Computers, Materials & Continua
DOI: 10.32604/cmc.2023.033006
Article

(α, γ)-Anti-Multi-Fuzzy Subgroups and Some of Its Properties

Memet Şahin[1], Vakkas Uluçay[2], S. A. Edalatpanah[3,*], Fayza Abdel Aziz Elsebaee[4] and Hamiden Abd El-Wahed Khalifa[5]

[1]Department of Mathematics, Gaziantep University, Gaziantep, Turkey
[2]Department of Mathematics, Kilis 7 Aralık University, Kilis, Turkey
[3]Department of Applied Mathematics, Ayandegan Institute of Higher Education, Tonekabon, Iran
[4]Department of Mathematics, Helwan University, Cairo, Egypt
[5]Department of Mathematics, Qassim University, Alasyah, Saudi Arabia
*Corresponding Author: S. A. Edalatpanah. Email: saedalatpanah@gmail.com
Received: 04 June 2022; Accepted: 09 August 2022

Abstract: Recently, fuzzy multi-sets have come to the forefront of scientists' interest and have been used in algebraic structures such as multi-groups, multi-rings, anti-fuzzy multigroup and (α, γ)-anti-fuzzy subgroups. In this paper, we first summarize the knowledge about the algebraic structure of fuzzy multi-sets such as (α, γ)-anti-multi-fuzzy subgroups. In a way, the notion of anti-fuzzy multigroup is an application of anti-fuzzy multi sets to the theory of group. The concept of anti-fuzzy multigroup is a complement of an algebraic structure of a fuzzy multi set that generalizes both the theories of classical group and fuzzy group. The aim of this paper is to highlight the connection between fuzzy multi-sets and algebraic structures from an anti-fuzzification point of view. Therefore, in this paper, we define (α, γ)-anti-multi-fuzzy subgroups, (α, γ)-anti-multi-fuzzy normal subgroups, (α, γ)-anti-multi-fuzzy homomorphism on (α, γ)-anti-multi-fuzzy subgroups and these been explicated some algebraic structures. Then, we introduce the concept (α, γ)-anti-multi-fuzzy subgroups and (α, γ)-anti-multi-fuzzy normal subgroups and of their properties. This new concept of homomorphism as a bridge among set theory, fuzzy set theory, anti-fuzzy multi sets theory and group theory and also shows the effect of anti-fuzzy multi sets on a group structure. Certain results that discuss the (α, γ) cuts of anti-fuzzy multigroup are explored.

Keywords: Fuzzy set; anti-fuzzy multi set; anti-fuzzy multi subgroup; anti-fuzzy multi normal subgroup

1 Introduction

Dresher et al. [1] laid the foundations of the theory of multigroup in 1938. Zadeh [2] introduced the concept of a fuzzy subset of a set, fuzzy set are a kind of useful mathematical structure to represent a collection of objects whose boundary is uncertainty in 1965. Therefore, on the basis of fuzzy set theory, Sebastian et al. [3] introduced Multi-Fuzzy Sets, Atanassov [4] proposed intuitionistic fuzzy set theory, Shinoj et al. [5] initiated intuitionistic fuzzy multisets. Recently, the above theories have developed in many directions and found its applications in a wide variety of fields including algebraic structures. For example, on fuzzy sets [6–8], on fuzzy multi sets [9–11] on anti-fuzzy group theory [12–17] are some of the selected works. Rosenfeld [18] defined the notion of fuzzy subgroup. Biswas [19] introduced the concept of anti-fuzzy subgroup of group. Yuan et al. [20] introduced the concept of fuzzy subgroup with thresholds. A fuzzy subgroup with thresholds lambda and mu is also called a (lambda, mu)-fuzzy subgroup. Yao [21] defined (lambda, mu)-fuzzy normal subgroups and (lambda, mu)-fuzzy quotient subgroups these examined some properties. On these studies, Shen [22] defined anti-fuzzy subgroups and Dong [23] introduced the product of anti-fuzzy subgroups. Then, Feng et al. [24] introduces the notion of (lambda, mu)-anti-fuzzy subgroups and discussed some properties. Since the idea of anti-multi fuzzy subgroup has been extended to multi fuzzy subgroups, it is expedient to explore the idea in (α, γ)-anti-multi-fuzzy subgroups setting. The motivation of this paper is to extend the notions of anti-multi fuzzy subgroups and (α, γ)-anti-multi-fuzzy subgroups to fuzzy multigroup environment and to present some new results. Moreover, this research proposes the generalization of the results known for (α, γ)-anti-multi-fuzzy subgroups. It is known that the notion of fuzzy multiset is well entrenched in solving many real-life problems. So, the algebraic structure defined concerning them in this paper could help to approach these issues from a different position. The benefit of this paper is the link found between algebraic structures and fuzzy multisets by introducing (α, γ)-anti-multi-fuzzy subgroups and studying their properties.

The outlines are presented as follows: Section 2 presents some foundational notions relevant to the study, whereas the main results are reported in Section 3. In Section 4, we make some concluding remarks and suggestions for future work.

2 Preliminary

In this paper, $\mathbb{G}$, $\mathbb{G}_1$ and $\mathbb{G}_2$ stands for groups with identities $1, 1_1$ and 1_2, respectively. In the rest of the article, we will always suppose that $0 \leq \alpha < \gamma \leq 1$.

Definition 2.1 [3] Let A be a fuzzy subset of $\mathbb{G}$. A is called a fuzzy subgroup of $\mathbb{G}$ if, for all $x.y \in \mathbb{G}$,

i) $A(xy) \geq A(x) \wedge A(y)$,
ii) $A\left(x^{-1}\right) \geq A(x)$.

Definition 2.2 [9] Let A be a fuzzy subset of $\mathbb{G}$. A is called a (α,γ)-anti-fuzzy subgroup of $\mathbb{G}$ if, for all $x.y, z \in \mathbb{G}$,

i) $A(xy) \wedge \gamma \leq (A(x) \vee A(y)) \vee \alpha$,
ii) $A\left(z^{-1}\right) \wedge \alpha \leq A(z) \vee \gamma$.

Definition 2.3 [10] Let E be a non-empty set and Q be the set of all crisp multisets drawn from the interval [0, 1]. A fuzzy multiset A drawn from E is represented by a function $CM_A \colon E \to Q$.

The value $CM_A(x)$, mentioned above, is a crisp multiset drawn from [0, 1]. For each $x \in E$, $CM_A(x)$, is defined as the decreasingly ordered sequence of elements and it is denoted by:

$\left(\mu_{\mathcal{A}}^{1}(x), \mu_{\mathcal{A}}^{2}(x), \ldots, \mu_{\mathcal{A}}^{p}(x)\right)$: $\mu_{\mathcal{A}}^{1}(x) \geq \mu_{\mathcal{A}}^{2}(x) \geq \ldots \geq \mu_{\mathcal{A}}^{p}(x)$.

A fuzzy set on a set E can be understood as a special case of fuzzy multiset where $CM_{A}(x) = \mu_{A}^{1}(x)$ for all $x \in E$.

3 (α, γ)-Anti-Multi Fuzzy Subgroups and Some of Its Properties

Definition 3.1 A fuzzy set $\mathbb{A}$ of a group $\mathbb{G}$ is called a (α, γ)-anti fuzzy multi subgroup of $\mathbb{G}$ if $\forall g_1, g_2, g_3 \in \mathbb{G}$

$$\mu_{\mathbb{G}}^{i}(g_1g_2) \wedge \gamma \leq \left(\mu_{\mathbb{G}}^{i}(g_1) \vee \mu_{\mathbb{G}}^{i}(g_2)\right) \vee \alpha \tag{1}$$

and

$$\mu_{\mathbb{G}}^{i}\left((g_3)^{-1}\right) \wedge \gamma \leq \mu_{\mathbb{G}}^{i}(g_3) \vee \alpha \tag{2}$$

where $(g_3)^{-1}$ is the inverse element of (g_3).

Proposition 3.2 If $\mathbb{A}$ is a (α, γ)-anti-fuzzy-multi-subgroup of a group $\mathbb{G}$, then

$$\mu_{\mathbb{G}}^{i}(1) \wedge \gamma \leq \mu_{\mathbb{G}}^{i}(g_1) \vee \alpha \tag{3}$$

$\forall g_1 \in \mathbb{G}$, where 1 is the identity of $\mathbb{G}$.

Proof $\forall g_1 \in \mathbb{G}$ and let $(g_1)^{-1}$ be the inverse element of (g_1). Then

$$\begin{aligned}
\mu_{\mathbb{G}}^{i}(1) \wedge \gamma = \mu_{\mathbb{G}}^{i}\left((g_1)^{-1} g_1\right) \wedge \gamma &\leq \left(\mu_{\mathbb{G}}^{i}\left((g_1)^{-1} g_1\right) \wedge \gamma\right) \wedge \gamma \\
&\leq \left(\left(\mu_{\mathbb{G}}^{i}(g_1) \vee \mu_{\mathbb{G}}^{i}(g_1)^{-1}\right) \vee \alpha\right) \wedge \gamma \\
&= \left(\mu_{\mathbb{G}}^{i}(g_1) \wedge \gamma\right) \vee \left(\mu_{\mathbb{G}}^{i}\left(g_1{}^{-1}\right) \wedge \gamma\right) \vee (\alpha \wedge \gamma) \\
&\leq \mu_{\mathbb{G}}^{i}(g_1) \vee \left(\mu_{\mathbb{G}}^{i}(g_1) \vee \alpha\right) \vee \alpha \\
&= \mu_{\mathbb{G}}^{i}(g_1) \vee \alpha
\end{aligned} \tag{4}$$

Theorem 3.3 Let $\mathbb{A}$ be multi fuzzy subset of a group $\mathbb{G}$. Then $\mathbb{A}$ is a (α, γ)-anti-fuzzy multi subgroup of

$$\mathbb{G} \Leftrightarrow \mu_{\mathbb{G}}^{i}\left((g_1)^{-1} g_2\right) \wedge \gamma \leq \left(\mu_{\mathbb{G}}^{i}(g_1) \vee \mu_{\mathbb{G}}^{i}(g_2)\right) \vee \alpha, \forall g_1, g_2 \in \mathbb{G}. \tag{5}$$

Proof Let $\mathbb{A}$ is a (α, γ)-anti-fuzzy multi group of $\mathbb{G}$, then

$$\begin{aligned}
\mu_{\mathbb{G}}^{i}\left((g_1)^{-1} g_2\right) \wedge \gamma &= \mu_{\mathbb{G}}^{i}\left((g_1)^{-1} g_2\right) \wedge \gamma \wedge \gamma \\
&\leq \left(\left(\mu_{\mathbb{G}}^{i}(g_2) \vee \mu_{\mathbb{G}}^{i}(g_1)^{-1}\right) \vee \alpha\right) \wedge \gamma \\
&= \left(\mu_{\mathbb{G}}^{i}(g_2) \vee \mu_{\mathbb{G}}^{i}\left(g_1{}^{-1}\right) \wedge \gamma\right) \vee (\alpha \wedge \gamma) \\
&\leq \mu_{\mathbb{G}}^{i}(g_2) \vee \left(\mu_{\mathbb{G}}^{i}(g_1) \vee \alpha\right) \vee \alpha \\
&= \left(\mu_{\mathbb{G}}^{i}(g_1) \vee \mu_{\mathbb{G}}^{i}(g_2)\right) \vee \alpha.
\end{aligned} \tag{6}$$

Conversely, assume

$$\mu_{\mathbb{G}}^{i}\left((g_1)^{-1} g_2\right) \wedge \gamma \leq \left(\mu_{\mathbb{G}}^{i}(g_1) \vee \mu_{\mathbb{G}}^{i}(g_2)\right) \vee \alpha, \quad \forall g_1, g_2 \in \mathbb{G}, \tag{7}$$

then

$$\mu^i_{\mathbb{G}}(1) \wedge \gamma = \mu^i_{\mathbb{G}}\left((g_1)^{-1} g_2\right) \wedge \gamma \leq \left(\mu^i_{\mathbb{G}}(g_1) \vee \mu^i_{\mathbb{G}}(g_1)\right) \vee \alpha = \mu^i_{\mathbb{G}}(g_1) \vee \alpha. \tag{8}$$

So

$$\begin{aligned}
\mu^i_{\mathbb{G}}\left((g_1)^{-1}\right) \wedge \gamma &= \mu^i_{\mathbb{G}}\left((g_1)^{-1} 1\right) \wedge \gamma \\
&= \mu^i_{\mathbb{G}}\left((g_1)^{-1} 1\right) \wedge \gamma \wedge \gamma \\
&= \left(\mu^i_{\mathbb{G}}(g_1) \vee \mu^i_{\mathbb{G}}(1) \vee \alpha\right) \wedge \gamma \\
&\leq \mu^i_{\mathbb{G}}(g_2) \vee \left(\mu^i_{\mathbb{G}}(g_1) \vee \alpha\right) \vee \alpha \\
&= \left(\mu^i_{\mathbb{G}}(g_1) \vee \mu^i_{\mathbb{G}}(g_2)\right) \vee \alpha.
\end{aligned} \tag{9}$$

In this way $\mathbb{A}$ is a (α, γ)-anti-fuzzy multi-subgroup of $\mathbb{G}$.

Theorem 3.4 Let $\mathbb{A}$ be a fuzzy multi subset of a group $\mathbb{G}$. Then the following are equivalent:

i) $\mathbb{A}_{(\delta)}$ is a multi-subgroup of $\mathbb{G}$, $\forall \delta \in (\alpha, \gamma]$, where $\mathbb{A}_{(\delta)} \neq \emptyset$;
ii) $\mathbb{A}$ is a (α, γ)-anti-fuzzy multi subgroup of $\mathbb{G}$.

Proof (i) $\Rightarrow$ (ii) let $\mathbb{A}_{(\delta)}$ is a multi-subgroup of $\mathbb{G}$. We need to prove that

$$\mu^i_{\mathbb{G}}\left((g_1)^{-1} g_2\right) \wedge \gamma \leq \mu^i_{\mathbb{G}}(g_1) \vee \mu^i_{\mathbb{G}}(g_2) \vee \alpha, \forall g_1, g_2 \in \mathbb{G}. \tag{10}$$

If there exists $\forall g_3, g_4 \in \mathbb{G}$ such that

$$\mu^i_{\mathbb{G}}\left((g_3)^{-1} g_4\right) \wedge \gamma = \delta > \mu^i_{\mathbb{G}}(g_3) \vee \mu^i_{\mathbb{G}}(g_4) \vee \alpha, \tag{11}$$

Then $\mu^i_{\mathbb{G}}(g_3) < \delta, \mu^i_{\mathbb{G}}(g_4) < \delta$ and $\delta \in (\alpha, \gamma]$. Thus $\mu^i_{\mathbb{G}}(g_3) \in \mathbb{A}_{(\delta)}, \mu^i_{\mathbb{G}}(g_4) \in \mathbb{A}_{(\delta)}$. But $\mu^i_{\mathbb{G}}\left((g_3)^{-1} g_4\right) \geq \delta$, that is $(g_3)^{-1} g_4 \notin \mathbb{A}_{(\delta)}$. This is a contradiction with that $\mathbb{A}_{(\delta)}$ is a multi-subgroup of $\mathbb{G}$. Hence

$$\mu^i_{\mathbb{G}}\left((g_1)^{-1} g_2\right) \wedge \gamma \leq \mu^i_{\mathbb{G}}(g_1) \vee \mu^i_{\mathbb{G}}(g_2) \vee \alpha, \tag{12}$$

Holds $\forall g_1, g_2 \in \mathbb{G}$. Therefore, $\mathbb{A}$ is a (α, γ)-anti-fuzzy multi subgroup of $\mathbb{G}$.

(ii) $\Rightarrow$ (i)

Let $\mathbb{A}$ is a (α, γ)-anti-fuzzy multi subgroup of $\mathbb{G}$. $\forall \delta \in (\alpha, \gamma]$, such that $\mathbb{A}_{(\delta)} \neq \emptyset$, we need to show that $(g_1)^{-1} g_2 \notin \mathbb{A}_{(\delta)}, \forall g_1, g_2 \in \mathbb{A}_{(\delta)}$. Since $\mu^i_{\mathbb{G}}(g_1) < \delta, \mu^i_{\mathbb{G}}(g_2) < \delta$ then

$$\begin{aligned}
\mu^i_{\mathbb{G}}\left((g_1)^{-1} g_2\right) \wedge \gamma &\leq \mu^i_{\mathbb{G}}(g_1) \vee \mu^i_{\mathbb{G}}(g_2) \vee \alpha \\
&< \delta \vee \delta \vee \alpha \\
&= \delta \vee \alpha = \alpha.
\end{aligned} \tag{13}$$

Note that $\delta < \gamma$, we have $\mu^i_{\mathbb{G}}\left((g_1)^{-1} g_2\right) < \delta$. Thus $(g_1)^{-1} g_2 \in \mathbb{A}_{(\delta)}$. We set $inf\ \emptyset = 1$, where $\emptyset$ is the empty set.

Theorem 3.5 Let $\mathbb{A}$ and $\mathbb{B}$ are two fuzzy multi-subsets of groups $\mathbb{G}_1$ and $\mathbb{G}_2$, respectively. The product of $\mathbb{A}$ and $\mathbb{B}$, denoted by $\mu^i_{\mathbb{G}_1} \times \mu^i_{\mathbb{G}_2}$s a (α, γ)-anti-fuzzy multi-subgroup of $\mathbb{G}_1 \times \mathbb{G}_2$, where

$$\mu^i_{\mathbb{G}_1} \times \mu^i_{\mathbb{G}_2}(g_1, g_2) = \mu^i_{\mathbb{G}_1}(g_1) \vee \mu^i_{\mathbb{G}_2}(g_2), \forall (g_1, g_2) \in \mathbb{G}_1 \times \mathbb{G}_2. \tag{14}$$

Proof Let ${g_{2_1}}^{-1}$ is the inverse element of g_{2_1} in $\mathbb{G}_2$ and ${g_{1_1}}^{-1}$ is the inverse element of g_{1_1} in $\mathbb{G}_1$. Then$\left({g_{1_1}}^{-1}, {g_{2_1}}^{-1}\right)$ be the inverse element of $\left(g_{1_1}, g_{2_1}\right) \in \mathbb{G}_1 \times \mathbb{G}_2$. Hence

$$\mu^i_{\mathbb{G}_1}\left({g_{1_1}}^{-1}\right) \wedge \gamma \leq \mu^i_{\mathbb{G}_1}\left(g_{1_1}\right) \vee \alpha \tag{15}$$

and

$$\mu^i_{\mathbb{G}_2}\left({g_{2_1}}^{-1}\right) \wedge \gamma \leq \mu^i_{\mathbb{G}_2}\left(g_{2_1}\right) \vee \alpha. \tag{16}$$

$\forall \left(g_{1_2}, g_{2_2}\right) \in \mathbb{G}_1 \times \mathbb{G}_2$. We have

$$\begin{aligned}
\left(\left(\mu^i_{\mathbb{G}_1} \times \mu^i_{\mathbb{G}_2}\right)\left(g_{1_1}, g_{2_1}\right)^{-1}, \left(g_{1_2}, g_{2_2}\right)\right) \wedge \gamma &= \left(\mu^i_{\mathbb{G}_1} \times \mu^i_{\mathbb{G}_2}\right)\left(\left({g_{1_1}}^{-1}, {g_{2_1}}^{-1}\right), \left(g_{1_2}, g_{2_2}\right)\right) \wedge \gamma \\
&= \left(\mu^i_{\mathbb{G}_1}\left({g_{1_1}}^{-1}, g_{1_2}\right) \vee \mu^i_{\mathbb{G}_2}\left({g_{2_1}}^{-1}, g_{2_2}\right)\right) \wedge \gamma \\
&= \left(\mu^i_{\mathbb{G}_1}\left({g_{1_1}}^{-1}, g_{1_2}\right) \wedge \gamma\right) \vee \left(\mu^i_{\mathbb{G}_2}\left({g_{2_1}}^{-1}, g_{2_2}\right) \wedge \gamma\right) \\
&\leq \left(\mu^i_{\mathbb{G}_1}\left(g_{1_1}\right) \vee \mu^i_{\mathbb{G}_1}\left(g_{1_2}\right) \vee \alpha\right) \vee \left(\mu^i_{\mathbb{G}_2}\left(g_{2_1}\right) \vee \mu^i_{\mathbb{G}_2}\left(g_{2_2}\right) \vee \alpha\right) \\
&\leq \left(\mu^i_{\mathbb{G}_1}\left(g_{1_1}\right) \vee \mu^i_{\mathbb{G}_1}\left(g_{1_2}\right) \vee \alpha\right) \vee \left(\mu^i_{\mathbb{G}_2}\left(g_{2_1}\right) \vee \mu^i_{\mathbb{G}_2}\left(g_{2_2}\right) \vee \alpha\right) \\
&= \left(\mu^i_{\mathbb{G}_1}\left(g_{1_1}\right) \vee \mu^i_{\mathbb{G}_2}\left(g_{2_1}\right)\right) \vee \left(\mu^i_{\mathbb{G}_1}\left(g_{1_2}\right) \vee \mu^i_{\mathbb{G}_2}\left(g_{2_2}\right)\right) \vee \alpha \\
&= \left(\mu^i_{\mathbb{G}_1} \times \mu^i_{\mathbb{G}_2}\left(g_{1_1}, g_{2_1}\right)\right) \vee \left(\mu^i_{\mathbb{G}_1} \times \mu^i_{\mathbb{G}_2}\left(g_{1_2}, g_{2_2}\right)\right) \vee \alpha.
\end{aligned} \tag{17}$$

Hence $\mu^i_{\mathbb{G}_1} \times \mu^i_{\mathbb{G}_2}$ s a (α, γ)-anti-fuzzy multi-subgroup of $\mathbb{G}_1 \times \mathbb{G}_2$.

Theorem 3.6 Let $\mathbb{A}$ and $\mathbb{B}$ are two fuzzy multi-subsets of groups $\mathbb{G}_1$ and $\mathbb{G}_2$, respectively. If $\mathbb{A} \times \mathbb{B}$ is a (α, γ)-anti-fuzzy multi-subgroup of $\mathbb{G}_1 \times \mathbb{G}_2$, then at least one of the following statements must hold.

$$\mu^i_{\mathbb{G}_1}(1_1) \wedge \gamma \leq \mu^i_{\mathbb{G}_2}(\mathbb{a}) \vee \alpha, \forall \mathbb{a} \in \mathbb{G}_2 \tag{18}$$

and

$$\mu^i_{\mathbb{G}_2}(1_2) \wedge \gamma \leq \mu^i_{\mathbb{G}_1}(g_1) \vee \alpha, \forall g_1 \in \mathbb{G}_1. \tag{19}$$

Proof Let $\mathbb{A} \times \mathbb{B}$ is a (α, γ)-anti-fuzzy multi-subgroup of $\mathbb{G}_1 \times \mathbb{G}_2$. By contraposition, assume that none of the statements holds. Then we can find $g_1 \in \mathbb{G}_1$ and $\mathbb{a} \in \mathbb{G}_2$ such that $\mu^i_{\mathbb{G}_1}(g_1) \vee \alpha < \mu^i_{\mathbb{G}_2}(1_2) \wedge \gamma$ and $\mu^i_{\mathbb{G}_2}(\mathbb{a}) \vee \alpha < \mu^i_{\mathbb{G}_1}(1_1) \wedge \gamma$. Now

$$\begin{aligned}
\mu^i_{\mathbb{G}_1} \times \mu^i_{\mathbb{G}_2}(g_1, \mathbb{a}) \vee \alpha &= \left(\mu^i_{\mathbb{G}_1}(g_1) \vee \mu^i_{\mathbb{G}_2}(\mathbb{a})\right) \vee \alpha, \\
&= \left(\mu^i_{\mathbb{G}_1}(g_1) \vee \alpha\right) \vee \left(\mu^i_{\mathbb{G}_2}(\mathbb{a}) \vee \alpha\right) \\
&< \left(\mu^i_{\mathbb{G}_1}(1_1) \wedge \gamma\right) \vee \left(\mu^i_{\mathbb{G}_2}(1_2) \wedge \gamma\right) \\
&= \mu^i_{\mathbb{G}_1} \times \mu^i_{\mathbb{G}_2}(1_1, 1_2) \wedge \gamma.
\end{aligned} \tag{20}$$

Therefore $\mathbb{A} \times \mathbb{B}$ is a (α, γ)-anti-fuzzy multi-subgroup of $\mathbb{G}_1 \times \mathbb{G}_2$ satisfying

$$\mu^i_{\mathbb{G}_1} \times \mu^i_{\mathbb{G}_2}(g_1, \mathbb{a}) \vee \alpha < \mu^i_{\mathbb{G}_1} \times \mu^i_{\mathbb{G}_2}(1_1, 1_2) \wedge \gamma. \tag{21}$$

This is a contradict with that $(1_1, 1_2)$ is the identity of $\mathbb{G}_1 \times \mathbb{G}_2$.

Theorem 3.7 Let $f\colon \mathbb{G}_1 \to \mathbb{G}_2$ is a homomorphism and let $\mathbb{A}$ is a (α, γ)-anti-fuzzy multi-subgroup of $\mathbb{G}_1$. Then $f\left(\mu^i_{\mathbb{G}_1}\right)$ is a (α, γ)-anti-fuzzy multi-subgroup of $\mathbb{G}_2$, where

$$f\left(\mu^i_{\mathbb{G}_1}\right)(g_2) = \inf_{g_1\in\mathbb{G}_1}\left\{\mu^i_{\mathbb{G}_1}(g_1)\colon f(g_1) = g_2\right\}, \forall g_2 \in \mathbb{G}_2 \tag{22}$$

Proof If $f^{-1}\left(g_{2_1}\right) = \emptyset$ or $f^{-1}\left(g_{2_2}\right) = \emptyset$ for any $g_{2_1}, g_{2_2} \in \mathbb{G}_2$, then

$$\left(f\left(\mu^i_{\mathbb{G}_1}\right)\left(\left(g_{2_1}\right)^{-1} g_{2_2}\right)\right) \wedge \gamma \le 1 = \left(f\left(\mu^i_{\mathbb{G}_1}\right)\left(g_{2_1}\right) \vee f\left(\mu^i_{\mathbb{G}_1}\right)\left(g_{2_2}\right)\right) \vee \alpha. \tag{23}$$

Assume that $f^{-1}\left(g_{2_1}\right) = \emptyset$ or $f^{-1}\left(g_{2_2}\right) = \emptyset$ for any $g_{2_1}, g_{2_2} \in \mathbb{G}_2$ then

$$\begin{aligned}
\left(f\left(\mu^i_{\mathbb{G}_1}\right)\left(\left(g_{21}\right)^{-1} g_{22}\right)\right) \wedge \gamma &= \inf_{k\in\mathbb{G}_1}\left\{\mu^i_{\mathbb{G}_1}(k)\colon f(k) = \left(g_{21}\right)^{-1} g_{22}\right\} \wedge \gamma \\
&= \inf_{k\in\mathbb{G}_1}\left\{\mu^i_{\mathbb{G}_1}(k) \wedge \gamma\colon f(k) = \left(g_{21}\right)^{-1} g_{22}\right\} \\
&\le \inf_{g_{11},g_{12}\in\mathbb{G}_1}\left\{\mu^i_{\mathbb{G}_1}\left(\left(g_{11}\right)^{-1} g_{12}\right) \wedge \gamma\colon f\left(g_{11}\right) = g_{21}, f\left(g_{12}\right) = g_{22}\right\} \\
&\le \inf_{g_{11},g_{12}\in\mathbb{G}_1}\left\{\left(\mu^i_{\mathbb{G}_1}\right)\left(g_{11}\right) \vee f\left(\mu^i_{\mathbb{G}_1}\right)\left(g_{12}\right) \vee \alpha\colon f\left(g_{11}\right) = g_{21}, f\left(g_{12}\right) = g_{22}\right\} \\
&= \left(\inf_{g_{11}\in\mathbb{G}_1}\left\{\left(\mu^i_{\mathbb{G}_1}\right)\left(g_{11}\right)\colon f\left(g_{11}\right) = g_{21}\right\} \vee \inf_{g_{12}\in\mathbb{G}_1}\left\{\left(\mu^i_{\mathbb{G}_1}\right)\left(g_{12}\right)\colon f\left(g_{12}\right) = g_{22}\right\}\right) \vee \alpha \\
&= \left(f\left(\mu^i_{\mathbb{G}_1}\right)\left(g_{21}\right) \vee f\left(\mu^i_{\mathbb{G}_1}\right)\left(g_{22}\right)\right) \vee \alpha
\end{aligned} \tag{24}$$

Thus $f\left(\mu^i_{\mathbb{G}_1}\right)$ is a (α, γ)-anti-fuzzy multi-subgroup of $\mathbb{G}_2$.

Theorem 3.8 Let $f\colon \mathbb{G}_1 \to \mathbb{G}_2$ is a homomorphism and let $\mathbb{A}$ is a (α, γ)-anti-fuzzy multi-subgroup of $\mathbb{G}_2$. Then $f^{-1}\left(\mu^i_{\mathbb{G}_2}\right)$ is a (α, γ)-anti-fuzzy multi-subgroup of $\mathbb{G}_1$, where

$$f^{-1}\left(\mu^i_{\mathbb{G}_2}\right)(g_1) = \mu^i_{\mathbb{G}_2}(f(g_1)), \quad \forall g_1 \in \mathbb{G}_1. \tag{25}$$

Proof For any $g_{1_1}, g_{1_2} \in \mathbb{G}_1$

$$\begin{aligned}
f^{-1}\left(\mu^i_{\mathbb{G}_2}\right)\left(\left(g_{1_1}\right)^{-1} g_{1_2}\right) \wedge \gamma &= \mu^i_{\mathbb{G}_2}\left(f\left(\left(g_{1_1}\right)^{-1} g_{1_2}\right)\right) \wedge \gamma \\
&= \mu^i_{\mathbb{G}_2}\left(f\left(\left(g_{1_1}\right)^{-1}\right) f\left(g_{1_2}\right)\right) \wedge \gamma \\
&\le \left(\mu^i_{\mathbb{G}_2}\left(f\left(g_{1_1}\right)\right) \vee \mu^i_{\mathbb{G}_2}\left(f\left(g_{1_2}\right)\right)\right) \vee \alpha \\
&= \left(f^{-1}\left(\mu^i_{\mathbb{G}_2}\right)\left(g_{1_1}\right) \vee f^{-1}\left(\mu^i_{\mathbb{G}_2}\right)\left(g_{1_2}\right)\right) \vee \alpha.
\end{aligned}$$

Thus $f^{-1}\left(\mu^i_{\mathbb{G}_2}\right)$ is a (α, γ)-anti-fuzzy multi-subgroup of $\mathbb{G}_1$.

Definition 3.9 Let $\mathbb{A}$ is a (α, γ)-anti fuzzy multi subgroup of $\mathbb{G}$. $\mathbb{A}$ is called a (α, γ)-anti fuzzy normal subgroup of $\mathbb{G}$ if, $\forall g_1, g_2 \in \mathbb{G}$

$$\mu^i_{\mathbb{G}}\left(g_1 g_2 g_1{}^{-1}\right) \wedge \gamma \le \mu^i_{\mathbb{G}}(g_2) \vee \alpha \tag{26}$$

Proposition 3.10 Let $\mathbb{A}$ is a (α, γ)-anti fuzzy multi subgroup of $\mathbb{G}$. $\mathbb{A}$ is a (α, γ)-anti fuzzy normal subgroup if and only if, $\forall g_1, g_2 \in \mathbb{G}$,

$$\mu^i_{\mathbb{G}}(g_1g_2) \wedge \gamma \leq \mu^i_{\mathbb{G}}(g_2g_1) \vee \alpha. \tag{27}$$

Proposition 3.11 Let $\mathbb{A}$ is a (α, γ)-anti fuzzy multi subset of $\mathbb{G}$. Then $\mathbb{A}$ is a (α, γ)-anti fuzzy multi normal subgroup of $\mathbb{G}$ if and only if $\mathbb{A}_\delta \neq \emptyset$ is a normal subgroup of $\mathbb{G}$ $\forall\delta \in (\alpha, \gamma]$.

Proposition 3.12 Let $\mathbb{A}$ is a (α, γ)-anti fuzzy multi normal subgroup of $\mathbb{G}$ and $g_1 \in \mathbb{G}$.

i) If $\mu^i_{\mathbb{G}}(g_1) \leq \alpha$, then $\mu^i_{\mathbb{G}}\left(g_1g_2g_1^{-1}\right) \leq \alpha$ for all $g_2 \in \mathbb{G}$.
ii) If $\gamma < \mu^i_{\mathbb{G}}(g_1) < \alpha$, then $\mu^i_{\mathbb{G}}\left(g_2g_1g_2^{-1}\right) = \mu^i_{\mathbb{G}}(g_1)$ for all $g_2 \in \mathbb{G}$.
iii) If $g_2 \in \mathbb{G}$ and $\gamma < \mu^i_{\mathbb{G}}(g_1g_2) < \alpha$, then $\mu^i_{\mathbb{G}}(g_1g_2) = \mu^i_{\mathbb{G}}(g_2g_1)$.
iv) If $g_2 \in \mathbb{G}$ and $\mu^i_{\mathbb{G}}(g_1g_2) \leq \alpha$, then, $\mu^i_{\mathbb{G}}(g_2g_1) \leq \alpha$
v) If $g_2 \in \mathbb{G}$ and $\mu^i_{\mathbb{G}}(g_1g_2) \geq \gamma$, then, $\mu^i_{\mathbb{G}}(g_2g_1) \geq \gamma$.

Proof (i) $\mu^i_{\mathbb{G}}(g_1) \leq \alpha$, then $g_1 \in \mathbb{A}_\alpha$. By proposition 9, $\mathbb{A}_\alpha$ is a fuzzy multi normal subgroup of $\mathbb{G}$ and thus $g_1g_2g_1^{-1} \in \mathbb{A}_\alpha$. Hence $\mu^i_{\mathbb{G}}\left(g_1g_2g_1^{-1}\right) \leq \alpha$. (ii) Let $\mu^i_{\mathbb{G}}(g_1) = \delta$. Then $\gamma < \delta < \alpha$. By proposition 9, $\mathbb{A}_\delta$ is a fuzzy multi normal subgroup of $\mathbb{G}$. Hence $g_1g_2g_1^{-1} \in \mathbb{A}_\delta$; that is $\mu^i_{\mathbb{G}}\left(g_1g_2g_1^{-1}\right) \leq \delta = \mu^i_{\mathbb{G}}(g_1)$. Suppose $\mu^i_{\mathbb{G}}\left(g_1g_2g_1^{-1}\right) < \delta$. Set $\delta_0 = min\left\{\mu^i_{\mathbb{G}}\left(g_1g_2g_1^{-1}\right), \alpha\right\}$. Then $\gamma < \delta_0 < \alpha$. By proposition 9, $\mathbb{A}_{\delta_0}$ is a fuzzy multi normal subgroup of $\mathbb{G}$, and thus $g_1g_2g_1^{-1} \in \mathbb{A}_{\delta_0}$. Therefore $g_1 = g_2^{-1}\left(g_1g_2g_1^{-1}\right)g_2 \in \mathbb{A}_{\delta_0}$; that is $\mu^i_{\mathbb{G}}\left(g_1g_2g_1^{-1}\right) \leq \delta_0 < \delta$ which is a contradiction to that $\mu^i_{\mathbb{G}}(g_1) = \delta$. As result, $\mu^i_{\mathbb{G}}\left(g_2g_1g_2^{-1}\right) = \mu^i_{\mathbb{G}}(g_1)$.

(iii) If $\gamma < \mu^i_{\mathbb{G}}(g_1g_2) < \alpha$, then $\mu^i_{\mathbb{G}}(g_2g_1) = \mu^i_{\mathbb{G}}\left(g_1^{-1}(g_2g_1)g_1\right) = \mu^i_{\mathbb{G}}(g_1g_2)$ by (ii); that is $\mu^i_{\mathbb{G}}(g_1g_2) = \mu^i_{\mathbb{G}}(g_2g_1)$.

(iv) $\mu^i_{\mathbb{G}}(g_1g_2) \leq \alpha$, then $g_1g_2 \in \mathbb{A}_\alpha$. Since $\mathbb{A}_\alpha$ is a fuzzy multi normal subgroup of $\mathbb{G}$ by Proposition 9, $(g_2g_1) = g_1^{-1}(g_1g_2)g_1 \in \mathbb{A}_\alpha$; that is $\mu^i_{\mathbb{G}}(g_2g_1) \leq \alpha$.

(v) Assume that $\mu^i_{\mathbb{G}}(g_2g_1) < \gamma$ on the contrary. If $\mu^i_{\mathbb{G}}(g_2g_1) \geq \alpha$, then, by (i) $\mu^i_{\mathbb{G}}(g_1g_2) \geq \alpha$, which is contradictory to that $\mu^i_{\mathbb{G}}(g_1g_2) > \gamma$. If $\mu^i_{\mathbb{G}}(g_2g_1) < \gamma$, then, by (iii), $\mu^i_{\mathbb{G}}(g_1g_2) = \mu^i_{\mathbb{G}}(g_2g_1) < \alpha$, which is contradictory to that $\mu^i_{\mathbb{G}}(g_1g_2) \geq \gamma$. Therefore $\mu^i_{\mathbb{G}}(g_2g_1) \geq \gamma$.

Proposition 3.13 Let $\mathbb{A}$ is a (α, γ)-anti fuzzy multi subgroup of $\mathbb{G}$. Then $\mathbb{A}$ is a (α, γ)-anti fuzzy multi normal subgroup of $\mathbb{G}$ if and only if

$$\mu^i_{\mathbb{G}}([g_1, g_2]) \wedge \gamma \leq \mu^i_{\mathbb{G}}(g_1) \vee \alpha \tag{28}$$

for all $g_1, g_2 \in \mathbb{G}$, where $[g_1, g_2] = g_1^{-1}g_2^{-1}g_1g_2$ is a commutator in $\mathbb{G}$.

Proof For any $g_1, g_2 \in \mathbb{G}$,

$$\begin{aligned}
\mu^i_{\mathbb{G}}([g_1, g_2]) \wedge \gamma &= \mu^i_{\mathbb{G}}\left(g_1^{-1}g_2^{-1}g_1g_2\right) \wedge \gamma \\
&= \mu^i_{\mathbb{G}}\left(g_1^{-1}\left(g_2^{-1}g_1g_2\right)\right) \wedge \gamma \wedge \gamma \\
&\leq \left(\mu^i_{\mathbb{G}}\left(g_1^{-1}\right) \vee \mu^i_{\mathbb{G}}\left(g_2^{-1}g_1g_2\right) \vee \alpha\right) \wedge \gamma \\
&= \left(\mu^i_{\mathbb{G}}\left(g_1^{-1}\right) \wedge \gamma\right) \vee \left(\mu^i_{\mathbb{G}}\left(g_2^{-1}g_1g_2\right) \wedge \gamma\right) \vee \alpha
\end{aligned} \tag{29}$$

Since $\mathbb{A}$ is a (α, γ)-anti fuzzy multi normal subgroup of $\mathbb{G}$, $\mu^i_{\mathbb{G}}\left(g_1^{-1}\right) \wedge \gamma \leq \mu^i_{\mathbb{G}}(g_1) \vee \alpha$ and $\mu^i_{\mathbb{G}}\left(g_2^{-1}g_1g_2\right) \wedge \gamma \leq \mu^i_{\mathbb{G}}(g_1) \vee \alpha$. Hence,

$$\mu^i_{\mathbb{G}}([g_1, g_2]) \wedge \gamma \leq \mu^i_{\mathbb{G}}(g_1) \vee \alpha \tag{30}$$

Conversely, if $\mu^{i}_{\mathbb{G}}([g_1, g_2]) \wedge \gamma \leq \mu^{i}_{\mathbb{G}}(g_1) \vee \alpha$, then

$$\begin{aligned}\mu^{i}_{\mathbb{G}}([g_1, g_2]) \wedge \gamma &= \mu^{i}_{\mathbb{G}}\left(g_1 g_1^{-1} g_2^{-1} g_1 g_2\right) \wedge \gamma \\ &= \mu^{i}_{\mathbb{G}}\left(g_1\left(g_1^{-1} g_2^{-1} g_1 g_2\right)\right) \wedge \gamma \wedge \gamma \\ &\leq \left(\mu^{i}_{\mathbb{G}}(g_1) \vee \mu^{i}_{\mathbb{G}}\left(g_1^{-1} g_2^{-1} g_1 g_2\right) \vee \alpha\right) \wedge \gamma \\ &= \left(\mu^{i}_{\mathbb{G}}(g_1) \wedge \gamma\right) \vee \left(\mu^{i}_{\mathbb{G}}\left(g_1^{-1} g_2^{-1} g_1 g_2\right) \wedge \gamma\right) \vee \alpha \\ &\leq \left(\mu^{i}_{\mathbb{G}}(g_1) \wedge \gamma\right) \vee \left(\mu^{i}_{\mathbb{G}}(g_1) \vee \alpha\right) \vee \alpha \\ &\leq \mu^{i}_{\mathbb{G}}(g_1) \vee \mu^{i}_{\mathbb{G}}(g_1) \vee \alpha \\ &= \mu^{i}_{\mathbb{G}}(g_1) \vee \alpha \end{aligned} \tag{31}$$

Therefore $\mathbb{A}$ is a (α, γ)-anti fuzzy multi normal subgroup of $\mathbb{G}$.

Proposition 3.14 If $\mathbb{G}$ is an abelian group and $\mathbb{A}$ is a (α, γ)-anti fuzzy multi subgroup of $\mathbb{G}$, then $\mathbb{A}$ is a (α, γ)-anti fuzzy multi normal subgroup of $\mathbb{G}$.

Proof Since $\mathbb{G}$ is an abelian group, we have $[g_1 g_2] = e$; hence

$$\mu^{i}_{\mathbb{G}}([g_1, g_2]) \wedge \gamma = \mu^{i}_{\mathbb{G}}(e) \wedge \gamma \leq \mu^{i}_{\mathbb{G}}(g_1) \vee \alpha \tag{32}$$

for all $g_1, g_2 \in \mathbb{G}$. By Proposition 11, $\mathbb{A}$ is a (α, γ)-anti fuzzy multi normal subgroup of $\mathbb{G}$.

4 Conclusions

The aim of this paper was to highlight the function between (α, γ)-anti-multi-fuzzy subgroups and algebraic structures from other a point of view. It is well known that the concept of fuzzy multi set is well established in dealing with many real-life problems. So, the algebraic structure defined concerning them in this paper would help to approach these problems with a different perspective.

In this paper, we have defined the notion of (α, γ)-anti-multi-fuzzy subgroups and this structure some algebraic properties were developed. In this article, we have discussed (α, γ)-anti-multi-fuzzy subgroups, (α, γ)-anti-multi-fuzzy normal subgroups and defined (α, γ)-anti-multi-fuzzy homomorphism on (α, γ)-anti-multi-fuzzy subgroups. Interestingly, it has been observed that (α, γ)-anti-multi-fuzzy concept adds another dimension to the defined anti-fuzzy multi normal subgroups. This concept can further be extended for new results.

Funding Statement: Yibin University Pre-research Project, Research on the coupling and coordinated development of manufacturing and logistics industry under the background of intelligent manufacturing, (2022YY001); Sichuan Provincial Department of Education Water Transport Economic Research Center, Research on the Development Path and Countermeasures of the Advanced Manufacturing Industry in the Sanjiang New District of Yibin under a "dual circulation" development pattern (SYJJ2020A06).

Conflicts of Interest: The authors declare that they have no conflicts of interest to report regarding the present study.

References

[1] M. Dresher and O. Ore, "Theory of multigroup," *American Journal of Mathematics*, vol. 60, no. 3, pp. 705–733, 1938.
[2] L. A. Zadeh, "Fuzzy sets," *Information Control*, vol. 8, no. 3, pp. 338–353, 1965.
[3] S. Sebastian and T. T. Ramakrishnan, "Multi-fuzzy sets," *International Mathematical Forum*, vol. 5, pp. 2471–2476, 2010.
[4] K. Atanassov, "Intuitionistic fuzzy sets," *Fuzzy Set System*, vol. 20, no. 1, pp. 87–96, 1986.
[5] T. K. Shinoj and S. S. John, "Intuitionistic fuzzy multisets and its application in medical diagnosis," *World Academy of Science, Engineering and Technology*, vol. 6, pp. 1418–1421, 2012.
[6] S. Abdullah and M. A. Naeem, "New type of interval valued fuzzy normal subgroups of groups," *New Trends in Mathematical Sciences*, vol. 3, pp. 62–77, 2015.
[7] J. N. Mordeson, K. R. Bhutani and A. Rosenfeld, "Fuzzy subsets and fuzzy subgroups," in *Fuzzy Group Theory*, 1st ed., vol. 1. Berlin, Heidelberg, Germany: Springer, pp. 1–39, 2005.
[8] Y. L. Liu, "Quotient groups induced by fuzzy subgroups," *Quasigroups Related System*, vol. 11, pp. 71–78, 2004.
[9] A. Baby, T. K. Shinoj and S. J. John, "On abelian fuzzy multi groups and orders of fuzzy multi groups," *Journal of New Theory*, vol. 5, pp. 80–93, 2015.
[10] R. Muthuraj and S. Balamurugan, "Multi-fuzzy group and its level subgroups," *Gen*, vol. 17, pp. 74–81, 2013.
[11] Y. Tella, "On algebraic properties of fuzzy membership sequenced multisets," *British Journal of Mathematics & Computer Science*, vol. 6, no. 2, pp. 146–164, 2015.
[12] P. A. Ejegwa, J. A. Awolola, J. M. Agbetayo and I. M. Adamu, "On the characterisation of anti-fuzzy multigroups," *Annals of Fuzzy Mathematics and Informatics*, vol. 21, no. 3, pp. 307–318, 2021.
[13] S. Hoskova-Mayerova and M. Al Tahan, "Anti-fuzzy multi-ideals of near ring," *Mathematics*, vol. 9, no. 5, pp. 494–504, 2021.
[14] S. Gayen, S. Jha, M. K. Singh and A. K. Prasad, "On anti-fuzzy subgroup," *Yugoslav Journal of Operations Research*, vol. 31, no. 4, pp. 539–546, 2021.
[15] R. Sumitha and S. Jayalakshmi, "Anti-fuzzy quasi-ideals of near subtraction semigroups," *Malaya Journal of Matematik*, vol. S, no. 1, pp. 261–264, 2021.
[16] K. R. Balasubramanian and R. Revathy, "Product of (λ, μ)-multifuzzy subgroups of a group," *Annals of the Romanian Society for Cell Biology*, vol. 25, no. 6, pp. 2448–2460, 2021.
[17] S. H. Asaad and A. S. Mohammed, "New properties of anti-fuzzy ideals of regular semigroups," *Ibn AL-Haitham Journal for Pure and Applied Sciences*, vol. 32, no. 3, pp. 109–116, 2019.
[18] A. Rosenfeld, "Fuzzy groups," *Journal of Mathematical Analysis and Applications*, vol. 35, no. 3, pp. 512–551, 1971.
[19] R. Biswas, "Fuzzy subgroups and anti-fuzzy subgroups," *Fuzzy Sets and Systems*, vol. 35, pp. 121–124, 1990.
[20] X. Yuan, C. Zhang and Y. Ren, "Generalized fuzzy groups and many-valued implications," *Fuzzy Sets System*, vol. 138, no. 1, pp. 205–211, 2003.
[21] B. Yao, "(λ, μ)-fuzzy normal subgroups and (λ, μ)-fuzzy quotient subgroups," *Journal of Fuzzy Mathematics*, vol. 13, no. 3, pp. 695–705, 2005.
[22] Z. Shen, "The anti-fuzzy subgroup of a group," *Journal of Liaoning Normat. Univ. (Nat. Sci.)*, vol. 18, no. 2, pp. 99–101, 1995.
[23] B. Dong, "Direct product of anti-fuzzy subgroups," *Journal of Shaoxing Teachers College*, vol. 5, pp. 29–34, 1992.
[24] Y. Feng and B. Yao, "On (λ, μ)-anti-fuzzy subgroups," *Journal of Inequalities and Applications*, vol. 1, pp. 1–5, 2012.