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ARTICLE

A New Modified Inverse Lomax Distribution: Properties, Estimation and Applications to Engineering and Medical Data

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ABSTRACT

In this paper, a modified form of the traditional inverse Lomax distribution is proposed and its characteristics are studied. The new distribution which called modified logarithmic transformed inverse Lomax distribution is generated by adding a new shape parameter based on logarithmic transformed method. It contains two shape and one scale parameters and has different shapes of probability density and hazard rate functions. The new shape parameter increases the flexibility of the statistical properties of the traditional inverse Lomax distribution including mean, variance, skewness and kurtosis. The moments, entropies, order statistics and other properties are discussed. Six methods of estimation are considered to estimate the distribution parameters. To compare the performance of the different estimators, a simulation study is performed. To show the flexibility and applicability of the proposed distribution two real data sets to engineering and medical fields are analyzed. The simulation results and real data analysis showed that the Anderson-Darling estimates have the smallest mean square errors among all other estimates. Also, the analysis of the real data sets showed that the traditional inverse Lomax distribution and some of its generalizations have shortcomings in modeling engineering and medical data. Our proposed distribution overcomes this shortage and provides a good fit which makes it a suitable choice to model such data sets.

KEYWORDS

Inverse lomax distribution; logarithmic transformed method; order statistics; maximum likelihood estimation; maximum product of spacing; manuscript; preparation; typeset; format

1 Introduction

Inverse Lomax (IL) distribution is a very important lifetime distribution which can be used as a good alternative to the well known distributions such as gamma, inverse Weibull, Weibull and Lomax distributions. It can be considered as a member of generalized beta family of distributions. It has different applications in modelling various types of data including economics and actuarial sciences data because its hazard rate can be decreasing and upside down bathtub shaped. The random variable X is said to have an IL distribution if its probability density function (PDF) is

$$h(x;\alpha,\theta)=\alpha\theta x^{-2}\left(1+\frac{\theta}{x}\right)^{-(\alpha+1)},\quad x>0,\ \alpha,\ \theta>0. \tag{1}$$

The cumulative distribution function (CDF) of (1) is

$$H(x; \alpha, \theta) = \left(1 + \frac{\theta}{x}\right)^{-\alpha}, \tag{2}$$

where α is the shape parameter and θ is the scale parameter. For more details about the inverse distributions one can refers to [1–4]. The traditional IL distribution does not give a good fit for modeling many real life data such as in reliability and biological studies. Recently, many generalizations of IL distribution have been developed. Hassan et al. [5] introduced Weibull IL distribution and studied some of its statistical properties. Maxwell et al. [6] proposed the Marshall-Olkin IL distribution by adding a new shape parameter for more flexibility. ZeinEldin et al. [7] used the alpha power method to add a new shape parameter to the IL distribution and proposed a new distribution which called alpha power IL (APIL) distribution. In spite of these generalizations, still a new generalization of IL is needed because these distributions are not able to model various types of data.

The main objective of this paper is to propose a new form of the IL distribution by adding a new shape parameter to the CDF in (2) using the same approach of [8]. They introduced a new method to add an extra shape parameter to an existing distributions which called the logarithmic transformed (LT) method. The LT method has the following CDF and PDF

$$F(x; \lambda) = 1 - \frac{\log[1 - (1 - \lambda)\bar{H}(x)]}{\log(\lambda)} \tag{3}$$

and

$$f(x; \lambda) = \frac{(\lambda - 1)h(x)}{\log(\lambda)[1 - (1 - \lambda)\bar{H}(x)]}. \tag{4}$$

respectively, with $\lambda > 0$, $\lambda \neq 1$. By taking $\bar{H}(x) = 1 - H(x)$ in (3) of the IL distribution CDF in (2) we obtain a modified logarithmic transformed IL (MLTIL) distribution. Many researchers used the LT method to propose a new continuous distributions by taking the baseline $H(x)$ of any known distribution. [9,10] introduced a generalizations for modified Weibull extension and generalized exponential distributions, respectively. [11] introduced a logarithmic inverse Lindley distribution by taking the $H(x)$ in (3) of the inverse Lindley distribution. Also, the logarithm transformed Fréchet distribution by [12]. The overall motivations to propose the MLTIL distribution are

1. To develop different shapes for the PDF and hazard rate function.
2. To increase the flexibility of the traditional IL distribution in modelling different phenomenons.
3. To model skewed data which can not be modeled by other traditional models.
4. To increase the flexibility of the traditional IL distribution properties like mean, variance, skewness and kurtosis.
5. Two applications showed that the MLTIL distribution provides a better fit than the traditional IL distribution and some of its generalizations.
6. Another motivation to this article is to use six classical estimation methods to estimate the parameters in order to recommend which method provide the best estimates based on mean square error criteria and via a simulation study.

The hazard rate function of the MLTIL distribution can has decreasing or upside-down shapes depending on its shape parameters which makes the distribution is quite effectively in modelling lifetime data. It can be used as an alternative to IL and inverse Weibull distributions. Simulation results reveal that the Anderson-Darling (AD) estimators perform better than other estimators in terms of minimum mean-squared errors. Finally, the analysis of engineering and medical data sets show the ability of the MLTIL distribution to provide a better fit than some other competitive models. The rest of the paper is organized as follows: In the next Section we describe the MLTIL distribution and a mixture representation of its density. Some of its statistical properties are discussed in Section 3. Six classical estimation methods are considered in Section 4. A simulation study is conducted in Section 5. Two applications are considered in Section 6. In Section 7, the paper is concluded.

2 Model Description

In this section we introduce the MLTIL distribution. Let the random variable X follows the IL distribution with PDF and CDF, respectively, given by (1), (2), then from (1)–(3) the PDF of the MLTIL distribution is given by

$$f(x;\lambda,\alpha,\theta)=\frac{(\lambda-1)\alpha\theta x^{-2}\left(1+\frac{\theta}{x}\right)^{-(\alpha+1)}}{\log(\lambda)\left[\lambda-(\lambda-1)\left(1+\frac{\theta}{x}\right)^{-\alpha}\right]},\quad x>0,\ \lambda,\ \alpha,\ \theta>0,\ \lambda\neq 1, \tag{5}$$

and its CDF is

$$F(x;\lambda,\alpha,\theta)=1-\frac{\log\left[\lambda-(\lambda-1)\left(1+\frac{\theta}{x}\right)^{-\alpha}\right]}{\log(\lambda)}. \tag{6}$$

The survival function (SF) is given by

$$S(x;\lambda,\alpha,\theta)=\frac{\log\left[\lambda-(\lambda-1)\left(1+\frac{\theta}{x}\right)^{-\alpha}\right]}{\log(\lambda)} \tag{7}$$

and the hazard rate functions is

$$h(x;\lambda,\alpha,\theta)=\frac{(\lambda-1)\alpha\theta x^{-2}\left(1+\frac{\theta}{x}\right)^{-(\alpha+1)}}{\left[\lambda-(\lambda-1)\left(1+\frac{\theta}{x}\right)^{-\alpha}\right]\log\left[\lambda-(\lambda-1)\left(1+\frac{\theta}{x}\right)^{-\alpha}\right]}. \tag{8}$$

For some selected values of the parameters λ, α and θ, Fig. 1 indicates the various shapes of the PDF of the MLTIL distribution, while Fig. 2 shows the different shapes of the hazard rate function. These Figure indicate the flexibility of the MLTIL distribution to model right skewed data as well as the data with decreasing and upside down bathtub shaped.

Now, we can obtain an useful representation for the PDF and CDF of the MLTIL distribution. Using the series representation in the form

$$(1-q)^{n-1}=\sum_{j=0}^{\infty}(-1)^j\binom{n-1}{j}q^j,\quad \text{if } |q|<1,\ n>0 \text{ positive real non-integer}. \tag{9}$$

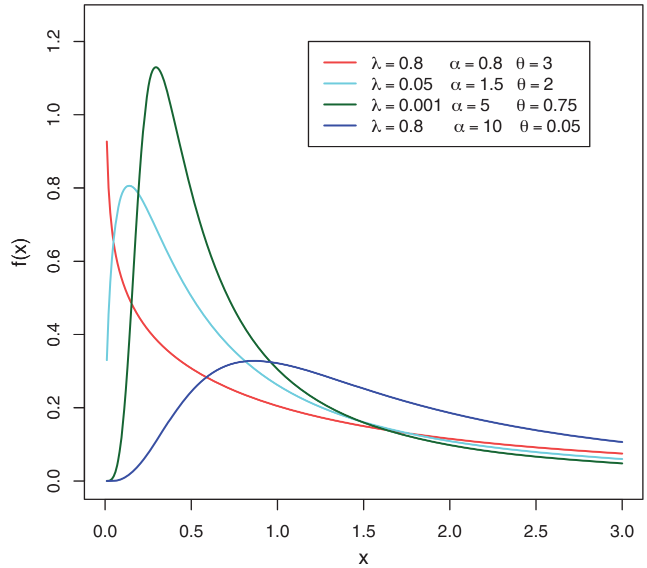

Figure 1: Plots of the MLTIL density for different values of λ, α and θ

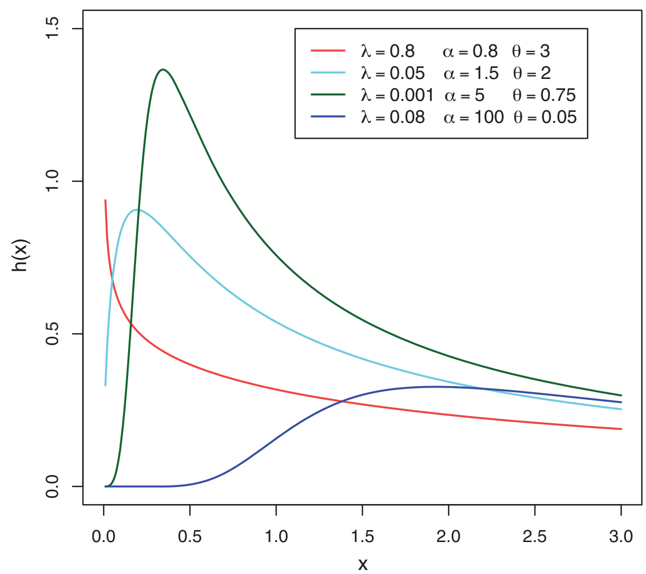

Figure 2: Plots of the MLTIL hazard rate function for different values of λ, α and θ

Applying (9) for the PDF in (1), we can obtain

$$f(x) = \sum_{k=0}^{\infty} \vartheta_k h(x; \alpha(k+1), \theta), \tag{10}$$

where $h(x; \alpha(k+1), \theta)$ denotes the PDF of the IL distribution with scale parameter θ and shape parameter $\alpha(k+1)$ and

$$\vartheta_k = \frac{\lambda}{(k+1)\log(\lambda)}\left(\frac{\lambda-1}{\lambda}\right)^{k+1}.$$

Different structural properties of the MLTIL distribution can be determined using this representation. By integrating (10), the CDF of X is given by

$$F(x) = \sum_{k=0}^{\infty} \vartheta_k H(x; \alpha(k+1), \theta), \tag{11}$$

where $H(.)$ is the CDF of the IL distribution with scale and shape parameters θ and shape parameter $\alpha(k+1)$, respectively.

3 Properties of the MLTIL Distribution

In this section some statistical properties of the MLTIL distribution are obtained including quintile function, moments, incomplete moments, conditional moments and entropies.

3.1 Quantile Function

For the MLTIL distribution the quantile function, say $x = Q_{MLTIL}(q)$, can be obtained by inverting (6) as

$$F^{-1}(q) = Q_{MLTIL}(q) = \frac{\theta}{\left(\frac{\lambda-1}{\lambda(1-\lambda^{-q})}\right)^{\frac{1}{\alpha}} - 1}. \tag{12}$$

We can easily generate X by taking q as a uniform random variable in (0, 1). Based on (12), one can compute the measures of skewness (Sk) and kurtosis (Ku) as follows:

$$Sk = \frac{Q(\frac{3}{4}) + Q(\frac{1}{4}) - 2Q(\frac{1}{2})}{Q(\frac{3}{4}) - Q(\frac{1}{4})}$$

and

$$Ku = \frac{Q(\frac{7}{8}) - Q(\frac{5}{8}) + Q(\frac{3}{8}) - Q(\frac{1}{8})}{Q(\frac{6}{8}) - Q(\frac{2}{8})},$$

respectively, where $Q(\cdot)$ refers to the quantile function.

3.2 Moments

The nth moment of MLTIL distribution can be derives as

$$\mu'_n = E(X^n) = \int_{-\infty}^{\infty} x^n f(x)dx$$

$$= \sum_{k=0}^{\infty} \vartheta_k \int_0^{\infty} x^n h(x; \alpha(k+1), \theta)dx$$

$$= \sum_{k=0}^{\infty} \vartheta_k \alpha(k+1)\theta^n B(1-n, \alpha(k+1)+n), \quad n = 1, 2, 3, \ldots$$

where $B(a, b)$ is the beta function, $\Gamma(0) = -\gamma$ and $\Gamma(-n) = \frac{(-1)^n}{n!}\phi(n) - \frac{(-1)^n}{n!}\gamma$ for $n = 1, 2, 3, \ldots, \gamma$ denotes Euler's constant and $\phi(n) = \sum_{j=1}^{n} \frac{1}{j}$. For more details see [13]. Similarly, for the MLTIL distribution we can obtain the nth inverse moment as follows:

$$E\left(\frac{1}{X^n}\right) = \sum_{k=0}^{\infty} \vartheta_k \frac{B(1+n, \alpha(k+1)-n)}{\theta^n}.$$

In particular,

$$\mu_1' = E(X) = \sum_{k=0}^{\infty} \vartheta_k \alpha(k+1)\theta\Gamma(0),$$

where γ denotes Euler's constant.

$$\mu_2' = E(X^2) = \sum_{k=0}^{\infty} \vartheta_k \alpha(k+1)\left(\alpha(k+1)+1\right)\theta^2\Gamma(-1)$$

and

$$var(X) = \mu_2' - (\mu_1')^2.$$

The rth central moment μ_r of X is derived as

$$\mu_n = E(X-\mu_1')^n = \sum_{k=0}^{n} (-1)^{r-k} \binom{n}{k} \mu_{n-k}' (\mu_1')^k.$$

The Sk and Ku measures can be computed using the following expressions:

$$\mathtt{Sk} = \frac{\mu_3' - 3\mu_2'\mu + 2\mu^3}{\left(\mu_2' - \mu^2\right)^{\frac{3}{2}}},$$

$$\mathtt{Ku} = \frac{\mu_4' - 4\mu_3'\mu + 6\mu_2'\mu^2 - 3\mu^4}{\left(\mu_2' - \mu^2\right)^2}.$$

The following propositions are a description of three different types of moments such as incomplete moments, moment generating function (mgf) and conditional moment.

prop 3.1. If $X \sim$ MLTIL, then the incomplete moments of X is given as follows:

$$I_X(t) = \sum_{k=0}^{\infty} \vartheta_k \int_{-\infty}^{t} x^n h(x; \alpha(k+1), \theta) dx$$

$$= \sum_{k=0}^{\infty} \vartheta_k \int_0^t x^n \left(\alpha\theta(k+1)x^{-2}\left(1+\frac{\theta}{x}\right)^{-\alpha(k+1)-1}\right) dx$$

$$= \alpha\theta \sum_{k=0}^{\infty} \vartheta_k (k+1) \int_0^t x^{\alpha(k+1)+n-1} (\theta + x)^{-\alpha(k+1)-1} dx$$

$$= \sum_{k=0}^{\infty} \vartheta_k \frac{\alpha(k+1)t^n \left(\frac{t}{\theta}\right)^{\alpha(k+1)}}{\alpha(k+1)+n} F_{2:1}\left(\alpha(k+1)+1;\, \alpha(k+1)+n;\, \alpha(k+1)+n+1;\, -\frac{t}{\theta}\right), \tag{13}$$

where, $F_{2:1}(u;\, v;\, c;\, y) = \frac{1}{B(u,\, c-v)} \int_0^1 \frac{z^{v-1}(1-z)^{c-v-1}}{(1-zy)^u} dz$ is the hypergeometric function see [14]. See also [15–17].

The first incomplete moment, $I_1(t)$, follows from Eq. (13) with $n = 1$.

prop 3.2. If $X \sim$ MLTIL, then the mgf of X is

$$M_X(t) = \sum_{j=0}^{\infty} \frac{t^j}{j!} E(X^j) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \vartheta_k \frac{(\theta t)^j}{j!} \alpha(k+1) B(1-j,\, \alpha(k+1)+j).$$

prop 3.3. If $X \sim$ MLTIL, then the characteristic generating function (cgf) of X is

$$M_X(t) = \sum_{j=0}^{\infty} \frac{(it)^j}{j!} E(X^j) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \vartheta_k \frac{(\theta it)^j}{j!} \alpha(k+1) B(1-j,\, \alpha(k+1)+j).$$

prop 3.4. If $X \sim$ MLTIL, then the conditional moment of X is

$$C_X(t) = E\left(E(X^n \mid X > x\right) = \frac{\psi_n(x)}{1-F(x)},$$

where

$$\psi_n(x) = \sum_{k=0}^{\infty} \varpi_k \int_x^{\infty} y^n h(x;\, \alpha(k+1),\, \theta) dy$$

$$= \sum_{k=0}^{\infty} \vartheta_k \alpha(k+1)\theta(-\theta)^{n-1} B_{\frac{-\theta}{x}}(1-n,\, -\alpha(k+1)),$$

where, $B_z(a,\, b) = \int_0^z y^{a-1}(1-y)^{b-1} dy$ is the incomplete beta function.

3.3 Entropies

Entropy has been used in areas like in physics (sparse kernel density estimation), medicin (molecular imaging of tumors) and engineering (measure the randomness of systems). The entropy is a measure of variation of the uncertainty. The Rényi entropy (RE) of order ρ is defined as

$$I_R(\rho) = \frac{1}{1-\rho} \log\left(\int f^{\rho}(x) dx\right). \tag{14}$$

For the MLTIL distribution in (1), we can write

$$I_R(\rho)=\frac{1}{1-\rho}\log\left\{\left(\frac{(\lambda-1)\alpha\theta}{\log(\lambda)}\right)^{\rho}\int_0^{\infty}\left(\frac{x^{-2}\left(1+\frac{\theta}{x}\right)^{-(\alpha+1)}}{\lambda-(\lambda-1)\left(1+\frac{\theta}{x}\right)^{-\alpha}}\right)^{\rho}dx\right\},\quad \rho\geq 0,\ \rho\neq 1$$
$$=\frac{1}{1-\rho}\log\left\{\left(\frac{(\lambda-1)\alpha}{\lambda\theta\log(\lambda)}\right)^{\rho}\sum_{k=0}^{\infty}\left(\frac{(\lambda-1)}{\lambda}\right)^{k}\frac{\Gamma(\rho+k)}{k!}B\left(2\rho-1,\alpha(k+\rho)-\rho+1\right).\right\}.$$

The ρ-entropy, $H_{\rho-R}(x)$, can be written as

$$H_{\rho-R}(\rho)=\frac{1}{1-\rho}\log\left(1-\int_0^{\infty}f^{\rho}(x)dx\right). \tag{15}$$

Shannon's entropy (SE) is defined as $E[-\log f(X)]$ and it is a special case of (15) for $\rho\uparrow 1$. Limiting $\rho\uparrow 1$ in (15) and using L̓Hospital̓s rule, one obtains

$$E[-\log f(X)]=E\left[-\log\left(\frac{(\lambda-1)\alpha\theta x^{-2}\left(1+\frac{\theta}{x}\right)^{-(\alpha+1)}}{\log(\lambda)\left[\lambda-(\lambda-1)\left(1+\frac{\theta}{x}\right)^{-\alpha}\right]}\right)\right]$$
$$=-\log\left[\frac{(\lambda-1)\alpha\theta}{\log(\lambda)}\right]-E\left[\log\left(\frac{x^{-2}\left(1+\frac{\theta}{x}\right)^{-(\alpha+1)}}{\lambda-(\lambda-1)\left(1+\frac{\theta}{x}\right)^{-\alpha}}\right)\right]$$
$$=\log\left[\frac{\log(\lambda)}{\alpha\theta(\lambda-1)}\right]+2E\left(\log(x)\right)+(\alpha+1)E\left(\log(1+\frac{\theta}{x})\right)$$
$$+E\left(\log\left[\lambda-(\lambda-1)\left(1+\frac{\theta}{x}\right)^{-\alpha}\right]\right).$$

3.4 Distribution of Order Statistics

Let $X_{(1)} < X_{(2)},\ldots,< X_{(n)}$ be the order statistics of a random sample of size n taken from the MLTIL distribution. In this case, we can write the PDF of the sth order statistic as follows

$$f_{X_{(s)}}(x)=\frac{f(x)}{B(s,n-s+1)}\sum_{k=0}^{n-s}(-1)^{k}\binom{n-s}{k}F^{k+s-1}(x), \tag{16}$$

where $B(.,.)$ is the beta function. Using (1), (6) and a power series expansion, we obtain

$$f(x)F^{k+s-1}(x)=\frac{\alpha\theta(\lambda-1)}{(\log(\lambda))^{k+s}}\frac{x^{-2}(1+\frac{\theta}{x})^{-(\alpha+1)}}{\lambda-(\lambda-1)(1+\frac{\theta}{x})^{-\alpha}}\left[\log\left(\frac{\lambda}{\lambda-(\lambda-1)(1+\frac{\theta}{x})^{-\alpha}}\right)\right]^{k+s-1}. \tag{17}$$

From (17) and (16), we can write

$$f_{X_{(s)}}(x) = \frac{\alpha\theta(\lambda-1)}{(B(s, n-s+1)} \sum_{k=0}^{n-s} \frac{(-1)^k}{(\log(\lambda))^{k+s}} \binom{n-s}{k} \frac{x^{-2}(1+\frac{\theta}{x})^{-(\alpha+1)}}{\left(\lambda-(\lambda-1)(1+\frac{\theta}{x})^{-\alpha}\right)}$$
$$\times \left[\log\left(\frac{\lambda}{\lambda-(\lambda-1)(1+\frac{\theta}{x})^{-\alpha}}\right)\right]^{k+s-1}. \tag{18}$$

Particularly, we can obtain the PDF of the first and last order statistics from (18), respectively, as

$$f_{X_{(1)}}(x) = \frac{\alpha\theta(\lambda-1)}{n} \sum_{k=0}^{n-1} \frac{(-1)^k}{(\log(\lambda))^{k+1}} \binom{n-1}{k} \frac{x^{-2}(1+\frac{\theta}{x})^{-(\alpha+1)}}{\left(\lambda-(\lambda-1)(1+\frac{\theta}{x})^{-\alpha}\right)}$$
$$\times \left[\log\left(\frac{\lambda}{\lambda-(\lambda-1)(1+\frac{\theta}{x})^{-\alpha}}\right)\right]^{k}, \tag{19}$$

and

$$f_{X_{(n)}}(x) = \frac{\alpha\theta(\lambda-1)}{n\,(\log(\lambda))^n} \frac{x^{-2}(1+\frac{\theta}{x})^{-(\alpha+1)}}{\left(\lambda-(\lambda-1)(1+\frac{\theta}{x})^{-\alpha}\right)} \left[\log\left(\frac{\lambda}{\lambda-(\lambda-1)(1+\frac{\theta}{x})^{-\alpha}}\right)\right]^{n-1}. \tag{20}$$

3.5 Probability Weighted Moments

The (u, v)th probability weighted moments of the random variable X is defined by

$$\rho_{u,v} = E\left\{X^u F(X)^v\right\} = \int_{-\infty}^{\infty} x^u f(x) F^v(X) dx. \tag{21}$$

Based on Eq. (21), we can obtain the probability weighted moments of the MLTIL distribution as

$$\rho_{u,v} = \frac{\alpha\theta(\lambda-1)\rho}{(\log(\lambda))^{v+1}}, \tag{22}$$

where

$$\rho = \int_0^{\infty} \frac{x^{u-2}(1+\frac{\theta}{x})^{-(\alpha+1)}}{\lambda-(\lambda-1)(1+\frac{\theta}{x})^{-\alpha}} \left[\log\left(\frac{\lambda}{\lambda-(\lambda-1)(1+\frac{\theta}{x})^{-\alpha}}\right)\right]^{v} dx. \tag{23}$$

3.6 Residual Life and Reversed Failure Rate Function

The nth moment of the residual life (RL) of the random variable X is defined as follows

$$R_n(t) = E((X-t)^n \mid X > t) = \frac{1}{1-F(t)} \int_t^\infty (x-t)^n f(x)dx, \quad n \geq 1$$

Based on Eq. (1) and applying the binomial expansion of $(x-t)^n$, we have

$$R_n(t) = \frac{1}{1-F(t)} \sum_{m=0}^{n} (-t)^{n-m} \binom{n}{m} \int_t^\infty x^n f(x)dx$$

$$= \frac{\alpha\theta(\lambda-1)}{\lambda} \sum_{m=0}^{n}\sum_{k=0}^{\infty} (-t)^{n-m}(-\theta)^{n-1} \binom{n}{m} \left(\frac{\lambda-1}{\lambda}\right)^k \frac{B_{\frac{-\theta}{t}}(1-n, -(k+1)\alpha)}{\log\left(\lambda-(\lambda-1)(1+\frac{\theta}{x})^{-\alpha}\right)}.$$

The nth moment of the reversed RL can be derived using the general formula

$$m_n(t) = E((t-X)^n \mid X \leq t) = \frac{1}{F(t)} \int_0^t (t-x)^n f(x)dx, \quad n \geq 1.$$

Based on Eq. (1) and applying the binomial expansion of $(t-x)^n$, we can write

$$m_n(t) = \frac{\alpha\theta(\lambda-1)}{\lambda\log(\theta)F(t)} \sum_{m=0}^{n} (-t)^{n-m} \binom{n}{m} \int_0^t x^n f(x)dx$$

$$= \frac{\alpha(\lambda-1)}{\lambda\log(\theta)F(t)} \sum_{m=0}^{n}\sum_{k=0}^{\infty} (-1)^{n-m} \binom{n}{m} \left(\frac{\lambda-1}{\lambda}\right)^k \frac{t^{2n-m+\alpha(k+1)}}{\theta^{\alpha(k+1)}(n+\alpha(k+1))}$$

$$\times F_{2:1}\left(a(k+1)+1; n+a(k+1); n+a(k+1)+1; -\frac{t}{\theta}\right).$$

3.7 Stress-Strength Model

Let $Y_1 \sim MLTIL(\lambda_1, \alpha_1, \theta_1)$ and $Y_2 \sim MLTIL(\lambda_2, \alpha_2, \theta_2)$. If Y_1 represents stress and Y_2 represents strength, then the stress-strength parameter, denoted by R, for the MLTIL distribution is given by

$$R = P(Y_2 > Y_1) = \int_0^\infty F_1(y)dF_2(y) = 1 - \int_0^\infty \bar{F}_1(y)dF_2(y)$$

$$= 1 - \frac{(\lambda_2-1)\alpha_2\theta_2}{\log(\lambda_1)\log(\lambda_2)} \int_0^\infty \frac{y^{-2}\left(1+\frac{\theta_2}{y}\right)^{-(\alpha_2+1)} \log\left[\lambda_1-(\lambda_1-1)\left(1+\frac{\theta_1}{x}\right)^{-\alpha_1}\right]}{\lambda_2-(\lambda_2-1)\left(1+\frac{\theta_2}{x}\right)^{-\alpha_2}} dy$$

$$= 1 - \frac{(\lambda_2-1)\alpha_2\theta_2}{\log(\lambda_1)\log(\lambda_2)} \int_0^\infty \frac{y^{-2}\left(1+\frac{\theta_2}{y}\right)^{-(\alpha_2+1)} \left[\log(\lambda_1)+\log\left(1-(1-\lambda_1)\left(1+\frac{\theta_1}{x}\right)^{-\alpha_1}\right)\right]}{\lambda_2-(\lambda_2-1)\left(1+\frac{\theta_2}{x}\right)^{-\alpha_2}} dy$$

$$= 1 - \frac{(\lambda_2 - 1)\alpha_2\theta_2}{\log(\lambda_2)} \int_0^\infty \frac{y^{-2}\left(1 + \frac{\theta_2}{y}\right)^{-(\alpha_2+1)}}{\lambda_2 - (\lambda_2 - 1)\left(1 + \frac{\theta_2}{x}\right)^{-\alpha_2}} dy$$

$$- \frac{(\lambda_2 - 1)\alpha_2\theta_2}{\log(\lambda_1)\log(\lambda_2)} \int_0^\infty \frac{y^{-2}\left(1 + \frac{\theta_2}{y}\right)^{-(\alpha_2+1)} \log\left(1 - (1 - \lambda_1)\left(1 + \frac{\theta_1}{x}\right)^{-\alpha_1}\right)}{\lambda_2 - (\lambda_2 - 1)\left(1 + \frac{\theta_2}{x}\right)^{-\alpha_2}} dy.$$

Using series expansion in the last equation, we obtain

$$R = 1 - \frac{(\lambda_2 - 1)\alpha_2\theta_2}{\log(\lambda_2)} \sum_{m=0}^{\infty} \left(\frac{\lambda_2 - 1}{\lambda_2}\right)^m \int_0^\infty y^{-2}\left(1 + \frac{\theta_2}{y}\right)^{-\alpha_2(m+1)-1} dy - \frac{(\lambda_2 - 1)\alpha_2\theta_2}{\log(\lambda_1)\log(\lambda_2)}$$

$$\times \sum_{m=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^{k-1}}{k} \left(\frac{\lambda_2 - 1}{\lambda_2}\right)^m \left(\frac{1 - \lambda_1}{\lambda_1}\right)^k \int_0^\infty y^{-2}\left(1 + \frac{\theta_1}{y}\right)^{-\alpha_1 k} \left(1 + \frac{\theta_2}{y}\right)^{-\alpha_2(m+1)-1} dy$$

$$= 1 - \frac{(\lambda_2 - 1)\alpha_2\theta_2}{\log(\lambda_2)} \sum_{m=0}^{\infty} \left(\frac{\lambda_2 - 1}{\lambda_2}\right)^m \frac{1}{\lambda_2\theta_2(m+1)} - \frac{(\lambda_2 - 1)\alpha_2\theta_2}{\log(\lambda_1)\log(\lambda_2)} \sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^{k-1}(-\theta_1)^j}{k}$$

$$\times \left(\frac{\lambda_2 - 1}{\lambda_2}\right)^m \left(\frac{1 - \lambda_1}{\lambda_1}\right)^k \binom{\alpha_1 k + j - 1}{j} \int_0^\infty y^{-2-j}\left(1 + \frac{\theta_2}{y}\right)^{-\alpha_2(m+1)-1} dy$$

$$= 1 - \frac{(\lambda_2 - 1)\alpha_2\theta_2}{\log(\lambda_2)} \sum_{m=0}^{\infty} \left(\frac{\lambda_2 - 1}{\lambda_2}\right)^m \frac{1}{\lambda_2\theta_2(m+1)} - \frac{(\lambda_2 - 1)\alpha_2}{\log(\lambda_1)\log(\lambda_2)} \sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^{k-1}(-\frac{\theta_1}{\theta_2})^j}{k}$$

$$\times \left(\frac{\lambda_2 - 1}{\lambda_2}\right)^m \left(\frac{1 - \lambda_1}{\lambda_1}\right)^k \binom{\alpha_1 k + j - 1}{j} B\left(j + 1, \alpha_2(m+1) - j\right).$$

4 Estimation

In this section, six estimation methods are considered to estimate the MLIIL distribution parameters.

4.1 Maximum Likelihood Estimation

Using a random sample of size m taken from the MLTIL distribution, then based on (5) the log-likelihood function can be written as

$$E_1(\boldsymbol{\varpi}) = m\log\left(\frac{(\lambda - 1)\alpha\theta}{\log(\lambda)}\right) - 2\sum_{j=1}^{m}\log(x_j) - (\alpha + 1)\sum_{j=1}^{m}\log\left(1 + \frac{\theta}{x_j}\right)$$

$$- \sum_{j=1}^{m}\log\left[\lambda - (\lambda - 1)\left(1 + \frac{\theta}{x_j}\right)^{-\alpha}\right], \quad (24)$$

where $\varpi = (\lambda, \alpha, \theta)^T$. The maximum likelihood estimates (MLEs) of λ, α, and θ, denoted by $\hat{\lambda}^{ML}$, $\hat{\alpha}^{ML}$, and $\hat{\theta}^{ML}$, are the solution of the following three equations

$$\frac{\partial E_1(\varpi)}{\partial\lambda} = \frac{m}{\lambda-1} - \frac{m}{\lambda\log(\lambda)} - \sum_{j=1}^{m} \frac{1-\left(1+\frac{\theta}{x_j}\right)^{-\alpha}}{\lambda-(\lambda-1)\left(1+\frac{\theta}{x_j}\right)^{-\alpha}} = 0,$$

$$\frac{\partial E_1(\varpi)}{\partial\alpha} = \frac{m}{\alpha} - \sum_{j=1}^{m}\log\left(1+\frac{\theta}{x_j}\right) - \sum_{j=1}^{m} \frac{(\lambda-1)\left(1+\frac{\theta}{x_j}\right)^{-\alpha}\log\left(1+\frac{\theta}{x_j}\right)}{\lambda-(\lambda-1)\left(1+\frac{\theta}{x_j}\right)^{-\alpha}} = 0$$

and

$$\frac{\partial E_1(\varpi)}{\partial\theta} = \frac{m}{\theta} - (\alpha+1)\sum_{j=1}^{m}\frac{1}{x_j\left(1+\frac{\theta}{x_j}\right)} + \alpha\sum_{j=1}^{m}\frac{(\lambda-1)x_j^{-1}\left(1+\frac{\theta}{x_j}\right)^{-\alpha-1}}{\lambda-(\lambda-1)\left(1+\frac{\theta}{x_j}\right)^{-\alpha}} = 0.$$

It is observed that these equations cannot be solved analytically for λ, α, and θ, therefore to obtain $\hat{\lambda}^{ML}$, α^{ML}, and θ^{ML} one can use any numerical technique for this purpose.

4.2 Percentile Estimation

Kao et al. [18,19] originally proposed the percentile method of estimation and recently used by many authors to obtain the parameters of some proposed distributions, see for example [20,21]. It is very useful when the distribution has a closed form quantile function. Since the MLTIL distribution has an explicit quantile function as in (12), then we can obtain the percentile estimates (PEs) of λ, α, and θ, denoted by $\hat{\lambda}^P$, $\hat{\alpha}^P$, and $\hat{\theta}^P$, by minimizing

$$E_2(\varpi) = \sum_{j=1}^{m}\left[x_{(j)} - \frac{\theta}{\left(\frac{\lambda-1}{\lambda(1-\lambda^{-q_j})}\right)^{\frac{1}{\alpha}}-1}\right]^2, \tag{25}$$

where $x_{(j)}$ is the ordered observation of x_j, $j = 1, \ldots, n$ and $q_j = j/(m+1)$. Instead of minimizing (25) with respect to λ, α, and θ, we can obtain $\hat{\lambda}^P$, $\hat{\alpha}^P$, and $\hat{\theta}^P$ by solving

$$\frac{\partial E_2(\varpi)}{\partial\lambda} = \sum_{j=1}^{m}\left[x_{(j)} - \frac{\theta}{\left(\frac{\lambda-1}{\lambda(1-\lambda^{-q_j})}\right)^{\frac{1}{\alpha}}-1}\right]\Phi_{1j}(\varpi) = 0,$$

$$\frac{\partial E_2(\varpi)}{\partial\alpha} = \sum_{j=1}^{m}\left[x_{(j)} - \frac{\theta}{\left(\frac{\lambda-1}{\lambda(1-\lambda^{-q_j})}\right)^{\frac{1}{\alpha}}-1}\right]\Phi_{2j}(\varpi) = 0$$

and

$$\frac{\partial E_2(\boldsymbol{\varpi})}{\partial \theta} = \sum_{j=1}^{m} \left[x_{(j)} - \frac{\theta}{\left(\frac{\lambda-1}{\lambda(1-\lambda^{-q_j})}\right)^{\frac{1}{\alpha}} - 1} \right] \Phi_{3j}(\boldsymbol{\varpi}) = 0,$$

where

$$\Phi_{1j}(\boldsymbol{\varpi}) = \frac{\theta \left(\frac{1-\lambda}{\lambda v_j}\right)^{1/\alpha - 1} \left[1 - \frac{\lambda-1}{\lambda} + \frac{q_j(\lambda-1)}{\lambda^{q_j+1} v_j}\right]}{\alpha \lambda v_j \left[\left(\frac{1-\lambda}{\lambda v_j}\right)^{1/\alpha} - 1\right]^2},$$

$$\Phi_{2j}(\boldsymbol{\varpi}) = \frac{\theta \left(\frac{1-\lambda}{\lambda v_j}\right)^{1/\alpha} \log\left(\frac{1-\lambda}{\lambda v_j}\right)}{\alpha^2 \left[\left(\frac{1-\lambda}{\lambda v_j}\right)^{1/\alpha} - 1\right]^2}$$

and

$$\Phi_{3j}(\boldsymbol{\varpi}) = \left[\left(\frac{1-\lambda}{\lambda v_j}\right)^{1/\alpha} - 1\right]^{-1},$$

where $v_j = \frac{1}{\lambda^{q_j}} - 1$.

4.3 Maximum Product of Spacing Estimation

Cheng et al. [22,23] introduced the method of maximum product of spacing (MPS) to obtain the estimates of the continuous univariate distributions parameters as an alternative to the method of maximum likelihood. Let $\Psi_i(\boldsymbol{\varpi})$ is the uniform spacing defined as follows:

$$\Psi_j(\boldsymbol{\varpi}) = F(x_{(j)} \mid \boldsymbol{\varpi}) - F(x_{(j-1)} \mid \boldsymbol{\varpi})$$
$$= \frac{\log\left[\lambda - (\lambda-1)\left(1 + \frac{\theta}{x_{(j-1)}}\right)^{-\alpha}\right] - \log\left[\lambda - (\lambda-1)\left(1 + \frac{\theta}{x_{(j)}}\right)^{-\alpha}\right]}{\log(\lambda)}.$$

To obtain the MPS estimates (MPSEs) of λ, α, and θ, denoted by $\hat{\lambda}^{MPS}$, $\hat{\alpha}^{MPS}$, and $\hat{\theta}^{MPS}$, we can maximize the following function

$$E_3(\boldsymbol{\varpi}) = \frac{1}{m+1} \sum_{j=1}^{m+1} \log[\Psi_i(\boldsymbol{\varpi})], \tag{26}$$

with respect to λ, α, and θ. Instead of maximizing (26) to obtain $\hat{\lambda}^{MPS}$, $\hat{\alpha}^{MPS}$, and $\hat{\theta}^{MPS}$, we can solve the following three equations to obtain these estimates

$$\frac{\partial E_3(\boldsymbol{\varpi})}{\partial\lambda} = \frac{1}{m+1}\sum_{j=1}^{m+1}\frac{\delta_{1j}(x_{(j)}\mid\boldsymbol{\varpi}) - \delta_{1j}(x_{(j-1)}\mid\boldsymbol{\varpi})}{\Psi_j(\boldsymbol{\varpi})} = 0,$$

$$\frac{\partial E_3(\boldsymbol{\varpi})}{\partial\alpha} = \frac{1}{m+1}\sum_{j=1}^{m+1}\frac{\delta_{2j}(x_{(j)}\mid\boldsymbol{\varpi}) - \delta_{2j}(x_{(j-1)}\mid\boldsymbol{\varpi})}{\Psi_j(\boldsymbol{\varpi})} = 0$$

and

$$\frac{\partial E_3(\boldsymbol{\varpi})}{\partial\theta} = \frac{1}{m+1}\sum_{j=1}^{m+1}\frac{\delta_{3j}(x_{(j)}\mid\boldsymbol{\varpi}) - \delta_{3j}(x_{(j-1)}\mid\boldsymbol{\varpi})}{\Psi_j(\boldsymbol{\varpi})} = 0,$$

where

$$\delta_{1j}(x_{(j)}\mid\boldsymbol{\varpi}) = \frac{\xi_j\log(\lambda) + \lambda\log(\lambda)c_j}{\lambda\log^2(\lambda)\xi_j}, \tag{27}$$

$$\delta_{2j}(x_{(j)}\mid\boldsymbol{\varpi}) = \frac{(1-\lambda)\log(1+\theta/x_{(j)})(1+\theta/x_{(j)})^{-\alpha}}{\log(\lambda)\xi_j} \tag{28}$$

and

$$\delta_{3j}(x_{(j)}\mid\boldsymbol{\varpi}) = \frac{\alpha(1-\lambda)(1+\theta/x_{(j)})^{-\alpha-1}}{x_{(j)}\log(\lambda)\xi_j}, \tag{29}$$

where $\xi_j = \lambda - (\lambda-1)\left(1+\frac{\theta}{x_{(j)}}\right)^{-\alpha}$ and $c_j = \left(1+\frac{\theta}{x_{(j)}}\right)^{-\alpha}$.

4.4 Least and Weighted Least Squares Estimation

Swain et al. [24] considered the least squares (LS) and weighted least squares (WLS) methods to estimate the beta distribution parameters. The LS estimates (LSEs) of λ, α, and θ, denoted by $\hat{\lambda}^{LS}$, $\hat{\alpha}^{LS}$, and $\hat{\theta}^{LS}$ can be obtained by minimizing

$$E_4(\boldsymbol{\varpi}) = \sum_{j=1}^{m}\left\{\eta_j - \frac{\log\left[\lambda-(\lambda-1)\left(1+\frac{\theta}{x_{(j)}}\right)^{-\alpha}\right]}{\log(\lambda)}\right\}^2,$$

with respect to λ, α, and θ, where $\eta_j = 1 - j/(m+1)$. The estimates $\hat{\lambda}^{LS}$, $\hat{\alpha}^{LS}$, and $\hat{\theta}^{LS}$ can be also obtained by solving

$$\frac{\partial E_4(\boldsymbol{\varpi})}{\partial\lambda} = \sum_{j=1}^{m}\left\{\eta_j - \frac{\log\left[\lambda-(\lambda-1)\left(1+\frac{\theta}{x_{(j)}}\right)^{-\alpha}\right]}{\log(\lambda)}\right\}\delta_{1j}(x_{(j)}\mid\boldsymbol{\varpi}) = 0,$$

$$\frac{\partial E_4(\boldsymbol{\varpi})}{\partial \alpha}=\sum_{j=1}^{m}\left\{\eta_j-\frac{\log\left[\lambda-(\lambda-1)\left(1+\frac{\theta}{x_{(j)}}\right)^{-\alpha}\right]}{\log(\lambda)}\right\}\delta_{2j}(x_{(j)}\mid\boldsymbol{\varpi})=0$$

and

$$\frac{\partial E_4(\boldsymbol{\varpi})}{\partial \theta}=\sum_{j=1}^{m}\left\{\eta_j-\frac{\log\left[\lambda-(\lambda-1)\left(1+\frac{\theta}{x_{(j)}}\right)^{-\alpha}\right]}{\log(\lambda)}\right\}\delta_{3j}(x_{(j)}\mid\boldsymbol{\varpi})=0,$$

where $\delta_{1j}(x_{(j)}\mid\boldsymbol{\varpi})$, $\delta_{2j}(x_{(j)}\mid\boldsymbol{\varpi})$ and $\delta_{3j}(x_{(j)}\mid\boldsymbol{\varpi})$ are given by (27), (28). Similarly, the WLS estimates (WLSEs) of λ, α, and θ, denoted by $\hat{\lambda}^{WLS}$, $\hat{\alpha}^{WLS}$, and $\hat{\theta}^{WLS}$ can be computed by minimizing

$$E_5(\boldsymbol{\varpi})=\sum_{j=1}^{m}\varsigma_j\left\{\eta_j-\frac{\log\left[\lambda-(\lambda-1)\left(1+\frac{\theta}{x_{(j)}}\right)^{-\alpha}\right]}{\log(\lambda)}\right\}^2, \tag{30}$$

with respect to λ, α, and θ, where $\varsigma_j=\frac{(m+1)^2(m+2)}{j(m-j+1)}$. Another way to obtain these estimates is to solve the following equations:

$$\frac{\partial E_5(\boldsymbol{\varpi})}{\partial \lambda}=\sum_{j=1}^{m}\varsigma_j\left\{\eta_j-\frac{\log\left[\lambda-(\lambda-1)\left(1+\frac{\theta}{x_{(j)}}\right)^{-\alpha}\right]}{\log(\lambda)}\right\}\delta_{1j}(x_{(j)}\mid\boldsymbol{\varpi})=0,$$

$$\frac{\partial E_5(\boldsymbol{\varpi})}{\partial \alpha}=\sum_{j=1}^{m}\varsigma_j\left\{\eta_j-\frac{\log\left[\lambda-(\lambda-1)\left(1+\frac{\theta}{x_{(j)}}\right)^{-\alpha}\right]}{\log(\lambda)}\right\}\delta_{2j}(x_{(j)}\mid\boldsymbol{\varpi})=0$$

and

$$\frac{\partial E_5(\boldsymbol{\varpi})}{\partial \theta}=\sum_{j=1}^{m}\varsigma_j\left\{\eta_j-\frac{\log\left[\lambda-(\lambda-1)\left(1+\frac{\theta}{x_{(j)}}\right)^{-\alpha}\right]}{\log(\lambda)}\right\}\delta_{3j}(x_{(j)}\mid\boldsymbol{\varpi})=0,$$

where $\delta_{1j}(x_{(j)}\mid\boldsymbol{\varpi})$, $\delta_{2j}(x_{(j)}\mid\boldsymbol{\varpi})$ and $\delta_{3j}(x_{(j)}\mid\boldsymbol{\varpi})$ are given by (27), (28).

4.5 Anderson-Darling Estimation

The AD method of estimation is a type of minimum distance estimator which obtained by minimizing an AD statistic. The AD estimates (ADEs) of λ, α, and θ, denoted by $\hat{\lambda}^{AD}$, $\hat{\alpha}^{AD}$, and $\hat{\theta}^{AD}$ can be obtained by minimizing

$$E_6(\boldsymbol{\varpi}) = -m - \frac{1}{m}\sum_{j=1}^{m}(2j-1)[\log(F(x_{(j)} \mid \boldsymbol{\varpi})) + \log(\bar{F}(x_{(m-j+1)} \mid \boldsymbol{\varpi}))],$$

with respect to λ, α, and θ. Another way to obtain these estimates is to solve the following equations:

$$\frac{\partial E_6(\boldsymbol{\varpi})}{\partial \lambda} = \sum_{j=1}^{m}(2j-1)\left[\frac{\delta_{1j}(x_{(j)} \mid \boldsymbol{\varpi})}{F(x_{(j)} \mid \boldsymbol{\varpi})} - \frac{\delta_{1j}(x_{(m-j+1)} \mid \boldsymbol{\varpi})}{\bar{F}(x_{(m-j+1)} \mid \boldsymbol{\varpi})}\right] = 0,$$

$$\frac{\partial E_6(\boldsymbol{\varpi})}{\partial \alpha} = \sum_{j=1}^{m}(2j-1)\left[\frac{\delta_{2j}(x_{(j)} \mid \boldsymbol{\varpi})}{F(x_{(j)} \mid \boldsymbol{\varpi})} - \frac{\delta_{2j}(x_{(m-j+1)} \mid \boldsymbol{\varpi})}{\bar{F}(x_{(m-j+1)} \mid \boldsymbol{\varpi})}\right] = 0$$

and

$$\frac{\partial E_6(\boldsymbol{\varpi})}{\partial \theta} = \sum_{j=1}^{m}(2j-1)\left[\frac{\delta_{3j}(x_{(j)} \mid \boldsymbol{\varpi})}{F(x_{(j)} \mid \boldsymbol{\varpi})} - \frac{\delta_{3j}(x_{(m-j+1)} \mid \boldsymbol{\varpi})}{\bar{F}(x_{(m-j+1)} \mid \boldsymbol{\varpi})}\right] = 0,$$

where $\delta_{1j}(x_{(j)} \mid \boldsymbol{\varpi})$, $\delta_{2j}(x_{(j)} \mid \boldsymbol{\varpi})$ and $\delta_{3j}(x_{(j)} \mid \boldsymbol{\varpi})$ are given by (27), (28).

5 Simulation Study

We cannot compare the performance of the different proposed estimators theoretically, therefore a simulation study is done in order to show the behavior of the various estimators in terms of mean square error (MSE) criteria. To conduct the simulation study, we choose two sets of the parameters values; Set I: $(\lambda, \alpha, \theta) = (1.5, 0.5, 0.5)$ and Set II: $(\lambda, \alpha, \theta) = (0.5, 2, 2)$. Also, different sample sizes are chosen, where $m = 20, 50, 100, 150, 200$ and 250. These values are selected to indicate the affect of the small, moderate and large sample sizes in the accuracy of the estimates. Each setting is replicated 1000 times and the average estimates and average MSEs are obtained. These values are tabulated for Sets I and II in Tabs. 1 and 2, respectively. The results in these Tables show that the MSEs decrease as the sample size increases in all the cases, which means hat these estimators are consistent. It is also observed that the MSEs tend to zero except the PEs especially in the small and moderate sample sizes. Comparing the performance of the different estimates in terms of minimum MSEe, we can conclude that the ADEs perform better than other estimates in most of the cases. Based on these results, we recommend to use the Anderson-Darling method to estimate the MLTIL distribution parameters.

Table 1: Estimates and MSEs for different estimates of $\lambda = 1.5$ and $\alpha = \theta = 0.5$

m	Par	MLEs	PEs	MPSEs	LSEs	WLSEs	ADEs
20	λ	1.1604	1.4832	2.2886	1.6726	1.7763	1.6367
		2.0005	1.7900	3.0197	2.0821	2.1711	1.7594
	α	0.8496	1.5604	0.6775	0.6466	0.6219	0.6995
		0.5289	3.4964	0.4734	0.2771	0.2380	0.2503
	θ	1.1459	1.9455	0.6493	1.0761	1.0240	0.9762
		1.5818	6.4506	0.5248	1.4903	1.3346	0.5239
50	λ	1.3942	1.7430	2.1332	1.5095	1.5987	1.4844
		1.5724	1.7501	2.2973	1.4051	1.3911	1.3195
	α	0.6236	1.5054	0.5288	0.5814	0.5782	0.5922
		0.0631	3.3498	0.0350	0.0780	0.2102	0.0315
	θ	0.9662	2.0991	0.7091	0.8991	0.8654	0.8724
		1.0591	5.7922	0.4954	0.7303	0.7310	0.4146
100	λ	1.4189	1.5875	1.8143	1.4026	1.4560	1.3970
		1.5217	1.1934	1.6017	1.0632	1.1859	1.0429
	α	0.5878	1.2454	0.5277	0.5654	0.5675	0.5694
		0.0368	2.3354	0.0217	0.0256	0.0287	0.0259
	θ	0.8933	2.0128	0.7322	0.8138	0.8228	0.8032
		0.6525	4.4985	0.3852	0.5253	0.4945	0.3550
150	λ	1.4509	1.4629	1.7503	1.4787	1.4674	1.4642
		0.9781	0.7618	0.8992	0.6851	0.7495	0.6072
	α	0.5434	1.2519	0.5050	0.5466	0.5383	0.5454
		0.0143	1.8435	0.0101	0.0155	0.0116	0.0100
	θ	0.7426	1.6745	0.6484	0.7334	0.7068	0.7283
		0.3446	3.1477	0.2432	0.3190	0.2813	0.2344
200	λ	1.4645	1.4999	1.5109	1.4550	1.4883	1.4380
		0.5327	0.3899	0.4489	0.3856	0.4176	0.4334
	α	0.5324	1.1448	0.5076	0.5249	0.5270	0.5256
		0.0057	1.3608	0.0047	0.0056	0.0050	0.0045
	θ	0.5978	1.4511	0.5545	0.5793	0.5919	0.5840
		0.0489	2.1426	0.0366	0.0498	0.0499	0.0356
250	λ	1.4734	1.4186	1.5640	1.4916	1.4830	1.4618
		0.3037	0.1825	0.2893	0.2405	0.2665	0.2273
	α	0.5137	1.0331	0.4957	0.5208	0.5183	0.5168
		0.0031	0.7968	0.0027	0.0065	0.0042	0.0016
	θ	0.5482	1.0652	0.5314	0.5476	0.5514	0.5470
		0.0304	1.0735	0.0260	0.0367	0.0347	0.0325

Table 2: Estimates and MSEs for different estimates of $\lambda = 0.5$ and $\alpha = \theta = 2$

m	Par	MLEs	PEs	MPSEs	LSEs	WLSEs	ADEs
20	λ	1.2809	1.3543	0.9743	1.3234	1.2492	1.1243
		2.0139	2.4882	1.2214	1.9640	1.8293	1.1965
	α	3.3311	3.2602	2.5432	2.3978	2.5560	2.6517
		3.8339	4.3882	2.4643	2.0164	2.2287	2.0110
	θ	1.4935	2.6633	2.3092	2.3737	2.2743	2.2358
		1.4649	3.9371	1.5313	1.9404	1.5747	1.4051
50	λ	1.1317	1.3206	0.8494	0.9740	0.9650	0.9242
		1.0138	1.3307	0.6400	0.7398	0.7357	0.6187
	α	2.7140	2.6147	2.1473	2.2559	2.3143	2.2754
		1.5975	2.8690	0.8521	1.1503	1.1863	0.8238
	θ	1.6998	2.5816	2.2930	2.1570	2.0636	2.0997
		1.3562	3.6776	1.2410	1.2698	1.1682	1.0776
100	λ	0.8234	1.1222	0.6592	0.8100	0.7650	0.7797
		0.4265	0.8565	0.2745	0.4146	0.3906	0.2487
	α	2.2291	2.6479	1.9685	2.0719	2.0772	2.1111
		0.3826	1.9660	0.2800	0.3953	0.3289	0.2660
	θ	1.9395	3.0038	2.3626	2.0385	2.0961	2.0336
		0.9087	2.3214	1.0649	0.5601	0.6380	0.5142
150	λ	0.7437	0.8321	0.6380	0.7068	0.7616	0.7487
		0.2403	0.3895	0.1988	0.2219	0.2553	0.1844
	α	2.1763	2.5740	1.9825	1.9978	2.0643	2.0714
		0.2268	1.1945	0.1585	0.2610	0.2258	0.1472
	θ	1.9431	2.6172	2.2111	2.0253	1.9159	1.9129
		0.5857	1.2741	0.6777	0.3739	0.3871	0.2687
200	λ	0.5175	0.7575	0.4761	0.5723	0.5499	0.5341
		0.1186	0.3659	0.0951	0.1494	0.1359	0.0943
	α	2.1460	2.3308	2.0230	2.0886	2.0479	2.0688
		0.1172	0.4712	0.0895	0.1548	0.1142	0.1085
	θ	2.0274	2.1949	2.2052	2.0030	2.0005	2.0426
		0.3106	0.4493	0.3647	0.1699	0.1822	0.1401
250	λ	0.5489	0.8514	0.5335	0.6153	0.5820	0.5787
		0.0747	0.3122	0.0729	0.1174	0.0926	0.0642
	α	2.0506	2.1351	1.9710	2.0067	2.0270	2.0271
		0.0645	0.1379	0.0575	0.0745	0.0693	0.0570
	θ	1.9786	2.1614	2.0857	1.9838	1.9876	1.9758
		0.0970	0.1514	0.1222	0.0749	0.0816	0.0649

6 Applications

To show the applicability of the proposed distribution and the different estimators derived in the previous sections two real data sets are analyzed. The first data set considered by [25] which represent life times (in days) of 39 liver cancers patients taken from Elminia cancer center Ministry of Health in Egypt. The second data set consists of 29 time between failures of a piece of construction equipment in chronological studied by [26]. These data are displayed in Tab. 3.

Table 3: Real data sets

Data I						Data II				
10	17	23	40	71	107	0.33	1.02	6.52	27.4	92.9
14	18	23	49	74	107	0.33	1.17	7.25	27.43	
14	20	24	51	75	116	0.5	1.72	8.58	31.93	
14	20	26	52	87	150	0.5	1.83	10.25	38.37	
14	20	30	60	96		0.5	3.2	11.58	40.02	
14	20	30	61	105		0.95	4.35	13.83	62.77	
15	20	31	67	107		1	5.25	15.93	88.27	

We compare the results of the MLTIL distribution with IL distribution, inverse Weibull (IW) distribution, APIL distribution by [7] and alpha power inverse Weibull (APIW) distribution by [27]. The density functions of these distributions (for $x > 0$) are as follow

$$f_{IW}(x; \alpha, \theta) = \alpha\theta^{\alpha} x^{-\alpha-1} e^{-(\theta/x)^{\alpha}},$$

$$f_{APIL}(x; \lambda, \alpha, \theta) = \frac{\log(\lambda)}{\lambda - 1}\alpha\theta x^{-2}(1+\theta/x)^{-\alpha-1}\lambda^{(1+\theta/x)^{-\alpha}},$$

and

$$f_{APIW}(x; \lambda, \alpha, \theta) = \frac{\log(\lambda)}{\lambda - 1}\alpha\theta x^{-\alpha-1} e^{-\theta x^{-\alpha}} \lambda^{e^{-\theta x^{-\alpha}}}.$$

We first obtain the MLEs of the competitive distributions. These estimates are presented in Tabs. 4 and 5 for datas 1 and 2, respectively. To compare the performance of the difference distributions to fit the data set, we use Kolmogorov-Smirnov (K-S) distance and the corresponding p-value. These statistics are also presented in Tabs. 4 and 5 for datas 1 and 2, respectively. From the results in these Tables it is observed that the MLTIL distribution has the smallest K-S distance with the highest p-value among other competitive distributions, therefore we conclude that the MLTIL distribution is the suitable model to fit these data. The fitted PDF, CDF, SF and P-P plot of the MLTIL distribution are presented in Figs. 3 and 4 for datas 1 and 2, respectively. These Figures support the results discussed before that the MLTIL distribution provides a close fit to these data sets.

Table 4: MLEs, K-S and p-value of real data 1

Model	Estimates			K-S	p-value
IL(α, θ)		35.616	1.594	0.3708	0.0000
IW(α, θ)		1.531	24.961	0.6187	0.0000
MLTIL(λ, α, θ)	0.00022	20.3948	6.8869	0.1172	0.6574
APIL(λ, α, θ)	1.552	148.947	1.594	0.1321	0.5044
APIW(λ, α, θ)	0.063	103.324	0.467	0.1250	0.5762

Table 5: MLEs, K-S and p-value of real data 2

Model	Estimates			K-S	p-value
IL(α, θ)		3.308	0.352	0.3595	0.0011
IW(α, θ)		0.634	2.195	0.1733	0.3486
MLTIL(λ, α, θ)	52.57694	41.521	0.01526	0.0958	0.9531
APIL(λ, α, θ)	3.408	1.218	0.703	0.1083	0.8855
APIW(λ, α, θ)	0.745	0.887	7.600	0.1217	0.7839

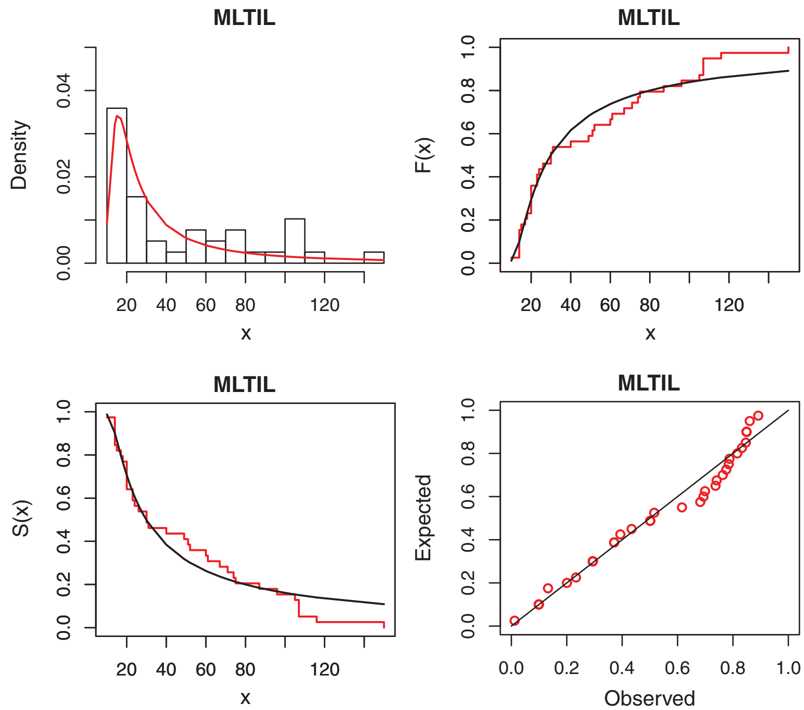

Figure 3: Fitted density, estimated CDF and SF, and P-P plots for real data 1

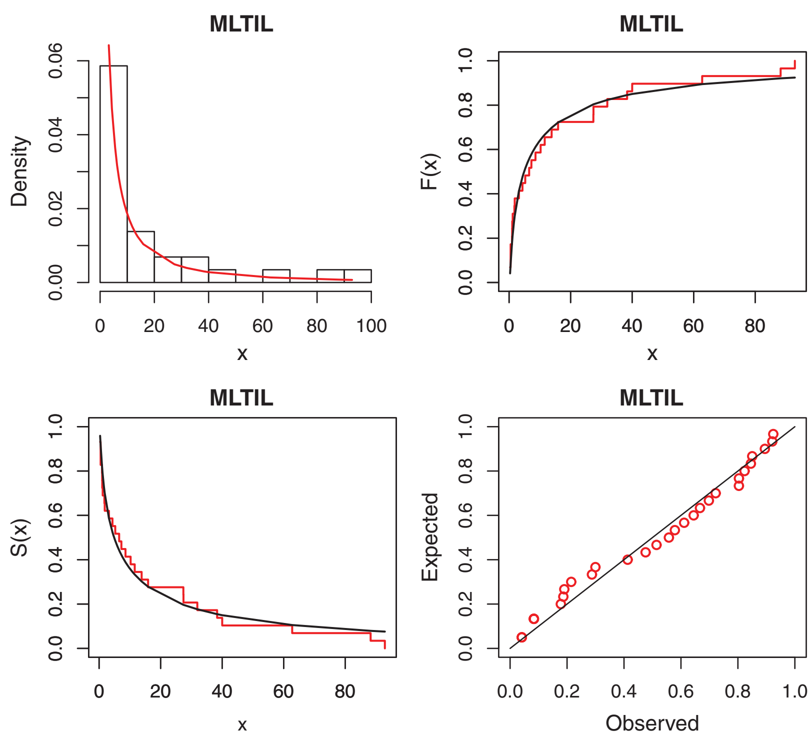

Figure 4: Fitted density, estimated CDF and SF, and P-P plots for real data 2

To see which estimation method provide a good fit to these data we compare the other estimation methods with the maximum likelihood methods based on K-S distance and its p-value. For the two data sets, the various estimates, K-S distance and its p-value are obtained in Tabs. 6 and 7, respectively. Comparing the fitting of the different methods we can observe that the ADEs provides a good fit based on minimum K-S distance with highest p-value. Therefore, we can recommend to use the AD method to estimate the MLTIL distribution parameters from these real data sets. Also, the histogram and the fitted MLTIL distribution based on the different methods are presented in Fig. 5 for the two data sets.

Table 6: Various estimates, K-S and p-value of real data 1

Model	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\theta}$	K-S	p-value
MLEs	0.00022	20.3948	6.8869	0.1172	0.6574
PEs	0.0009	128.205	0.3341	0.5432	0.0000
MPSEs	0.1975	137.030	0.3359	0.1621	0.2571
LSEs	0.0774	191.701	0.2528	0.1221	0.6062
WLSEs	0.0361	108.2163	0.5261	0.1150	0.6810
ADEs	0.0009	20.8313	5.6234	0.1148	0.6827

Table 7: Various estimates, K-S and p-value of real data 2

Model	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\theta}$	K-S	p-value
MLEs	52.57694	41.521	0.01526	0.0958	0.9531
PEs	0.1265	51.1731	0.1911	0.3590	0.0011
MPSEs	23.8235	2.175	0.5229	0.1337	0.6781
LSEs	0.0097	1.035	49.1779	0.0974	0.9460
WLSEs	0.0017	1.4282	42.379	0.0837	0.9872
ADEs	0.0010	1.5226	42.4419	0.0833	0.9878

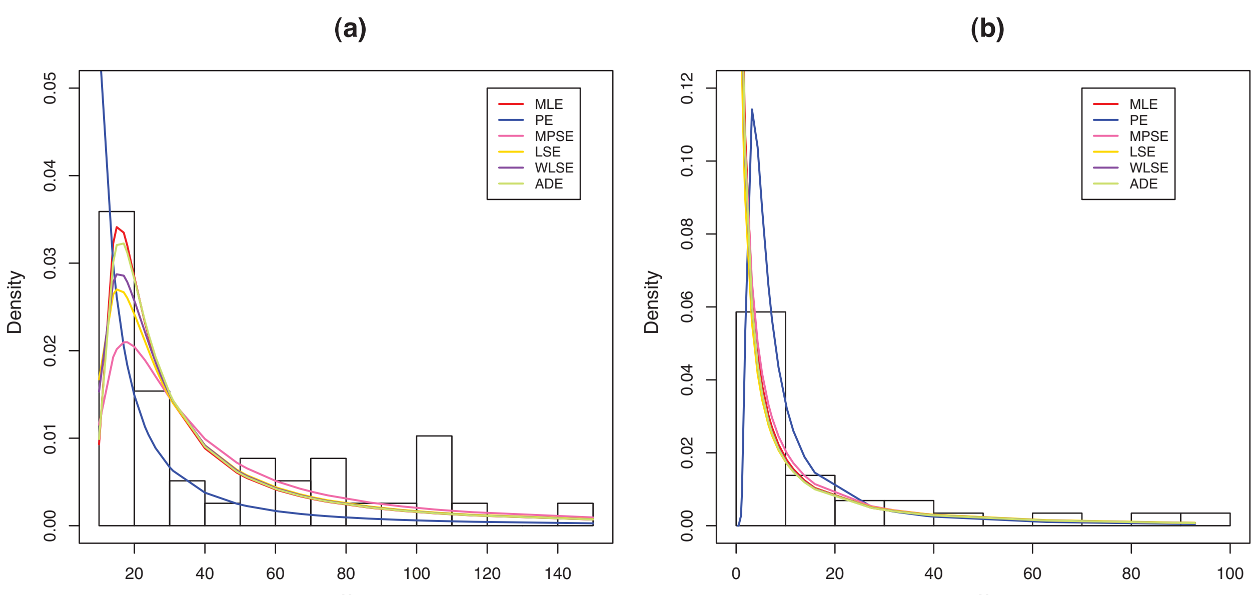

Figure 5: Histogram and the fitted MLTIL density using different estimation methods for (a) data 1 and (b) data 2

7 Conclusion

In this paper, we have considered and studied a new generalization of the traditional inverse Lomax distribution by adding a new shape parameter. We have used the logarithmic transformed method for this purpose and a new three parameters inverse Lomax distribution which called modified logarithmic transformed inverse Lomax distribution is introduced. The new distribution has different failure rate shapes, so it can be used in analyzing lifetime data. Some statistical properties of the new distribution are derived including quantiles, moments, probability weighted moments, entropies, residual life, stress-strength parameter and order statistics. To estimate the parameters of the proposed distribution, six classical methods are considered. To compare the efficiency of these methods a simulation study is performed and the performance of the different estimators is compared. To show the applicability of the new distribution, two real data sets are analyzed which indicate that our new distribution perform better than some other competitive distributions. Also, the numerical illustration revealed that the Anderson Darling estimation method is the best method to estimate the proposed distribution parameters. In this study, the proposed distribution shows its ability in modelling engineering and medical data sets where traditional and some recently proposed models cannot be used for this purpose. We hope that this model attract wider sets of applications in the other different fields.

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