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Article

A Practical Quantum Network Coding Protocol Based on Non-Maximally Entangled State

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Abstract: In many earlier works, perfect quantum state transmission over the butterfly network can be achieved via quantum network coding protocols with the assist of maximally entangled states. However, in actual quantum networks, a maximally entangled state as auxiliary resource is hard to be obtained or easily turned into a non-maximally entangled state subject to all kinds of environmental noises. Therefore, we propose a more practical quantum network coding scheme with the assist of non-maximally entangled states. In this paper, a practical quantum network coding protocol over grail network is proposed, in which the non-maximally entangled resource is assisted and even the desired quantum state can be perfectly transmitted. The achievable rate region, security and practicability of the proposed protocol are discussed and analyzed. This practical quantum network coding protocol proposed over the grail network can be regarded as a useful attempt to help move the theory of quantum network coding towards practicability.

Keywords: Quantum network coding; non-maximally entangled state; quantum grail network; practical protocol

1 Introduction

Classical network coding (CNC) [1], with many years of development, has made significant advances in classical network communications [2–4]. As a breakthrough technology, CNC can effectively improve the network communication efficiency since it can achieve the maximum flow network communication and reduce the bandwidth resource consumption. In 2007, Hayashi et al. [5] first introduced this idea into quantum networks, creating a new technology called quantum network coding (QNC). QNC has now become an important research direction related to the field of quantum communication and quantum information processes. Just like the CNC, QNC can solve the transmission congestion over quantum networks, gaining higher quantum communication efficiency [6–8] and achieving larger quantum network throughput [9–11] than the traditional technology of routing.

In Hayashi et al. foundation work [5] of QNC, it is proved that quantum states can not be perfectly transmitted through the network without the assistance of auxiliary resources. Thus, in recent years, there have been more researches on the perfect QNC assisted with auxiliary resources. In general, the representative resources introduced into the QNC schemes mainly include prior entanglement [12–14] and classical communication [15–17]. For the prior entanglement, in 2007, Hayashi [18] first introduced this kind of auxiliary resources into the QNC scheme over the butterfly network. Afterwards, several different kinds of perfect QNC schemes assisted with prior entanglement were proposed in [19,20]. For classical communication, in 2009, Kobayashi et al. [21] first explored the perfect QNC scheme assisted with this kind of auxiliary resources, based on the linear CNC. Subsequently, various QNC schemes assisted with classical communication have been proposed in [22,23] to achieve perfect transmission of quantum states. In 2019, Li et al. [24] proposed an efficient quantum state transmission scheme via perfect quantum network coding, in which auxiliary resources of both maximally entangled state and classical communication are assisted. Through the analysis of the amounts of the introduced auxiliary resources including prior entanglement and classical communication, the QNC scheme in [24] reached the highest level of quantum communication efficiency so far.

However, on the one hand, the network models including butterfly network and quantum k-pair network studied in [18–24] are homogeneous, since the quantum k-pair network is virtually extended from the butterfly network. On the other hand, in the QNC schemes of [18–20,24], the ideal situation was considered, where the maximally entangled state was introduced as the auxiliary entanglement resource. Hence, we have been trying to propose a more practical QNC scheme without reducing quantum communication efficiency. It is well known, as a kind of general entanglement with representation, non-maximally entangled state is more common in practice and hard to be distinguished. Therefore, it is reasonable to believe that non-maximally entangled state is contributed to improving the availability and security of the QNC.

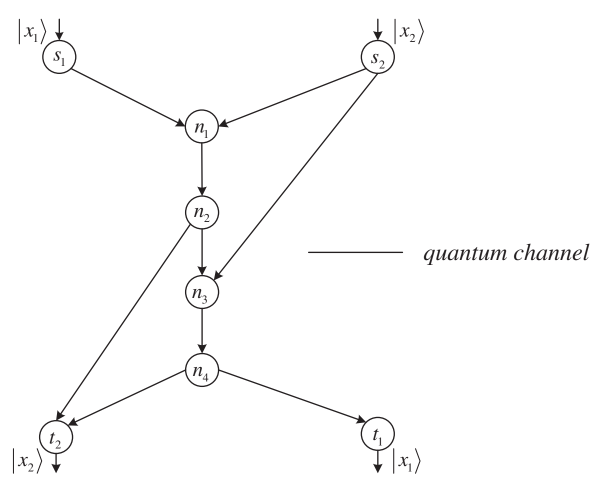

Figure 1: Quantum grail network

This work emphasizes on the proposal of a practical QNC scheme over the quantum grail network illustrated in Fig. 1 with the assist of non-maximally entangled state and classical communication. From the network model, the quantum grail network we considered is rarely studied but fairly imperative since it is another fundamental primitive network [25]. From the non-maximally entangled state, it is a kind of entanglement resource that can be more easily obtained in practice, which helps our QNC scheme better suited to applications. Besides, by the use of our proposed QNC scheme, the desired quantum states can be perfectly transmitted through the network, helping to expand the existed theory of QNC.

2 A Practical QNC Protocol Based on Non-Maximally Entangled State

In [25], grail network is viewed as a fundamental primitive network for CNC like butterfly network. Also like "butterfly network," the network is named "grail network" because the network model is shaped like a "grail." A typical communication task for CNC over grail network can be treated as the bottleneck problem like butterfly network. Applying that analogy to quantum network, the quantum communication task for QNC over quantum grail network can be treated as the quantum bottleneck problem. The specific quantum network model is illustrated in Fig. 1. It can be considered as a directed acyclic network (DAN). This DAN consists of a directed acyclic graph (DAG) $G=(V, E)$ and the edge quantum capacity function $c\colon E \to \mathbb{Z}^{+}$, where V is the set of nodes while E is the set of edges that connect pairs of nodes in V. Herein, we discuss the practical QNC scheme over this quantum grail network on d-dimension Hilbert space $\boldsymbol{\mathcal{H}}=\mathbb{C}^{d}$ directly. According to the communication task of QNC, two source nodes s_1, s_2 needs to transmit two arbitrary qudit state $|x_1\rangle$, $|x_2\rangle \in \boldsymbol{\mathcal{H}}$ to the sink nodes t_1, t_2 simultaneously and respectively through the network under the condition that $c(e) \equiv 1$, $e \in E$, i.e., each edge of the network can transmit no more than one qudit state over $\boldsymbol{\mathcal{H}}$.

Suppose in the quantum grail network, for $i \in \{1, 2\}$, each of the source nodes s_i possesses one quantum register S_i while each of the sink nodes t_i possesses one quantum register T_i. Quantum register S_i can be considered to be received from a virtual incoming edge and T_i can be considered to be transmitted to a virtual outcoming edge. Before proposing our QNC protocol, the auxiliary entanglement resources of two identical non-maximally entangled states are formed as

$$\begin{aligned} |\phi\rangle_{N_1N_2} &= \sum_{m\in\mathbb{Z}_d} \beta_m |m, m\rangle_{N_1N_2} \\ |\phi\rangle_{N_3N_4} &= \sum_{n\in\mathbb{Z}_d} \gamma_n |n, n\rangle_{N_3N_4} \end{aligned} \tag{1}$$

are pre-shared between the intermediate nodes n_1 and n_2 (n_3 and n_4) respectively, where the β_m (γ_n) are unequal complex numbers such that $\sum_{m\in\mathbb{Z}_d} \beta_m = 1$ ($\sum_{n\in\mathbb{Z}_d} \gamma_n = 1$), and the N_1, N_2, N_3, N_4 represent the four quantum registers introduced at the corresponding nodes. Besides, for convenience, the two arbitrary qudit states initially possessed at the two source nodes can be written as an entire quantum system formed as

$$|\Psi\rangle_S = \sum_{x_1, x_2\in Z_d} \alpha_{x_1, x_2} |x_1 x_2\rangle_{S_1 S_2}, \tag{2}$$

where the coefficients α_{x_1, x_2} are complex numbers such that $\sum_{x_1, x_2\in\mathbb{Z}_d} |\alpha_{x_1, x_2}|^2 = 1$. Then, the initial state over the whole network before the transmission can be written as

$$|\Psi\rangle_0 = \sum_{m, n\in\mathbb{Z}_d} \beta_m \gamma_n |\Psi\rangle_S \otimes |m, m\rangle_{N_1, N_2} \otimes |n, n\rangle_{N_3 N_4}. \tag{3}$$

Next, we will describe the specific processes of the practical QNC protocol based on the non-maximally entangled state over the quantum grail network in detail. The corresponding QNC model over the grail network is illustrated in Fig. 2.

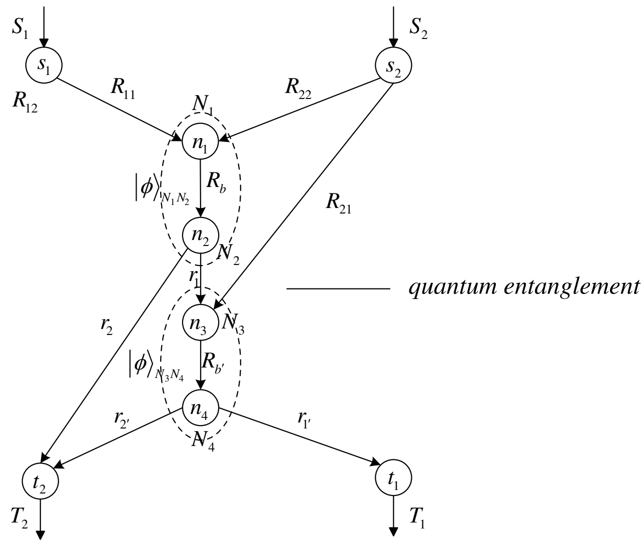

Figure 2: QNC model over quantum grail network

2.1 Encoding

In this process, the object is to make the particles in the quantum registers mutually entangled in the network topological order. Here, the quantum circuit of encoding is shown in Fig. 3 and the detailed steps are given below.

(S1) For $i, j \in \{1, 2\}$, quantum registers R_{ij}, each initialized to $|0_{\mathcal{H}}\rangle$, are introduced at each source node s_i, and then the operator $\widetilde{CX}_{S_i \to R_{ii}}$ is applied to the registers S_i and R_{ii}, operator $\widetilde{CR}_{S_i \to R_{ij}}$ is applied to the registers S_i and $R_{ij}\,(j \neq i)$. Here, quantum operator $\widetilde{CX}_{A \to B}$ is defined as $\widetilde{CX}_{A \to B} := \sum_{i \in \mathbb{Z}_d} |i\rangle\langle i|_A \otimes X_B^i$, where $X|i\rangle = |i \oplus 1 \mod d\rangle$ is an analogue on qudits of the unitary Pauli operator σ_x on qubits [26]. Quantum operator $\widetilde{CR}_{A \to B}$ is defined as $\widetilde{CR}_{A \to B} := \sum_{i \in \mathbb{Z}_d} |i\rangle\langle i|_A \otimes R_B^i$, where $R|i\rangle = |i - 1 \mod d\rangle$ is the reverse transformation of X on qudits. Thus, the whole quantum system state becomes

$$|\Psi\rangle_1 = \sum_{x_1, x_2, m, n \in \mathbb{Z}_d} \alpha_{x_1, x_2} \beta_m \gamma_n |m, m, n, n\rangle_{N_1 N_2 N_3 N_4} \bigotimes_{i,j=1, j \neq i}^{2} |x_i\rangle_{S_i} |x_i\rangle_{R_{ii}} |-x_i\rangle_{R_{ij}}. \tag{4}$$

Then, quantum registers R_{ii} are sent from each node s_i to the intermediate node n_1, register R_{21} is sent to the intermediate node n_3, register R_{12} is kept at node s_1, and registers S_i are kept at node s_i. Meanwhile, ancillary register R_b initialized to $|0_{\mathcal{H}}\rangle$ is introduced at node n_1.

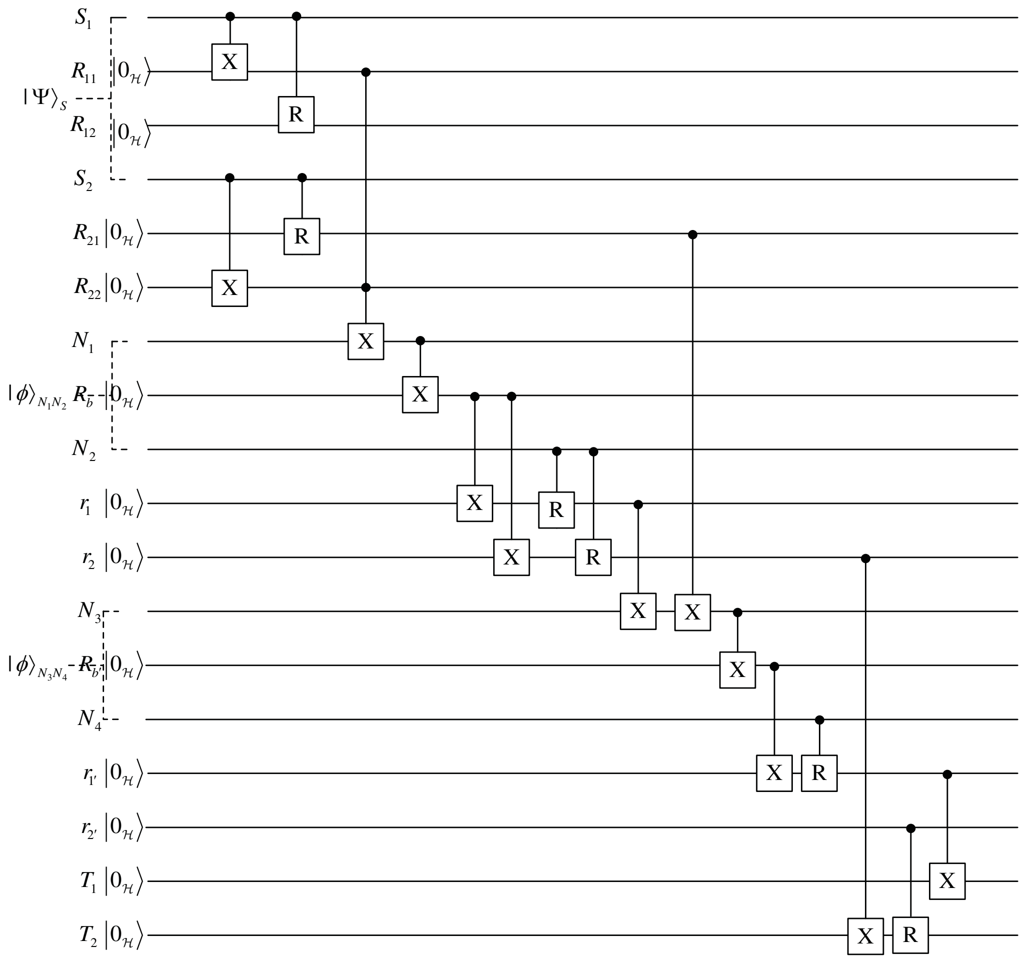

Figure 3: Quantum circuit of encoding

(S2) For $i \in \{1, 2\}$, applying $\widetilde{CX}_{R_{ii}\to N_1}$ on the registers R_{ii} and N_1, then $\widetilde{CX}_{N_1\to R_b}$ on the registers N_1 and R_b at the intermediate node n_1, we have the quantum state

$$|\Psi\rangle_2 = \sum_{x_1, x_2, m, n \in \mathbb{Z}_d} \alpha_{x_1, x_2}\beta_m\gamma_n |\overline{\mathbf{X}}\rangle_{N_1, R_b, N_2} |n, n\rangle_{N_3N_4} \bigotimes_{i,j=1, j\neq i}^{2} |x_i\rangle_{S_i} |x_i\rangle_{R_{ii}} |-x_i\rangle_{R_{ij}}, \tag{5}$$

where $|\overline{\mathbf{X}}\rangle_{N_1, R_b, N_2} = |x_1 \oplus x_2 \oplus m,\ x_1 \oplus x_2 \oplus m,\ m\rangle_{N_1, R_b, N_2}$. Then, quantum register R_b is sent from the node n_1 to n_2, registers R_{ii} and N_1 are kept at n_1.

(S3) At the intermediate node n_2, quantum registers r_i $(i = 1, 2)$, each initialized to $|0_{\mathcal{H}}\rangle$, are introduced; then the quantum operator $\widetilde{CX}_{R_b\to r_i}$ is applied to the registers R_b and r_i, and $\widetilde{CR}_{N_2\to r_i}$ is applied to the registers N_2 and r_i. Thus, the quantum state becomes

$$|\Psi\rangle_3 = \sum_{x_1, x_2, m, n \in \mathbb{Z}_d} \alpha_{x_1, x_2}\beta_m\gamma_n |\overline{\mathbf{X}}\rangle_{N_1, R_b, N_2} |n, n\rangle_{N_3N_4} \bigotimes_{i,j=1, j\neq i}^{2} |x_1 \oplus x_2\rangle_{r_i} |x_i\rangle_{S_i} |x_i\rangle_{R_{ii}} |-x_i\rangle_{R_{ij}}. \tag{6}$$

Then, quantum register r_1, r_2 are transmitted from the node n_2 to node n_3 and to the sink node t_2 respectively, the registers R_b, N_2 are maintained at n_2.

(S4) At the intermediate node n_3, quantum registers $R_{b'}$ initialized to $|0_{\mathcal{H}}\rangle$ is introduced. Applying quantum operator $\widetilde{CX}_{r_1\to N_3}$ and $\widetilde{CX}_{R_{21}\to N_3}$ on the registers r_1, R_{21} and N_3, and then $\widetilde{CX}_{N_3\to R_{b'}}$ on the registers N_3 and $R_{b'}$, we have the quantum state

$$|\Psi\rangle_4 = \sum_{x_1,x_2,m,n\in\mathbb{Z}_d} \alpha_{x_1,x_2}\beta_m\gamma_n|\overline{\mathbf{X}}\rangle_{N_1,R_b,N_2}|\overline{\mathbf{Y}}\rangle_{N_3,R_{b'},N_4} \bigotimes_{i,j=1,j\neq i}^{2} |x_1\oplus x_2\rangle_{r_i}|x_i\rangle_{S_i}|x_i\rangle_{R_{ii}}|-x_i\rangle_{R_{ij}}, \tag{7}$$

where $|\overline{\mathbf{Y}}\rangle_{N_3,R_{b'},N_4} = |x_1\oplus n, x_1\oplus n, n\rangle_{N_3,R_{b'},N_4}$. Then, quantum register $R_{b'}$ is sent from the node n_3 to n_4, registers r_1, N_3 and R_{21} are kept at n_3.

(S5) At the intermediate node n_4, quantum registers $r_{i'}$ $(i\in\{1, 2\})$, each initialized to $|0_{\mathcal{H}}\rangle$, are introduced; then the quantum operator $\widetilde{CX}_{r_{b'}\to r_{i'}}$ and $\widetilde{CR}_{N_4\to r_{i'}}$ is applied to the registers $r_{b'}$, N_4 and $r_{1'}$. Thus, the quantum state becomes

$$|\Psi\rangle_5 = \sum_{x_1,x_2,m,n\in\mathbb{Z}_d} \alpha_{x_1,x_2}\beta_m\gamma_n|\overline{\mathbf{X}}\rangle_{N_1,R_b,N_2}|\overline{\mathbf{Y}}\rangle_{N_3,R_{b'},N_4} \bigotimes_{i,j=1,j\neq i}^{2} |x_1\oplus x_2\rangle_{r_i}|x_1\rangle_{r_{i'}}|x_i\rangle_{S_i}|x_i\rangle_{R_{ii}}|-x_i\rangle_{R_{ij}}. \tag{8}$$

Then, quantum register $r_{1'}, r_{2'}$ are transmitted from the node n_4 to the sink node t_1 and t_2 respectively, the registers $R_{b'}$, N_4 are maintained at n_4.

(S6) For each sink node $(i\in\{1, 2\})$, the quantum register T_i initialized to $|0_{\mathcal{H}}\rangle$ is introduced. Remembering that t_2 has received register r_2 in step (S3) and register $r_{2'}$ in step (S5), the quantum operator $\widetilde{CX}_{r_2\to T_2}$ is applied to r_2 and T_2, $\widetilde{CR}_{r_{2'}\to T_2}$ is applied to $r_{2'}$ and T_2 at the sink node t_2. Simultaneously, the quantum operator $\widetilde{CX}_{r_{1'}\to T_1}$ is applied to $r_{1'}$ and T_1 at the sink node t_1.

Hence, the resulting state becomes

$$|\Psi\rangle_6 = \sum_{x_1,x_2,m,n\in\mathbb{Z}_d} \alpha_{x_1,x_2}\beta_m\gamma_n|\overline{\mathbf{X}}\rangle_{N_1,R_b,N_2}|\overline{\mathbf{Y}}\rangle_{N_3,R_{b'},N_4} \bigotimes_{i,j=1,j\neq i}^{2} |x_1\oplus x_2\rangle_{r_i}|x_1\rangle_{r_{i'}}|x_i\rangle_{T_i}|x_i\rangle_{S_i}|x_i\rangle_{R_{ii}}|-x_i\rangle_{R_{ij}}. \tag{9}$$

2.2 Decoding

In this process, the object is to remove all the entangled particles in the network topological order. Here, the quantum circuit of decoding is shown in Fig. 4 and the detailed steps are given as below.

(T1) Considering the owned registers $R_{b'}$, N_4, the intermediate node n_4 performs the quantum operation $\widetilde{CX}_{R_{b'}\to N_4}$, followed by the Bell measurement on the two qudits, providing the measurement result u_1u_2. Here, it is worth mentioning that in the quantum system $\mathcal{H}=\mathbb{C}^d$, the Bell states are represented as follows:

$$|\phi\,(u_1, u_2)\rangle = \frac{1}{\sqrt{d}}\sum_{j=0}^{d-1} e^{2\pi\iota j u_1/d}|j, j\oplus u_2\rangle,\ u_1, u_2\in\mathbb{Z}_d, \quad \textit{where}\ \iota^2=-1. \tag{10}$$

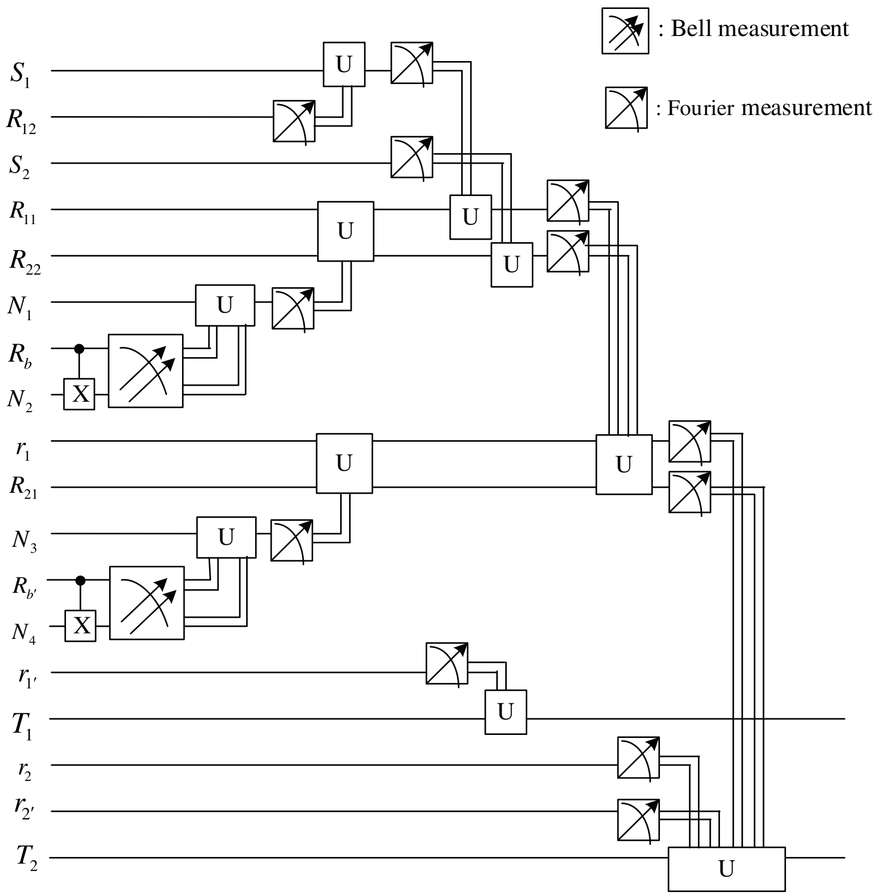

Figure 4: Quantum circuit of decoding

Then, the basis states $\{|\phi(u_1, u_2)\rangle\}_{u_1, u_2 \in \mathbb{Z}_d}$ are called the Bell basis, and the quantum measurement in the Bell basis is called the Bell measurement.

Hence after the Bell measurement, we obtain the quantum state

$$|\Psi\rangle_7 = \sum_{x_1, x_2, m \in \mathbb{Z}_d} \alpha_{x_1, x_2} \beta_m e^{-2\pi\iota(x_1 \oplus u_2)u_1/d} |\overline{\mathbf{X}}\rangle_{N_1, R_b, N_2} |x_1 \oplus u_2\rangle_{N_3} \bigotimes_{i,j=1, j \neq i}^{2} |x_1 \oplus x_2\rangle_{r_i} |x_1\rangle_{r_{i'}} |x_i\rangle_{T_i} |x_i\rangle_{S_i}$$
$$\otimes |x_i\rangle_{R_{ii}} |-x_i\rangle_{R_{ij}} \tag{11}$$

Then, classical information $u_1 u_2$ are transmitted from the node n_4 to n_3 through the bottleneck channel.

(T2) Upon receiving the information $u_1 u_2$, the node n_3 applies the quantum unitary operator on its register N_3, mapping the state $|x\rangle$ to $e^{2\pi\iota u_1 x/d}|x - u_2\rangle$ for each $x \in \mathbb{Z}_d$. Thus, the phase resulting from the Bell measurement in (T1) is corrected. Next, quantum Fourier measurement is performed on N_3, providing the measurement result l. Here, it is worth mentioning that in the quantum system $\mathcal{H} = \mathbb{C}^d$, quantum Fourier transform $\mathcal{F}$ is a unitary transformation that transforms the computing basis states $\{|k\rangle\}_{k\in\mathbb{Z}_d}$ to the Fourier basis as follows:

$$|w_k\rangle = \mathcal{F}|k\rangle = \frac{1}{\sqrt{d}}\sum_{l=0}^{d-1} e^{2\pi\iota kl/d}|l\rangle, \quad where\ \iota^2 = -1. \tag{12}$$

Thus the basis states $\{|w_k\rangle\}_{k\in\mathbb{Z}_d}$ are called the quantum Fourier basis, and the quantum measurement in the Fourier basis is called the quantum Fourier measurement. Hence after the quantum Fourier measurement, we obtain the quantum state

$$|\Psi\rangle_{8'} = \sum_{x_1,x_2,m\in\mathbb{Z}_d} \alpha_{x_1,x_2}\beta_m e^{-2\pi\iota x_1 l/d}|\overline{\mathbf{X}}\rangle_{N_1,R_b,N_2} \bigotimes_{i,j=1,j\neq i}^{2} |x_1\oplus x_2\rangle_{r_i}|x_1\rangle_{r_{i'}}|x_i\rangle_{T_i}|x_i\rangle_{S_i}|x_i\rangle_{R_{ii}}|-x_i\rangle_{R_{ij}}. \tag{13}$$

Then, the phase introduced is corrected as followings: the node n_3 applies the unitary operator on its registers r_1 and R_{21}, mapping the state $|x_1\oplus x_2, -x_2\rangle$ to the state $e^{2\pi\iota lx_1/d}|x_1\oplus x_2, -x_2\rangle$ for any $x_1, x_2 \in \mathbb{Z}_d$. Consequently, the state then becomes

$$|\Psi\rangle_8 = \sum_{x_1x_2,m\in\mathbb{Z}_d} \alpha_{x_1,x_2}\beta_m|\overline{\mathbf{X}}\rangle_{N_1,R_b,N_2} \bigotimes_{i,j=1,j\neq i}^{2} |x_1\oplus x_2\rangle_{r_i}|x_1\rangle_{r_{i'}}|x_i\rangle_{T_i}|x_i\rangle_{S_i}|x_i\rangle_{R_{ii}}|-x_i\rangle_{R_{ij}}. \tag{14}$$

(T3) The intermediate node n_2 performs the quantum operation $\widetilde{CX}_{R_b\to N_2}$, followed by the Bell measurement on the two qudits, providing the measurement result $u_{1'}u_{2'}$. Thus, we obtain the quantum state

$$|\Psi\rangle_9 = \sum_{x_1,x_2\in\mathbb{Z}_d} \alpha_{x_1,x_2} e^{-2\pi\iota(x_1\oplus x_2\oplus u_{2'})u_{1'}/d}|x_1\oplus x_2\oplus u_{2'}\rangle_{N_1} \bigotimes_{i,j=1,j\neq i}^{2} |x_1\oplus x_2\rangle_{r_i}|x_1\rangle_{r_{i'}}|x_i\rangle_{T_i}$$
$$\otimes |x_i\rangle_{S_i}|x_i\rangle_{R_{ii}}|-x_i\rangle_{R_{ij}}. \tag{15}$$

Then, classical information $u_{1'}u_{2'}$ are transmitted from the node n_2 to n_1 through the bottleneck channel.

(T4) Once receiving the information $u_{1'}u_{2'}$, node n_1 applies the quantum unitary operator on its register N_1, mapping the state $|x\rangle$ to $e^{2\pi\iota u_{1'}x/d}|x-u_{2'}\rangle$ for each $x\in\mathbb{Z}_d$. Then, quantum Fourier measurement is performed on registers and N_1 , producing the measurement result l'. Hereafter, The phase introduced is corrected as followings: the node n_1 applies the unitary operator on its registers R_{ii} $(i = 1, 2)$, mapping the state $|x_1, x_2\rangle$ to the state $e^{2\pi\iota(x_1\oplus x_2)l'/d}|x_1, x_2\rangle$. Then, the resulting state becomes

$$|\Psi\rangle_{10} = \sum_{x_1,x_2\in\mathbb{Z}_d} \alpha_{x_1,x_2} \bigotimes_{i,j=1,j\neq i}^{2} |x_1\oplus x_2\rangle_{r_i}|x_1\rangle_{r_{i'}}|x_i\rangle_{T_i}|x_i\rangle_{S_i}|x_i\rangle_{R_{ii}}|-x_i\rangle_{R_{ij}}. \tag{16}$$

(T5) At the source node s_1, first the quantum Fourier measurement is applied to register R_{12}, and then the phase introduced is corrected at the register S_1. Afterwards, quantum Fourier measurements are simultaneously applied to the registers S_i $(i=1,2)$, returning the measurement results h_i. As result, the whole quantum state becomes

$$|\Psi\rangle_{11}=\sum_{x_1,x_2\in\mathbb{Z}_d}\alpha_{x_1,x_2}\prod_{i=1}^{2}e^{-2\pi\iota h_i x_i/d}\bigotimes_{i,j=1,j\neq i}^{2}|x_1\oplus x_2\rangle_{r_i}|x_1\rangle_{r_{i'}}|x_i\rangle_{T_i}|x_i\rangle_{R_{ii}}|-x_2\rangle_{R_{21}}. \tag{17}$$

Then, h_i are transmitted from the node s_i to n_1 respectively.

(T6) Upon receiving h_i, the intermediate node n_1 corrects the phase by performing the quantum unitary operator mapping on its register R_{ii}, wherein the state $|x_i\rangle$ is mapped to $e^{2\pi\iota h_i x_i/d}|x_i\rangle$ for each $x_i\in\mathbb{Z}_d$. Hereafter, quantum Fourier measurements are applied to the registers R_{ii} respectively, thereby producing the measurement results g_i. Thus the state then becomes

$$|\Psi\rangle_{12}=\sum_{x_1,x_2\in\mathbb{Z}_d}\alpha_{x_1,x_2}\prod_{i=1}^{2}e^{-2\pi\iota g_i x_i/d}\bigotimes_{i,j=1,j\neq i}^{2}|x_1\oplus x_2\rangle_{r_i}|x_1\rangle_{r_{i'}}|x_i\rangle_{T_i}|-x_2\rangle_{R_{21}}. \tag{18}$$

Then, g_i are transmitted from the node n_1 to n_3 past n_2 respectively.

(T7) At the intermediate node n_3, to correct the phase produced by the measurements, it applies the unitary operator on its register r_1 and R_{21}, mapping the state $|x_1\oplus x_2,-x_2\rangle$ to the state $e^{2\pi\iota[g_1(x_1\oplus x_2)-(g_1-g_2)x_2]/d}|x_1\oplus x_2,-x_2\rangle$. Hereafter, quantum Fourier measurements are applied to the registers r_1 and R_{21} respectively, then after the measurement results' transmission, the sink node t_2 correct the introduced phase. Afterwards, the sink node t_1 and t_2 applies quantum Fourier measurements on the registers $r_{1'}$ and r_2, $r_{2'}$ respectively. Finally, the introduced phases are corrected at the two sink node. Thus, the final quantum state becomes the desired state, as follows:

$$|\Psi\rangle_{13}=\sum_{x_1,x_2\in\mathbb{Z}_d}\alpha_{x_1,x_2}|x_1x_2\rangle_{T_1T_2}. \tag{19}$$

That is, the state of the quantum system over every source node is perfectly transmitted to the corresponding sink node through the quantum grail network.

3 Protocol Analysis

3.1 Correctness

The correctness of the proposed QNC protocol can be verified by the specific encoding and decoding steps. From Section 2, in the encoding process, the particles at every network node are entangled to the whole quantum system by applying relevant quantum operators on them. The resulting quantum state after the entanglement of each time is presented in the ending of every encoding steps. In the decoding process, by applying relevant quantum measurements, all the unnecessary particles are disentangled from the whole quantum system and leave alone the certain particles on the two sink nodes. The resulting quantum state after the disentanglement of each time is presented in the ending of every decoding steps. Thus, after all the encoding and decoding steps, the final quantum state at the two sink nodes formed $|\Psi\rangle_{13}=\sum_{x_1,x_2\in Z_d}\alpha_{x_1,x_2}|x_1x_2\rangle_{T_1T_2}$ is exactly equal to the initial source state $|\Psi\rangle_S=\sum_{x_1,x_2\in Z_d}\alpha_{x_1,x_2}|x_1x_2\rangle_{S_1S_2}$ at the two source nodes. Therefore, according to all the calculating procedure and numerical results, the correctness of the proposed QNC protocol is verified.

3.2 Achievable Rate Region

It is known that the communication rate [25] between s_i and t_i in n network uses is defined as $r_i^{(n)} = \frac{1}{n} log |\mathcal{H}_i|$, where $\mathcal{H}_i$ denotes the Hilbert space of the transmitted quantum state owned by s_i, and $|\cdot|$ denotes the dimension of the Hilbert space. Also, an edge capacity constraint [27], i.e., $log |\mathcal{H}_{(u,v)}| \leq n \cdot c((u, v))$, exists when the quantum state is transmitted with the fidelity of one over the edge $(u, v) \in E$ in n uses.

Accordingly, in our protocol presented above, the perfect transmission of the quantum state over the quantum grail network can be achieved in one use of the network, which means that the 1-flow [25] value reaches

$$r_1^{(1)} + r_2^{(1)} = log |\mathcal{H}_1| + log |\mathcal{H}_2| \leq \sum_{i=1}^{2} c((u, v)) = \sum_{i=1}^{2} 1 = 2, \quad (20)$$

under the condition that the capacity $c((u, v))$ of each edge (u, v) always remains equal to 1 according to the quantum grail network model. In fact, the 1-max flow is the supremum of 1-flow over all achievable rate. Hence, 1-max flow of value 2 is achievable through our PQNC protocol, and then the achievable rate region [25,28] can be written as $\{(r_1, r_2) | r_1 + r_2 \leq 2\}$.

3.3 Security

As is well known, the non-maximally entangled state is a kind of generalized entangled state, and is hard to be distinguished [29–31]. In the actual quantum communications, it is difficult for adversaries to launch attacks by forging the non-maximally entangled state. Therefore, the non-maximally entangled states which are pre-shared over the network can effectively improve the security of the whole quantum network communications.

3.4 Practicability

In terms of the network model, the quantum grail network we considered is rarely studied but fairly imperative since it is also a fundamental primitive network [25] like butterfly network. And the proposed protocol over quantum grail network can also be applied to the butterfly network. Thus, it is applicable to the communication scenarios of practically complex quantum networks. On the other hand, in terms of the non-maximally entangled state, it is a kind of entanglement resource that can be more easily obtained in practice, which helps our QNC scheme better suited to applications.

4 Protocol Comparison

In this section, our proposed QNC protocol is compared with the existed QNC protocols [18,19,24,25] from the network model, the entanglement resource type, the amount of entanglement resource, and the success probability. The comparison result is shown in Tab. 1 as below.

From the comparison result, it can be seen that for butterfly network, Hayashi's protocol [18] and Li et al. [24] protocol show that maximally entangled states can be used as the assisted resource to obtain the perfect quantum state transmission with success probability 1. Ma et al. [19] protocol shows the success probability of which assisted by non-maximally entangled states is less than 1. For grail network, Akibue et al. [25] protocol shows that maximally entangled states also can be assisted to obtain the perfect quantum state transmission with success probability 1 but consumed more. However, our protocol shows that non-maximally entangled states can also be assisted to obtain the perfect quantum state transmission with success probability 1, and even the resource consumption is lower. Therefore, compared with the existed protocols, our protocol expresses a certain advantage.

Table 1: Comparison result of different QNC protocols

QNC protocols	Network model	Entanglement resource type	The amount of entanglement resource	Success probability
Hayashi [18]	Butterfly network	Maximal	2 Pairs	1
Li et al. [24]	Butterfly network	Maximal	1 Pair	1
Ma et al. [19]	Butterfly network	Non-maximal	2 Pairs	<1
Akibue et al. [25]	Grail network	Maximal	9 Pairs	1
Ours	Grail network	Non-maximal	2 Pairs	1

5 Conclusions

In this paper, we propose a practical QNC scheme with the assist of the non-maximally entangled state over the grail network. Firstly, in terms of the network model, the grail network is another fundamental primitive network [25]. The research on the QNC scheme over grail network can effectively enrich the existing theory of QNC. Secondly, our proposed QNC scheme with the assist of non-maximally entangled state can also achieve the perfect quantum state transmission and 1-max flow quantum communications. Moreover, due to the security and practicability of the non-maximally entangled state, our QNC scheme is more applicable for actual quantum network communications.

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