > 1$ which is a contradiction. Hence, for all $0 \le i \le k-1$, if $d \in D(G_i, G_i)$ then $-d \notin D(G_i, G_i)$.

(iii) For any $0 \le i \le k-1$, let $n/2 \in D(G_i, G_i)$.

So there exist two edges $e_1 = (x, x+l)$, $e_2 = (x+n/2, x+n/2+l)$ belong to $E(G_i)$ with the same length l and $D(e_1, e_2) = n/2$.

Then $G_i + n/2$ contains also $e_1 = (x, x + l)$, $e_2 = (x + n/2, x + n/2 + l)$ that means $|G_i \cap G_i + n/2| > 1$ which is a contradiction. Hence, for all $0 \le i \le k-1$, if n is even then $n/2 \notin D(G_i, G_i)$.

Therefore, $\lambda n \equiv 0 \, mod\{e\}$ is a necessary condition of the existence of the $(K_{n,n}, B, \lambda; K_2)$-*GD* generator.

Lemma 3.4 *Let* $\psi(G) = (A_0, A_1, \ldots, A_{n-1})$ *is a pre-starter of* $(K_{n,n}, B, \lambda; K_2)$*-GD and*

$Max\{|A_i| : 0 \le i \le n-1\} = m$.

Then

$n > 2\binom{m}{2}$ if n even,

$n \ge 2\binom{m}{2} + 1$ if n odd.

Proof. Case 1. For n is even;

for $n \le 2\binom{m}{2}$,

then $n/2 \le \binom{m}{2}$ (the number of differences of the edges of length i), then $D(G, G)$ is a multiset set. This is a contradiction, then $n > 2\binom{m}{2}$.

Case 2. For n is odd;

for $n < 2\binom{m}{2} + 1$,

then $(n-1)/2 < \binom{m}{2}$ (the number of differences of the edges of length i), then $D(G, G)$ is a multiset set. This is a contradiction, then $n \ge 2\binom{m}{2} + 1$.

Proposition 3.5 *Let* $m \ge 2$ and $n \ge 1$ *be any integers*, B *is a set of graphs of size* e. *If there exists a* $(K_{m,m}, B, \lambda; K_2)$*-GD generator of* $K_{m,m}$ *by* B, *then there exists a* $(K_{mn,mn}, B, \lambda; K_2)$*-GD generator of* $K_{mn,mn}$ *by* B.

Proof. Here, the element $(s, t) \in \mathbb{Z}_m \times \mathbb{Z}_n$ is written as st. Let $\Omega = \{G_0, G_1, \ldots, G_{k-1}\}$ be a $(K_{m,m}, B, \lambda; K_2)$-GD generator of $K_{m,m}$ by B with respect to $\mathbb{Z}_m$ that is every edge in $K_{m,m}$ appears λ times in Ω and $D(G^p, G^q)$ for all $0 \le p \le q \le k-1$, are sets, and $k = \lambda m/e$.

For all $i \in \mathbb{Z}_m$ and $0 \leq s \leq k-1$, let $\psi(G_s) = (A_0^s, A_1^s, \ldots, A_{(m-1)}^s)$ is a pre-starter graph $G_s \in \Omega$, that is $\psi_i(G_s) = A_i^s$ and $E(G_s) = \{(a, a+i) \colon a \in A_i^s\}$.

Let the set $D(G_p, G_p) = D_1$ and the set $D(G_p, G_q) = D_2$ and $p \neq q$.

For all $i \in \mathbb{Z}_m$, and for all $j, t \in \mathbb{Z}_n$, define $\Omega^* = \{G_{0j}, G_{1j}, \ldots, G_{kj}\}$ by

$$\psi_{ij}(G_{st}) = \begin{cases} A_i^s \times \{0\} & \text{if } j = t, \\ \varnothing & \text{otherwise.} \end{cases}$$

Then $E(G_{st}) = \left\{(a0, a0+it) : a \in A_i^s\right\}$. Then every edge in $K_{mn,mn}$ (its vertices are $\mathbb{Z}_m \times \mathbb{Z}_n \times \mathbb{Z}_2$) appears λ times in Ω^*. For any two graphs $G, H \in \Omega^*, D(G,G) = D_1 \times \{0\} \subseteq \mathbb{Z}_m \times \mathbb{Z}_n \times \{0\}$ which is a set and $D(G,H) = D_2 \times \{0\} \subseteq \mathbb{Z}_m \times \mathbb{Z}_n \times \{0\}$ which is a set then Ω^* is a $(\boldsymbol{K_{mn,mn}}, \boldsymbol{B}, \boldsymbol{\lambda}; \boldsymbol{K_2})$-GD generator of $\boldsymbol{K_{mn,mn}}$ by $\boldsymbol{B}$ with respect to $\mathbb{Z}_m \times \mathbb{Z}_n$.

4 Coding Error Detection and Correction Application

The rows or columns of the incedence matrix of the GDs can be used as binary codes because all of its entries are 0 or 1. Let us define the GD's Incedence matrix $\mathcal{J}$ as follows.

For the $(K_{n,n}, G, \lambda; K_2)$-GD, since $K_{n,n}$ has n^2 edges and we have s blocks (GD subgraphs), define $\mathcal{J}$ as $s \times n^2$ integer matrix where its elements are 0 or 1 and displays the relation between the edges and the blocks where every row corresponds to a block (GD subgraph G_i) and every column corresponds to an edge (e_j) in the graph $K_{n,n}$.

$$\mathcal{J}_{ij} = \begin{cases} 1 & \text{if } e_j \in G_i \\ 0, & \text{otherwise.} \end{cases}$$

GD Incidence Matrix has the following properties:

As the incidence matrix $\mathcal{J}$ of a $(K_{n,n}, G, \lambda; K_2)$-GD has the following properties.

1. Every row has n number of 1s,
2. Every column has λ number of 1s,
3. Two distinct columns both have 1s in at most 1 rows.

For illustration, the following example is produced.

The blocks of $(K_{3,3}, S_3, 2; K_2)$-GD is constructed as:

$\{G_1 = \{00, 01, 02\},\ G_2 = \{10, 11, 12\},\ G_3 = \{20, 21, 22\},$

$G_4 = \{00, 10, 20\},\ G_5 = \{01, 11, 21\},\ G_6 = \{02, 12, 22\}\}$

where ab is an edge between vertex a_0 and vertex b_1, see Fig. 1. The incedence matrix of this GD is

$$\mathcal{J} = \begin{matrix} G_0 \\ G_1 \\ G_2 \\ G_3 \\ G_4 \\ G_5 \end{matrix} \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

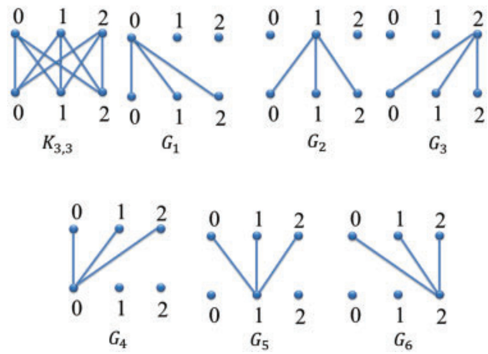

Figure 1: $(K_{3,3}, S_3, 2; K_2)$-GD

When a GD is transformed into an incidence matrix, the rows and the columns can be both viewed as a binary nonlinear code. The binary codes formed from the row denoted as $\mathcal{S}_{row}$ and binary codes from the column will be referred as $\mathcal{S}_{column}$. As mentioned previously, by conversion of GD to incidence matrix, the incidence matrix of a GD retains certain properties that are inherited from GD. Using these properties, results can be obtained to evaluate the minimum Hamming distance (number of different bits in two codes) between codes from $\mathcal{S}_{row}$ or $\mathcal{S}_{column}$. Where

$$\mathcal{S}_{row} = \{111000000, 000111000, 000000111, 100100100, 010010010, 001001001\}$$

and

$$\mathcal{S}_{column} = \{100100, 100010, 100001, 010100, 010010, 010001, 001100, 001010, 001001\}.$$

The minimum Hamming distance $\delta(\mathcal{S}_{row}) = 4$ and $\delta(\mathcal{S}_{column}) = 2$.

Distance in binary codes detects the number of errors a code can detect or correct [25]. As proved in [26], we have

- a binary code $\mathcal{S}$ can be detected up to q errors iff the minimum distance δ is greater or equivalent to $q+1$.
- a binary code $\mathcal{S}$ can be corrected up to q errors iff the minimum distance δ is greater or equivalent to $2q+1$.

Then for our example $\mathcal{S}_{row}$ can detect upto 3 errors and correct upto *one* error.

Efficiency factor E is the the quality estimation of the design efficiency. The efficiency factor E is a numerical value lies between 0 and 1. The quality of a design is "good" if E is greater than 0.75 The efficiency of the (v, b, r, k, λ)-BIBD design codes [27] is calculated as $E = \dfrac{v(k-1)}{k(v-1)}$ which can be simplified for our graph design as $E = \dfrac{n^2(n-1)}{n(n^2-1)} = \dfrac{n}{n+1}$ (put $v = n^2$, the size of $K_{n,n}$ and $k = n$, the size of G) which will be always greater than 0.75 where n is the size of the GD blocks. Then the efficiency of the codes from the GDs are very good and can be safely used in coding processes. For more details about the design efficiency, see [27]. For more applications of networks, see [28–30].

To clear the proposed application, we use the above $\mathcal{S}_{row}$ for coding the following words shown in Tab. 1 and assuming that there is a possibility of occurring an error in at most two positions. From the structure of the corresponding GD, the number of ones must be 3 in any code.

Table 1: Words' codes

Words	Codes
Go	111000000
Stop	000111000
Forward	000000111
Back	100100100
Left	010010010
Right	001001001

If the code 111100001 is received. Since number of ones must be 3, the error is detected. To correct the error, the code with the minimum Hamming distance from the received one can be chosen that is 111000000. Then the message is "go," and so on.

5 Graph Designs $(K_{n,n}, B, \lambda; K_2)$-GD's

Here, we use the above representation of graphs to construct $(K_{n,n}, B, \lambda; K_2)$-GD for $\lambda \in \{2, 3, 4\}$ by certain graph classes B.

5.1 Graph Designs $(K_{n,n}, \{C_m\}, 2; K_2)$-GD's

Lemma 5.1 *Let $t \geq 1$ be a positive integer. There exists $(K_{6t,6t}, \{C_6\}, 2; K_2)$-GD.*

Proof. For $n = 6$, define $\Omega = \{G_0, G_1\}$ by

$$\psi(G_0) = (\{0,1\}, \{1,5\}, \{1\}, \varnothing, \{5\}, \varnothing) \quad \text{and} \quad \psi(G_1) = (\varnothing, \varnothing, \{1\}, \{0,2\}, \{2\}, \{0,1\})$$

Then all graphs in Ω are isomorphic to C_6 and

$$rd_\Omega\text{-matrix} = \begin{bmatrix} \{1,4\} & \{0,3\} \\ \{0,3\} & \{1,2\} \end{bmatrix}$$

Since every cell of the rd_Ω-matrix is a set satisfying Lemma 3.3, then Ω is a $(K_{6,6}, \{C_6\}, 2; K_2)$-GD generator. Applying Proposition 3.5 completes the proof. ■

Lemma 5.2 *Let $t \geq 1$ be a positive integer. There exists $(K_{10t,10t}, \{C_{10}\}, 2; K_2)$-GD.*

Proof. For $n = 10$, define $\Omega = \{G_0, G_1\}$ by

$$\psi(G_0) = (\{0,9\}, \{8\}, \{0\}, \{9\}, \{6\}, \varnothing, \{8\}, \{6\}, \{5\}, \{5\}),$$

$$\psi(G_1) = (\varnothing, \{0\}, \{0\}, \{7\}, \{5\}, \{5,7\}, \{8\}, \{6\}, \{5\}, \{5\})$$

Then all graphs in Ω are isomorphic to C_{10} and

$$rd_\Omega\text{-matrix} = \begin{bmatrix} \{1,9\} & \{0,6,7,9,5,4,2,8\} \\ \{0,6,7,9,5,4,2,8\} & \{2,8\} \end{bmatrix}$$

Since every cell of the rd_Ω-matrix is a set satisfying Lemma 3.3, then Ω is a $(K_{10t,10t},\{C_{10}\},2;K_2)$-GD generator. Applying Proposition 3.5 completes the proof. ■

5.2 Graph Designs $(K_{n,n},B,3;K_2)$-GD's

The existence of $(K_{4,4},\{P_5\},\lambda;K_2)$-GD still open for $\lambda\geq 3$. Nevertheless, we can record the following result as:

Lemma 5.3 *For* $\lambda\geq 3$. *There is no* $(K_{4,4},\{P_5\},\lambda;K_2)$-*GD generator*.

Proof. Let P_5 is a spanning subgraph of $K_{4,4}$. Then the following vectors and all of its translates are the all possible pre-starter vectors of P_5 shown in Tab. 2. By careful inspection, we find that there are no $\lambda\geq 2$ mutually orthogonal pre-starter vectors inside this collection, then the proof is complete.

Table 2: All possible pre-starter vectors of P_5

$(\{0,1\},\{0\},\{1\},\varnothing)$	$(\{0,1\},\{1\},\{0\},\varnothing)$	$(\{1,1\},\{0\},\{2\},\varnothing)$
$(\{0,1\},\varnothing,\{3\},\{1\})$	$(\{0,1\},\varnothing,\{2\},\{2\})$	$(\{0,1\},\{3\},\{3\},\varnothing)$
$(\{0\},\{0,1\},\varnothing,\{1\})$	$(\{1\},\{0,1\},\varnothing,\{0\})$	$(\varnothing,\{0,1\},\{0\},\{1\})$
$(\varnothing,\{0,1\},\{1\},\{0\})$	$(\{1\},\{0,1\},\varnothing,\{3\})$	$(\varnothing,\{0,1\},\{0\},\{2\})$
$(\{1\},\{0\},\{0,1\},\varnothing)$	$(\{0\},\{1\},\{0,1\},\varnothing)$	$(\{3\},\{1\},\{0,1\},\varnothing)$
$(\{2\},\varnothing,\{0,1\},\{0\})$	$(\{3\},\varnothing,\{0,1\},\{3\})$	$(\{2\},\{2\},\{0,1\},\varnothing)$
$(\{2\},\{0,1\},\{2\},\varnothing)$	$(\varnothing,\{0,1\},\{3\},\{3\})$	$(\{0\},\{1\},\varnothing,\{0,1\})$
$(\{1\},\{0\},\varnothing,\{0,1\})$	$(\varnothing,\{0\},\{1\},\{0,1\})$	$(\varnothing,\{1\},\{0\},\{0,1\})$
$(\varnothing,\{3\},\{1\},\{0,1\})$	$(\varnothing,\{2\},\{2\},\{0,1\})$	$(\{3\},\{3\},\varnothing,\{0,1\})$
$(\{0\},\{0\},\{3\},\{1\})$	$(\{0\},\{3\},\{3\},\{0\})$	$(\{0\},\{3\},\{1\},\{0\})$

■

Proposition 5.4 *Let* $n\geq 3$ *be a positive integer. There exists a* $(K_{n,n},\{P_4\},3;K_2)$-*GD*.

Proof. Define $\Omega=\{G_0,G_1,\ldots,G_{n-1}\}$ as follows.

For all $i,j\in\mathbb{Z}_n$.

$$\psi_i(G_j)=\begin{cases}\{0,1\} & \text{if } i=j,\\ \{0\} & \text{if } i=j+1,\\ \varnothing & \text{otherwise.}\end{cases}$$

Then all graphs in Ω are isomorphic to P_4 and $E(G_j)=\{(0,i),(i,i),(0,i+1)\}$, and

$$D(G_i,G_j)=\begin{cases}\{1\} & \text{if } i=j,\\ \{0,1\} & \text{if } j=i+1,\\ \varnothing & \text{otherwise.}\end{cases}$$

Since every cell of the rd_Ω-matrix is a set satisfying Lemma 3.3, then Ω is a $(K_{n,n},\{P_4\},3;K_2)$-GD generator. ■

Lemma 5.5 *Let* $t\geq 1$ *be a positive integer. There exists a* $(K_{8t,8t},\{C_4\},3;K_2)$-*GD*.

Proof. For $n = 8$, define $\Omega = \{G_0, G_1, G_2, G_3, G_4, G_5\}$ as.

$\psi(G_0) = (\{0,1\}, \{0\}, \varnothing, \varnothing, \varnothing, \varnothing, \varnothing, \{1\})$, $\psi(G_1) = (\varnothing, \{1\}, \{0,1\}, \{0\}, \varnothing, \varnothing, \varnothing, \varnothing)$,

$\psi(G_2) = (\varnothing, \varnothing, \varnothing, \{1\}, \{0,1\}, \{0\}, \varnothing, \varnothing)$, $\psi(G_3) = (\varnothing, \varnothing, \varnothing, \varnothing, \varnothing, \{1\}, \{0,1\}, \{0\})$,

$\psi(G_4) = (\{0\}, \{5\}, \varnothing, \{5\}, \varnothing, \varnothing, \{0\}, \varnothing)$, $\psi(G_5) = (\varnothing, \varnothing, \{0\}, \varnothing, \{0\}, \{5\}, \varnothing, \{5\})$.

Then all graphs in Ω are isomorphic to C_4 and

$$rd_\Omega\text{-matrix} = \begin{bmatrix} \{1\} & \{1\} & \varnothing & \{7\} & \{0,7,5\} & \{4\} \\ \{1\} & \{1\} & \{1\} & \varnothing & \{4,5\} & \{0,7\} \\ \varnothing & \{1\} & \{1\} & \{1\} & \{4\} & \{0,7,5\} \\ \{7\} & \varnothing & \{1\} & \{1\} & \{0,7\} & \{4,5\} \\ \{0,7,5\} & \{4,5\} & \{4\} & \{0,7\} & \varnothing & \varnothing \\ \{4\} & \{0,7\} & \{0,7,5\} & \{4,5\} & \varnothing & \varnothing \end{bmatrix}$$

Since every cell of the rd_Ω-matrix is a set satisfying Lemma 3.3, then Ω is a $(K_{8,8}, \{C_4\}, 3; K_2)$-GD generator. Applying Proposition 3.5 completes the proof. ■

Lemma 5.6 *Let $t \geq 1$ be a positive integer. There exists $(K_{6t,6t}, \{P_7\}, 3; K_2)$-GD.*

Proof. For $n = 6$, define $\Omega = \{G_0, G_1, G_2\}$ as.

$\psi(G_0) = (\{0,1\}, \{0,4\}, \varnothing, \{3\}, \{1\}, \varnothing)$, $\psi(G_1) = (\{1\}, \varnothing, \{0,1\}, \{0,2\}, \varnothing, \{2\})$,

$\psi(G_2) = (\varnothing, \{5\}, \{0\}, \varnothing, \{4,5\}, \{4,0\})$.

Then all graphs in Ω are isomorphic to P_7 and

$$rd_\Omega\text{-matrix} = \begin{bmatrix} \{1,4\} & \{1,0,3,5\} & \{1,5,2,3\} \\ \{1,0,3,5\} & \{1,2\} & \{0,1,2,4\} \\ \{1,5,2,3\} & \{0,1,2,4\} & \{1,4\} \end{bmatrix}$$

Since every cell of the rd_Ω-matrix is a set satisfying Lemma 3.3, then Ω is a $(K_{6,6}, \{P_7\}, 3; K_2)$-GD generator, Applying Proposition 3.5 completes the proof. ■

Lemma 5.7 *Let $t \geq 1$ be a positive integer. There exists a $(K_{6t,6t}, \{C_4 \cup S_2\}, 3; K_2)$-GD.*

Proof. For $n = 6$, define $\Omega = \{G_0, G_1, G_2\}$ by

$\psi(G_0) = (\{0,5\}, \varnothing, \{0,4\}, \varnothing, \{4\}, \{5\})$, $\psi(G_1) = (\{1\}, \{0,5\}, \varnothing, \{3,5\}\}, \varnothing, \{3\})$

$\psi(G_2) = (\varnothing, \{4\}, \{3\}, \{0\}, \{0,5\}, \{5\})$.

Then all graphs in Ω are isomorphic to $C_4 \cup S_2$ and

$$rd_\Omega\text{-matrix} = \begin{bmatrix} \{4,5\} & \{1,2,4\} & \{3,5,2,1,0\} \\ \{1,2,4\} & \{2,5\} & \{4,5,3,1,2\} \\ \{3,5,2,1,0\} & \{4,5,3,1,2\} & \{5\} \end{bmatrix}$$

Since every cell of the rd_Ω-matrix is a set satisfying Lemma 3.3, then Ω is a $(K_{6,6}, \{C_4 \cup S_2\}, 3; K_2)$-GD generator. Applying Proposition 3.5 completes the proof. ■

Lemma 5.8 *Let* $t \geq 1$ *be a positive integer. There exists a* $(K_{6t,6t}, \{C_6, P_4 \cup P_3 \cup P_2\}, 3; K_2)$*-GD.*

Proof. For $n = 6$, define $\Omega = \{G_0, G_1, G_2\}$ by

$$\psi(G_0) = (\{0,1\}, \varnothing, \{5\}, \varnothing, \{0\}, \{1,5\}), \quad \psi(G_1) = (\{1\}, \{0,1\}, \varnothing, \{0\}, \{4\}, \{4\}),$$
$$\psi(G_2) = (\varnothing, \{4\}, \{0,1\}, \{0,2\}, \{3\}, \varnothing).$$

Then $\{G_0, G_1\}$ are isomorphic to C_6, G_2 is isomorphic to $P_4 \cup P_3 \cup P_2$ and

$$rd_\Omega\text{-matrix} = \begin{bmatrix} \{1,4\} & \{1,0,4,3,5\} & \{1,2,3\} \\ \{1,0,4,3,5\} & \{1\} & \{4,3,0,2,5\} \\ \{1,2,3\} & \{4,3,0,2,5\} & \{1,2\} \end{bmatrix}$$

Since every cell of the rd_Ω-matrix is a set satisfying Lemma 3.3, then Ω is a $(K_{6,6}, \{C_6, P_4 \cup P_3 \cup P_2\}, 3; K_2)$-GD generator. Applying Proposition 3.5 completes the proof.

Lemma 5.9 *Let* $t \geq 1$ *be a positive integer. There exists a* $(K_{6t,6t}, \{C_6, P_5 \cup 2P_2\}, 3; K_2)$*-GD.*

Proof. For $n = 6$, define $\Omega = \{G_0, G_1, G_2\}$ by

$$\psi(G_0) = (\{4\}, \{1\}, \{0,1\}, \varnothing, \{0\}, \{4\}), \quad \psi(G_1) = (\{0\}, \varnothing, \varnothing, \{0,1\}, \{5\}, \{1,5\}),$$
$$\psi(G_2) = (\{5\}, \{1,4\}, \{5\}, \{1\}, \{2\}, \varnothing).$$

Then $\{G_0, G_1\}$ are isomorphic to C_6, G_2 is isomorphic to $P_5 \cup 2P_2$ and

$$rd_\Omega\text{-matrix} = \begin{bmatrix} \{1\} & \{2,5,3,1\} & \{1,0,3,5,4,2\} \\ \{2,5,3,1\} & \{1,4\} & \{5,1,0,3\} \\ \{1,0,3,5,4,2\} & \{5,1,0,3\} & \{1,2\} \end{bmatrix}$$

Since every cell of the rd_Ω-matrix is a set satisfying Lemma 3.3, then Ω is a $\left(K_{6,6}, \{C_6, P_5 \cup 2P_2\}, 3; K_2\right)$-GD generator. Applying Proposition 3.5 completes the proof. ■

Lemma 5.10 *Let* $t \geq 1$ *be a positive integer and* G *is the class of the spanning sub-graphs isomorphic to the graph with vertices* $\{a,b,c,d,e,r,s\}$ *and the* 6 *edges* $\{(a,b),(c,b),(c,d),(e,b),(e,d),(r,s)\}$*. There exists a* $(K_{6t,6t}, \{C_6, G\}, 3; K_2)$*-GD.*

Proof. For $n = 6$, define $\Omega = \{G_0, G_1, G_2\}$ by

$$\psi(G_0) = (\{4\}, \{1\}, \{0,1\}, \varnothing, \{0\}, \{4\}), \quad \psi(G_1) = (\{0\}, \varnothing, \varnothing, \{0,1\}, \{5\}, \{1,5\}),$$
$$\psi(G_2) = (\{1\}, \{0,1\}, \{2\}, \{4\}, \{4\}, \varnothing).$$

Then $\{G_0, G_1\}$ are isomorphic to C_6, G_2 is isomorphic to G and

$$rd_\Omega\text{-matrix} = \begin{bmatrix} \{0,1\} & \{2,5,3,1\} & \{3,5,0,2,1,4\} \\ \{2,5,3,1\} & \{1,4\} & \{1,4,3,5\} \\ \{3,5,0,2,1,4\} & \{4,3,0,2,5\} & \{1\} \end{bmatrix}$$

Since every cell of the $rd_\Omega-$matrix is a set satisfying Lemma 3.3, then Ω is a $\left(K_{6,6}, \{C_6, G\}, 3; K_2\right)$-GD generator. Applying Proposition 3.5 completes the proof. ■

Lemma 5.11 *Let* $t \geq 1$ *be a positive integer. There exists a* $(K_{8t,8t}, \{C_6\}, 3; K_2)$*-GD.*

Proof. For $n = 8$, define $\Omega = \{G_0, G_1, G_2, G_3\}$ as.

$$\psi(G_0) = (\varnothing, \varnothing, \{6\}, \varnothing, \varnothing, \{0\}, \{1\}, \{0,1,6\}), \quad \psi(G_1) = (\{0,1,6\}, \{0\}, \{6\}, \varnothing, \varnothing, \{1\}, \varnothing, \varnothing),$$
$$\psi(G_2) = (\varnothing, \{0\}, \{1\}, \{0,1,6\}, \varnothing, \varnothing, \{6\}, \varnothing), \quad \psi(G_3) = (\varnothing, \{1\}, \varnothing, \varnothing, \{0,1,6\}, \{0\}, \{6\}, \varnothing).$$

Then all graphs in Ω are isomorphic to C_6 and

$$rd_{\Omega}\text{-matrix} = \begin{bmatrix} \{1,2,3\} & \{0,1\} & \{5,-5\} & \{0,5\} \\ \{0,1\} & \{1,2,3\} & \{0,5\} & \{1,-1\} \\ \{5,-5\} & \{0,5\} & \{1,2,3\} & \{0,1\} \\ \{0,5\} & \{1,-1\} & \{0,1\} & \{1,2,3\} \end{bmatrix}$$

Since every cell of the rd_{Ω}-matrix is a set satisfying Lemma 3.3, then Ω is a $\left(K_{8,8}, \{C_6\}, 3; K_2\right)$-GD generator. Applying Proposition 3.5 completes the proof. ■

Lemma 5.12 *Let* $t \geq 1$ *be a positive integer and* G *is a graph containing a cycle* C_4 *in addition to an edge* K_2*such that they share a vertex. There exists* $(K_{5t,5t}, \{C_4 \cup K_2, G\}, 3; K_2)$*-GD.*

Proof. For $n = 5$, define $\Omega = \{G_0, G_1, G_2\}$ as:

$$\psi(G_0) = (\{0,3\}, \varnothing, \{3\}, \{0\}, \{2\}), \quad \psi(G_1) = (\{3\}, \{2,3\}, \{2,0\}, \varnothing, \varnothing),$$
$$\psi(G_2) = (\varnothing, \{2\}, \varnothing, \{3,4\}, \{2,4\}).$$

Then $\{G_0, G_1\}$ are isomorphic to C_4, G_2 is isomorphic to G and

$$rd_{\Omega}\text{-matrix} = \begin{bmatrix} \{3\} & \{3,0,4,2\} & \{3,4,0,2\} \\ \{3,0,4,2\} & \{1,3\} & \{0,4\} \\ \{3,4,0,2\} & \{0,4\} & \{1,2\} \end{bmatrix}$$

Since every cell of the rd_{Ω}-matrix is a set satisfying Lemma 3.3, then Ω is a $\left(K_{5,5}, \{C_4 \cup K_2, G\}, 3; K_2\right)$-GD generator. Applying Proposition 3.5 completes the proof. ■

Lemma 5.13 *Let* $t \geq 1$ *be a positive integer. There exists a* $(K_{5t,5t}, \{P_6\}, 3; K_2)$*-GD.*

Proof. For $n = 5$, define $\Omega = \{G_0, G_1, G_2\}$ as:

$$\psi(G_0) = (\{0,3\}, \{3,4\}, \varnothing, \{0\}, \varnothing), \quad \psi(G_1) = (\varnothing, \{2\}, \{0,2\}, \varnothing, \{0,4\}),$$
$$\psi(G_2) = (\{4\}, \varnothing, \{4\}, \{0,3\}, \{0\}).$$

Then all graphs in Ω are isomorphic to P_6 and

$$rd_{\Omega}\text{-matrix} = \begin{bmatrix} \{3,1\} & \{4,3\} & \{4,1,0,3\} \\ \{4,3\} & \{2,4\} & \{4,2,0,1\} \\ \{4,1,0,3\} & \{4,2,0,1\} & \{3\} \end{bmatrix}$$

Since every cell of the rd_{Ω}-matrix is a set satisfying Lemma 3.3, then Ω is a $\left(K_{5,5}, \{P_6\}, 3; K_2\right)$-GD generator. Applying Proposition 3.5 completes the proof.

Lemma 5.14 *Let* $t \geq 1$ *be a positive integer. There exists a* $(K_{5t,5t}, \{P_6, P_4 \cup 2K_2\}, 3; K_2)$*-GD.*

Proof. For $n = 5$, define $\Omega = \{G_0, G_1, G_2\}$ as.

$\psi(G_0) = (\{0,1\}, \{0\}, \{1\}, \varnothing, \{4\})$, $\psi(G_1) = (\varnothing, \{3\}, \{1\}, \{0,1\}, \{3\})$,

$\psi(G_2) = (\{4\}, \{1\}, \{3\}, \{0\}, \{4\})$.

Then $\{G_0, G_1\}$ are isomorphic to P_6, G_2 is isomorphic to $P_4 \cup 2K_2$ and

$$rd_\Omega\text{-matrix} = \begin{bmatrix} \{1\} & \{3,0,4\} & \{4,3,1,2,0\} \\ \{3,0,4\} & \{1\} & \{3,2,0,4,1\} \\ \{4,3,1,2,0\} & \{3,2,0,4,1\} & \varnothing \end{bmatrix}$$

Since every cell of the rd_Ω-matrix is a set satisfying Lemma 3.3, then Ω is a $\left(K_{5,5}, \{P_6, P_4 \cup 2K_2\}, 3; K_2\right)$-GD generator. Applying Proposition 3.5 completes the proof. ■

5.3 Graph Designs $(K_{n,n}, B, 4; K_2)$*-GD's*

Lemma 5.15 *Let* $t \geq 1$ *be a positive integer. There exists a* $(K_{5t,5t}, \{P_4 \cup P_3, P_5 \cup K_2\}, 4; K_2)$*-GD.*

Proof. For $n = 5$, define $\Omega = \{G_0, G_1, G_2, G_3\}$ as:

$\psi(G_0) = (\{0,2\}, \varnothing, \{4\}, \{2,3\}, \varnothing)$, $\psi(G_1) = (\{0,1\}, \{2,4\}, \{1\}, \varnothing, \varnothing)$,

$\psi(G_2) = (\varnothing, \{4,0\}, \varnothing, \{4\}, \{0,3\})$, $\psi(G_3) = (\varnothing, \varnothing, \{0,2\}, \{2\}, \{0,4\})$.

Then $\{G_0, G_1, G_2\}$ are isomorphic to $P_4 \cup P_3$, G_3 is isomorphic to $P_4 \cup P_3$ and

$$rd_\Omega\text{-matrix} = \begin{bmatrix} \{2,1\} & \{0,3,1,4,2\} & \{2,1\} & \{1,3,0,4\} \\ \{0,3,1,4,2\} & \{1,2\} & \{2,0,3,1\} & \{2,4\} \\ \{2,1\} & \{2,0,3,1\} & \{1,3\} & \{4,0,2,1,3\} \\ \{1,3,0,4\} & \{2,4\} & \{4,0,2,1,3\} & \{2,4\} \end{bmatrix}$$

Since every cell of the rd_Ω-matrix is a set satisfying Lemma 3.3, then Ω is a $\left(K_{5,5}, \{P_4 \cup P_3, P_5 \cup K_2\}, 4; K_2\right)$-GD generator. Applying Proposition 3.5 completes the proof. ■

Lemma 5.16 *Let* $t \geq 1$ *be a positive integer. There exists a* $(K_{5t,5t}, \{P_4 \cup K_2\}, 4; K_2)$*-GD.*

Proof. For $n = 5$, define $\Omega = \{G_0, G_1, G_2, G_3\}$ as:

$\psi(G_0) = (\{0,2\}, \varnothing, \varnothing, \{1,2\}, \varnothing)$, $\psi(G_1) = (\{0,1\}, \varnothing, \varnothing, \varnothing, \{1,3\})$,

$\psi(G_2) = (\varnothing, \{2,3\}, \varnothing, \{0,2\}, \varnothing)$, $\psi(G_3) = (\varnothing, \varnothing, \{0,2\}, \varnothing, \{0,4\})$,

$\psi(G_4) = (\varnothing, \{0,2\}, \{1,2\}, \varnothing, \varnothing)$.

Then all graphs in Ω are isomorphic to $P_4 \cup K_2$ and

$$rd_\Omega\text{-matrix} = \begin{bmatrix} \{2,1\} & \{0,3,1,4\} & \varnothing & \varnothing & \varnothing \\ \{0,3,1,4\} & \{1,2\} & \varnothing & \{4,2,3,1\} & \varnothing \\ \varnothing & \varnothing & \{1,2\} & \varnothing & \{3,2,0,4\} \\ \varnothing & \{4,2,3,1\} & \varnothing & \{2,4\} & \{1,4,2,0\} \\ \varnothing & \varnothing & \{3,2,0,4\} & \{1,4,2,0\} & \{2,1\} \end{bmatrix}$$

Since every cell of the rd_Ω-matrix is a set satisfying Lemma 3.3, then Ω is a $\left(K_{5,5}, \{P_4 \cup K_2\}, 4; K_2\right)$-GD generator. Applying Proposition 3.5 completes the proof. ■

6 Graph Designs $(H, B, \lambda; K_2)$-GD's

Definition 6.1 *Let $\Omega = \{G_0, G_1, \ldots, G_g\}$ be a collection of spanning subgraphs of $H =$ r-regular $Circ(\mathbb{Z}_n, A = A^+ \cup -A^+)$ where $A^+ = A \cap \{1, 2, \ldots, \lfloor n/2 \rfloor\}$. We call Ω a $(H, B, \lambda; K_2)$-GD generator if it satisfies the following conditions:*

(1) *Every element of A^+ appears exactly λ times in the sum of the multisets $L(G_i)$, $i = 0, 1, 2, \ldots, g-1$.*
(2) *For all pairs i,j with $0 \le i \le j \le g-1$, the cells of the rd_Ω matrix are sets, that is $D(G_i, G_j)$ are all sets.*

The elements of the generator Ω are called $(H, B, \lambda; K_2)$-GD *pre-starters* graphs.

Theorem 6.2 *Let $\Omega = \{G_0, G_1, \ldots, G_g\}$ be a $(H, B, \lambda; K_2)$-GD generator. Then for all $0 \le i \le g-1$, the collection of all the translates of $G_i + \alpha$ for all $x \in \mathbb{Z}_n$, forms a $(H, B, \lambda; K_2)$-GD by B.*

Proof. It is clear that the collection of all translates covers every edge of H exactly λ times. Now, It is to show that the collection of all translates are mutually orthogonal, that is any two graphs of the collection of all translates share at most one edge. Consider two translates $G_i + \alpha$ and $G_j + \beta$ where $\alpha, \beta \in \mathbb{Z}_n$ and assume that they share two edges $e_1 = (x, y)$ with length $l_1 = y - x$ and $e_2 = (u, v)$ with length $l_2 = v - u$. Then the two edges $(x - \alpha, y - \alpha)$, $(u - \alpha, v - \alpha) \in G_i$ with lengths l_1, l_2 respectively and $(x - \beta, y - \beta)$, $(u - \beta, v - \beta) \in G_j$ with lengths l_1, l_2 respectively. Then the distance between the two edges with length l_1 in G_i and G_j is $\alpha - \beta$, and also the distance between the two edges with length l_1 in G_i and G_j is $\alpha - \beta$ and then $D(G_i, G_j)$ is not a set. This is a contradiction of the second condition in the Definition 6.1 of the $(H, B, \lambda; K_2)$-*GD* generator. Consequently, all subgraphs in the collection of all translates of *GD*-generator are mutually orthogonal, that is a $(H, B, \lambda; K_2)$-GD. ■

Lemma 6.3 *Let $\Omega = \{G_0, G_1, \ldots, G_{g-1}\}$ be a $(H, B, \lambda; K_2)$-GD generator, then*

(i) the number of pre-starters in Ω is$g = \lambda nr/2e$,
(ii) For all $0 \le i \le g-1$, if $d \in D(G_i, G_i)$ then $-d \notin D(G_i, G_i)$,
(iii) For all $0 \le i \le g-1$, if n is even then $n/2 \notin D(G_i, G_i)$.

Proof. (i) $\Sigma = \{G_0, G_1, \ldots, G_s\} = (H, B, \lambda; K_2)$-GD. Since the $s = gn$ then $gne = \lambda nr/2$ and hence $g = \lambda r/2e$.

(ii) Let $D(G_i, G_i)$ contains $\pm d$ then G_i contains four edges each pair of them has the same length l_1 and l_2, that is $(x, x + l_1)$, $(x + d, x + d + l_1)$, $(u, u + l_2)$, $(u - d, u - d + l_2) \in G_i$.

Then $G_i + d$ contains $(x + d, x + d + l_1)$, $(x + 2d, x + 2d + l_1)$, $(u + d, u + d + l_2)$, $(u, u + l_2)$ which imply that $|G_i \cap G_i + d| > 1$ which is a contradiction. Hence, for all $0 \le i \le g-1$, if $d \in D(G_i, G_i)$ then $-d \notin D(G_i, G_i)$

(iii) For any $0 \le i \le g$, let $n/2 \in D(G_i, G_i)$.

So there exist two edges $e_1 = (x, x + l)$, $e_2 = (x + n/2, x + n/2 + l)$ belong to $E(G_i)$ with the same length l and $D(e_1, e_2) = n/2$.

Then $G_i + n/2$ contains also $e_1 = (x, x + l)$, $e_2 = (x + n/2, x + n/2 + l)$ that means $|G_i \cap G_i + n/2| > 1$ which is a contradiction. Hence, for all $0 \le i \le g-1$, if n is even then $n/2 \notin D(G_i, G_i)$.

Therefore, $\lambda r/2 \equiv 0 \, mod\{e\}$ is a necessary condition of the existence of the $(H, B, \lambda; K_2)$-*GD* generator. ■

Proposition 6.4 *Let* $m \geq 2$ *and* $n \geq 2m+1$ *be integers and let* $H = 2m$*-regular* $Circ(\mathbb{Z}_n, A)$ *where* $A = A^+ \cup -A^+)$
where $A^+ = A \cap \{1, 2, \ldots, \lfloor n/2 \rfloor\} = \{l_0, l_1, \ldots, l_{m-1}\}$. *Then there exists* $(H, \{P_4\}, 3; K_2)$*-GD.*

Proof. Define $\Omega = \{G_0, G_1, \ldots, G_{m-1}\}$ as:

For all $j \in \mathbb{Z}_m$ and for all $i \in \mathbb{Z}_m$

$$\psi_{l_i}(G_j) = \begin{cases} \{0, l_{j+1}\} & \text{if } i = j, \\ \{l_j\} & \text{if } i = j+1, \\ \varnothing & \text{otherwise.} \end{cases}$$

Then all graphs in Ω are isomorphic to P_4 and

$$D(G_i, G_j) = \begin{cases} \left\{l_{j+1}\right\} & \text{if } i = j, \\ \left\{-l_j, l_{j+2} - l_j\right\} & \text{if } i = j-1, \\ \varnothing & \text{otherwise.} \end{cases}$$

Since every cell of the rd_Ω-matrix is a set satisfying Lemma 6.3, then Ω is a $(H, \{P_4\}, 3; K_2)$-GD generator. ■

Proposition 6.5 *Let* $n \geq 9$ *be an integer and let* $H = 8$*-regular* $Circ(\mathbb{Z}_n, A)$ *where* $A = A = A^+ \cup -A^+)$ *where* $A^+ = A \cap \{1, 2, \ldots, \lfloor n/2 \rfloor\} = \{l_1, l_2, l_3, l_4\}$. *Then there exists* $(H, \{P_5\}, 3; K_2)$*-GD.*

Proof. Define $\Omega = \{G_0, G_1, G_2\}$ as:

$$\psi_i(G_0) = \begin{cases} \{0, l_1, l_4 - l_1\} & \text{if } i = l_1, \\ \{0\} & \text{if } i = l_4, \\ \varnothing & \text{otherwise.} \end{cases}$$

$$\psi_i(G_1) = \begin{cases} \{0, l_2, l_3 - l_2\} & \text{if } i = l_2, \\ \{0\} & \text{if } i = l_3, \\ \varnothing & \text{otherwise.} \end{cases}$$

$$\psi_i(G_2) = \begin{cases} \{0, l_3 + l_4\} & \text{if } i = l_3, \\ \{0, l_3\} & \text{if } i = l_4, \\ \varnothing & \text{otherwise.} \end{cases}$$

Then all graphs in Ω are isomorphic to P_5 and

$$rd_\Omega\text{-matrix} = \begin{bmatrix} \{l_1, l_4 - l_1, l_4 - 2l_1\} & \varnothing & \{0, l_3\} \\ \varnothing & \{1\} & \{0, l_3 + l_4\} \\ \{0, l_3\} & \{0, l_3 + l_4\} & \varnothing \end{bmatrix}$$

Since every cell of the rd_Ω-matrix is a set satisfying Lemma 6.3, then Ω is a $(H, \{P_5\}, 3; K_2)$-GD generator. ■

Proposition 6.6 *Let* $n \geq 7$ *be a positive integer and* H *be* 4*-regular* $Circ(\mathbb{Z}_n, A)$ *where* $A = A = A^+ \cup -A^+)$ *where* $A^+ = A \cap \{1, 2, \ldots, \lfloor n/2 \rfloor\} = \{l_1, l_2\}$ *such that* $\{l_2, l_1 - l_2, l_1 - 2l_2\}$, $\{l_1, l_1 + l_2, 2l_1 + l_2\}$, $\{0, -l_1, l_1 + l_2, l_1, l_1 - l_2, l_2\}$, $\{l_1 - l_2, l_1, 0, l_2, 2l_2\}$ *and* $\{-l_1, 0, l_1, l_1 + l_2, 2l_1 + l_2\}$ *are all sets* (i.e., *all have different elements*). *Then there exists* $(H, \{P_5\}, 4; K_2)$*-GD.*

Proof. Define $\Omega = \{G_0, G_1\}$ as

$$\psi_i(G_0) = \begin{cases} \{0, -l_1, l_1 + l_2\} & \text{if } i = l_1, \\ \{l_1\} & \text{if } i = l_2, \\ \varnothing & \text{otherwise.} \end{cases}$$

$$\psi_i(G_1) = \begin{cases} \{0\} & \text{if } i = l_1 \\ \{0, l_2, l_1 - l_2\} & \text{if } i = l_2, \\ \varnothing & \text{otherwise.} \end{cases}$$

Since $\{l_1 - l_2, l_1, 0, l_2, 2l_2\}$ and $\{-l_1, 0, l_1, l_1 + l_2, 2l_1 + l_2\}$ are sets then all graphs in Ω are isomorphic to P_5 and

$$rd_\Omega\text{-matrix} = \begin{bmatrix} \{l_2, l_1 - l_2, l_1 - 2l_2\} & \{0, -l_1, l_1 + l_2, l_1, l_1 - l_2, l_2\} \\ \{0, -l_1, l_1 + l_2, l_1, l_1 - l_2, l_2\} & \{l_1, l_1 + l_2, 2l_1 + l_2\} \end{bmatrix}$$

Since every cell of the rd_Ω-matrix is a set satisfying Lemma 6.3, then Ω is a $(H, \{P_5\}, 4; K_2)$-GD generator. For illustration, at $n = 7$ take $l_1 = 1$ and $l_2 = 3$.

Table 3: New graph designs

H	B	λ	F	H	B	λ	F
$K_{6t,6t}$	$\{C_6\}$	2	K_2	$K_{5t,5t}$	$\{P_6, P_4 \cup 2K_2\}$	3	K_2
$K_{10t,10t}$	$\{C_{10}\}$	2	K_2	$K_{6t,6t}$	$\{C_6, P_5 \cup 2P_2\}$	3	K_2
$K_{8t,8t}$	$\{C_4\}$	3	K_2	$K_{6t,6t}$	$\{C_6, G\}$	3	K_2
$K_{6t,6t}$	$\{P_7\}$	3	K_2	$K_{8t,8t}$	$\{C_6\}$	3	K_2
$K_{6t,6t}$	$\{C_4 \cup S_2\}$	3	K_2	$K_{n,n}$	$\{P_4\}$	3	K_2
$K_{6t,6t}$	$\{C_6, P_4 \cup P_3 \cup P_2\}$	3	K_2	$K_{5t,5t}$	$\{C_4 \cup K_2, G\}$	3	K_2
$K_{6t,6t}$	$\{C_6, P_5 \cup 2P_2\}$	3	K_2	$K_{5t,5t}$	$\{P_6\}$	3	K_2
$K_{8t,8t}$	$\{C_6\}$	3	K_2	$K_{5t,5t}$	$\{P_6, P_4 \cup 2K_2\}$	3	K_2
$K_{n,n}$	$\{P_4\}$	3	K_2	$K_{5t,5t}$	$\{P_4 \cup P_3, P_5 \cup K_2\}$	3	K_2
$K_{5t,5t}$	$\{P_6\}$	3	K_2	$K_{5t,5t}$	$\{P_4 \cup K_2\}$	3	K_2
$K_{5t,5t}$	$\{P_6, P_4 \cup 2K_2\}$	3	K_2	H	$\{P_4\}$	3	K_2
H	$\{P_5\}$	4	K_2	H	$\{P_5\}$	3	K_2

7 Conclusion

In this paper, we have studied the group generated graph designs. A new representation of graphs has been proposed that help in constructing new graph designs $(H, \mathrm{B}, \lambda; \mathrm{F})$-GD that can be summerized in Tab. 3. Where H is certain circulant graph. In addition, an efficient coding method has been proposed using the constructed graph designs which may open a new door to produce more research in this area. Finally, we can state that the constructed GD's can be efficiently used to generate a code set.

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