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ARTICLE

Power Aggregation Operators and Similarity Measures Based on Improved Intuitionistic Hesitant Fuzzy Sets and their Applications to Multiple Attribute Decision Making

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ABSTRACT

Intuitionistic hesitant fuzzy set (IHFS) is a mixture of two separated notions called intuitionistic fuzzy set (IFS) and hesitant fuzzy set (HFS), as an important technique to cope with uncertain and awkward information in realistic decision issues. IHFS contains the grades of truth and falsity in the form of the subset of the unit interval. The notion of IHFS was defined by many scholars with different conditions, which contain several weaknesses. Here, keeping in view the problems of already defined IHFSs, we will define IHFS in another way so that it becomes compatible with other existing notions. To examine the interrelationship between any numbers of IHFSs, we combined the notions of power averaging (PA) operators and power geometric (PG) operators with IHFSs to present the idea of intuitionistic hesitant fuzzy PA (IHFPA) operators, intuitionistic hesitant fuzzy PG (IHFPG) operators, intuitionistic hesitant fuzzy power weighted average (IHFPWA) operators, intuitionistic hesitant fuzzy power ordered weighted average (IHFPOWA) operators, intuitionistic hesitant fuzzy power ordered weighted geometric (IHFPOWG) operators, intuitionistic hesitant fuzzy power hybrid average (IHFPHA) operators, intuitionistic hesitant fuzzy power hybrid geometric (IHFPHG) operators and examined as well their fundamental properties. Some special cases of the explored work are also discovered. Additionally, the similarity measures based on IHFSs are presented and their advantages are discussed along examples. Furthermore, we initiated a new approach to multiple attribute decision making (MADM) problem applying suggested operators and a mathematical model is solved to develop an approach and to establish its common sense and adequacy. Advantages, comparative analysis, and graphical representation of the presented work are elaborated to show the reliability and effectiveness of the presented works.

KEYWORDS

Intuitionistic fuzzy sets; intuitionistic hesitant fuzzy sets; power aggregation operators; similarity measures; multiple attribute decision making

1 Introduction

In modern decision science, multi-attribute decision making (MADM) is a vital investigation area on how to choose the correct option corresponding to many prominent attributes [1–3]. Usually, the decision-makers (DMs) utilize crisp figures to express the favorites regarding the alternative in conventional multi-attribute decision making difficulties. But, because of shortage of data, lack of time, deficiency of information and quality values, particularly, for subjective attribute values, usually may not be shown by real numbers, and few of them are simpler to be stated by fuzzy data. Since Zadeh [4] introduced the notion of fuzzy set, several expansions of fuzzy sets (FS) were presented by scholars [5–7]. A FS contains an ordered pair of an element and a membership (MS) function, which gives grade of MS to every component of universal set X in the closed interval from 0 to 1. The model of fuzzy set is applied in many areas, mainly wherever traditional numerical methods restrict effectiveness, involving organic and social sciences, linguistics, psychology and mostly soft sciences. In these areas, variables are hard to evaluate and conditions among variables are so ill-defined. Further, Atanassov [8,9] gave the idea of intuitionistic fuzzy set IFS in 1986. IFS is an expansion of FS to cope with doubtful and complicated data. In IFS every object is indicated by an ordered pair set where every ordered pair set described a grade of MS as well as a grade of NMS.

The total of the grade of MS and the grade of NMS of each ordered pair set is smaller than or equivalent to 1 and greater than or equivalent to 0. The IFS has been receiving more consideration since its arrival [10–20]. Intuitionistic fuzzy set is extra influential in managing with vagueness than fuzzy set which only provides a grade of MS to every component. Undoubtedly, IF data aggregation performs a crucial part in intuitionistic fuzzy set, that is an attractive study direction. Zhao et al. [21] established few elementary arithmetic aggregation operators, whereas IF weighted averaging operator, IF ordered weighted averaging operator and IF hybrid averaging operator for aggregating IFSs. Xu et al. [22] established few basic geometric aggregation operators whereas IF weighted geometric operator, IF ordered weighted geometric operator, and IF hybrid geometric operator and enforced them to MADM established on IFS. Furthermore, Torra et al. [23,24] presented the hesitant fuzzy set HFS. An HFS is a direct simplification of FS. The theory of hesitant fuzzy set is extensively utilized in many problems. Many researchers gave a serious analysis on HF information aggregation methods and their implications in decision making [25–32].

An HFS allows the MS taking a set of conceivable values for example, in order to obtain a sensible decision outcome, a decision association, containing many DMs, which is approved to assess the grade that an alternative should fulfill a criterion. Consider there are three situations, few DMs offer 0.2, few offer 0.4, and the rest offer 0.9, and these units may not convince one another, thus the grade that the alternative should fulfill the criterion can be signified by an HF $\{0.2, 0.4, 0.9\}$. It is observed that the HF $\{0.2, 0.4, 0.9\}$ may define the above condition more quantitatively than the interval-valued FS $[0.2, 0.4]$, due to grades that alternatives fulfil the condition out of the convex of 0.2 and 0.9 or the interval between 0.2 and 0.9. Thereafter, several multi attribute decision making techniques [33–38] and procedures containing relationship, distance, and similarity have offered for hesitant fuzzy set by various investigators. Liao et al. [39,40] introduced the subtraction and division operations, hybrid arithmetical averaging for hesitant fuzzy sets, and hybrid arithmetical geometric for HFSs. Zhang [41] introduced power aggregation operators for HFS. In everyday life, DMs would think ranking among unlike conditions. To manage this type of position, Yager [42] established PA operator and implements it to multi-attribute decision making difficulties. Liu et al. [43] introduced POWA operator to manage the fuzzy data.

Mostly, this is noted that one fuzzy framework is not enough to deal with practical problems. There is a common trend of combining two or more fuzzy frameworks. Therefore, by mixing IFS and HFS established the theory of IHFS. IHFS is also described by the grade of MS and the grade of NMS, whose summation is smaller than or equivalent to 1 and greater than or equal to 0. IHFS has emerged as a powerful instrument for illustrating vagueness of the MADM difficulties. The determination of the article is to present the idea of power aggregation operators based on IHFS by combining the theory IFS and HFS. We found that two different definitions of IHFS which were proposed by Beg et al. [44] and Geetha et al. [45] are not compatible with the other existing notions. So, to make the IHFS compatible with the other existing notions we have defined IHFS in another way. In some situations the theories of HFS and IFS cannot deal effectively, for instance, when a decision maker gives $\{0.6, 0.4, 0.3\}$ for the grade of truth and $\{0.3, 0.2, 0.1\}$ for the grade of falsity then the condition of proposed improved IHFS is an important technique to cope with uncertain and unreliable information in realistic decision issues. The conditions of Beg et al. [44] and Geetha et al. [45] are that the sum of maximum (also for minimum) of the truth grade and the minimum (also for maximum) of the falsity grade cannot exceed from unit interval and the sum of the maximum of the truth grade and the maximum of the falsity exceeds from unit interval. To resolve such kinds of issues, we redefined the theory of IHFS with the new condition that the sum of the maximum of the truth grade and the maximum of the falsity grade cannot exceed from unit interval. Additionally, we have established the sequence of IHF power aggregation operators, which has weighting vectors varing by input reasons as well as permit data being aggregated to assist everyone and examine the required characteristics. Motivation and achievements of the article are shown as follows:

1. Enlarge several PA operators, as IHFPA operator, IHFPWA operator, IHFPOWA operator, IHFPHA operator, IHFPG operator, IHFPWG operator, IHFPOWG operator, IHFPHG operator and check their characteristics.
2. Explore the similarity measures based on IHFSs and justified with the help of numerical example.
3. Describe a new DM method consists over the proposal operations.
4. Provide some numerical to demonstrate the reliability and supremacy of described techniques.

The making of article is followed as in portion 2, it gives few fundamental notions as well as in this section we reviewed the definition of IHFS which are established by Beg et al. [44] and Geetha et al. [45]. In Section 3, we established few IHF power aggregation operators and calculated their suitable characteristics. In Section 4, we explored the similarity measures based on IHFSs. In Section 5, we utilized these operators to establish few forms for multi attribute decision making challenges founded by IHFPWA operator and IHFPWG operator with intuitionistic hesitant fuzzy data. Additionally, we mentioned a practical problem for examining efficiency of the suggested operators. In Section 6, we summarized this article and wrote few comments.

2 Another View of Intuitionistic Hesitant Fuzzy Sets

In this study, we review the idea of IHFS which was established by Beg et al. [44] and established by Geetha et al. [45]. Then we redefine IHFS to make it compatible with other existing notions [46].

Definition 1: [44] An IHFS on X are functions μ and v that when applied to X return the subsets of $[0,1]$, which can be represented as the following:

$$P = \{(x, \mu(x), v(x)) \mid x \in X\} \tag{1}$$

where μ(x) and $v(x)$ are sets of some values in $[0,1]$, denoting the possible membership degrees and non-membership degrees of the element $x \in X$ to the set P with the conditions: $max(\mu(x)) + min(v(x)) \leq 1$ and $min(\mu(x)) + max(v(x)) \leq 1$. For convenience, $(\mu(x), v(x))$ is an intuitionistic hesitant fuzzy element (IHFE).

Definition 2: [45] An intuitionistic hesitant fuzzy set P on X is represented by using the two functions μ and v. Mathematically, it is represented by following expression:

$$P = \{(x, \mu(x), v(x)) \mid x \in X\} \tag{2}$$

where μ(x) and $v(x)$ are sets of some values in $[0,1]$, denoting the possible membership degrees and non-membership degrees of the element $x \in X$ to the set P with the condition that $0 \leq max(\mu(x)) + max(v(x)) \geq 1$. For convenience, $(\mu(x), v(x))$ is an intuitionistic hesitant fuzzy element (IHFE).

Definition 3: An intuitionistic hesitant fuzzy set P on X is represented by using the two functions μ and v. Mathematically, it is represented by following expression:

$$P = \{(x, \mu(x), v(x)) \mid x \in X\} \tag{3}$$

where μ(x) and $v(x)$ are sets of some values in $[0,1]$, denoting the possible membership degrees and non-membership degrees of the element $x \in X$ to the set E with the condition that $0 \leq max(\mu(x)) + max(v(x)) \leq 1$. For convenience, $(\mu(x), v(x))$ an intuitionistic hesitant fuzzy element (IHFE). In this manuscript we will follow throughout the IHFS:

$$P = \{(x, \mu(x), v(x)) \mid x \in X\} \tag{4}$$

satisfying $0 \leq max(\mu(x)) + max(v(x)) \leq 1$.

Definition 4: For any IHFE $P = (\mu_P, v_P)$, the score function and accuracy function are stated by:

$$S(P) = \frac{(S(\mu_P) - S(v_P))}{2} \tag{5}$$

$$H(P) = \frac{(S(\mu_P) + S(v_P))}{2} \tag{6}$$

where $(\mu_P) = \frac{sum\ of\ all\ elements\ in\ (\mu_P)}{order\ of\ (\mu_P)}$, $S(v_P) = \frac{sum\ of\ all\ elements\ in\ (v_P)}{order\ of\ (v_P)}$, $S(P) \in [-1,1]$, $H(P) \in [0,1]$.

Definition 5: For any two IHFEs $P_1 = (\mu_1, v_1)$ and $P_2 = (\mu_2, v_2)$, then

$$P_1 \oplus P_2 = \bigcup_{\begin{pmatrix} l_1 \varepsilon \mu_1 \\ l_2 \varepsilon \mu_2 \\ m_1 \varepsilon v_1 \\ m_2 \varepsilon v_2 \end{pmatrix}} (\{l_1 + l_2 - l_1 l_2\}, \{m_1 m_2\}) \tag{7}$$

$$P_1 \otimes P_2 = \bigcup_{\begin{pmatrix} l_1 \varepsilon \mu_1 \\ l_2 \varepsilon \mu_2 \\ m_1 \varepsilon v_1 \\ m_2 \varepsilon v_2 \end{pmatrix}} (\{l_1 l_2\}, \{m_1 + m_2 - m_1 m_2\}) \tag{8}$$

$$\lambda P_1 = \bigcup_{\begin{pmatrix} l_1 \varepsilon \mu_1 \\ m_1 \varepsilon v_1 \end{pmatrix}} \left(1 - (1 - l_1)^{\lambda}, {m_1}^{\lambda}\right), \quad \lambda > 0 \tag{9}$$

$$P_1^{\lambda} = \bigcup_{\begin{pmatrix} l_1 \varepsilon \mu_1 \\ m_1 \varepsilon v_1 \end{pmatrix}} \left({l_1}^{\lambda}, 1 - (1 - m_1)^{\lambda}\right), \quad \lambda > 0 \tag{10}$$

$$P_1^c = (l_1, m_1) \tag{11}$$

Definition 6: Power aggregation (PA) operator is defined as:

$$PA\,(P_1, P_2, \ldots P_n) = \frac{\left(\sum_{i=1}^{n} (1 + T(P_i))P_i\right)}{\left(\sum_{i=1}^{n} (1 + T(P_i))\right)} \tag{12}$$

where

$$T\,(P_i) = \sum_{\substack{j=1 \\ j \neq i}}^{n} Sup(P_i, P_j) \tag{13}$$

and $Sup(P_1, P_2)$ is the Sup for P_1 from P_2, which meets the given properties:

$Sup\,(P_1, P_2) \in [0, 1]$

$Sup\,(P_1, P_2) = Sup\,(P_2, P_1)$

$Sup(P_1, P_2) \geq Sup(X, Y), \quad if\ \ |P_1 - P_2| < |X - Y|$

The support (Sup) amount is basically a similarity indicator.

Definition 7: Power geometric (PG) operator is defined as:

$$PG(P_1, P_2, \ldots, P_n) = \prod_{i=1}^{n} {P_i}^{\frac{1+T(P_i)}{\sum_{i=1}^{n}(1+T(P_i))}} \tag{14}$$

Based upon intuitionistic hesitant fuzzy PA operators and PG, we will describe few IHFPG aggregation operators. Next, we will establish few intuitionistic hesitant fuzzy power arithmetic aggregation operators.

3 Intuitionistic Hesitant Fuzzy Power Aggregation Operators

The purpose of this section is to establish few novel aggregation operators for IHFSs which are IHFPA, IHFPG, IHFPWA, IHFPWG, IHFPOWA, IHFPOWG, IHFPHA, IHFPHG operators and verify their fundamental properties. The mentioned operators are not only developed in this section but also their characteristics have been studied and their fitness is established using induction phenomenon.

Definition 8: Suppose $P_j = \left(\mu_j, \nu_j\right)$ is a gathering of IHFSs, then we describe the IHFPA operator as follow:

$$IHFPA\,(P_1, P_2, \ldots, P_n) = \frac{\bigoplus_{j=1}^{n}((1 + T(P_j)P_j)}{\sum_{j=1}^{n}(1 + T(P_j))} \tag{15}$$

where

$$T\left(P_j\right) = \sum_{\substack{i=1\\ i\neq j}}^{n} Sup(P_j, P_i) \tag{16}$$

and $Sup(P_j, P_i)$ is the support for P_j from P_i, with the conditions:

1. $Sup(P_j, P_i) \in [0, 1]$;
2. $Sup(P_j, P_i) = Sup(P_i, P_j)$;
3. $Sup\left(P_j, P_i\right) \geq Sup\,(P_s, P_t)$ if $d\left(P_j, P_i\right) < d\,(P_s, P_t)$, wherever d be a distance measure.

If we will choose the grade of truth and falsity in the form of singleton sets then the Eq. (15) will be converted for intuitionistic fuzzy sets. Similarly, if we choose the values of falsity grade is zero then the Eq. (15) will be converted for hesitant fuzzy sets.

Theorem 1: The aggregated objects by utilizing intuitionistic hesitant fuzzy power average (IHFPA) operator is as well an IHFS, wherever

$$IHFPA\,(P_1, P_2, \ldots, P_n) = \frac{\bigoplus_{j=1}^{n}\left(\left(1 + T\left(P_j\right)\right)P_j\right)}{\sum_{j=1}^{n}\left(1 + T\left(P_j\right)\right)}$$
$$= \bigcup_{\substack{l_j \varepsilon \mu_j\\ m_j \varepsilon \nu_j}} \left(1 - \prod_{j=1}^{n}(1 - (l_j)^{\frac{(1+T(P_j)))}{\sum_{j=1}^{n}(1+T(P_j))}}, \prod_{j=1}^{n}(m_j)^{\frac{(1+T(P_j))}{\sum_{j=1}^{n}(1+T(P_j))}}\right) \tag{17}$$

where

$$T\left(P_j\right) = \bigcup_{\substack{\mu_j \varepsilon P_j\\ \nu_j \varepsilon P_j}} \left(\sum_{\substack{i=1\\ i\neq j}}^{n} Sup(P_j, P_i)\right) \tag{18}$$

If we will choose the grade of truth and falsity in the form of singleton sets then the Eq. (17) will be converted for intuitionistic fuzzy sets. Similarly, if we choose the values of falsity grade is zero then the Eq. (17) will be converted for hesitant fuzzy sets.

Definition 9: Let $P_j = \left(\mu_j, \nu_j\right)$ be a group of IHFS and $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is weight vector of P_j, $\omega_j > 0$ and $\sum_{j=1}^{n} \omega_j = 1$, $(j = 1, 2, 3 \ldots, n)$ The IHFPWA operator is a function IHFPWA: $P^n \to P$ where

$$IHFPWA_\omega(P_1, P_2, \ldots, P_n) = \frac{\bigoplus_{j=1}^{n} (\omega_j \left(1 + T\left(P_j\right) P_j\right)}{\sum_{j=1}^{n} \omega_j \left(1 + T\left(P_j\right)\right)}$$

$$= \bigcup_{\substack{l_j \varepsilon \mu_j \\ m_j \varepsilon \nu_j}} \left(1 - \prod_{j=1}^{n} (1 - (l_j)^{\frac{\omega_j((1+T(P_j))}{\sum_{j=1}^{n} \omega_j (1+T(P_j))}}, \prod_{j=1}^{n} \left(m_j\right)^{\frac{\omega_j (1+T(P_j))}{\sum_{j=1}^{n} \omega_j (1+T(P_j))}} \right) \tag{19}$$

where

$$T\left(P_j\right) = \bigcup_{\substack{\mu_j \varepsilon P_j \\ \nu_j \varepsilon P_j}} \left(\sum_{\substack{i=1 \\ i \neq j}}^{n} \omega_j Sup(P_j, P_i) \right) \tag{20}$$

If we will choose the grade of truth and falsity in the form of singleton sets then the Eq. (19) will be converted for intuitionistic fuzzy sets. Similarly, if we choose the values of falsity grade to be zero then the Eq. (19) will be converted for hesitant fuzzy sets.

Property 1: (Idempotency) When P_j are equivalent, $P_j = P$ for every $j\,(j = 1, 2, 3 \ldots, n)$, then

$$IHFPWA_\omega\left(\mathrm{P}_1, \mathrm{P}_2, \ldots, \mathrm{P}_\mathrm{n}\right) = P \tag{21}$$

Property 2: (Boundedness) Let P_j be a family of IHFSs, and allows

$$P^- = \overset{min}{j} P_j,\ P^+ = \overset{max}{j} P_j\,(j = 1, 2, \ldots, \mathrm{n})$$

then

$$P^- \le \mathrm{IHFPW}A_\omega\left(P_1, P_2, \ldots, P_n\right) \le P^+ \tag{22}$$

Property 3: (Monotonicity) Let P_j and P_j' be two sets of intuitionistic hesitant fuzzy sets (IHFSs), if $P_j \le P_j'$ for all j, then

$$\mathrm{IHFPW}A_\omega\left(P_1, P_2, \ldots, P_n\right) \le \mathrm{IH}FPWA_\omega\left(P_1', P_2', \ldots, P_n'\right) \tag{23}$$

Further, we give an IHFPOWA operator as follows:

Definition 10: Suppose $P_j = (\mu_j, \nu_j)$ is family of IHFSs, the IHFPOWA operator of dimension n a function IHFPOWA: $P^n \to P$, associated with weight vector $\omega = (\omega_1, \omega_2, \ldots, \omega_j)^T$ such that $\omega_j > 0$ and $\sum_{j=1}^{n} \omega_j = 1$. Furthermore

$$\text{IHFPOW}A_\omega(P_1,P_2,\ldots,P_n)=\frac{\bigoplus_{j=1}^{n}(\omega_j(1+T(P_{\sigma(j)})P_{\sigma(j)})}{\sum_{j=1}^{n}\omega_j(1+T(P_{\sigma(j)}))}$$
$$=\bigcup_{\substack{l_{\sigma(j)}\varepsilon\mu_j\\ m_{\sigma(j)}\varepsilon v_j}}\left(1-\prod_{j=1}^{n}(1-(l_{\sigma(j)})^{\frac{(\omega_j(1+T(P_{\sigma(j)}))}{\sum_{j=1}^{n}\omega_j(1+T(P_{\sigma(j)}))}},\prod_{j=1}^{n}\left(m_{\sigma(j)}\right)^{\frac{(\omega_j(1+T(P_{\sigma(j)}))}{\sum_{j=1}^{n}\omega_j(1+T(P_{\sigma(j)}))}}\right) \tag{24}$$

where $\sigma(1),\sigma(2),\ldots,\sigma(n)$ indicates permutation of $(1,2,\ldots,n)$, where $P_{\sigma(j-1)}\geq P_{\sigma(j)}$, ω_j $(j=1,2,\ldots,n)$ is family of weights in such a way that

$$\omega_j=g\left(\frac{R_j}{TV}\right)-g\left(\frac{R_{j-1}}{TV}\right),\quad R_j=\sum_{i=1}^{j}V_{\sigma(i)},\quad TV=\sum_{i=1}^{n}V_{\sigma(i)},\quad V_{\sigma(i.)}=1.+T(P_{\sigma(i)}) \tag{25}$$

where $T(P_{\sigma(i)})$ implies the *Sup* of jth main IHFS $T(P_{\sigma(i)})$ by all the other (IHFSs), that is,

$$T\left(P_{\sigma(i)}\right)=\bigcup_{\substack{\mu_{\sigma(j)}\varepsilon P_j\\ v_{\sigma(j)}\varepsilon P_j}}\left(\sum_{\substack{i=1\\ i\neq j}}^{n}Sup(P_{\sigma(j)},P_{\sigma(i)})\right) \tag{26}$$

where $\bigcup_{\substack{\mu_{\sigma(j)}\varepsilon P_j\\ v_{\sigma(j)}\varepsilon P_j}}\left(\sum_{\substack{i=1\\ i\neq j}}^{n}Sup(P_{\sigma(j)},P_{\sigma(i)})\right)$ shows the *Sup* of jth is the biggest intuitionistic hesitant fuzzy set (IHFS) $P_{\sigma(j)}$, for the ith largest intuitionistic hesitant fuzzy set (IHFS) $P_{\sigma(i)}$. If we will choose the grade of truth and falsity in the form of singleton sets then the Eq. (24) will be converted for intuitionistic fuzzy sets. Similarly, if we choose the values of falsity grade is zero then the Eq. (24) will be converted for hesitant fuzzy sets. Some characteristics of IHFPOWA operator are as follow:

Property 4: (Idempotency) when each P_j is equivalent, which is, $P_j=P$ for every $j\,(j=1,2,3\ldots,n)$, so

$$IHFPOWA_\omega(P_1,P_2,\ldots,P_n)=P \tag{27}$$

Property 5: (Boundedness) Suppose P_j is family of IHFSs, suppose

$$P^-=\overset{min}{\text{j}}P_j,\ P^+=\overset{max}{j}P_j$$

then

$$P^-\leq \text{IHFPOW}A_\omega(P_1,P_2,\ldots,P_n)\leq P^+ \tag{28}$$

Property 6: (Monotonicity) Let P_j and P_j' be IHFSs, if $P_j\leq P_j'$ for all j. Then

$$\text{IHFPOW}A_\omega(P_1,P_2,\ldots,P_n)\leq \text{IHFPOW}A_\omega\left(P_1',P_2',\ldots,P_n'\right) \tag{29}$$

Property 7: (Commutativity) Let P_j and P'_j be IHFSs, if $P_j \le P'_j$ for all j. Then

$$\mathrm{IHFPOW}A_\omega(P_1, P_2, \ldots, P_n) = \mathrm{IHFPOW}A_\omega\left(P'_1, P'_2, \ldots, P'_n\right) \tag{30}$$

where P'_j be a permutation of P_j.

Definition 11: Let $P_j = (\mu_j, \nu_j)$ be family of IHFSs, the intuitionistic hesitant fuzzy power hybrid averaging (IHFPHA) operator of elements n a function IHFPHA: $P^n \to P$, such that

$$IHFPHA_\omega(P_1, P_2, \ldots, P_n) = \frac{\bigoplus_{j=1}^{n}(\omega_j\left(1 + T\left(\dot{P}_{\sigma(j)}\right)\dot{P}_{\sigma(j)}\right)}{\sum_{j=1}^{n}\omega_j\left(1 + T\left(\dot{P}_{\sigma(j)}\right)\right)}$$
$$= \bigcup_{\substack{\dot{l}_{\sigma(j)}\varepsilon\dot{\mu}_j \\ \dot{m}_{\sigma(j)}\varepsilon\dot{\nu}_j}}\left(1 - \prod_{j=1}^{n}(1 - (\dot{l}_{\sigma(j)})^{\frac{(\omega_j(1+T(\dot{P}_{\sigma(j)}))}{\sum_{j=1}^{n}\omega_j(1+T(\dot{P}_{\sigma(j)}))}}, \prod_{j=1}^{n}\left(\dot{m}_{\sigma(j)}\right)^{\frac{(\omega_j(1+T(\dot{P}_{\sigma(j)}))}{\sum_{j=1}^{n}\omega_j(1+T(\dot{P}_{\sigma(j)}))}}\right) \tag{31}$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_j)^T$ is a mapped weight vector, such that $\omega_j \in [0\,1]$ and $\sum_{j=1}^{n}\omega_j = 1$ and $\dot{P}_{\sigma(j)}$ is the jth biggest element in intuitionistic hesitant fuzzy arguments $\dot{P}_j(\dot{P} = (n\omega_j)P_j$, $j = 1, 2, \ldots, n)$, $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ be the weighting vector of IHF arguments $P_j(j = 1, 2, \ldots, n)$, $\omega_j \in [0\,1]$ and $\sum_{j=1}^{n}\omega_j = 1$. And ω_j be a family such that

$$\omega_j = g\left(\frac{R_j}{TV}\right) - g\left(\frac{R_{j-1}}{TV}\right), \quad R_j = \sum_{i=1}^{j}V_{\sigma(i)}, \quad TV = \sum_{i=1}^{n}V_{\sigma(i)}, \quad V_{\sigma(i)} = 1 + T(\dot{P}_{\sigma(i)}) \tag{32}$$

where $T(\dot{P}_{\sigma(i)})$ is the *Sup* of jth biggest IHFSs $T(\dot{P}_{\sigma(i)})$ by all the other (IHFSs), that is,

$$T\left(\dot{P}_{\sigma(i)}\right) = \bigcup_{\substack{\dot{\mu}_{\sigma(j)}\varepsilon\dot{P}_j \\ \dot{\nu}_{\sigma(j)}\varepsilon\dot{P}_j}}\left(\sum_{\substack{i=1 \\ i\neq j}}^{n} Sup(\dot{P}_{\sigma(j)}, \dot{P}_{\sigma(i)})\right) \tag{33}$$

where $\bigcup_{\substack{\dot{\mu}_{\sigma(j)}\varepsilon\dot{P}_j \\ \dot{\nu}_{\sigma(j)}\varepsilon\dot{P}_j}}\left(\sum_{\substack{i=1 \\ i\neq j}}^{n} Sup(\dot{P}_{\sigma(j)}, \dot{P}_{\sigma(i)})\right)$ shows the *Sup* of jth biggest IHFS $\dot{P}_{\sigma(j)}$, for the ith biggest IHFS $\dot{P}_{\sigma(i)}$. Particularly, IHFPHA is decreased to IHFPWA operator if $\omega = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$ and IHFPHA is decreased to IHFPOWA operator if $\omega = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$. If we will choose the grade of truth and falsity in the form of singleton sets then the Eq. (31) will be converted for intuitionistic fuzzy sets. Similarly, if we choose the values of falsity grade is zero then the Eq. (31) will be converted for hesitant fuzzy sets.

Definition 12: Suppose $P_j = \left(\mu_j, \nu_j\right)$ is family of IHFSs, intuitionistic hesitant fuzzy power geometric (IHFPG) operator defined as a function IHFPG: $P^n \to P$ where

$$IHFPG(P_1, P_2, \ldots, P_n) = \bigoplus_{j=1}^{n}(P)^{\frac{1+T(P_j)}{\sum_{i=1}^{n}(1+T(P_j))}} \tag{34}$$

where

$$T\left(P_j\right) = \sum_{\substack{i=1 \\ i \neq j}}^{n} Sup(P_j, P_i) \tag{35}$$

where $Sup(P_j, P_i)$ is the support for P_j from P_i, with the conditions

1. $Sup(P_j, P_i) \in [0, 1]$;
2. $Sup(P_j, P_i) = Sup(P_i, P_j)$;
3. $Sup\left(P_j, P_i\right) \geq Sup\left(P_s, P_t\right)$ if $d(P_j, P_i) < d(P_s, P_t)$, such that d is a distance measure.

If we will choose the grade of truth and falsity in the form of singleton sets then the Eq. (34) will be converted for intuitionistic fuzzy sets. Similarly, if we choose the values of falsity grade is zero then the Eq. (34) will be converted for hesitant fuzzy sets.

Theorem 2: The aggregated elements by utilizing IHFPG operator define an IHFS, wherever

$$IHFPG\left(P_1, P_2, \ldots, P_n\right) = \bigoplus_{j=1}^{n} (P)^{\frac{1+T\left(P_j\right)}{\sum_{i=1}^{n}\left(1+T\left(P_j\right)\right)}}$$

$$= \bigcup_{\substack{l_j \varepsilon \mu_j \\ m_j \varepsilon \nu_j}} \left(\prod_{j=1}^{n} \left(l_j\right)^{\frac{(1+T(P_j))}{\sum_{j=1}^{n}(1+T(P_j))}}, 1 - \prod_{j=1}^{n} (1 - \left(m_j\right)^{\frac{(1+T(P_j))}{\sum_{j=1}^{n}(1+T(P_j))}} \right) \tag{36}$$

where

$$T\left(P_j\right) = \bigcup_{\substack{\mu_j \varepsilon P_j \\ \nu_j \varepsilon P_j}} \left(\sum_{\substack{i=1 \\ i \neq j}}^{n} Sup\left(P_j, P_i\right) \right) \tag{37}$$

Definition 13: Let $P_j = \left(\mu_j, \nu_j\right)$ $(j = 1, 2, \ldots, n)$ be family of IHFSs, $\omega = (\omega_1, \omega_2, \ldots, \omega_j)^T$ is weight vector of P_j, $\omega_j > 0, \sum_{j=1}^{n} \omega_j = 1$. The IHFPWG operator defined as mapping IHFPWG: $P^n \rightarrow P$ where

$$IHFPWG(P_1, P_2, \ldots, P_n) = \bigoplus_{j=1}^{n} (P)^{\frac{(\omega_j\left(1+T\left(P_j\right)\right)}{\sum_{j=1}^{n} \omega_j\left(1+T\left(P_j\right)\right)}}$$

$$= \bigcup_{\substack{l_j \varepsilon \mu_j \\ m_j \varepsilon \nu_j}} \left(\prod_{j=1}^{n} \left(l_j\right)^{\frac{\omega_j(1+T(P_j))}{\sum_{j=1}^{n} \omega_j(1+T(P_j))}}, 1 - \prod_{j=1}^{n} (1 - \left(m_j\right)^{\frac{\omega_j((1+T(P_j))}{\sum_{j=1}^{n} \omega_j(1+T(P_j))}} \right) \tag{38}$$

where

$$T\left(P_j\right) = \bigcup_{\substack{\mu_j \varepsilon P_j \\ \nu_j \varepsilon P_j}} \left(\sum_{\substack{i=1 \\ i \neq j}}^{n} \omega_j Sup\left(P_j, P_i\right) \right) \tag{39}$$

If we will choose the grade of truth and falsity in the form of singleton sets then the Eq. (36) will be converted for intuitionistic fuzzy sets. Similarly, if we choose the values of falsity grade is zero then the Eq. (36) will be converted for hesitant fuzzy sets. IHFPWG operator has following characteristics.

Property 8: (Idempotency) when every P_j $(j=1,2,\ldots,n)$ is equivalent, such that $P_j=P$ for every j, then

$$IHFPWG_{\omega}(P_1,P_2,\ldots,P_n)=P \tag{40}$$

Property 9: (Boundedness) Suppose P_j is family of IHFSs, also suppose

$$P^{-}=\min_{j} P_j,\ \ P^{+}=\max_{j} P_j$$

then

$$P^{-}\leq \text{IHFPW}G_{\omega}(P_1,P_2,\ldots,P_n)\leq P^{+} \tag{41}$$

Property 10: (Monotonicity) Let P_j $(j=1,2,\ldots,n)$ and P'_j be IHFSs, if $P_j\leq P'_j$ for all j, then

$$\text{IHFPW}G_{\omega}(P_1,P_2,\ldots,P_n)\leq \text{IHFPW}G_{\omega}\left(P'_1,P'_2,\ldots,P'_n\right) \tag{42}$$

Further, we give an IHFPOWG operator below:

Definition 14: Suppose $P_j=(\mu_j,\nu_j)$ is family of IHFSs, IHFPOWG operator of dimension n is mapping IHFPOWG: $P^n\to P$, with an associated weight vector $\omega=(\omega_1,\omega_2,\ldots,\omega_j)^T$ such that $\omega_j>0$ and $\sum_{j=1}^{n}\omega_j=1$. Furthermore

$$IHFPOWG_{\omega}(P_1,P_2,\ldots,P_n)=\bigoplus_{j=1}^{n}(P_{\sigma(j)})^{\frac{(\omega_j\left(1+T\left(P_{\sigma(j)}\right)\right)}{\sum_{j=1}^{n}\omega_j\left(1+T\left(P_{\sigma(j)}\right)\right)}}$$

$$=\bigcup_{\substack{l_{\sigma(j)}\varepsilon\mu_j\\ m_{\sigma(j)}\varepsilon\nu_j}}\left(\prod_{j=1}^{n}(l_{\sigma(j)})^{\frac{(\omega_j(1+T(P_{\sigma(j)}))}{\sum_{j=1}^{n}\omega_j(1+T(P_{\sigma(j)}))}},1-\prod_{j=1}^{n}(1-(m_{\sigma(j)})^{\frac{(\omega_j(1+T(P_{\sigma(j)}))}{\sum_{j=1}^{n}\omega_j(1+T(P_{\sigma(j)}))}}\right) \tag{43}$$

Such that $(\sigma(1),\sigma(2),\ldots,\sigma(n))$ be permutation of $1,2,\ldots,n$, where $P_{\sigma(j-1)}\geq P_{\sigma(j)}$ for every $j=1,2,\ldots,n$, $\omega_j\,(j=1,2,\ldots,n)$ is family of weights where

$$\omega_j=g\left(\frac{R_j}{TV}\right)-g\left(\frac{R_{j-1}}{TV}\right),\quad R_j=\sum_{i=1}^{j}V_{\sigma(i)},\quad TV=\sum_{i=1}^{n}V_{\sigma(i)},\quad V_{\sigma(i)}=1+T(P_{\sigma(i)}) \tag{44}$$

where $T(P_{\sigma(i)})$ indicates the sup of jth biggest intuitionistic hesitant fuzzy sets (IHFSs) $T(P_{\sigma(i)})$ by all the other (IHFSs), that is,

$$T\left(P_{\sigma(i)}\right)=\bigcup_{\substack{\mu_{\sigma(j)}\varepsilon P_j\\ \nu_{\sigma(j)}\varepsilon P_j}}\left(\sum_{\substack{i=1\\ i\neq j}}^{n}Sup(P_{\sigma(j)},P_{\sigma(i)}\right) \tag{45}$$

where $\sum_{\substack{i=1 \\ i \neq j}}^{n} Sup(P_{\sigma(j)}, P_{\sigma(i)})$ shows the *sup* of jth biggest IHFS $P_{\sigma(j)}$,for the ith biggest IHFS $P_{\sigma(i)}$. If we will choose the grade of truth and falsity in the form of singleton sets then the Eq. (43) will be converted for intuitionistic fuzzy sets. Similarly, if we choose the values of falsity grade is zero then the Eq. (43) will be converted for hesitant fuzzy sets. IHFPOWG operator provides the following characteristics.

Property 11: (Idempotency) when every P_j $(j = 1, 2, \ldots, n)$ is equivalent, such that, $P_j = P$ for every j, so

$$IHFPOWG_{\omega}(P_1, P_2, \ldots, P_n) = P \tag{46}$$

Property 12: (Boundedness) Suppose P_j is family of IHFSs, suppose

$$P^- = \min_j P_j, \ \ P^+ = \max_j P_j$$

then

$$P^- \leq \text{IHFPOW}G_{\omega}(P_1, P_2, \ldots, P_n) \leq P^+ \tag{47}$$

Property 13: (Monotonicity) Let P_j and P'_j be of intuitionistic hesitant fuzzy sets (IHFSs), if $P_j \leq P'_j$ for all j, then

$$\text{IHFPOW}G_{\omega}(P_1, P_2, \ldots, P_n) \leq \text{IHFPOW}G_{\omega}\left(P'_1, P'_2, \ldots, P'_n\right) \tag{48}$$

Property 14: (Commutativity) Let P_j and P'_j be two intuitionistic hesitant fuzzy sets (IHFSs), if $P_j \leq P'_j$ for all j, then

$$\text{IHFPOW}G_{\omega}(P_1, P_2, \ldots, P_n) = \text{IHFPOW}G_{\omega}\left(P'_1, P'_2, \ldots, P'_n\right), \ (j = 1, 2, \ldots, n) \tag{49}$$

where P'_j be permutation of P_j.

Definition 15: Let $P_j = (\mu_j, \nu_j)$ be family of IHFSs, the IHFPHG operator of elements n is the function IHFPHG: $P^n \to P$, where

$$IHFPHG_{\omega}(P_1, P_2, \ldots, P_n) = \bigoplus_{j=1}^{n} (\dot{P}_{\sigma(j)})^{\frac{\bigoplus_{j=1}^{n} (\omega_j\left(1+T\left(\dot{P}_{\sigma(j)}\right)\dot{P}_{\sigma(j)}\right)}{\sum_{j=1}^{n} \omega_j\left(1+T\left(\dot{P}_{\sigma(j)}\right)\right)}}$$

$$= \bigcup_{\substack{\dot{l}_{\sigma(j)} \varepsilon \dot{\mu}_j \\ \dot{m}_{\sigma(j)} \varepsilon \dot{\nu}_j}} \left(\prod_{j=1}^{n} \left(\dot{l}_{\sigma(j)}\right)^{\frac{(\omega_j(1+T(\dot{P}_{\sigma(j)}))}{\sum_{j=1}^{n} \omega_j(1+T(\dot{P}_{\sigma(j)}))}}, 1 - \prod_{j=1}^{n} (1 - (\dot{m}_{\sigma(j)})^{\frac{(\omega_j(1+T(\dot{P}_{\sigma(j)}))}{\sum_{j=1}^{n} \omega_j(1+T(\dot{P}_{\sigma(j)}))}} \right) \tag{50}$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_j)^T$ is a related weight vector, where $\omega_j \in [01]$ and $\sum_{j=1}^{n} \omega_j = 1$. $\dot{P}\sigma(j)$ is the jth biggest element of the intuitionistic hesitant fuzzy arguments $\dot{P}_j$ $(\dot{P} = \left(P_j\right)^{n\omega_j}, j = 1, 2, \ldots, n)$,

$\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ is weighting vector of IHF arguments P_j where $\omega_j \in [0\,1]$, $\sum_{j=1}^{n} \omega_j = 1$ and n is a matching factor, and $\omega_j (j = 1, 2, \ldots, n)$ is the collection of weights such that

$$\omega_j = g\left(\frac{R_j}{TV}\right) - g\left(\frac{R_{j-1}}{TV}\right), \quad R_j = \sum_{i=1}^{j} V_{\sigma(i)}, \quad TV = \sum_{i=1}^{n} V_{\sigma(i)}, \quad V_{\sigma(i)} = 1 + T(\dot{P}_{\sigma(i)}) \tag{51}$$

where $T(\dot{P}_{\sigma(i)})$ indicates the *Sup* of jth biggest IHFSs $T(\dot{P}_{\sigma(i)})$ by all the other (IHFSs), that is,

$$T\left(\dot{P}_{\sigma(i)}\right) = \bigcup_{\substack{\dot{\mu}_{\sigma(j)} \varepsilon \dot{P}_j \\ \dot{v}_{\sigma(j)} \varepsilon \dot{P}_j}} \left(\sum_{\substack{i=1 \\ i \neq j}}^{n} Sup(\dot{P}_{\sigma(j)}, \dot{P}_{\sigma(i)}) \right) \tag{52}$$

where $\bigcup_{\substack{\dot{\mu}_{\sigma(j)} \varepsilon \dot{P}_j \\ \dot{v}_{\sigma(j)} \varepsilon \dot{P}_j}} \left(\sum_{\substack{i=1 \\ i \neq j}}^{n} Sup(\dot{P}_{\sigma(j)}, \dot{P}_{\sigma(i)}) \right)$ shows the *Sup* of jth biggest IHFS $\dot{P}_{\sigma(j)}$ and the ith biggest IHFS $\dot{P}_{\sigma(i)}$. Particularly, IHFPHA is decreased to the IHFPWA operator when $\omega = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$ IHFPHA is decreased to the IHFPOWA operator when $\omega = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$. If we will choose the grade of truth and falsity in the form of singleton sets then the Eq. (50) will be converted for intuitionistic fuzzy sets. Similarly, if we choose the values of falsity grade to be zero then the Eq. (50) will be converted for hesitant fuzzy sets.

4 Similarity Measures Based on Intuitionistic Hesitant Fuzzy Sets

The VSM is one of the important tools for the similarity degree between objects. We straightforwardly utilized Jaccard, Dice and Cosine SM. Presently in this segment we characterize VSMs and weighted VSMs (WVSMs) for IHFSs.

Definition 16: Suppose that $P = (\mu_P, \nu_P)$ and $Q = \left(\mu_Q, \nu_Q\right)$ are two IHFSs on X, then the Jaccard similarity measure (JSM) between P and Q is denoted and defined as follows:

$$Jac\,(P, Q) = \frac{1}{n} \sum_{k=1}^{n} \left(\frac{\frac{1}{\mathscr{g}} \sum_{\mathscr{p}=1}^{\mathscr{g}} \mu_{P_{\mathscr{p}}}(x_k) \cdot \mu_{Q_{\mathscr{p}}}(x_k) + \frac{1}{\hbar} \sum_{\mathscr{q}=1}^{\hbar} \nu_{P_{\mathscr{p}}}(x_k) \cdot \nu_{Q_{\mathscr{p}}}(x_k)}{\begin{array}{c} \frac{1}{\mathscr{g}} \sum_{\mathscr{p}=1}^{\mathscr{g}} \mu^2_{P_{\mathscr{p}}}(x_k) + \frac{1}{\mathscr{g}} \sum_{\mathscr{p}=1}^{\mathscr{g}} \mu^2_{Q_{\mathscr{p}}}(x_k) + \\ \frac{1}{\hbar} \sum_{\mathscr{q}=1}^{\hbar} \nu^2_{P_{\mathscr{p}}}(x_k) + \frac{1}{\hbar} \sum_{\mathscr{q}=1}^{\hbar} \nu^2_{Q_{\mathscr{p}}}(x_k) - \\ \left(\frac{1}{\mathscr{g}} \sum_{\mathscr{p}=1}^{\mathscr{g}} \mu_{P_{\mathscr{p}}}(x_k) \cdot \mu_{Q_{\mathscr{p}}}(x_k) + \frac{1}{\hbar} \sum_{\mathscr{q}=1}^{\hbar} \nu_{P_{\mathscr{p}}}(x_k) \cdot \nu_{Q_{\mathscr{p}}}(x_k) \right) \end{array}} \right) \tag{53}$$

JSMs fulfill the following axioms:

1. $0 \le Jac\,(P, Q) \le 1$;
2. $Jac\,(P, Q) = Jac\,(Q, P)$;
3. $Jac\,(P, Q) = 1$, if $P = Q$.

If we will choose the grade of truth and falsity in the form of singleton sets then the Eq. (53) will be converted for intuitionistic fuzzy sets. Similarly, if we choose the values of falsity grade to be zero then the Eq. (53) will be converted for hesitant fuzzy sets.

Definition 17: Suppose that $P = (\mu_P, \nu_P)$ and $Q = \left(\mu_Q, \nu_Q\right)$ are two IHFSs on X, then the weighted JSM (WJSM) between P and Q is denoted and defined as follows:

$$Jac_w(P,Q) = \sum_{k=1}^{n} w_k \left(\frac{\frac{1}{\mathcal{g}}\sum_{\mathcal{p}=1}^{\mathcal{g}} \mu_{P_{\mathcal{p}}}(x_k) \cdot \mu_{Q_{\mathcal{p}}}(x_k) + \frac{1}{\hbar}\sum_{\mathcal{q}=1}^{\hbar} \nu_{P_{\mathcal{p}}}(x_k) \cdot \nu_{Q_{\mathcal{p}}}(x_k)}{\begin{array}{c} \frac{1}{\mathcal{g}}\sum_{\mathcal{p}=1}^{\mathcal{g}} \mu^2_{P_{\mathcal{p}}}(x_k) + \frac{1}{\mathcal{g}}\sum_{\mathcal{p}=1}^{\mathcal{g}} \mu^2_{Q_{\mathcal{p}}}(x_k) + \\ \frac{1}{\hbar}\sum_{\mathcal{q}=1}^{\hbar} \nu^2_{P_{\mathcal{p}}}(x_k) + \frac{1}{\hbar}\sum_{\mathcal{q}=1}^{\hbar} \nu^2_{Q_{\mathcal{p}}}(x_k) - \\ \left(\frac{1}{\mathcal{g}}\sum_{\mathcal{p}=1}^{\mathcal{g}} \mu_{P_{\mathcal{p}}}(x_k) . \mu_{Q_{\mathcal{p}}}(x_k) + \frac{1}{\hbar}\sum_{\mathcal{q}=1}^{\hbar} \nu_{P_{\mathcal{p}}}(x_k) . \nu_{Q_{\mathcal{p}}}(x_k)\right) \end{array}} \right) \tag{54}$$

WJSMs fulfill the following axioms:

1. $0 \leq Jac_w(P,Q) \leq 1$;
2. $Jac_w(P,Q) = Jac_w(Q,P)$;
3. $Jac_w(P,Q) = 1$, if $P = Q$.

where $w = (w_1, w_2, \ldots, w_n)^T$ speaks to the weight vector of every component $x_k\,(k = 1,2,3,\ldots,n)$ contained in IHFS and the weight vector fulfills $w_k \in [0,1]$ for each $k = 1,2,3,\ldots,n$, $\sum_{k=1}^{n} w_k = 1$. When we assume the weight vector be $w = \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)^T$, at that point the WJSM will change into JSM. Otherwise speaking when $w_k = \frac{1}{n}$, $k = 1,2,3,\ldots,n$ then $Jac_w(P,Q) = Jac(P,Q)$. If we will choose the grade of truth and falsity in the form of singleton sets then the Eq. (54) will be converted for intuitionistic fuzzy sets. Similarly, if we choose the values of falsity grade to be zero then the Eq. (54) will be converted for hesitant fuzzy sets.

Definition 18: Suppose that $P = (\mu_P, \nu_P)$ and $Q = \left(\mu_Q, \nu_Q\right)$ are two IHFSs on X, then the Dice similarity measure (DSM) between P and Q is denoted and defined as follows:

$$Dic(P,Q) = \frac{1}{n}\sum_{k=1}^{n} \left(\frac{\frac{2}{\mathcal{g}}\sum_{\mathcal{p}=1}^{\mathcal{g}} \mu_{P_{\mathcal{p}}}(x_k) \cdot \mu_{Q_{\mathcal{p}}}(x_k) + \frac{2}{\hbar}\sum_{\mathcal{q}=1}^{\hbar} \nu_{P_{\mathcal{p}}}(x_k) \cdot \nu_{Q_{\mathcal{p}}}(x_k)}{\frac{1}{\mathcal{g}}\sum_{\mathcal{p}=1}^{\mathcal{g}} \mu^2_{P_{\mathcal{p}}}(x_k) + \frac{1}{\mathcal{g}}\sum_{\mathcal{p}=1}^{\mathcal{g}} \mu^2_{Q_{\mathcal{p}}}(x_k) + \frac{1}{\hbar}\sum_{\mathcal{q}=1}^{\hbar} \nu^2_{P_{\mathcal{p}}}(x_k) + \frac{1}{\hbar}\sum_{\mathcal{q}=1}^{\hbar} \nu^2_{Q_{\mathcal{p}}}(x_k)} \right) \tag{55}$$

DSMs fulfills the following axioms:

1. $0 \leq Dic(P,Q) \leq 1$;
2. $Dic(P,Q) = Dic(Q,P)$;
3. $Dic(P,Q) = 1$, if $P = Q$.

If we will choose the grade of truth and falsity in the form of singleton sets then the Eq. (55) will be converted for intuitionistic fuzzy sets. Similarly, if we choose the values of falsity grade to be zero then the Eq. (55) will be converted for hesitant fuzzy sets.

Definition 19: Suppose that $\mathcal{S}P=(\mu_P,\nu_P)$ and $Q=\left(\mu_Q,\nu_Q\right)$ are two IHFSs on X, then the weighted DSM (WDSM) between P and Q is denoted and defined as follows

$$Dic_w\left(P,Q\right)=\sum_{k=1}^{n}w_k\left(\frac{\frac{2}{g}\sum_{p=1}^{g}\mu_{P_p}\left(x_k\right)\cdot\mu_{Q_p}\left(x_k\right)+\frac{2}{\hbar}\sum_{q=1}^{\hbar}\nu_{P_p}\left(x_k\right)\cdot\nu_{Q_p}\left(x_k\right)}{\frac{1}{g}\sum_{p=1}^{g}\mu_{P_p}^2\left(x_k\right)+\frac{1}{g}\sum_{p=1}^{g}\mu_{Q_p}^2\left(x_k\right)+\frac{1}{\hbar}\sum_{q=1}^{\hbar}\nu_{P_p}^2\left(x_k\right)+\frac{1}{\hbar}\sum_{q=1}^{\hbar}\nu_{Q_p}^2\left(x_k\right)}\right) \tag{56}$$

WDSMs fulfills the following axioms

1. $0\leq Dic_w\left(P,Q\right)\leq 1$;
2. $Dic_w\left(P,Q\right)=Dic_w\left(Q,P\right)$;
3. $Dic_w\left(P,Q\right)=1$, if $P=Q$.

where $w=(w_1,w_2,\ldots,w_n)^T$ speaks to the weight vector of every component $x_k\left(k=1,2,3,\ldots,n\right)$ contained in IHFS and the weight vector fulfills $w_k\in[0,1]$ for each $k=1,2,3,\ldots,n$, $\sum_{k=1}^{n}w_k=1$. When we assume the weight vector be $w=\left(\frac{1}{n},\frac{1}{n},\ldots,\frac{1}{n}\right)^T$, at that point the WDSM will change into DSM. Otherwise speaking when $w_k=\frac{1}{n}$, $k=1,2,3,\ldots,n$ then $Dic_w\left(P,Q\right)=Dic\left(P,Q\right)$. If we will choose the grade of truth and falsity in the form of singleton sets then the Eq. (56) will be converted for intuitionistic fuzzy sets. Similarly, if we choose the values of falsity grade to be zero then the Eq. (56) will be converted for hesitant fuzzy sets.

5 Multiple Attribute Decision Making Technique Based on Intuitionistic Hesitant Fuzzy Sets

In the portion, we use IHF power aggregation operators to multiple attribute DM through intuitionistic hesitant fuzzy data. Following hypotheses or concepts are utilized to signify the multiple attribute DM difficulties for possible calculation of developing technology commercialization with intuitionistic hesitant fuzzy data. Consider $A=\{A_1,A_2,\ldots,A_m\}$ is distinct set of alternatives, $G=\{G_1,G_2,\ldots,G_n\}$ is the set of attributes. Consider $\omega=(\omega_j)$ $(j=1,2,\ldots n)$ is weight vector of attributes, such that $\omega_j\geq 0$, $\sum_{j=1}^{n}\omega_j=1$. Then we will be going to apply the IHFPWA or PFPWG operator to the multiple attribute DM difficulties for possible calculation of developing technology commercialization by IHF data.

Step 1. Compute the supports:

$$Sup\left(P_{ij},P_{ik}\right)=1-d\left(P_{ij},P_{ik}\right),\ (j,k=1,2,\ldots,n), \tag{57}$$

where justify Sup terms (1)–(3) in portion 3. Here, with no loss of generalization, we compute $d(P_{ij},P_{ik})$ with the normalized Hamming distance

$$d\left(P_{ij},P_{ik}\right)=\frac{1}{m}\sum_{i=1}^{m}\left(\frac{1}{2f}\sum_{s=1}^{f}\left(\left|l_{ij}{}^{\sigma(s)}-l_{ik}{}^{\sigma(s)}\right|+\left|m_{ij}{}^{\sigma(s)}-m_{ik}{}^{\sigma(s)}\right|\right)\right) \tag{58}$$

where $l_{ij}\in\mu_{ij}$, $l_{ik}\in\mu_{ik}$, $m_{ij}\in\nu_{ij}$ and $m_{ik}\in\nu_{ik}$.

Step 2. Using the weights ω_j of attribute G_j to compute weighted sup $T(P_{ij})$ of the IHFS P_{ij} by other IHFS P_{ik} $(j,\ k = 1, 2, \ldots, n,\ k \neq j)$

$$T(P_{ij}) = \bigcup_{\begin{pmatrix} \mu_{ij} \varepsilon P_{ij} \\ \mu_{ik} \varepsilon P_{ik} \\ v_{ij} \varepsilon P_{ij} \\ v_{ik} \varepsilon P_{ik} \end{pmatrix}} \left(\sum_{\substack{k=1 \\ k \neq j}}^{n} \omega_j Sup(P_{ij}, P_{ik}) \right) \tag{59}$$

where calculated weight ξ_{ij} is connected with the IHPFS P_{ij}, $(j = 1, 2, \ldots, n,\ i = 1, 2, \ldots, m)$

$$\xi_{ij} = \frac{\omega_j((1 + T(P_{ij}))}{\sum_{j=1}^{n} \omega_j (1 + T(P_{ij}))} \tag{60}$$

where $\xi_{ij} \geq 0,\ \sum_{j=1}^{n} \xi_{ij} = 1$.

Step 3. Use decision data provided in Tab. 1, IHFPWA operator

$$P_i = IHFPWA_{\omega}(P_{i1}, P_{i2}, \ldots, P_{in}) = \frac{\bigoplus_{j=1}^{n} (\omega_j \left(1 + T\left(P_{ij}\right) P_{ij}\right)}{\sum_{j=1}^{n} \omega_j \left(1 + T\left(P_{ij}\right)\right)}$$

$$= \bigcup_{\substack{l_{ij} \varepsilon \mu_{ij} \\ m_{ij} \varepsilon v_{ij}}} \left(1 - \prod_{j=1}^{n} (1 - (l_{ij})^{\frac{\omega_j((1+T(P_{ij}))}{\sum_{j=1}^{n} \omega_j (1+T(P_{ij}))}}, \prod_{j=1}^{n} (m_{ij})^{\frac{\omega_j (1+T(P_{ij}))}{\sum_{j=1}^{n} \omega_j (1+T(P_{ij}))}} \right) \tag{61}$$

or

$$P_i = IHFPWG(P_{i1}, P_{i2}, \ldots, P_{in}) = \bigoplus_{j=1}^{n} (P_{ij})^{\frac{(\omega_j \left(1+T\left(P_{ij}\right)\right)}{\sum_{j=1}^{n} \omega_j \left(1+T\left(P_{ij}\right)\right)}}$$

$$= \bigcup_{\substack{l_{ij} \varepsilon \mu_{ij} \\ m_{ij} \varepsilon v_{ij}}} \left(\prod_{j=1}^{n} (l_{ij})^{\frac{\omega_j (1+T(P_{ij}))}{\sum_{j=1}^{n} \omega_j (1+T(P_{ij}))}}, 1 - \prod_{j=1}^{n} (1 - (m_{ij})^{\frac{\omega_j((1+T(P_{ij}))}{\sum_{j=1}^{n} \omega_j (1+T(P_{ij}))}} \right) \tag{62}$$

to receive the total preference objects P_i of the alternative $A_i\, (i = 1, 2, \ldots, m)$.

Step 4. Compute scores $S(P_i)$ of the whole IHFSs P_i to rank each the A_i then to select the top one(s). If two scores $S(P_i)$ and $S(P_j)$ have no difference then we want to compute the accuracy grades $H(P_i)$, $H(P_j)$ of the whole IHFSs P_i, P_j, respectively, classify the alternatives A_i, A_j consistent with accuracy grades $H(P_i)$ and $H(P_j)$.

Step 5. Ranking whole alternatives A_i and choose the greatest one(s) in accord by (P_i) $(i = 1, 2, \ldots, m)$.

Step 6. The end.

Example 1: Therefore, in the portion we give a mathematical model to illustrate the possible estimation of developing technology commercialization by intuition hesitant fuzzy data illustrating the technique recommended in this article. There is the board with five possible developing technologies enterprises $A_i (i = 1, \ldots, 5)$ to choose. Specialists choose four attributes to calculate the five possible developing technology enterprises: (i) G_1 is the technical development (ii) G_2 is the potential market and market risk; (*iii*) G_3 is the industrialized structure, human resource management, and economic circumstances (*iv*) G_4 is the job creation and the development of science and technology. The five possible developing technology enterprises $A_i (i = 1, 2, 3, 4, 5)$ are to be estimated utilizing the IHF data by the decision maker in accordance with proposed attributes and weighting vector $\omega = (0.4, 0.2, 0.1, 0.3)^T$ shown in Tab. 1.

Table 1: Original decision matrix

G_1	G_2	G_3	G_4
$\{\{\mathbf{0.1, 0.3}\}, \{\mathbf{0.1, 0.4}\}\}$	$\{\{0.0, 0.3\}, \{0.3, 0.4\}\}$	$\{\{0.0, 0.3\}, \{0.1, 0.1\}\}$	$\{\{0.2, 0.4\}, \{0.1, 0.2\}\}$
$\{\{\mathbf{0.1, 0.0}\}, \{\mathbf{0.2, 0.2}\}\}$	$\{\{0.0, 0.1\}, \{0.1, 0.2\}\}$	$\{\{0.1, 0.1\}, \{0.1, 0.3\}\}$	$\{\{0.1, 0.2\}, \{0.1, 0.3\}\}$
$\{\{\mathbf{0.3, 0.2}\}, \{\mathbf{0.2, 0.1}\}\}$	$\{\{0.1, 0.2\}, \{0.0, 0.1\}\}$	$\{\{0.2, 0.5\}, \{0.2, 0.1\}\}$	$\{\{0.0, 0.6\}, \{0.2, 0.1\}\}$
$\{\{\mathbf{0.3, 0.1}\}, \{\mathbf{0.2, 0.5}\}\}$	$\{\{0.3, 0.5\}, \{0.1, 0.1\}\}$	$\{\{0.1, 0.0\}, \{0.1, 0.2\}\}$	$\{\{0.3, 0.2\}, \{0.2, 0.2\}\}$
$\{\{\mathbf{0.2, 0.1}\}, \{\mathbf{0.5, 0.1}\}\}$	$\{\{0.1, 0.4\}, \{0.1, 0.2\}\}$	$\{\{0.2, 0.2\}, \{0.5, 0.2\}\}$	$\{\{0.3, 0.5\}, \{0.2, 0.2\}\}$

Next, we use the method established to indicate potential evaluation of developing technology commercialization of four possible developing technology enterprises, see Tab. 2.

Table 2: Aggregated values by using the formulas of IHFPWA and IHFPWG

	IHFPWA	***IHFPWG***
A_1	$(0.3315, 0.1203)$	$(0.3269, 0.1372)$
A_2	$(0.0879, 0.1377)$	$(0, 0.1476)$
A_3	$(0.3707, 0)$	$(0.2965, 0.1)$
A_4	$(0.2877, 0.1696)$	$(0.2781, 0.1773)$
A_5	$(0.3012, 0.1452)$	$(0.2150, 0.1552)$

Step 1. Compute the weight ξ_{ij} $(i = 1, \ldots, 5,\ j = 1, \ldots, 4)$ that is related by IHFN $\tilde{p}$ $(i = 1, \ldots 5,$ $j = 1, \ldots, 4)$, that included $R = \left(\tilde{r}_{ij}\right)_{5\times 4}$

$$\xi = \begin{bmatrix} 0.4636 & 0.1683 & 0.0687 & 0.2992 \\ 0.4621 & 0.1689 & 0.0686 & 0.3001 \\ 0.4628 & 0.1685 & 0.0688 & 0.2997 \\ 0.4616 & 0.1684 & 0.0688 & 0.3009 \\ 0.4616 & 0.1694 & 0.0689 & 0.3000 \end{bmatrix}$$

Step 2. Corresponding to ξ and IHFN $\tilde{p}_{ij}$ $(i = 1, \ldots, 5,\ j = 1, \ldots, 4)$, compute the whole IHFNs $\tilde{p}_{ij}$ $(i = 1, 2, 3, 4, 5,\ j = 1, 2, 3, 4)$ by utilizing the IHFPWA (IHFPWG) operator to get the whole IHFNs $\tilde{p}_i$ $(i = 1, 2, 3, 4, 5)$ of the developing technology enterprise A_i. The aggregating values are reflected in Tab. 1.

By using the formula of score value, we examine the score values of the aggregated values of Tab. 2, see Tab. 3.

Step 1. In accordance with the aggregating values presented in Tab. 2 and the score functions of the developing technology enterprises are presented in Tab. 3.

Table 3: Score values of the aggregated values

	IHFPWA	*IHFPWG*
A_1	0.2112	0.1897
A_2	−0.0498	−0.1476
A_3	0.3706	0.196
A_4	0.118	0.1008
A_5	0.156	0.1196

Further, we examine ranking results of the score values.

Step 2. Approve the score functions presented in the Tab. 3, and compare the formula of score functions, the ranking of the developing technology enterprises as presented in Tab. 4. Remember this the greater than sign "$>$" implies "preference."

Table 4: Ranking results

Methods	Ranking
IHFPWA	$A_3 > A_1 > A_5 > A_4 > A_2$
IHFPWG	$A_3 > A_1 > A_5 > A_4 > A_2$

From above analysis, we get as best option alternative A_3.

Example 2: The amount from developments by an organization is legitimately corresponding to the standard of building substances they use. Appropriate review of building substance before development is the confirmation of good building measures. The building substances to be utilized ought to be carefully checked before applying. The best possible check and equalization arrangement of investigation approves the manufacturers to utilize the correct substances for developments to improve the standard of their task. Let five known building substances $P_r\,(r = 1, 2, 3, 4, 5)$ be as given in the IHFSs structure as follows:

$$P_1 = \left\{ \begin{array}{c} (x_1, \{\{0.2, 0.3, 0.5\}, \{0.3, 0.2\}\}), (x_2, \{\{0.3\}, \{0.6, 0.4\}\}), \\ (x_3, \{\{0.6\}, \{0.2\}\}), (x_4, \{\{0.15, 0.6\}, \{0.35\}\}) \\ (x_5, \{\{0.4\}, \{0.3\}\}) \end{array} \right\}$$

$$P_2 = \left\{ \begin{array}{c} (x_1, \{\{0.25\}, \{0.6, 0.3\}\}), (x_2, \{\{0.35\}, \{0.5\}\}), \\ (x_3, \{\{0.35, 0.45, 0.6\}, \{0.3, 0.1\}\}), (x_4, \{\{0.6, 0.5\}, \{0.2, 0.25\}\}) \\ (x_5, \{\{0.5\}, \{0.25, 0.4\}\}) \end{array} \right\}$$

$$P_3 = \left\{ \begin{array}{c} (x_1, \{\{0.3, 0.2\}, \{0.4, 0.5\}\}), (x_2, \{\{0.45, 0.5\}, \{0.15\}\}), \\ (x_3, \{\{0.6\}, \{0.35\}\}), (x_4, \{\{0.7\}, \{0.25, 0.15\}\}) \\ (x_5, \{\{0.1, 0.25, 0.4\}, \{0.5, 0.2\}\}) \end{array} \right\}$$

$$P_4 = \left\{ \begin{array}{c} (x_1, \{\{0.4, 0.6\}, \{0.15, 0.3\}\}), (x_2, \{\{0.3\}, \{0.6\}\}), \\ (x_3, \{\{0.4, 0.2, 0.6\}, \{0.2, 0.15\}\}), (x_4, \{\{0.5, 0.3\}, \{0.4\}\}) \\ (x_5, \{\{0.6, 0.55, 0.3\}, \{0.35, 0.15, 0.1\}\}) \end{array} \right\}$$

$$P_5 = \left\{ \begin{array}{c} (x_1, \{\{0.3, 0.2\}, \{0.5, 0.4\}\}), (x_2, \{\{0.45, 0.7\}, \{0.25\}\}), \\ (x_3, \{\{0.2\}, \{0.6\}\}), (x_4, \{\{0.7\}, \{0.15, 0.25\}\}) \\ (x_5, \{\{0.4, 0.25, 0.1\}, \{0.5, 0.5, 0.3\}\}) \end{array} \right\}$$

and

$$P = \left\{ \begin{array}{c} (x_1, \{\{\{0.7, 0.3\}, \{0.2, 0.1\}\}\}), (x_2, \{\{0.7\}, \{0.2\}\}), \\ (x_3, \{\{0.6, 0.5, 0.4\}, \{0.3, 0.1\}\}), (x_4, \{0.4, 0.6\}, \{0.15\}) \\ (x_5, \{\{0.5, 0.4, 0.2\}, \{0.3, 0.1, 0.2\}\}) \end{array} \right\}$$

By using the Eq. (56), we get the following values, which are summarized based on $(0.1, 0.15, 0.3, 0.2, 0.25)$:

$Dic_w(P_1, P) = 0.5723$, $Dic_w(P_2, P) = 0.7724$, $Dic_w(P_3, P) = 0.4536$,
$Dic_w(P_4, P) = 0.674$, $Dic_w(P_5, P) = 0.6457$

Ranking values of the above measures are summarized as follows:

$P_2 \geq P_4 \geq P_5 \geq P_1 \geq P_3$

Table 5: Comparative analysis of the explored and existing measures

Methods	Results	Ranking
Peng et al. [47]	$Dic_w(P_1, P) = 0.7165$, $Dic_w(P_2, P) = 0.8946$, $Dic_w(P_3, P) = 0.5674$, $Dic_w(P_4, P) = 0.8423$, $Dic_w(P_5, P) = 0.7728$	$P_2 \geq P_4 \geq P_5 \geq P_1 \geq P_3$
Beg et al. [44]	$Dic_w(P_1, P) = 0.7257$, $Dic_w(P_2, P) = 0.8979$, $Dic_w(P_3, P) = 0.5721$, $Dic_w(P_4, P) = 0.8472$, $Dic_w(P_5, P) = 0.7727$	$P_2 \geq P_4 \geq P_5 \geq P_1 \geq P_3$
Explored measures	$Dic_w(P_1, P) = 0.5723$, $Dic_w(P_2, P) = 0.7724$, $Dic_w(P_3, P) = 0.4536$, $Dic_w(P_4, P) = 0.674$, $Dic_w(P_5, P) = 0.6457$	$P_2 \geq P_4 \geq P_5 \geq P_1 \geq P_3$

The best option is P_2. Additionally, if we choose the intuitionistic hesitant fuzzy types of information's with existing conditions that are the sum of the maximum (also for minimum) of the truth grade and minimum (also for maximum) of the falsity grade cannot exceed from unit interval, and the sum of the maximum of the truth grade and falsity grade exceeds the unit interval, then it is very difficult to cope with such types of issues. But, when we choose the condition as in this explorative study then the sum of the maximum of the truth grade and falsity grade cannot exceed from the unit interval. The theories of intuitionistic fuzzy set and hesitant fuzzy set describe the foundation of the intuitionistic hesitant fuzzy set. When we choose the intuitionistic fuzzy types of information's or hesitant fuzzy types of information then the explored approach easily copes with it. But, if we choose the intuitionistic hesitant fuzzy types of information's, then the existing types of theories are cannot able to cope with it.

The comparative analysis of the explored measures with selected existing measures are summarized in Tab. 5.

From above analysis, the three different measures above share the same ranking values and the best option is P_2. The graphical representation for the information of Tab. 5, we explained with the help of Fig. 1.

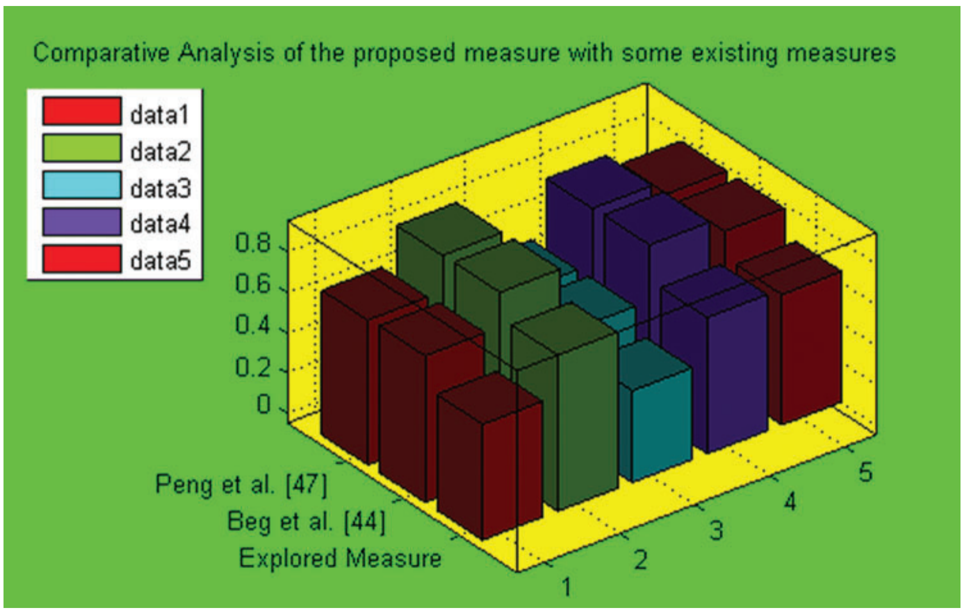

Figure 1: Graphical representation of the explored and existing measures

Fig. 1 represents the family of proposed and existing ideas and contains five types of values for each operator showing the family of alternatives. The alternative two provides the best values for all operators. For simplicity we have drawn the Fig. 1.

From the above analysis, the explored measures and operators based on IHFSs are more perfect and more proficient then existing methods and measures.

6 Conclusion

We explored the improved intuitionistic hesitant fuzzy set with a new condition that is the sum of the maximum of the truth grade and maximum of the falsity grade which cannot exceed from the unit interval. Additionally, we examine the multi-attribute decision making challenge built upon power aggregating operators with IHF data. So, inspired from the model of power aggregating operators, we established few power aggregation operators for aggregating IHF information: IHFPA operator, IHFPG operator, IHFPWA operator, IHFPWG operator, IHFPOWA operator, IHFPOWG operator, IHFPHA operator, and IHFPHG operator. Additionally, some similarity measures based on IHFSs are also explored and their special cases discussed. Outstanding feature of these recommended operators are examined. So, we used operators to establish few methods to resolve the IHF multi attribute DM difficulties. A helpful example is presented to confirm the established methodology and to determine its practicability and efficiency. The advantages, comparative analysis, and geometrical representation of the presented works are also discussed in detailed.

Notably, that the article results of the article can be expanded to the IvIHF situation and further fuzzy situations. In superior study, it is enough to get the implementation of these operators to resolve the actual DM drawbacks as fuzzy investigation, unsure programming and image recognition, etc. We must also deal with few new operators for the foundation of PHFNs for example, modify them to complex q-rung fuzzy aggregation operators [48–50], complex Pythagorean fuzzy set [51], spherical fuzzy operators [52].

Data Availability: The data used in this article are artificial and hypothetical, and anyone can use these data before prior permission by just citing this article.

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