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ARTICLE

Frenet Curve Couples in Three Dimensional Lie Groups

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ABSTRACT

In this study, we examine the possible relations between the Frenet planes of any given two curves in three dimensional Lie groups with left invariant metrics. We explain these possible relations in nine cases and then introduce the conditions that must be met to coincide with the planes of these curves in nine theorems.

KEYWORDS

Curves; lie groups; curvatures; Frenet plane

1 Introduction

The theory of curves has an important role in differential geometry studies. In the theory of curves, one of the interesting problems is to investigate the relations between two curves. The Frenet elements of the curves have an effective role in the solution of the problem.

For example, if the principal normal vectors coincide at the corresponding points of the curves α and β, the curve couple $\{\alpha, \beta\}$ is called the Bertrand curve couple in a three-dimensional Euclidean space [1,2]. Similarly, if tangent vectors coincide, the curve couple $\{\alpha, \beta\}$ is called the involute-evolute curve couple. Also, if the normal vector of the curve α coincides with the bi-normal vector of the curve β, the curve couple $\{\alpha, \beta\}$ is called the Mannheim curve couple [3].

In a three-dimensional Lie group G with a bi-invariant metric, Çiftçi has defined general helices [4]. Also Okuyucu et al. have obtained slant helices and Bertrand curves in G [5,6]. Gök et al. have investigated Mannheim curves in G [7]. Recently, Yampolsky et al. have examined helices in three dimensional Lie group G, with left-invariant metric [8]. Also, many applications of curves theory are studied and still have been investigated in three dimensional Lie groups (see [9–12], etc.).

Karakuş et al. have examined the possibility of whether any Frenet plane of a given space curve in a three-dimensional Euclidean space is also any Frenet plane of another space curve in the same space [13].

In this study, we examine the possible relations between the Frenet planes of given two curves in three dimensional Lie groups with left invariant metrics.

2 Preliminaries

Let G be a three dimensional Lie group with left-invariant metric $\langle , \rangle$ and let $\mathfrak{g}$ denote the Lie algebra of G which consists of the all smooth vector fields of G invariant under left translation. There are two classes of three dimensional Lie groups:

1. If the group is unimodular, we have a (positively oriented) orthonormal frame of left-invariant vector fields $\{e_1, e_2, e_3\}$, such that the brackets satisfy

$$[e_1, e_2] = \lambda_3 e_3, \quad [e_1, e_3] = \lambda_2 e_2, \quad [e_2, e_3] = \lambda_1 e_1,$$

where λ_i are called structure constants. The constants

$$\mu_i = \frac{1}{2}(\lambda_1 + \lambda_2 + \lambda_3) - \lambda_i,$$

are called connection coefficients.

2. If the group is nonunimodular, we have an othonormal frame $\{e_1, e_2, e_3\}$, such that

$$[e_1, e_2] = \alpha e_2 + \beta e_3, \quad [e_1, e_3] = -\beta e_2 + \delta e_3, \quad [e_2, e_3] = 0,$$

see [14].

Using the Koszul formula the covariant derivatives $\nabla_{e_i} e_j$ can be found as in the following tables:

∇	e_1	e_2	e_3
e_1	0	$\mu_1 e_3$	$-\mu_1 e_2$
e_2	$-\mu_2 e_3$	0	$\mu_2 e_1$
e_3	$\mu_3 e_2$	$-\mu_3 e_1$	0

and

∇	e_1	e_2	e_3
e_1	0	βe_3	$-\beta e_2$
e_2	$-\alpha e_2$	αe_1	0
e_3	$-\delta e_3$	0	δe_1

for unimodular and nonunimodular cases, respectively.

The cross-products of the vectors e_1, e_2, e_3 are defined by the following equalities:

$$e_1 \times e_2 = e_3, \quad e_2 \times e_3 = e_1, \quad e_3 \times e_1 = e_2,$$

in three dimensional case. For unimodular and nonunimodular groups, we have $\nabla_{e_i} e_k = \mu(e_i) \times e_k$, and so

$$\nabla_X e_k = \mu(X) \times e_k,$$

for any vector field X, where μ is a affine transformation as follows:

$$\mu(X) = \begin{cases} \mu_1 X^1 e_1 + \mu_2 X^2 e_2 + \mu_3 X^3 e_3, \\ \beta X^1 e_1 + \delta X^3 e_2 - \alpha X^2 e_3, \end{cases}$$

for unimodular and nonunimodular cases, respectively (see [8]).

We have $\nabla_{e_i} e_k = \mu(e_i) \times e_k$ for both groups, and so

$$\nabla_X e_k = \mu(X) \times e_k, \tag{1}$$

for any vector field X.

Let γ be an arc-lengthed curve on the group and $T = \dot{\gamma}$ be the unit tangent vector field. For any vector field $\xi \circ \gamma$, using the Eq. (1), we get

$$\nabla_T\xi = \dot{\xi}^k e_k + \mu(T) \times \xi, \tag{2}$$

where the vector field $\dot{\xi} = \frac{d\xi_i}{ds} e_i$ is dot-derivative of the vector field ξ along the curve γ. If ξ is the restriction of a left-invariant vector field to the curve γ the $\dot{\xi} = 0$ (see [8]).

Let T, N and B be the vectors of the standard Frenet frame of the curve γ. We get the following equations with the help of Eq. (2):

$$\nabla_T T = \dot{T} + \mu(T) \times T, \quad \nabla_T B = \dot{B} + \mu(T) \times B, \quad \nabla_T N = \dot{N} + \mu(T) \times N. \tag{3}$$

Along the curve γ, we can define a new frame $\{\tau, \nu, \beta\}$, which is called dot-Frenet Frame, by

$$\tau = T, \quad \nu = \frac{1}{k_0}\dot{\tau}, \quad \beta = \tau \times \nu,$$

where $k_0 = |\dot{T}|$. By definition $\varkappa_0 = |\dot{\beta}|$.

Proposition 2.1. The dot-Frenet frame $\{\tau, \nu, \beta\}$ satisfies dot-Frenet formulas, namely

$$\dot{\tau} = k_o\nu, \quad \dot{\nu} = -k_0\tau + \varkappa_0\beta, \quad \dot{\beta} = -\varkappa_0\nu.$$

The Frenet and the dot-Frenet frames are connected by

$$\tau = T, \quad \nu = \cos\alpha N + \sin\alpha B, \quad \beta = -\sin\alpha N + \cos\alpha B,$$

where $\alpha = \alpha(s)$ is the angle function (see [8]).

Proposition 2.2. The transformation $\mu(T)$ can be given by

$$\mu(T) = (\varkappa + \dot{\alpha} - \varkappa_0)T + k_0 sin\alpha N + (k - k_0 cos\alpha)B, \tag{4}$$

with respect to the Frenet frame $\{T, N, B\}$.

Define a group-curvature k_G and a group-torsion $\varkappa_G$ of a curve by

$$k_G = |\mu(T) \times T|, \quad \varkappa_G = |\mu(T) \times B|,$$

respectively. As a consequence of (4), the dot-curvature and the dot-torsion of a curve can be expressed in terms of the group-curvature k_G, group-torsion $\varkappa_G$ of a curve, and angle function α by

$$k_G^2 = (k - k_0)^2 + 4kk_0 sin^2(\alpha/2), \quad \varkappa_G^2 = k_0^2 sin\alpha^2 + (\varkappa - \varkappa_0 + \dot{\alpha})^2$$

(see [8]).

3 Frenet Curve Couples in Three Dimensional Lie Groups

Let $\zeta : I \subseteq \mathbb{R} \to G$ and $\eta : \bar{I} \subseteq \mathbb{R} \to G$ be curves with arc-length parameter s and $\bar{s}$, respectively, in three dimensional Lie group G with left-invariant metric. We denote Frenet apparatus of the curves ζ and η with $\{T, N, B, k_0, \varkappa_0, \alpha\}$ and $\{\bar{T}, \bar{N}, \bar{B}, \bar{k}_0, \bar{\varkappa}_0, \bar{\alpha}\}$, respectively. We know that $Sp\{T, N\}$, $Sp\{N, B\}$ and $Sp\{T, B\}$ are the osculating plane, the normal plane and the rectifying plane of the curve ζ, respectively. And similarly, $Sp\{\bar{T}, \bar{N}\}$, $Sp\{\bar{N}, \bar{B}\}$ and $Sp\{\bar{T}, \bar{B}\}$ are the osculating plane, the normal plane and the rectifying plane of the curve η, respectively. Now we investigate the possible cases and the relations between the Frenet planes of any given two curves in three dimensional Lie groups with left invariant metrics in a step by step manner:

Case 1: We assume that, osculating plane of the curve ζ is the osculating plane of the curve η, that is $Sp\{T, N\} = Sp\{\bar{T}, \bar{N}\}$. As in Fig. 1, this relationship exists at the corresponding points of along the curves ζ and η.

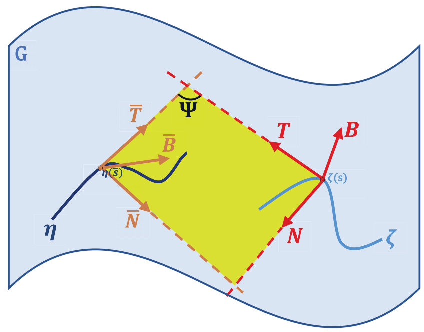

Figure 1: Osculating planes of the curves ζ and η

So we have following relation between the curves ζ and η:

$$\eta(\bar{s}) = \zeta(s) + aT(s) + bN(s), \quad a \neq 0, b \neq 0, \tag{5}$$

where a and b are real valued non-zero functions of s.

Calculating the dot-derivative of the Eq. (5) with the help of Eqs. (3) and (4), we get

$$\bar{T}(\bar{s}) = (1 + \dot{a} - bk_0 cos\alpha)\frac{1}{r}T(s) + (ak_0 cos\alpha + \dot{b})\frac{1}{r}N(s) + (ak_0 sin\alpha + b(-\dot{\alpha} + \varkappa_0))\frac{1}{r}B(s), \tag{6}$$

where $r = \dfrac{d\bar{s}}{ds}$.

We know that B is parallel to $\bar{B}$, since $B^{\perp} = Sp\{T, N\} = Sp\{\bar{T}, \bar{N}\} = \bar{B}^{\perp}$. If we multiply the Eq. (6) with B, we get

$$(ak_0 sin\alpha + b(-\dot{\alpha} + \varkappa_0))\frac{1}{r} = 0 \quad or \quad ak_0 sin\alpha + b(-\dot{\alpha} + \varkappa_0) = 0.$$

And so, we have

$$\bar{T}(\bar{s}) = (1 + \dot{a} - bk_0 cos\alpha)\frac{1}{r}T(s) + (ak_0 cos\alpha + \dot{b})\frac{1}{r}N(s). \tag{7}$$

By using Eq. (7), we can set

$$\begin{aligned} \bar{T}(\bar{s}) &= \cos\psi(s)T(s) + \sin\psi(s)N(s), \\ \bar{N}(\bar{s}) &= -\sin\psi(s)T(s) + \cos\psi(s)N(s), \end{aligned} \tag{8}$$

where ψ is smooth angle function between T and $\bar{T}$ on I and

$$\cos\psi(s) = (1 + \dot{a} - bk_0 cos\alpha)\frac{1}{r}, \tag{9}$$

$$\sin\psi(s) = (ak_0 cos\alpha + \dot{b})\frac{1}{r}. \tag{10}$$

By using the Eqs. (9) and (10), we obtain

$$r = \sqrt{(1 + \dot{a} - bk_0 cos\alpha)^2 + (ak_0 cos\alpha + \dot{b}),^2}$$

and

$$bk_0 \cos\alpha - \dot{a} + \cot\psi(ak_0 \cos\alpha + \dot{b}) = 1.$$

Calculating the dot-derivative of the Eq. (8) with the help of Eqs. (3) and (4), we get

$$\begin{aligned} r(\bar{k}_0 \cos\bar{\alpha}\bar{N} + \bar{k}_0 \sin\bar{\alpha}\bar{B}) =& (-\dot{\psi}\sin\psi - \sin\psi k_0 \cos\alpha)T \\ &+ (\cos\psi k_0 \cos\alpha + \dot{\psi}\cos\psi)N \\ &+ (\cos\psi k_0 \sin\alpha + \sin\psi(-\dot{\alpha} + \varkappa_0))B. \end{aligned} \tag{11}$$

If we multiply the Eq. (11), with $\bar{N}$ and $\bar{B}$, respectively, we get

$$r\bar{k}_0 \cos\bar{\alpha} = \dot{\psi} + k_0 \cos\alpha, \tag{12}$$

$$r\bar{k}_0 \sin\bar{\alpha} = \cos\psi k_0 \sin\alpha + \sin\psi(-\dot{\alpha} + \varkappa_0). \tag{13}$$

By using Eqs. (12) and (13), we obtain

$$\dot{\psi} + k_0 \cos\alpha - \cot\bar{\alpha}(\cos\psi k_0 \sin\alpha + \sin\psi(-\dot{\alpha} + \varkappa_0)) = 0.$$

Thus we introduce the following theorem:

Theorem 3.1. Let $\zeta : I \subseteq \mathbb{R} \to G$ and $\eta : \bar{I} \subseteq \mathbb{R} \to G$ be two arc-length parametrized curves with the Frenet apparatus $\{T, N, B, k_0, \varkappa_0, \alpha\}$ and $\{\bar{T}, \bar{N}, \bar{B}, \bar{k}_0, \bar{\varkappa}_0, \bar{\alpha}\}$, respectively, in three dimensional Lie group G with left-invariant metric. The osculating planes of these curves coincide if and only if there exist real valued non-zero functions a and b on I, such that

$$ak_0 sin\alpha + b(-\dot{\alpha} + \varkappa_0) = 0, \tag{i}$$

$$(1 + \dot{a} - bk_0 cos\alpha)^2 + (ak_0 cos\alpha + \dot{b})^2 \neq 0, \tag{ii}$$

$$bk_0 \cos\alpha - \dot{a} + \cot\psi(ak_0 \cos\alpha + \dot{b}) = 1, \tag{iii}$$

$$\dot{\psi} + k_0 \cos\alpha - \cot\bar{\alpha}(\cos\psi k_0 \sin\alpha + \sin\psi(-\dot{\alpha} + \varkappa_0)) = 0. \tag{iv}$$

where ψ is the angle between T and $\bar{T}$ at the corresponding points of ζ and η.

Case 2: We assume that, osculating plane of the curve ζ is the normal plane of the curve η, that is $Sp\{T, N\} = Sp\{\bar{N}, \bar{B}\}$. As in Fig. 2, this relationship exists at the corresponding points of along the curves ζ and η.

Thus, we have following relation between the curves ζ and η:

$$\eta(\bar{s}) = \zeta(s) + aT(s) + bN(s), \quad a \neq 0, b \neq 0, \tag{14}$$

where a and b are real valued non-zero functions of s.

Calculating the dot-derivative of the Eq. (14) with the help of Eqs. (3) and (4), we get

$$\bar{T}(\bar{s}) = (1 + \dot{a} - bk_0 cos\alpha)\frac{1}{r}T(s) + (ak_0 cos\alpha + \dot{b})\frac{1}{r}N(s) + (ak_0 sin\alpha + b(-\dot{\alpha} + \varkappa_0))\frac{1}{r}B(s), \tag{15}$$

where $r = \dfrac{d\bar{s}}{ds}$.

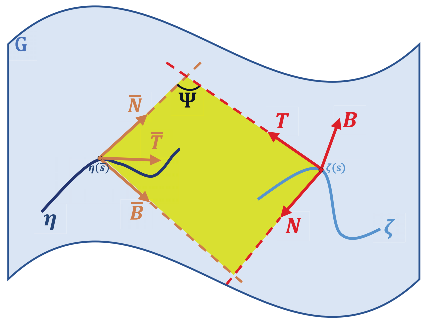

Figure 2: Osculating plane of the curve ζ and normal plane of the curve η

We know that B is parallel to $\bar{T}$, since $B^{\perp} = Sp\{T, N\} = Sp\{\bar{N}, \bar{B}\} = \bar{T}^{\perp}$. If we multiply the Eq. (15) with T, N and B, respectively, we get

$$1 + \dot{a} - bk_0cos\alpha = 0,$$

$$ak_0cos\alpha + \dot{b} = 0,$$

$$ak_0sin\alpha + b(-\dot{\alpha} + \varkappa_0) = r.$$

And so, we have the equation $\bar{T}(\bar{s}) = B(s)$. If we calculate the dot-derivative of this equation with the help of Eqs. (3) and (4), we get

$$r(\bar{k}_0cos\bar{\alpha}\bar{N} + \bar{k}_0sin\bar{\alpha}\bar{B}) = (\dot{\alpha} - \varkappa_0)N - k_0sin\alpha T. \tag{16}$$

If we multiply the Eq. (16) with $\bar{N}$ and $\bar{B}$, respectively, we get

$$r\bar{k}_0cos\bar{\alpha} = \sin\psi(\dot{\alpha} - \varkappa_0) - cos\psi k_0sin\alpha, \tag{17}$$

$$r\bar{k}_0sin\bar{\alpha} = \cos\psi(\dot{\alpha} - \varkappa_0) + sin\psi k_0sin\alpha, \tag{18}$$

where ψ is the smooth angle function between T and $\bar{N}$. By using the Eqs. (17) and (18), we obtain

$$cot\bar{\alpha} = \frac{\sin\psi(\dot{\alpha} - \varkappa_0) - cos\psi k_0sin\alpha}{\cos\psi(\dot{\alpha} - \varkappa_0) + sin\psi k_0sin\alpha}$$

that is

$$\sin\psi(\dot{\alpha} - \varkappa_0) - cos\psi k_0sin\alpha - cot\bar{\alpha}(\cos\psi(\dot{\alpha} - \varkappa_0) + sin\psi k_0sin\alpha) = 0.$$

Thus we introduce the following theorem:

Theorem 3.2. Let $\zeta : I \subseteq \mathbb{R} \to G$ and $\eta : \bar{I} \subseteq \mathbb{R} \to G$ be two arc-length parametrized curves with the Frenet apparatus $\{T, N, B, k_0, \varkappa_0, \alpha\}$ and $\{\bar{T}, \bar{N}, \bar{B}, \bar{k}_0, \bar{\varkappa}_0, \bar{\alpha}\}$, respectively, in three dimensional Lie group G with left-invariant metric. The osculating plane of the curve ζ and the normal plane of the curve η coincide if and only if there exist real valued non-zero functions a and b on I, such that:

$$1 + \dot{a} - bk_0 \cos\alpha = 0, \tag{i}$$

$$ak_0 \cos\alpha + \dot{b} = 0, \tag{ii}$$

$$ak_0 \sin\alpha + b(-\dot{\alpha} + \varkappa_0) \neq 0, \tag{iii}$$

$$\sin\psi(\dot{\alpha} - \varkappa_0) - \cos\psi k_0 \sin\alpha - \cot\bar{\alpha}(\cos\psi(\dot{\alpha} - \varkappa_0) + \sin\psi k_0 \sin\alpha) = 0, \tag{iv}$$

where ψ is the angle between T and $\bar{N}$ at the corresponding points of ζ and η.

Case 3: We assume that, osculating plane of the curve ζ is the rectifying plane of the curve η, that is $Sp\{T, N\} = Sp\{\bar{T}, \bar{B}\}$. As in Fig. 3, this relationship exists at the corresponding points of along the curves ζ and η.

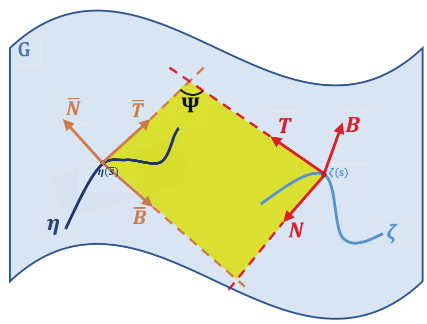

Figure 3: Osculating plane of the curve ζ and rectifying plane of the curve η

Thus, we have following relation between the curves ζ and η:

$$\eta(\bar{s}) = \zeta(s) + aT(s) + bN(s), \; a \neq 0, b \neq 0, \tag{19}$$

where a and b are real valued non-zero functions of s.

Calculating the dot-derivative of the Eq. (19) with the help of Eqs. (3) and (4), we get

$$\bar{T}(\bar{s}) = (1 + \dot{a} - bk_0 cos\alpha)\frac{1}{r}T(s) + (ak_0 cos\alpha + \dot{b})\frac{1}{r}N(s) + (ak_0 sin\alpha + b(-\dot{\alpha} + \varkappa_0))\frac{1}{r}B(s), \tag{20}$$

where $r = \dfrac{d\bar{s}}{ds}$.

We know that B is parallel to $\bar{N}$, since $B^{\perp} = Sp\{T, N\} = Sp\{\bar{T}, \bar{B}\} = \bar{N}^{\perp}$. If we multiply the Eq. (20) with B, we get

$$(ak_0 sin\alpha + b(-\dot{\alpha} + \varkappa_0))\frac{1}{r} = 0 \quad \text{or} \quad ak_0 sin\alpha + b(-\dot{\alpha} + \varkappa_0) = 0.$$

And so, we have

$$\bar{T}(\bar{s}) = (1 + \dot{a} - bk_0 cos\alpha)\frac{1}{r}T(s) + (ak_0 cos\alpha + \dot{b})\frac{1}{r}N(s). \tag{21}$$

By Eq. (21), we can set

$$\begin{aligned} \bar{T}(\bar{s}) &= \cos\psi(s)T(s) + \sin\psi(s)N(s), \\ \bar{B}(\bar{s}) &= -\sin\psi(s)T(s) + \cos\psi(s)N(s), \end{aligned} \tag{22}$$

where ψ is smooth angle function between T and $\bar{T}$ on I and

$$\cos\psi(s) \quad = (1 + \dot{a} - bk_0 cos\alpha)\frac{1}{r}, \tag{23}$$

$$\sin\psi(s) \quad = (ak_0 cos\alpha + \dot{b})\frac{1}{r}. \tag{24}$$

By using the Eqs. (23) and (24), we obtain

$$r = \sqrt{(1 + \dot{a} - bk_0 cos\alpha)^2 + (ak_0 cos\alpha + \dot{b})^2}$$

and

$$bk_0\cos\alpha - \dot{a} + \cot\psi(ak_0\cos\alpha + \dot{b}) = 1.$$

Calculating the dot-derivative of the Eq. (22) with the help of Eqs. (3) and (4), we get

$$\begin{aligned} r(\bar{k}_0\cos\bar{\alpha}\bar{N} + \bar{k}_0\sin\bar{\alpha}\bar{B}) = {} & (-\dot{\psi}\sin\psi - \sin\psi k_0\cos\alpha)T \\ & + (\cos\psi k_0\cos\alpha + \dot{\psi}\cos\alpha)N \\ & + (\cos\psi k_0\sin\alpha + \sin\psi(-\bar{\alpha} + \varkappa_0))B. \end{aligned} \tag{25}$$

If we multiply the Eq. (25), with $\bar{N}$ and $\bar{B}$, respectively, we get

$$r\bar{k}_0\cos\bar{\alpha} \quad = \cos\psi k_0\sin\alpha + \sin\psi(-\dot{\alpha} + \varkappa_0), \tag{26}$$

$$r\bar{k}_0\sin\bar{\alpha} \quad = \dot{\psi} + k_0\cos\alpha. \tag{27}$$

By using Eqs. (26) and (27), we obtain

$$\cos\psi k_0\sin\alpha + \sin\psi(-\dot{\alpha} + \varkappa_0) - \cot\bar{\alpha}(\dot{\psi} + k_0\cos\alpha) = 0.$$

Thus we introduce the following theorem:

Theorem 3.3. Let $\zeta : I \subseteq \mathbb{R} \to G$ and $\eta : \bar{I} \subseteq \mathbb{R} \to G$ be two arc-length parametrized curves with the Frenet apparatus $\{T, N, B, k_0, \varkappa_0, \alpha\}$ and $\{\bar{T}, \bar{N}, \bar{B}, \bar{k}_0, \bar{\varkappa}_0, \bar{\alpha}\}$, respectively, in three dimensional Lie group G with left-invariant metric. The osculating plane of the curve ζ and the rectifying plane of the curve η coincide if and only if there exist real valued non-zero functions a and b on I, such that

$$ak_0 sin\alpha + b(-\dot{\alpha} + \varkappa_0) = 0, \tag{i}$$

$$(1 + \dot{a} - bk_0 cos\alpha)^2 + (ak_0 cos\alpha + \dot{b})^2 \neq 0, \tag{ii}$$

$$bk_0\cos\alpha - \dot{a} + \cot\psi(ak_0\cos\alpha + \dot{b}) = 1, \tag{iii}$$

$$\cos\psi k_0\sin\alpha + \sin\psi(-\dot{\alpha} + \varkappa_0) - \cot\bar{\alpha}(\dot{\psi} + k_0\cos\alpha) = 0, \tag{iv}$$

where ψ is the angle between T and $\bar{T}$ at the corresponding points of ζ and η.

Case 4: We assume that, normal plane of the curve ζ is the osculating plane of the curve η, that is $Sp\{N, B\} = Sp\{\bar{T}, \bar{N}\}$. As in Fig. 4, this relationship exists at the corresponding points of along the curves ζ and η.

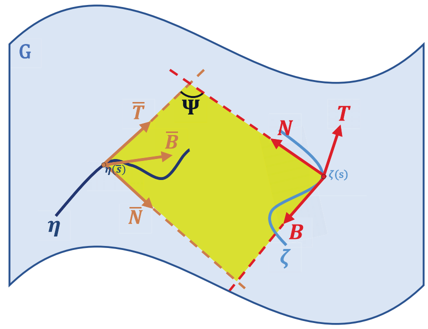

Figure 4: Normal plane of the curve ζ and osculating plane of the curve η

So we have following relation between the curves ζ and η:

$$\eta(\bar{s}) = \zeta(s) + aN(s) + bB(s), \quad a \neq 0, b \neq 0, \tag{28}$$

where a and b are real valued non-zero functions of s.

Calculating the dot-derivative of the Eq. (28) with the help of Eqs. (3) and (4), we get

$$\bar{T}(\bar{s}) = (1 - ak_0 cos\alpha - bk_0 \sin\alpha)\frac{1}{r}T(s) + (\dot{a} + b(\dot{\alpha} - \varkappa_0))\frac{1}{r}N(s) + (a(-\dot{\alpha} + \varkappa_0) + \dot{b})\frac{1}{r}B(s), \tag{29}$$

where $r = \dfrac{d\bar{s}}{ds}$.

We know that T is parallel to $\bar{B}$, since $T^{\perp} = Sp\{N, B\} = Sp\{\bar{T}, \bar{N}\} = \bar{B}^{\perp}$. If we multiply the Eq. (29) with T, we get

$$(1 - ak_0 cos\alpha - bk_0 \sin\alpha)\frac{1}{r} = 0 \quad \text{or} \quad ak_0 cos\alpha + bk_0 \sin\alpha = 1.$$

And so, we have

$$\bar{T}(\bar{s}) = (\dot{a} + b(\dot{\alpha} - \varkappa_0))\frac{1}{r}N(s) + (a(-\dot{\alpha} + \varkappa_0) + \dot{b})\frac{1}{r}B(s). \tag{30}$$

By the Eq. (30), we can set

$$\begin{aligned} \bar{T}(\bar{s}) &= \cos\psi(s)N(s) + \sin\psi(s)B(s), \\ \bar{N}(\bar{s}) &= -\sin\psi(s)N(s) + \cos\psi(s)B(s), \end{aligned} \tag{31}$$

where ψ is smooth angle function between $\bar{T}$ and N on I and

$$\cos\psi(s) = (\dot{a} + b(\dot{\alpha} - \varkappa_0))\frac{1}{r}, \tag{32}$$

$$\sin\psi(s) = (a(-\dot{\alpha} + \varkappa_0) + \dot{b})\frac{1}{r}. \tag{33}$$

By using Eqs. (32) and (33), we obtain

$$r = \sqrt{(\dot{a} + b(\dot{\alpha} - \varkappa_0))^2 + (a(-\dot{\alpha} + \varkappa_0) + \dot{b})^2}$$

and

$$\dot{a} + b(\dot{\alpha} - \varkappa_0) + \cot\psi(a(-\dot{\alpha} + \varkappa_0) + \dot{b}) = 0.$$

Calculating the dot-derivative of the Eq. (31) with the help of Eqs. (3) and (4), we get

$$\begin{aligned} r(\bar{k}_0 \cos\bar{\alpha}\bar{N} + \bar{k}_0 \sin\bar{\alpha}\bar{B}) &= (-\cos\psi k_0 \cos\alpha - \sin\psi k_0 \sin\alpha)T \\ &+ (-\dot{\psi}\sin\psi + \sin\psi(\dot{\alpha} - \varkappa_0))N \\ &+ (\cos\psi(-\dot{\alpha} + \varkappa_0) + \dot{\psi}\cos\psi)B. \end{aligned} \quad (34)$$

If we multiply the Eq. (34), with $\bar{N}$ and $\bar{B}$, respectively, we get

$$r\bar{k}_0 \cos\bar{\alpha} = \dot{\psi} + (-\dot{\alpha} + \varkappa_0), \quad (35)$$

$$r\bar{k}_0 \sin\bar{\alpha} = -\cos\psi k_0 \cos\alpha - \sin\psi k_0 \sin\alpha. \quad (36)$$

By using Eqs. (35) and (36), we obtain

$$\dot{\psi} + (-\dot{\alpha} + \varkappa_0) + \cot\bar{\alpha} k_0 \cos(\psi - \alpha) = 0.$$

Thus we introduce the following theorem:

Theorem 3.4. Let $\zeta : I \subseteq \mathbb{R} \to G$ and $\eta : \bar{I} \subseteq \mathbb{R} \to G$ be two arc-length parametrized curves with the Frenet apparatus $\{T, N, B, k_0, \varkappa_0, \alpha\}$ and $\{\bar{T}, \bar{N}, \bar{B}, \bar{k}_0, \bar{\varkappa}_0, \bar{\alpha}\}$, respectively, in three dimensional Lie group G with left-invariant metric. The normal plane of the curve ζ and the osculating plane of the curve η coincide if and only if there exist real valued non-zero functions a and b on I, such that

$$ak_0 cos\alpha + bk_0 \sin\alpha = 1, \quad \text{(i)}$$

$$(\dot{a} + b(\dot{\alpha} - \varkappa_0))^2 + (a(-\dot{\alpha} + \varkappa_0) + \dot{b})^2 \neq 0, \quad \text{(ii)}$$

$$\dot{a} + b(\dot{\alpha} - \varkappa_0) + \cot\psi(a(-\dot{\alpha} + \varkappa_0) + \dot{b}) = 0, \quad \text{(iii)}$$

$$\dot{\psi} + (-\dot{\alpha} + \varkappa_0) + \cot\bar{\alpha} k_0 \cos(\psi - \alpha) = 0, \quad \text{(iv)}$$

where ψ is the angle between $\bar{T}$ and N at the corresponding points of ζ and η.

Case 5: We assume that, normal plane of the curve ζ is the normal plane of the curve η, that is $Sp\{N, B\} = Sp\{\bar{N}, \bar{B}\}$. As in Fig. 5, this relationship exists at the corresponding points of along the curves ζ and η.

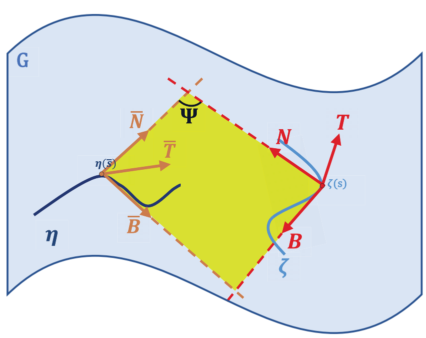

Figure 5: Normal planes of the curves ζ and η

Thus, we have following relation between the curves ζ and η:

$$\eta(\bar{s}) = \zeta(s) + aN(s) + bB(s), \quad a \neq 0, b \neq 0, \tag{37}$$

where a and b are real valued non-zero functions of s.

Calculating the dot-derivative of the Eq. (37) with the help of Eqs. (3) and (4), we get

$$\bar{T}(\bar{s}) = (1 - ak_0 cos\alpha - bk_0 \sin\alpha)\frac{1}{r}T(s) + (\dot{a} + b(\dot{\alpha} - \varkappa_0))\frac{1}{r}N(s) + (a(-\dot{\alpha} + \varkappa_0) + \dot{b})\frac{1}{r}B(s), \tag{38}$$

where $r = \dfrac{d\bar{s}}{ds}$.

We know that T is parallel to $\bar{T}$, since $T^{\perp} = Sp\{N, B\} = Sp\{\bar{N}, \bar{B}\} = \bar{T}^{\perp}$. If we multiply the Eq. (38) with T, N and B, respectively, we get

$$ak_0 \cos\alpha + bk_0 \sin\alpha = 1 - r,$$

$$\dot{a} + b(\dot{\alpha} - \varkappa_0) = 0,$$

$$a(-\dot{\alpha} + \varkappa_0) + \dot{b} = 0.$$

And so, we have the equation $\bar{T} = T$. If we calculate the dot-derivative of this equation with the help of Eqs. (3) and (4), we get

$$r(\bar{k}_0 \cos\bar{\alpha}\bar{N} + \bar{k}_0 \sin\bar{\alpha}\bar{B}) = k_0 \cos\alpha N + k_0 \sin\alpha B. \tag{39}$$

If we multiply the Eq. (39) with $\bar{N}$ and $\bar{B}$, respectively, we get

$$r\bar{k}_0 \cos\bar{\alpha} \quad = \cos\psi k_0 \cos\alpha + \sin\psi k_0 \sin\alpha, \tag{40}$$

$$r\bar{k}_0 \sin\bar{\alpha} \quad = -\sin\psi k_0 \cos\alpha + \cos\psi k_0 \sin\alpha, \tag{41}$$

where ψ is the smooth angle function between N and $\bar{N}$. By using the Eqs. (40) and (41), we obtain

$$\cot\bar{\alpha} = \frac{\cos\psi k_0 \cos\alpha + \sin\psi k_0 \sin\alpha}{-\sin\psi k_0 \cos\alpha + \cos\psi k_0 \sin\alpha}$$

that is

$$\cot\bar{\alpha} = \frac{\cos(\psi - \alpha)}{\sin(\alpha - \psi)}.$$

This means that,

$$\alpha - \psi = \bar{\alpha} + k\pi, \quad k \in \mathbb{Z}.$$

Thus we introduce the following theorem:

Theorem 3.5. Let $\zeta : I \subseteq \mathbb{R} \to G$ and $\eta : \bar{I} \subseteq \mathbb{R} \to G$ be two arc-length parametrized curves with the Frenet apparatus $\{T, N, B, k_0, \varkappa_0, \alpha\}$ and $\{\bar{T}, \bar{N}, \bar{B}, \bar{k}_0, \bar{\varkappa}_0, \bar{\alpha}\}$, respectively, in three dimensional Lie group G with left-invariant metric. The normal planes of these curves coincide if and only if there exist real valued non-zero functions a and b on I, such that:

$$1 - ak_0 cos\alpha - bk_0 \sin\alpha \neq 0, \tag{i}$$

$$\dot{a} + b(\dot{\alpha} - \varkappa_0) = 0, \tag{ii}$$

$$a(-\dot{\alpha} + \varkappa_0) + \dot{b} = 0, \tag{iii}$$

$$\alpha - \psi = \bar{\alpha} + k\pi, \quad k \in \mathbb{Z}, \tag{iv}$$

where ψ is the angle between N and $\bar{N}$ at the corresponding points of ζ and η.

Case 6: We assume that, normal plane of the curve ζ is the rectifying plane of the curve η, that is $Sp\{N, B\} = Sp\{\bar{T}, \bar{B}\}$. As in Fig. 6, this relationship exists at the corresponding points of along the curves ζ and η.

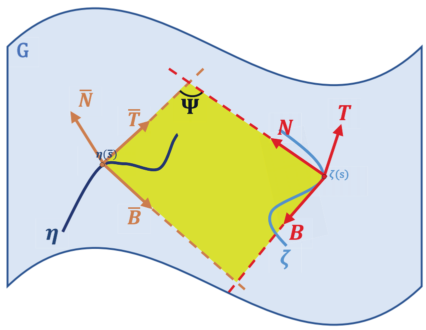

Figure 6: Normal plane of the curve ζ and rectifying plane of the curve η

Thus, we have following relations between the curves ζ and η:

$$\eta(\bar{s}) = \zeta(s) + aN(s) + bB(s), \quad a \neq 0, b \neq 0, \tag{42}$$

where a and b are real valued non-zero functions of s.

Calculating the dot-derivative of the Eq. (42) with the help of Eqs. (3) and (4), we get

$$\bar{T}(\bar{s}) = (1 - ak_0 cos\alpha - bk_0 \sin\alpha)\frac{1}{r}T(s) + (\dot{a} + b(\dot{\alpha} - \varkappa_0))\frac{1}{r}N(s) + (a(-\dot{\alpha} + \varkappa_0) + \dot{b})\frac{1}{r}B(s), \tag{43}$$

where $r = \dfrac{d\bar{s}}{ds}$.

We know that T is parallel to $\bar{N}$, since $T^{\perp} = Sp\{N, B\} = Sp\{\bar{T}, \bar{B}\} = \bar{N}^{\perp}$. If we multiply the Eq. (43) with T, we get

$$(1 - ak_0 cos\alpha - bk_0 \sin\alpha)\frac{1}{r} = 0 \quad or \quad ak_0 cos\alpha + bk_0 \sin\alpha = 1.$$

And so, we have

$$\bar{T}(\bar{s}) = (\dot{a} + b(\dot{\alpha} - \varkappa_0))\frac{1}{r}N(s) + (a(-\dot{\alpha} + \varkappa_0) + \dot{b})\frac{1}{r}B(s). \tag{44}$$

By the Eq. (44), we can set

$$\begin{aligned} \bar{T}(\bar{s}) &= \cos\psi(s)N(s) + \sin\psi(s)B(s), \\ \bar{B}(\bar{s}) &= -\sin\psi(s)N(s) + \cos\psi(s)B(s), \end{aligned} \tag{45}$$

where ψ is smooth angle function between $\bar{T}$ and N on I and

$$\cos\psi(s) \quad = (\dot{a} + b(\dot{\alpha} - \varkappa_0))\frac{1}{r}, \tag{46}$$

$$\sin\psi(s) \quad = (a(-\dot{\alpha} + \varkappa_0) + \dot{b})\frac{1}{r}. \tag{47}$$

By using Eqs. (46) and (47), we obtain

$$r = \sqrt{(\dot{a} + b(\dot{\alpha} - \varkappa_0))^2 + (a(-\dot{\alpha} + \varkappa_0) + \dot{b})^2}$$

and

$$\dot{a} + b(\dot{\alpha} - \varkappa_0) + \cot\psi(a(-\dot{\alpha} + \varkappa_0) + \dot{b}) = 0.$$

Calculating the dot-derivative of the Eq. (45) with the help of Eqs. (3) and (4), we get

$$\begin{aligned} r(\bar{k}_0 \cos\bar{\alpha}\bar{N} + \bar{k}_0 \sin\bar{\alpha}\bar{B}) =& (-\cos\psi k_0 \cos\alpha - \sin\psi k_0 \sin\alpha)T \\ &+ (-\dot{\psi}\sin\psi + \sin\psi(\dot{\alpha} - \varkappa_0))N \\ &+ (\cos\psi(-\dot{\alpha} + \varkappa_0) + \dot{\psi}\cos\psi)B. \end{aligned} \tag{48}$$

If we multiply the Eq. (48), with $\bar{N}$ and $\bar{B}$, respectively, we get

$$r\bar{k}_0 \cos\bar{\alpha} \quad = -\cos\psi k_0 \cos\alpha - \sin\psi k_0 \sin\alpha, \tag{49}$$

$$r\bar{k}_0 \sin\bar{\alpha} \quad = \dot{\psi} + (-\dot{\alpha} + \varkappa_0). \tag{50}$$

By using Eqs. (49) and (50), we obtain

$$k_0 \cos(\psi - \alpha) + \cot\bar{\alpha}(\dot{\psi} + (-\dot{\alpha} + \varkappa_0)) = 0.$$

Thus we introduce the following theorem:

Theorem 3.6. Let $\zeta : I \subseteq \mathbb{R} \to G$ and $\eta : \bar{I} \subseteq \mathbb{R} \to G$ be two arc-length parametrized curves with the Frenet apparatus $\{T, N, B, k_0, \varkappa_0, \alpha\}$ and $\{\bar{T}, \bar{N}, \bar{B}, \bar{k}_0, \bar{\varkappa}_0, \bar{\alpha}\}$, respectively, in three dimensional Lie group G with left-invariant metric. The normal plane of the curve ζ and the rectifying plane of the curve η coincide if and only if there exist real valued non-zero functions a and b on I, such that:

$$ak_0 cos\alpha + bk_0 \sin\alpha = 1, \tag{i}$$

$$(\dot{a} + b(\dot{\alpha} - \varkappa_0))^2 + (a(-\dot{\alpha} + \varkappa_0) + \dot{b})^2 \neq 0, \tag{ii}$$

$$\dot{a} + b(\dot{\alpha} - \varkappa_0) + \cot\psi(a(-\dot{\alpha} + \varkappa_0) + \dot{b}) = 0, \tag{iii}$$

$$k_0 \cos(\psi - \alpha) + \cot\bar{\alpha}(\dot{\psi} + (-\dot{\alpha} + \varkappa_0)) = 0, \tag{iv}$$

where ψ is the angle between $\bar{T}$ and N at the corresponding points of ζ and η.

Case 7: We assume that, rectifying plane of the curve ζ is the osculating plane of the curve η, that is $Sp\{T, B\} = Sp\{\bar{T}, \bar{N}\}$. As in Fig. 7, this relationship exists at the corresponding points of along the curves ζ and η.

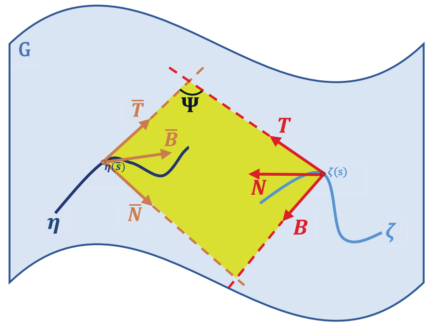

Figure 7: Rectifying plane of the curve ζ and osculating plane of the curve η

So we have following relation between the curves ζ and η:

$$\eta(\bar{s}) = \zeta(s) + aT(s) + bB(s), \quad a \neq 0, b \neq 0, \tag{51}$$

where a and b are real valued non-zero functions of s.

Calculating the dot-derivative of the Eq. (51) with the help of Eqs. (3) and (4), we get

$$\bar{T}(\bar{s}) = (1 + \dot{a} - bk_0 \sin\alpha)\frac{1}{r}T(s) + (ak_0\cos\alpha + b(\dot{\alpha} - \varkappa_0))\frac{1}{r}N(s) + (ak_0\sin\alpha + \dot{b})\frac{1}{r}B(s), \tag{52}$$

where $r = \dfrac{d\bar{s}}{ds}$.

We know that N is parallel to $\bar{B}$, since $N^{\perp} = Sp\{T, B\} = Sp\{\bar{T}, \bar{N}\} = \bar{B}^{\perp}$. If we multiply the Eq. (52) with N, we get

$$(ak_0\cos\alpha + b(\dot{\alpha} - \varkappa_0))\frac{1}{r} = 0 \quad or \quad ak_0\cos\alpha + b(\dot{\alpha} - \varkappa_0) = 0.$$

And so, we have

$$\bar{T}(\bar{s}) = (1 + \dot{a} - bk_0\sin\alpha)\frac{1}{r}T(s) + (ak_0\sin\alpha + \dot{b})\frac{1}{r}B(s). \tag{53}$$

By the Eq. (53), we can set

$$\begin{aligned} \bar{T}(\bar{s}) \quad &= \cos\psi(s)T(s) + \sin\psi(s)B(s), \\ \bar{N}(\bar{s}) \quad &= -\sin\psi(s)T(s) + \cos\psi(s)B(s), \end{aligned} \tag{54}$$

where ψ is smooth function between T and $\bar{T}$ on I and

$$\cos\psi(s) \quad = (1 + \dot{a} - bk_0\sin\alpha)\frac{1}{r}, \tag{55}$$

$$\sin\psi(s) \quad = (ak_0\sin\alpha + \dot{b})\frac{1}{r}. \tag{56}$$

By using Eqs. (55) and (56), we obtain

$$r = \sqrt{(1 + \dot{a} - bk_0\sin\alpha)^2 + (ak_0\sin\alpha + \dot{b})^2}$$

and

$$bk_0\sin\alpha - \dot{a} + \cot\psi(ak_0\sin\alpha + \dot{b}) = 1.$$

Calculating the dot-derivative of the Eq. (54) with the help of Eqs. (3) and (4), we get

$$r(\bar{k}_0 \cos\bar{\alpha}\bar{N} + \bar{k}_0 \sin\bar{\alpha}\bar{B}) = (-\dot{\psi}\sin\psi - \sin\psi k_0 \sin\alpha)T + (\cos\psi k_0 \cos\alpha + \sin\psi(\dot{\alpha} - \varkappa_0))N + (\cos\psi k_0 \sin\alpha + \dot{\psi}\cos\psi)B. \tag{57}$$

If we multiply the Eq. (57) with $\bar{N}$ and $\bar{B}$, respectively, we get

$$r\bar{k}_0 \cos\bar{\alpha} = \dot{\psi} + k_o \sin\alpha, \tag{58}$$

$$r\bar{k}_0 \sin\bar{\alpha} = \cos\psi k_0 \cos\alpha + \sin\psi(\dot{\alpha} - \varkappa_0). \tag{59}$$

By using Eqs. (58) and (59), we obtain

$$\dot{\psi} + k_o \sin\alpha - \cot\bar{\alpha}(\cos\psi k_0 \cos\alpha + \sin\psi(\dot{\alpha} - \varkappa_0)) = 0.$$

Thus we introduce the following theorem:

Theorem 3.7. Let $\zeta : I \subseteq \mathbb{R} \to G$ and $\eta : \bar{I} \subseteq \mathbb{R} \to G$ be two arc-length parametrized curves with the Frenet apparatus $\{T, N, B, k_0, \varkappa_0, \alpha\}$ and $\{\bar{T}, \bar{N}, \bar{B}, \bar{k}_0, \bar{\varkappa}_0, \bar{\alpha}\}$, respectively, in three dimensional Lie group G with left-invariant metric. The rectifying plane of the curve ζ and the osculating plane of the curve η coincide if and only if there exist real valued non-zero functions a and b on I, such that:

$$ak_0 \cos\alpha + b(\dot{\alpha} - \varkappa_0) = 0, \tag{i}$$

$$(1 + \dot{a} - bk_0 \sin\alpha)^2 + (ak_0 \sin\alpha + \dot{b})^2 \neq 0, \tag{ii}$$

$$bk_0 \sin\alpha - \dot{a} + \cot\psi(ak_0 \sin\alpha + \dot{b}) = 1, \tag{iii}$$

$$\dot{\psi} + k_o \sin\alpha - \cot\bar{\alpha}(\cos\psi k_0 \cos\alpha + \sin\psi(\dot{\alpha} - \varkappa_0)) = 0, \tag{iv}$$

where ψ is the angle between T and $\bar{T}$ at the corresponding points of ζ and η.

Case 8: We assume that, rectifying plane of the curve ζ is the normal plane of the curve η, that is $Sp\{T, B\} = Sp\{\bar{N}, \bar{B}\}$. As in Fig. 8, this relationship exists at the corresponding points of along the curves ζ and η.

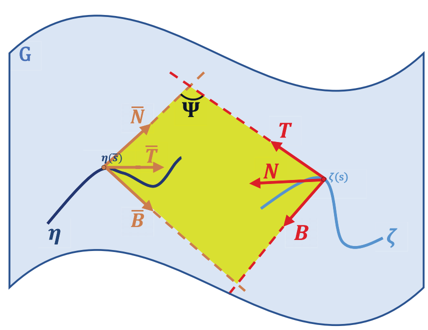

Figure 8: Rectifying plane of the curve ζ and normal plane of the curve η

Thus, we have following relation between the curves ζ and η:

$$\eta(\bar{s}) = \zeta(s) + aT(s) + bB(s), \quad a \neq 0, b \neq 0, \tag{60}$$

where a and b are real valued non-zero functions of s.

Calculating the dot-derivative of the Eq. (60) with the help of Eqs. (3) and (4), we get

$$\bar{T}(\bar{s}) = (1 + \dot{a} - bk_0 \sin\alpha)\frac{1}{r}T(s) + (ak_0 \cos\alpha + b(\dot{\alpha} - \varkappa_0))\frac{1}{r}N(s) + (ak_0 \sin\alpha + \dot{b})\frac{1}{r}B(s), \tag{61}$$

where $r = \dfrac{d\bar{s}}{ds}$.

We know that N is parallel to $\bar{T}$, since $N^{\perp} = Sp\{T, B\} = Sp\{\bar{N}, \bar{B}\} = \bar{T}^{\perp}$. If we multiply the Eq. (61) with T, N and B, respectively, we get

$$bk_0 \sin\alpha - \dot{a} = 1,$$

$$ak_0 \cos\alpha + b(\dot{\alpha} - \varkappa_0) = r,$$

$$ak_0 \sin\alpha + \dot{b} = 0.$$

And so, we have the equation $\bar{T}(\bar{s}) = N(s)$. If we calculate the dot derivative of this equation with the help of Eqs. (3) and (4), we get

$$r(\bar{k}_0 \cos\bar{\alpha}\bar{N}) + \bar{k}_0 \sin\bar{\alpha}\bar{B}) = (-\dot{\alpha} + \varkappa_0)B - k_0 \cos\alpha T. \tag{62}$$

If we multiply the Eq. (62) with $\bar{N}(\bar{s})$ and $\bar{B}(\bar{s})$, respectively, we get

$$r\bar{k}_0 \cos\bar{\alpha} \quad = \sin\psi(-\dot{\alpha} + \varkappa_0) - \cos\psi k_0 \cos\alpha, \tag{63}$$

$$r\bar{k}_0 \sin\bar{\alpha} \quad = \cos\psi(-\dot{\alpha} + \varkappa_0) + \sin\psi k_0 \cos\alpha, \tag{64}$$

where ψ is the smooth angle function between the T and $\bar{N}$. By using Eqs. (63) and (64), we obtain

$$\cot\bar{\alpha} = \frac{\sin\psi(-\dot{\alpha} + \varkappa_0) - \cos\psi k_0 \cos\alpha}{\cos\psi(-\dot{\alpha} + \varkappa_0) + \sin\psi k_0 \cos\alpha}$$

that is

$$\sin\psi(-\dot{\alpha} + \varkappa_0) - \cos\psi k_0 \cos\alpha - \cot\bar{\alpha}(\cos\psi(-\dot{\alpha} + \varkappa_0) + \sin\psi k_0 \cos\alpha) = 0.$$

Thus we introduce the following theorem:

Theorem 3.8. Let $\zeta : I \subseteq \mathbb{R} \to G$ and $\eta : \bar{I} \subseteq \mathbb{R} \to G$ be two arc-length parametrized curves with the Frenet apparatus $\{T, N, B, k_0, \varkappa_0, \alpha\}$ and $\{\bar{T}, \bar{N}, \bar{B}, \bar{k}_0, \bar{\varkappa}_0, \bar{\alpha}\}$, respectively, in three dimensional Lie group G with left-invariant metric. The rectifying plane of the curve ζ and the normal plane of the curve η coincide if and only if there exist real valued non-zero functions a and b on I, such that

$$bk_0 \sin\alpha - \dot{a} = 1, \tag{i}$$

$$ak_0 \cos\alpha + b(\dot{\alpha} - \varkappa_0) \neq 0, \tag{ii}$$

$$ak_0 \sin\alpha + \dot{b} = 0, \tag{iii}$$

$$\sin\psi(-\dot{\alpha} + \varkappa_0) - \cos\psi k_0 \cos\alpha - \cot\bar{\alpha}(\cos\psi(-\dot{\alpha} + \varkappa_0) + \sin\psi k_0 \cos\alpha) = 0, \tag{iv}$$

where ψ is the angle between T and $\bar{N}$ at the corresponding points of ζ and η.

Case 9: We assume that, rectifying plane of the curve ζ is the rectifying plane of the curve η, that is $Sp\{T, B\} = Sp\{\bar{T}, \bar{B}\}$. As in Fig. 9, this relationship exists at the corresponding points of along the curves ζ and η.

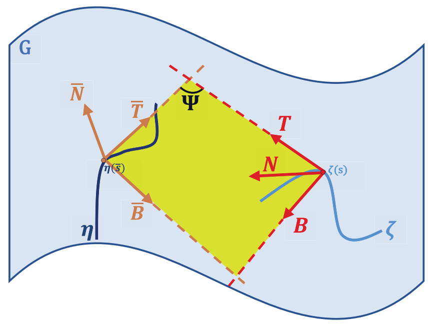

Figure 9: Rectifying planes of the curves ζ and η

Thus, we have following relation between the curves ζ and η:

$$\eta(\bar{s}) = \zeta(s) + aT(s) + bB(s), \quad a \neq 0, b \neq 0, \tag{65}$$

where a and b are real-valued non-zero functions of s.

Calculating the dot-derivative of the Eq. (65) with the help of Eqs. (3) and (4), we get

$$\bar{T}(\bar{s}) = (1 + \dot{a} - bk_0 \sin\alpha)\frac{1}{r}T(s) + (ak_0 \cos\alpha + b(\dot{\alpha} - \varkappa_0))\frac{1}{r}N(s) + (ak_0 \sin\alpha + \dot{b})\frac{1}{r}B(s), \tag{66}$$

where $r = \dfrac{d\bar{s}}{ds}$.

We know that N is parallel to $\bar{N}$, since $N^{\perp} = Sp\{T, B\} = Sp\{\bar{T}, \bar{B}\} = \bar{B}^{\perp}$. If we multiply the Eq. (66) with N, we get

$$(ak_0 \cos\alpha + b(\dot{\alpha} - \varkappa_0))\frac{1}{r} = 0 \quad or \quad ak_0 \cos\alpha + b(\dot{\alpha} - \varkappa_0) = 0.$$

And so, we have

$$\bar{T}(\bar{s}) = (1 + \dot{a} - bk_0 \sin\alpha)\frac{1}{r}T(s) + (ak_0 \sin\alpha + \dot{b})\frac{1}{r}B(s) \tag{67}$$

By the Eq. (67), we can set

$$\begin{aligned} \bar{T}(\bar{s}) &= \cos\psi(s)T(s) + \sin\psi(s)B(s), \\ \bar{B}(\bar{s}) &= -\sin\psi(s)T(s) + \cos\psi(s)B(s), \end{aligned} \tag{68}$$

where ψ is smooth angle function between T and $\bar{T}$ on I and

$$\cos\psi(s) = (1 + \dot{a} - bk_0 \sin\alpha)\frac{1}{r}, \tag{69}$$

$$\sin\psi(s) = (ak_0 \sin\alpha + \dot{b})\frac{1}{r}. \tag{70}$$

By using Eqs. (69) and (70), we obtain

$$r = \sqrt{(1 + \dot{a} - bk_0 \sin\alpha)^2 + (ak_0 \sin\alpha + \dot{b})^2}$$

and

$bk_0 \sin\alpha - \dot{a} + \cot\psi(ak_0 \sin\alpha + \dot{b}) = 1.$

Calculating the dot-derivative of the Eq. (68) with the help of Eqs. (3) and (4), we get

$$\begin{aligned} r(\bar{k}_0 \cos\bar{\alpha}\bar{N} + \bar{k}_0 \sin\bar{\alpha}\bar{B}) =& (-\dot{\psi}\sin\psi - \sin\psi k_0 \sin\alpha)T \\ &+ (\cos\psi k_0 \cos\alpha + \sin\psi(\dot{\alpha} - \varkappa_0))N \\ &+ (\cos\psi k_0 \sin\alpha + \dot{\psi}\cos\psi)B. \end{aligned} \tag{71}$$

If we multiply the Eq. (71) with $\bar{N}$ and $\bar{B}$, respectively, we get

$$r\bar{k}_0 \cos\bar{\alpha} = \cos\psi k_0 \cos\alpha + \sin\psi(\dot{\alpha} - \varkappa_0), \tag{72}$$

$$r\bar{k}_0 \sin\bar{\alpha} = \dot{\psi} + k_o \sin\alpha. \tag{73}$$

By using Eqs. (72) and (73), we obtain

$\cos\psi k_0 \cos\alpha + \sin\psi(\dot{\alpha} - \varkappa_0) - \cot\bar{\alpha}(\dot{\psi} + k_o \sin\alpha) = 0.$

Thus we introduce the following theorem:

Theorem 3.9. Let $\zeta : I \subseteq \mathbb{R} \to G$ and $\eta : \bar{I} \subseteq \mathbb{R} \to G$ be two arc-length parametrized curves with the Frenet apparatus $\{T, N, B, k_0, \varkappa_0, \alpha\}$ and $\{\bar{T}, \bar{N}, \bar{B}, \bar{k}_0, \bar{\varkappa}_0, \bar{\alpha}\}$, respectively, in three dimensional Lie group G with left-invariant metric. The rectifying planes of these curves coincide if and only if there exist real valued non-zero functions a and b on I, such that

$$ak_0 \cos\alpha + b(\dot{\alpha} - \varkappa_0) = 0, \tag{i}$$

$$(1 + \dot{a} - bk_0 \sin\alpha)^2 + (ak_0 \sin\alpha + \dot{b})^2 \neq 0, \tag{ii}$$

$$bk_0 \sin\alpha - \dot{a} + \cot\psi(ak_0 \sin\alpha + \dot{b}) = 1, \tag{iii}$$

$$\cos\psi k_0 \cos\alpha + \sin\psi(\dot{\alpha} - \varkappa_0) - \cot\bar{\alpha}(\dot{\psi} + k_o \sin\alpha) = 0, \tag{iv}$$

where ψ is the angle between T and $\bar{T}$ at the corresponding points of ζ and η.

4 Conclusions

It is well known that every smooth curves have a moving Frenet frame. This paper examines the relations between Frenet planes of two smooth curves in three dimensional Lie groups with left-invariant metric. There are nine possible relations that can occur. For each cases, we give conditions by nine theorems as above. These results are generalizations for relations between Frenet planes of two curves in three dimensional Euclidean spaces. By the paper's results, one will be able to investigate of special curve couples in three-dimensional Lie groups with left-invariant metric and correlate their results.

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