

DOI: 10.32604/cmes.2022.017010



ARTICLE

Attractive Multistep Reproducing Kernel Approach for Solving Stiffness Differential Systems of Ordinary Differential Equations and Some Error Analysis

Radwan Abu-Gdairi¹, Shatha Hasan², Shrideh Al-Omari^{3,*}, Mohammad Al-Smadi^{2,4} and Shaher Momani^{4,5}

¹Department of Mathematics, Faculty of Science, Zarqa University, Zarqa, 13132, Jordan

²Department of Applied Science, Ajloun College, Al-Balqa Applied University, Ajloun, 26816, Jordan

³Department of Physics and Basic Sciences, Faculty of Engineering Technology, Al-Balqa Applied University, Amman, 11134, Jordan

⁴Nonlinear Dynamics Research Center (NDRC), Ajman University, Ajman, 20550, United Arab Emirates

⁵Department of Mathematics, Faculty of Science, The University of Jordan, Amman, 11942, Jordan

*Corresponding Author: Shrideh Al-Omari. Email: shridehalomari@bau.edu.jo

Received: 19 April 2021 Accepted: 27 July 2021

ABSTRACT

In this paper, an efficient multi-step scheme is presented based on reproducing kernel Hilbert space (RKHS) theory for solving ordinary stiff differential systems. The solution methodology depends on reproducing kernel functions to obtain analytic solutions in a uniform form for a rapidly convergent series in the posed Sobolev space. Using the Gram-Schmidt orthogonality process, complete orthogonal essential functions are obtained in a compact field to encompass Fourier series expansion with the help of kernel properties reproduction. Consequently, by applying the standard RKHS method to each subinterval, approximate solutions that converge uniformly to the exact solutions are obtained. For this purpose, several numerical examples are tested to show proposed algorithm's superiority, simplicity, and efficiency. The gained results indicate that the multi-step RKHS method is suitable for solving linear and nonlinear stiffness systems over an extensive duration and giving highly accurate outcomes.

KEYWORDS

Multi-step approach; reproducing kernel Hilbert space method; stiffness system; error analysis; numerical solution

1 Introduction

During studying and modeling many basic physical phenomena, such as chemical kinematics, aerodynamics, electrical circuits, ballistics, control models, and missile guidance, a type of differential equations appears that is difficult to solve through traditional numerical procedures, called differential stiffness system, which was first highlighted in the work of Curtiss and Hirschfelder [1-6]. Mathematical stiffness models reflect the different growth rates and various dynamic processes of the considered physical systems. It arises when some of the solution components decay



much more rapidly than other components because they contain the term $e^{-\lambda t}$, $\lambda > 0$. Anyhow, in the last few years, several analytical methods have been exploited to give approximate solutions for ordinary differential stiff systems and physical models, including residual power series method, homotopy perturbation method, multistep second derivative method, variational iteration method, differential transform method, and block method [7–16]. Among the effective numerical methods that used to handle several types of ordinary differential equations is the reproducing kernel Hilbert space (RKHS) method. In this direction, to see more information, definitions, kinds of operators, and applications of RKHS method in solving differential and integrodifferential equations of different types and orders, the reader is asked to refer to [17–24]. By and large, when trying to apply numerical and analytical methods to solve stiff system, it fails. Indeed, the approximate solutions are valid only for a very small interval with slowly convergent as well as completely diverge in a longer interval. More specifically, consider the following stiff system:

$$x'(t) = -x(t) - 15y(t) + 15e^{-t},$$
(1)

$$y'(t) = 15x(t) - y(t) - 15e^{-t},$$
(1)

subject to the initial conditions

$$x(0) = y(0) = 1, \quad t \in [0, 5].$$
 (2)

By classical RKHS method to solve this system with 100 nodes, the numerical results will be achieved under the rapid increase of the error as shown in Fig. 1. Simultaneously, increasing the number of nodes leads to the need for large computer memory along with long operating times, and cumulative errors may also affect the accuracy of the solutions. However, to overcome such drawbacks, an advanced numerical algorithm will be formulated based on dividing any temporal interval into small subintervals and then applying the standard RKHS method on each subinterval. This technique is called a multi-step reproducing Hilbert space (MS-RKHS) method. It is worth noting here that there are advanced methods in the literature to address many different engineering and physical problems, for more detail we refer to [25–30] and references therein.



Figure 1: (a) Exact (-) and classic RKHS approximate (---) solutions of system (1), (2); (b) Absolute error

By and large, there are no conventional analytical or semi-approximate methods that produce precise approximate or closed-form solutions for stiff differential systems. Therefore, there has become an urgent need for effective numerical algorithms to find accurate solutions for such models especially for large periods of time. This gives us the incentive, in this work, to explore effective accurate solutions [31–39]. Motivated by the previous discussion, our study aims to design a novel iterative algorithm to generate an analytical solution to stiff models of ordinary differential equations over a large duration through the use of a multi-step technique. To begin with, two reproducing kernel functions are established to generate a complete orthonormal basis in the Hilbert space. Based on reproducing kernel property, linear, bounded, and invertible differential operator is defined to create an analytical solution of the proposed model over a dense partition of the time period. In this direction, the approximate solution converges uniformly to the analytical solution. Error analysis is discussed as well. Lastly, some numerical examples are presented to illustrate the reliability and efficiency of the suggested multi-step approach. This paper is organized in five sections including the introduction. In Section 2, a brief description of the RKHS method is given. In Section 3, the MS-RKHS method is presented. In Section 4, several examples are given. Finally, a short conclusion is presented in Section 5.

2 Reproducing Kernel Hilbert Space Method

A reproducing kernel space is a Hilbert space H of functions defined on an abstract set X so that for each $x \in X$, the evaluation functional for $x, \delta_x(f) = f(x), f \in H$, is continuous on H. The Riesz representation theorem gives a unique function $K: X \times X \to \mathbb{C}$ such that:

- (1) $\forall x \in X, K(., x) \in H$,
- (2) $\langle f, K(., x) \rangle = f(x)$ for all $x \in X$ and for all $f \in H$,

where K is the reproducing kernel function of H.

The reproducing kernel function possesses many nice properties, including it being unique, positive definite, conjugate symmetric. For the theory and applications of RKHS, we refer to [40–45].

Definition 2.1 [46]: The function space $W_2^1[a,b]$ is defined as

$$W_2^1[a,b] = \{ u: [a,b] \to \mathbb{R} : u \in AC[a,b], u' \in L_2[a,b] \}.$$

The inner product for $u, v \in W_2^1[a, b]$ is given by $\langle u, v \rangle_{W_2^1} = u(a) v(a) + \int_a^b u'(t) v'(t) dt$ and the norm of u is $||u||_{W_2^1} = \sqrt{\langle u(t), u(t) \rangle_{W_2^1}}$.

Theorem 1.1 [46]: The space $W_2^1[a, b]$ is a complete RKHS of the reproducing kernel function $G_t(s)$ so that $G_t(s) = \begin{cases} 1-a+t, & s \le t, \\ 1-a+s, & s > t \end{cases}$.

Definition 2.2 [47]: The function space $W_2^2[a, b]$ is defined by

 $W_2^2[a,b] = \{u: u, u' \in AC[a,b], u'' \in L_2[a,b], u(a) = 0\},\$

with inner product for $u, v \in W_2^2[a, b]$ given by $\langle u, v \rangle_{W_2^2} = u'(a) v'(a) + \int_a^b u''(t) v''(t) dt$, and the norm of u is $||u||_{W_2^2} = \sqrt{\langle u(t), u(t) \rangle_{W_2^2}}$.

(3)

Theorem 2.2 [47]: The space $W_2^2[a,b]$ is a complete RKHS with reproducing function

$$K_t(s) = \begin{cases} \frac{1}{6} (s-a) \left(2a^2 - s^2 + 3t \left(2+s\right) - a \left(6+3t+s\right) \right), & s \le t, \\ \frac{1}{6} \left(t-a\right) \left(2a^2 - t^2 + 3s \left(2+t\right) - a \left(6+3s+t\right) \right), & s > t. \end{cases}$$

Hereinafter, we consider the following stiff system:

$$x'_{i}(t) = f_{i}(t, x_{1}, x_{2}, \dots, x_{m}), \quad t \in [a, b], \quad i = 1, 2, \dots, m, \quad m \in \mathbb{N},$$

along with the following initial conditions (ICs)

$$x_i(t_0) = c_i, \quad t_0 \ge 0.$$
 (4)

To solve system (3)–(4) using the RKHS method, we first homogenize ICs (4) in light of the following transformation:

$$y_i(t) = x_i(t) - c_i, \quad i = 1, 2, 3, \dots, m,$$
(5)

which leads to the following system:

$$y'_{i}(t) = f_{i}(t, y_{1} + c_{1}, y_{2} + c_{2}, \dots, y_{m} + c_{m}), \quad y_{i}(t_{0}) = 0.$$
 (6)

Subsequently, we define the differential operator $L: W_2^2[a,b] \to W_2^1[a,b]$ such that $Ly_i(t) = y'_i(t), i = 1, 2, ..., m$. Hence, the system (6) can be rewritten as follows:

$$Ly_i(t) = f_i(t, y_1 + c_1, y_2 + c_2, \dots, y_m + c_m), \quad y_i(t_0) = 0, \quad t_0 \ge 0, \quad i = 1, 2, \dots, m.$$
(7)

Herein, we have to construct an orthogonal function system of the space $W_2^2[a,b]$. To do this, consider the countable dense set $\{t_i\}_{i=1}^{\infty}$ of [a,b], and let $\varphi_i(t) = G_{t_i}(t)$ and $\psi_i(t) = L^*\varphi_i(t)$, where L^* is the adjoint operator of L. In terms of the properties of the reproducing kernel $G_t(.)$, we obtain

$$\left\langle u_{j}(t),\psi_{i}(t)\right\rangle_{W_{2}^{2}} = \left\langle u_{j}(t),L^{*}\varphi_{i}(t)\right\rangle_{W_{2}^{2}} = \left\langle Lu_{j}(t),\varphi_{i}(t)\right\rangle_{W_{2}^{1}} = Lu_{j}(t_{i}), i = 1,2,\ldots,j = 1,2,\ldots,m$$

Next, we will use the Gram-Schmidt orthogonalization process on $\{\psi_i(t)\}_{i=1}^{\infty}$ to form the orthonormal function system $\{\overline{\psi_i}(t)\}_{i=1}^{\infty}$ of $W_2^2[a,b]$.

Let $\overline{\psi_i}(t) = \sum_{l=1}^{i} \beta_{il} \psi_l(t), i = 1, 2, 3, ...,$ where β_{il} are the orthogonalization coefficients which are given by

$$\beta_{11} = \frac{1}{\|\psi_1\|_{W_2^2}}, \ \beta_{ii} = \frac{1}{\sqrt{\|\psi_i\|_{W_2^2}^2 - \sum_{p=1}^{i-1} \langle\psi_i(t), \overline{\psi_p}(t)\rangle_{W_2^2}^2}}, \ \text{and} \ \beta_{il} = \frac{-\sum_{p=1}^{l-1} \langle\psi_i(t), \psi_p(t)\rangle_{W_2^2}^2 \beta_{pl}}{\sqrt{\|\psi_i\|_{W}^2 - \sum_{p=1}^{i-1} \langle\psi_i(t), \overline{\psi_p}(t)\rangle_{W_2^2}^2}},$$

. . .

for i > l.

The following theorems give the form of the solution of system (6).

Theorem 2.3: If $\{t_i\}_{i=1}^{\infty}$ is dense on [a, b] and the solution of system (6) is unique, then it has the form $y_i(t) = \sum_{i=1}^{\infty} \sum_{l=1}^{i} \beta_{il} f_i(t_l, y_1 + c_1, y_2 + c_2, \dots, y_m + c_m) \overline{\psi_i}(t)$.

The N-term approximate solution $y_i^N(t)$ of system (6) is given by the finite sum:

$$y_i^N(t) = \sum_{i=1}^N \sum_{l=1}^i \beta_{il} f_i(t_l, y_1(t_l) + c_1, y_2(t_l) + c_2, \dots, y_m(t_l) + c_m) \overline{\psi_i}(t).$$
(8)

Eventually, the solution of system (3) is obtained as $x_i^N(t) = y_i^N(t) + c_i, i = 1, 2, ..., m$.

3 Multi-Step Reproducing Kernel Hilbert Space Method

To clarify the MS-RKHS method that used in this work to find approximate solutions of system (3), we divide the interval [a,b] into equal length subintervals $[t_{n-1},t_n], n = 1, 2, ..., M$, $h = \frac{b-a}{M}$ and nodes $t_0 = a, t_n = a + nh$. Thus, to construct the approximate solution of system (3), we first apply the standard RKHS method to the problem:

$$x'_{1i}(t) = f_i(t, x_{11}, x_{12}, \dots, x_{1m}), x_{1i}(a) = c_i, t \in [a, t_1], i = 1, 2, \dots, m,$$

to get the approximate solution:

$$x_{1i}^{N}(t) = \sum_{i=1}^{N} \sum_{l=1}^{i} \beta_{1il} f_i(t_l, x_{11}(t_l), x_{12}(t_l), \dots, x_{1m}(t_l)) \overline{\psi_{1i}}(t) + c_i.$$
(9)

Second, we apply the standard RKHS method again to the problem:

$$x'_{2i}(t) = f_i(t, x_{21}, x_{22}, \dots, x_{2m}), x_{2i}(t_1) = x_{1i}^N(t_1), t \in [t_1, t_2], i = 1, 2, \dots, m,$$

to get the approximate solution:

$$x_{2i}^{N}(t) = \sum_{i=1}^{N} \sum_{l=1}^{i} \beta_{2il} f_{i}(t_{l}, x_{21}(t_{l}), x_{22}(t_{l}), \dots, x_{2m}(t_{l})) \overline{\psi_{2i}}(t) + x_{1i}^{N}(t_{1}).$$
(10)

Continuing this process at each subinterval and applying the standard RKHS method to the problems:

$$x'_{ni}(t) = f_i(t, x_{n1}, x_{n2}, \dots, x_{nm}), x_{ni}(t_{n-1}) = x^N_{(n-1)i}(t_{n-1}), t \in [t_{n-1}, t_n], i = 1, 2, \dots, m, m \in \mathbb{N},$$

to get the approximate solutions:

$$x_{ni}^{N}(t) = \sum_{i=1}^{N} \sum_{l=1}^{l} \beta_{il} f_{i}(t_{l}, x_{n1}(t_{l}), x_{n2}(t_{l}), \dots, x_{nm}(t_{l})) \overline{\psi_{ni}}(t) + x_{(n-1)i}^{N}(t_{n-1}), n = 1, 2, \dots, M$$

So, the solution of system (3) using the MS-RKHS method is given by

$$x_{ni}^{N}(t) = \begin{cases} x_{1i}^{N}(t), & t \in [a, t_{1}], \\ x_{2i}^{N}(t), & t \in [t_{1}, t_{2}], \\ \vdots & \vdots \\ x_{Mi}^{N}(t), & t \in [t_{M-1}, b]. \end{cases}$$
(11)

4 Numerical Examples

In this section, we will apply the MS-RKHS method described in Section 3 to solve some linear and nonlinear examples of stiff systems. In each example, we compare the exact solution with the approximate one when N = 100. The results are given in tables and graphs. Computations will be performed via Mathematica 10.0 software package.

Example 4.1: Consider the following homogeneous linear stiff system:

$$x'(t) = -x(t) + 95y(t),$$

$$y'(t) = -x(t) - 97y(t),$$
(12)

subject to the initial conditions x(0) = y(0) = 1.

The exact solution of system (12) is $x(t) = \frac{1}{47}(95e^{-2t} - 48e^{-96t})$ and $y(t) = \frac{1}{47}(48e^{-96t} - e^{-2t})$. Table 1 and Fig. 2 show the numerical results for $t \in [0, 1]$. While, If we take $t \in [0, 5]$, then the numerical and graphical results presented in Table 2 and Fig. 3 show the efficiency of the multi-step method to solve stiff system (12).

Table 1: Numerical results of Example 4.1 for $t \in [0, 1]$ with N = 21 and step size 0.1 using multistep RKHSM

t _i	Exact	solution	Approxim	ate solution	Absolute error		
	x(t)	<i>y</i> (<i>t</i>)	$x^{N}(t)$	$y^{\mathbf{N}}(t)$	$\left x\left(t\right)-x^{N}\left(t\right)\right $	$ y(t) - y^{N}(t) $	
0.0	1.0	1.0	1.0	1.0	0.0	0.0	
0.1	1.6548121396	-0.0173506335	1.654805717	-0.017348590	6.422×10^{-6}	2.043×10^{-6}	
0.2	1.3549022160	-0.0142621240	1.354894969	-0.014262047	7.247×10^{-6}	7.656×10^{-8}	
0.3	1.1093001155	-0.0116768433	1.109291216	-0.011676750	$8.900 imes 10^{-6}$	9.368×10^{-8}	
0.4	0.9082181190	-0.0095601907	0.908208404	-0.009560088	9.715×10^{-6}	1.023×10^{-7}	
0.5	0.7435861045	-0.0078272222	0.743576162	-0.007827117	9.943×10^{-6}	1.047×10^{-7}	
0.6	0.6087968113	-0.0064083875	0.608787043	-0.006408285	9.768×10^{-6}	1.028×10^{-7}	
0.7	0.4984406718	-0.0052467440	0.498431341	-0.005246646	9.331×10^{-6}	9.822×10^{-8}	
0.8	0.4080887066	-0.0042956706	0.408079976	-0.004295579	8.731×10^{-6}	9.190×10^{-8}	
0.9	0.3341147741	-0.0035169976	0.334106733	-0.003516913	8.042×10^{-6}	8.465×10^{-8}	
1.0	0.2735500406	-0.0028794741	0.273542725	-0.002879397	7.315×10^{-6}	$7.700 imes 10^{-8}$	

t_i	Exact	solution	Approxim	ate solution	Absolute error		
	$\overline{x(t)}$ $y(t)$		$x^{N}(t)$	$y^{\mathbf{N}}(t)$	$\left x\left(t\right)-x^{N}\left(t\right)\right $	$ y(t) - y^{\mathbf{N}}(t) $	
0.0	1.0	1.0	1.0	1.0	0.0	0.0	
0.5	0.743586104	-0.00782722	0.74356076	-0.007826955	2.535×10^{-5}	2.66788×10^{-7}	
1.0	0.273550041	-0.00287947	0.27353139	-0.002879278	1.865×10^{-5}	1.96290×10^{-7}	
1.5	0.100633436	-0.00105930	0.10062315	-0.001059191	1.029×10^{-5}	1.08315×10^{-7}	
2.0	0.037020972	-0.00038969	0.03701593	-0.000389641	5.047×10^{-6}	5.31282×10^{-8}	
2.5	0.013619254	-0.00014336	0.013616933	-0.000143336	2.321×10^{-6}	2.44306×10^{-8}	
3.0	0.005010244	-0.00005274	0.005009219	-0.000052729	1.025×10^{-6}	1.07848×10^{-8}	
3.5	0.001843166	-0.00001940	0.001842726	-0.000019397	4.397×10^{-7}	4.62868×10^{-9}	
4.0	0.000678063	-0.00000714	0.000677878	-7.136×10^{-6}	1.849×10^{-7}	1.94602×10^{-9}	
4.5	0.000249445	-0.00000263	0.000249369	-2.625×10^{-6}	$7.651 imes 10^{-8}$	8.05375×10^{-10}	
5.0	0.000091766	-9.66×10^{-7}	0.000091735	-9.656×10^{-7}	3.127×10^{-8}	3.29195×10^{-10}	

Table 2: Numerical results of Example 4.1 for $t \in [0, 5]$ with N = 56 and step size 0.5



Figure 2: Plots of exact and approximate solutions and absolute error of Example 4.1 for $t \in [0, 1]$ with step size 0.1: (a) x(t), (b) $|x(t) - x^{N}(t)|$, (c) y(t), (d) $|y(t) - y^{N}(t)|$



Figure 3: Plots of exact and approximate solutions and absolute error of Example 4.1 for $t \in [0, 5]$ with step size 0.5: (a) x(t), (b) $|x(t) - x^{N}(t)|$, (c) y(t), (d) $|y(t) - y^{N}(t)|$

Example 4.2: Consider the following nonhomogeneous linear stiff system:

$$x'(t) = -x(t) - 15y(t) + 15e^{-t},$$

$$y'(t) = 15x(t) - y(t) - 15e^{-t},$$
(13)

subject to the initial conditions x(0) = y(0) = 1.

The exact solution of system (13) is $x(t) = y(t) = e^{-t}$. Numerical and graphical results of system (13) for $t \in [0, 5]$ with step size 0.5 are shown in Table 3 and Fig. 4. Furthermore, a comparison between the absolute error results is provided in Table 4 by using the classical RKHS method and MS-RKHS for $t \in [0, 1]$ with step size 0.1 and N = 100 to illustrate the superiority of the proposed multi-step approach.

t_i	$t_i \qquad \frac{\text{Exact solution}}{x(t) \qquad y(t)}$		Approxima	te solution	Absolute error		
			$x^{N}(t)$	$y^{N}(t)$	$\left x\left(t ight)-x^{N}\left(t ight) ight $	$ y(t) - y^{\mathbf{N}}(t) $	
0.0	1.0	1.0	1.0	1.0	0.0	0.0	
0.5	0.606530660	0.606530660	0.60609391	0.60634775	4.36751×10^{-4}	1.82911×10^{-4}	
1.0	0.367879441	0.367879441	0.36763189	0.36748098	$2.47550 imes 10^{-4}$	3.98458×10^{-4}	
1.5	0.223130160	0.223130160	0.22314860	0.22284073	1.84386×10^{-5}	2.89434×10^{-4}	
2.0	0.135335283	0.135335283	0.13540774	0.13524695	7.24519×10^{-5}	8.83309×10^{-5}	
2.5	0.082084999	0.082084999	0.08209116	0.08208412	$6.15998 imes 10^{-6}$	8.83126×10^{-7}	
3.0	0.049787068	0.049787068	0.04975296	0.04977541	3.41069×10^{-5}	1.16543×10^{-5}	
3.5	0.030197383	0.030197383	0.03017548	0.03016635	$2.18995 imes 10^{-5}$	3.10316×10^{-5}	
4.0	0.018315639	0.018315639	0.01831587	0.01829131	2.34129×10^{-7}	2.43286×10^{-5}	
4.5	0.011108997	0.011108997	0.01111501	0.01110090	$6.01794 imes 10^{-6}$	8.09984×10^{-6}	
5.0	0.006737947	0.006737947	0.00673895	0.00673774	1.00748×10^{-6}	2.02362×10^{-7}	

Table 3: Numerical results of Example 4.2 for $t \in [0, 5]$ with N = 100 and step size 0.5

Table 4: Comparison between the absolute error of Example 4.2

t _i	MS-RKH	S method	Classical RKHS method		
	$ x(t) - x^{\mathbf{N}}(t) $	$ y(t) - y^{\mathbf{N}}(t) $	$\left x\left(t\right)-x^{\mathrm{N}}\left(t\right)\right $	$ y(t) - y^{\mathbf{N}}(t) $	
0.0	0.0	0.0	0.0	0.0	
0.1	2.08021×10^{-5}	1.89750×10^{-5}	2.37317×10^{-3}	1.594412×10^{-3}	
0.2	3.021354×10^{-6}	3.71606×10^{-5}	1.39728×10^{-3}	2.219421×10^{-3}	
0.3	1.63228×10^{-5}	2.06367×10^{-5}	2.14532×10^{-3}	1.149373×10^{-3}	
0.4	4.262052×10^{-6}	6.39627×10^{-7}	8.62032×10^{-4}	1.995360×10^{-3}	
0.5	1.30944×10^{-5}	$8.91258 imes 10^{-6}$	1.78439×10^{-3}	$6.742894 imes 10^{-4}$	
0.6	5.40758×10^{-6}	2.38993×10^{-5}	5.70963×10^{-4}	1.642269×10^{-3}	
0.7	9.81349×10^{-6}	1.68221×10^{-5}	1.55884×10^{-3}	4.327679×10^{-4}	
0.8	5.48313×10^{-6}	1.63801×10^{-6}	$2.67113 imes 10^{-4}$	1.429285×10^{-3}	
0.9	7.51799×10^{-6}	3.68070×10^{-6}	1.26306×10^{-3}	1.670565×10^{-4}	
1.0	$5.61491 imes 10^{-6}$	1.47367×10^{-5}	$1.22320 imes 10^{-4}$	1.146724×10^{-3}	

Example 4.3: Consider the following nonlinear stiff system:

$$x'(t) = -1002x(t) + 1000y^{2}(t),$$

$$y'(t) = x(t) - y(t) - y^{2}(t),$$
(14)

subject to the initial conditions x(0) = y(0) = 1.

The exact solution of system (14) is $x(t) = e^{-2t}$, $y(t) = e^{-t}$. Herein, Table 5 and Fig. 5 show the numerical results for $t \in [0, 1]$ with step size 0.1 and N = 50.



Figure 4: Plots of exact and approximate solutions and absolute error of Example 4.2 for $t \in [0, 5]$ with step size 0.5: (a) x(t), (b) $|x(t) - x^{N}(t)|$, (c) y(t), (d) $|y(t) - y^{N}(t)|$

t _i	Exact solution		Approxima	te solution	Absolute error		
	x(t)	y(t)	$x^{N}(t)$	$y^{\mathbf{N}}(t)$	$\left x\left(t\right)-x^{N}\left(t\right)\right $	$ y(t) - y^{N}(t) $	
0.0	1.0	1.0	1.0	1.0	0.0	0.0	
0.1	0.81873075	0.90483742	0.8187307822	0.9048372323	2.9116×10^{-8}	1.8569×10^{-7}	
0.2	0.67032005	0.81873075	0.6703198314	0.8187302073	2.1460×10^{-7}	5.4579×10^{-7}	
0.3	0.54881164	0.74081822	0.5488109551	0.7408173855	6.8103×10^{-7}	8.3518×10^{-7}	
0.4	0.44932896	0.67032005	0.4493279923	0.6703189816	9.7177×10^{-7}	1.0645×10^{-6}	
0.5	0.36787944	0.60653066	0.3678783066	0.6065294173	1.1346×10^{-6}	1.2425×10^{-6}	
0.6	0.30119421	0.54881164	0.3011930055	0.5488102592	1.2064×10^{-6}	1.3769×10^{-6}	
0.7	0.24659696	0.49658530	0.2465957491	0.4965838294	1.2148×10^{-6}	1.4744×10^{-6}	
0.8	0.20189652	0.44932896	0.2018953375	0.4493274232	1.1805×10^{-6}	1.5409×10^{-6}	
0.9	0.16529889	0.40656966	0.1652977696	0.4065680784	1.1187×10^{-6}	1.5813×10^{-6}	
1.0	0.13533528	0.36787944	0.1353342428	0.3678778411	1.0404×10^{-6}	1.6001×10^{-6}	

Table 5: Numerical results for Example 4.3 for $t \in [0, 1]$ with N = 50 and step size 0.1



Figure 5: Plots of exact and approximate solutions and absolute error of Example 4.3 for $t \in [0, 1]$ with step size 0.1: (a) x(t), (b) $|x(t) - x^{N}(t)|$, (c) y(t), (d) $|y(t) - y^{N}(t)|$

Example 4.4: Consider the following stiff system:

$$x'(t) = -20x(t) - 0.25y(t) - 19.75z(t),$$

$$y'(t) = 20x(t) - 20.25y(t) + 0.25z(t),$$

$$z'(t) = 20x(t) - 19.75y(t) - 0.25z(t),$$
(15)

subject to the initial conditions x(0) = 1, y(0) = 0, z(0) = -1.

The exact solution of system (15) is given as

$$\begin{aligned} x(t) &= \frac{1}{2} \left(e^{-0.5t} + e^{-20t} \left(\cos 20t + \sin 20t \right) \right), \\ y(t) &= \frac{1}{2} \left(e^{-0.5t} + e^{-20t} \left(\cos 20t - \sin 20t \right) \right), \\ z(t) &= -\frac{1}{2} \left(e^{-0.5t} + e^{-20t} \left(\cos 20t - \sin 20t \right) \right). \end{aligned}$$

In the following, some numerical and graphical simulation of Example 4.4 are performed in Table 6 and Fig. 6 for $t \in [0, 2]$ with step size 0.2.

t_i	Exact solution			Approximate solution			Absolute error		
	x(t)	y(t)	$z\left(t ight)$	$x^{N}(t)$	$y^{\mathbf{N}}\left(t ight)$	$z^{\mathbf{N}}\left(t ight)$	Er[x(t)]	Er[y(t)]	Er[z(t)]
0.0	1.0	0.0	-1.0	1.0	0.0	-1.0	0.0	0.0	0.0
0.2	0.4395021	0.4514740	-0.4533634	0.4395192	0.4514716	-0.4533659	1.71×10^{-5}	2.44×10^{-6}	2.45×10^{-6}
0.4	0.4095069	0.4095557	-0.4091750	0.4095064	0.4095553	-0.4091754	$4.69 imes 10^{-7}$	4.07×10^{-7}	4.24×10^{-7}
0.6	0.3704101	0.3704049	-0.3704134	0.3704101	0.3704049	-0.3704133	1.10×10^{-8}	$2.85 imes 10^{-8}$	$6.27 imes 10^{-9}$
0.8	0.3351560	0.3351601	-0.3351600	0.3351600	0.3351601	-0.3351600	$1.37 imes 10^{-8}$	1.31×10^{-8}	$1.37 imes 10^{-8}$
1.0	0.3032653	0.3032653	-0.3032653	0.3032653	0.3032653	-0.3032653	$1.51 imes 10^{-8}$	$1.51 imes 10^{-8}$	1.51×10^{-8}
1.2	0.2744058	0.2744058	-0.2744058	0.2744058	0.2744058	-0.2744058	2.74×10^{-9}	2.74×10^{-9}	2.74×10^{-9}
1.4	0.2482927	0.2482927	-0.2482927	0.2482927	0.2482927	-0.2482927	4.96×10^{-9}	4.96×10^{-9}	4.96×10^{-9}
1.6	0.2246645	0.2246644	-0.2246645	0.2246645	0.2246645	-0.2246645	$6.73 imes 10^{-9}$	6.73×10^{-9}	6.73×10^{-9}
1.8	0.2032848	0.2032848	-0.2032848	0.2032848	0.2032848	-0.2032848	8.12×10^{-9}	8.12×10^{-9}	8.12×10^{-9}
2.0	0.1839397	0.1839397	-0.1839397	0.1839397	0.1839397	-0.1839397	9.18×10^{-9}	9.18×10^{-9}	9.18×10^{-9}

Table 6: Numerical results of Example 4.4 for $t \in [0, 2]$ with N = 50 and step size 0.2



Figure 6: Plots of the exact and approximate solutions of Example 4.4 for $t \in [0, 2]$ with step size 0.2: (a) x(t), (b) y(t), and (c) z(t)

5 Conclusion

In this work, a modified multi-step algorithm, the MS-RKHS method, has been lucratively implemented based on the standard RKHS method to obtain approximate solutions of stiff systems of ordinary differential equations. Several examples of linear and non-linear stiff systems have been given to show the efficiency of the proposed method. The achieved results have been presented numerically and graphically as well. By comparing our results with the exact solutions and classical RKHS method, we observe that the MS-RKHS method yields accurate approximations. Moreover, using the MS-RKHS method, the intervals of convergence for the series solution will increase without needing large computer memory, which takes less time to give accurate numerical results. For the near future work, the presented multi-step approach will be applied for solving stiff systems of fractional and partial differential equations along with nonclassical conditions.

Acknowledgement: Authors would like to thank reviewers for their insightful comments and suggestions. The first author is grateful for the support provided by Zarqa University, Jordan.

Funding Statement: The authors received no specific funding for this study.

Conflicts of Interest: The authors declare that they have no conflicts of interest to report regarding the present study.

References

- 1. Shalashilin, V., Kuznetsov, E. (2003). *Parametric continuation and optimal parametrization in applied mathematics and mechanics*. Dordrech: Springer Science & Business Media. DOI 10.1007/978-94-017-2537-8.
- 2. Curtiss, C., Hirschfelder, J. (1952). Integration of stiff equations. *Proceedings of the National Academy of Sciences of the United States of America*, 38(3), 235. DOI 10.1073/pnas.38.3.235.
- Aminikhah, H., Hemmatnezhad, M. (2011). An effective modification of the homotopy perturbation method for stiff systems of ordinary differential equations. *Applied Mathematics Letters*, 24(9), 1502–1508. DOI 10.1016/j.aml.2011.03.032.
- 4. Akinfenwa, O., Akinnukawe, B., Mudasiru, S. (2015). A family of continuous third derivative block methods for solving stiff systems of first ordinary differential equations. *Journal of Nigerian Mathematical Society*, 34(2), 160–168. DOI 10.1016/j.jnnms.2015.06.002.
- 5. Yakubu, D., Markus, S. (2016). The efficiency of second derivative multistep methods for the numerical integration of stiff systems. *Journal of Nigerian Mathematical Society*, 35(1), 107–127. DOI 10.1016/j.jnnms.2016.02.002.
- 6. Atay, M., Kilic, O. (2013). The semianalytical solutions for stiff systems of ordinary differential equations by using variational iteration method and modified variational iteration method with comparison to exact solutions. *Mathematical Problems in Engineering*, 2013(1–2), 143915. DOI 10.1155/2013/143915.
- Al-Smadi, M., Freihat, A., Khalil, H., Momani, S., Khan, R. A. (2017). Numerical multistep approach for solving fractional partial differential equations. *International Journal of Computational Methods*, 14(3), 1750029. DOI 10.1142/S0219876217500293.
- 8. Al-Smadi, M., Abu Arqub, O., Momani, S. (2020). Numerical computations of coupled fractional resonant Schrödinger equations arising in quantum mechanics under conformable fractional derivative sense. *Physica Scripta*, 95(7), 75218. DOI 10.1088/1402-4896/ab96e0.
- 9. Yang, X. J. (2019). New non-conventional methods for quantitative concepts of anomalous rheology. *Thermal Science*, 23(6B), 427. DOI 10.2298/TSCI191028427Y.
- 10. Momani, S., Abu Arqub, O., Freihat, A., Al-Smadi, M. (2016). Analytical approximations for Fokker-Planck equations of fractional order in multistep schemes. *Applied and Computational Mathematics*, 15(3), 319–330.
- 11. Komashynska, I., Al-Smadi, M., Ateiwi, A., Al-Obaidy, S. (2016). Approximate analytical solution by residual power series method for system of Fredholm integral equations. *Applied Mathematics and Information Sciences, 10(3),* 975–985. DOI 10.18576/amis/100315.

- Al-Smadi, M. H., Gumah, G. N. (2014). On the homotopy analysis method for fractional SEIR epidemic model. *Research Journal of Applied Sciences, Engineering and Technology*, 7(18), 3809–3820. DOI 10.19026/rjaset.7.738.
- 13. Momani, S., Freihat, A., Al-Smadi, M. (2014). Analytical study of fractional-order multiple chaotic FitzHugh-Nagumo neurons model using multistep generalized differential transform method. *Abstract and Applied Analysis, 2014(4), 276279.* DOI 10.1155/2014/276279.
- Shqair, M., Al-Smadi, M., Momani, S., El-Zahar, E. (2020). Adaptation of conformable residual power series scheme in solving nonlinear fractional quantum mechanics problems. *Applied Sciences*, 10(3), 890. DOI 10.3390/app10030890.
- Jafari, H., Tajadodi, H., Kadkhoda, N., Baleanu, D. (2013). Fractional sub equation method for Cahn-Hilliard and Klein-Gordon equations. *Abstract and Applied Analysis*, 2013(1), 587179. DOI 10.1155/2013/587179.
- Abdeljawad, T., Ullah, K., Ahmad, J., de la Sen, M., Khan, M. N. (2021). Some convergence results for a class of generalized nonexpansive mappings in Banach spaces. *Advances in Mathematical Physics*, 2021, 1–6. DOI 10.1155/2021/8837317.
- 17. Abdeljawad, T. (2019). Fractional difference operators with discrete generalized Mittag-Leffler kernels. *Chaos, Solitons and Fractals, 126(2),* 315–324. DOI 10.1016/j.chaos.2019.06.012.
- Al-Smadi, M., Abu Arqub, O., Momani, S. (2013). A computational method for two-point boundary value problems of fourth-order mixed integrodifferential equations. *Mathematical Problems in Engineering*, 2013(4), 832074. DOI 10.1155/2013/832074.
- 19. Al-Smadi, M., Abu Arqub, O., Shawagfeh, N., Momani, S. (2016). Numerical investigations for systems of second-order periodic boundary value problems using reproducing kernel method. *Applied Mathematics and Computation*, 291(9), 137–148. DOI 10.1016/j.amc.2016.06.002.
- Harrouche, N., Momani, S., Hasan, S., Al-Smadi, M. (2021). Computational algorithm for solving drug pharmacokinetic model under uncertainty with nonsingular kernel type Caputo-Fabrizio fractional derivative. *Alexandria Engineering Journal*, 60(5), 4347–4362. DOI 10.1016/j.aej.2021.03.016.
- Hasan, S., Al-Smadi, M., El-Ajou, A., Momani, S., Hadid, S. et al. (2021). Numerical approach in the Hilbert space to solve a fuzzy Atangana-Baleanu fractional hybrid system. *Chaos, Solitons and Fractals,* 143(7), 110506. DOI 10.1016/j.chaos.2020.110506.
- 22. Al-Smadi, M. (2018). Simplified iterative reproducing kernel method for handling time-fractional BVPs with error estimation. *Ain Shams Engineering Journal*, 9(4), 2517–2525. DOI 10.1016/j.asej.2017.04.006.
- 23. Djeddi, N., Hasan, S., Al-Smadi, M., Momani, S. (2020). Modified analytical approach for generalized quadratic and cubic logistic models with Caputo-Fabrizio fractional derivative. *Alexandria Engineering Journal*, 59(6), 5111–5122. DOI 10.1016/j.aej.2020.09.041.
- 24. Al-Smadi, M. (2019). Reliable numerical algorithm for handling fuzzy integral equations of second kind in Hilbert spaces. *Filomat*, 33(2), 583–597. DOI 10.2298/FIL1902583A.
- 25. Habenom, H., Suthar, D. L., Baleanu, D., Purohit, S. D. (2021). A numerical simulation on the effect of vaccination and treatments for the fractional hepatitis B model. *ASME Journal of Computational and Nonlinear Dynamics*, *16(1)*, 11004. DOI 10.1115/1.4048475.
- 26. Agarwal, R., Yadav, M. P., Baleanu, D., Purohit, S. D. (2020). Existence and uniqueness of miscible flow equation through porous media with a nonsingular fractional derivative. *AIMS Mathematics*, 5(2), 1062–1073. DOI 10.3934/math.2020074.
- 27. Al-Smadi, M., Djeddi, N., Momani, S., Al-Omari, S., Araci, S. (2021). An attractive numerical algorithm for solving nonlinear Caputo-Fabrizio fractional Abel differential equation in a Hilbert space. *Advances in Difference Equations*, 2021(1), 271. DOI 10.1186/s13662-021-03428-3.
- 28. Kritika, Agarwal, R., Purohit, S. D. (2020). Mathematical model for anomalous subdiffusion using conformable operator. *Chaos Solitons Fractals*, 140(2), 110199. DOI 10.1016/j.chaos.2020.110199.
- 29. Suthar, D. L., Purohit, S. D., Araci, S. (2020). Solution of fractional kinetic equations associated with the (p, q)-Mathieu-type series. *Discrete Dynamics in Nature and Society*, 2020(1), 8645161. DOI 10.1155/2020/8645161.

- Al-Smadi, M. (2021). Fractional residual series for conformable time-fractional Sawada-Kotera-Ito, Lax, and Kaup-Kupershmidt equations of seventh order. *Mathematical Methods in the Applied Sciences* (in Editing). DOI 10.1002/mma.7507.
- 31. Alabedalhadi, M., Al-Smadi, M., Al-Omari, S., Baleanu, D., Momani, S. (2020). Structure of optical soliton solution for nonliear resonant space-time Schrödinger equation in conformable sense with full nonlinearity term. *Physica Scripta*, *95*(*10*), 105215. DOI 10.1088/1402-4896/abb739.
- Kumar, S., Kumar, A., Samet, B., Dutta, H. (2021). A study on fractional host-parasitoid population dynamical model to describe insect species. *Numerical Methods for Partial Differential Equations*, 37(2), 1673–1692. DOI 10.1002/num.22603.
- Al-Smadi, M., Abu Arqub, O., Zeidan, D. (2021). Fuzzy fractional differential equations under the Mittag-Leffler kernel differential operator of the ABC approach: Theorems and applications. *Chaos, Solitons and Fractals, 146(16)*, 110891. DOI 10.1016/j.chaos.2021.110891.
- 34. Kumar, S., Kumar, R., Agarwal, R. P., Samet, B. (2020). A study of fractional Lotka-Volterra population model using Haar wavelet and Adams–Bashforth–Moulton methods. *Mathematical Methods in the Applied Sciences*, *43*(*8*), 5564–5578. DOI 10.1002/mma.6297.
- 35. Kumar, S., Kumar, R., Cattani, C., Samet, B. (2020). Chaotic behaviour of fractional predator-prey dynamical system. *Chaos, Solitons and Fractals*, 135(17), 109811. DOI 10.1016/j.chaos.2020.109811.
- Al-Smadi, M., Abu Arqub, O., Hadid, S. (2020). Approximate solutions of nonlinear fractional Kundu– Eckhaus and coupled fractional massive Thirring equations emerging in quantum field theory using conformable residual power series method. *Physica Scripta*, 95(10), 105205. DOI 10.1088/1402-4896/abb420.
- Al-Smadi, M., Abu Arqub, O., Hadid, S. (2020). An attractive analytical technique for coupled system of fractional partial differential equations in shallow water waves with conformable derivative. *Communications* in *Theoretical Physics*, 72(8), 85001. DOI 10.1088/1572-9494/ab8a29.
- Al-Smadi, M., Dutta, H., Hasan, S., Momani, S. (2021). On numerical approximation of Atangana-Baleanu-Caputo fractional integro-differential equations under uncertainty in Hilbert space. *Mathematical Modelling of Natural Phenomena*, 16, 41. DOI 10.1051/mmnp/2021030.
- 39. Dutta, H., Akdemir, A., Atangana, A. (2020). *Fractional order analysis: Theory, methods and applications*. Hoboken, USA: John Wiley and Sons, Ltd.
- 40. Al-Smadi, M., Abu Arqub, O., El-Ajou, A. (2014). A numerical iterative method for solving systems of first-order periodic boundary value problems. *Journal of Applied Mathematics*, 2014(1), 135465. DOI 10.1155/2014/135465.
- Gumah, G., Moaddy, K., Al-Smadi, M., Hashim, I. (2016). Solutions to uncertain Volterra integral equations by fitted reproducing kernel Hilbert space method. *Journal of Function Spaces*, 2016(50), 2920463. DOI 10.1155/2016/2920463.
- 42. Al-Smadi, M., Abu Arqub, O. (2019). Computational algorithm for solving fredholm time-fractional partial integrodifferential equations of dirichlet functions type with error estimates. *Applied Mathematics and Computation*, 342, 280–294. DOI 10.1016/j.amc.2018.09.020.
- 43. Hasan, S., El-Ajou, A., Hadid, S., Al-Smadi, M., Momani, S. (2020). Atangana–Baleanu fractional framework of reproducing kernel technique in solving fractional population dynamics system. *Chaos, Solitons and Fractals*, 133(3), 109624. DOI 10.1016/j.chaos.2020.109624.
- 44. Al-Smadi, M., Abu Arqub, O., Gaith, M. (2021). Numerical simulation of telegraph and Cattaneo fractional-type models using adaptive reproducing kernel framework. *Mathematical Methods in the Applied Sciences*, 44(10), 8472–8489. DOI 10.1002/mma.6998.
- 45. Aronszajn, N. (1950). Theory of reproducing kernels. *Transaction of the American Mathematical Society*, 68(3), 337–404. DOI 10.1090/S0002-9947-1950-0051437-7.
- 46. Cui, M., Lin, Y. (2009). *Nonlinear numerical analysis in the reproducing kernel space*. New York, NY, USA: Nova Science.
- 47. Li, C., Cui, M. (2003). The exact solution for solving a class nonlinear operator equations in the reproducing kernel space. *Applied Mathematics and Computation*, 143(2-3), 393-399. DOI 10.1016/S0096-3003(02)00370-3.