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ARTICLE

Note on a New Construction of Kantorovich Form q-Bernstein Operators Related to Shape Parameter λ

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ABSTRACT

The main purpose of this paper is to introduce some approximation properties of a Kantorovich kind q-Bernstein operators related to Bézier basis functions with shape parameter $\lambda \in [-1, 1]$. Firstly, we compute some basic results such as moments and central moments, and derive the Korovkin type approximation theorem for these operators. Next, we estimate the order of convergence in terms of the usual modulus of continuity, for the functions belong to Lipschitz-type class and Peetre's K-functional, respectively. Lastly, with the aid of Maple software, we present the comparison of the convergence of these newly defined operators to the certain function with some graphical illustrations and error estimation table.

KEYWORDS

q-calculus; (λ, q)-Bernstein polynomials; order of convergence; Lipschitz-type function; Peetre's K-functional

1 Introduction

One of the uncomplicated and most elegant proof of the Weierstrass approximation theorem, the polynomials introduced by Bernstein [1], still guides many studies. Very recently, the quantum calculus shortly (q-calculus), which has numerous applications in various disciplines, has attracted the attention of many authors working on approximation theory. Firstly, Lupaş [2] investigated several approximation properties of the generalizations of q-Bernstein polynomials. Afterward, Phillips [3] obtained some convergence theorems and Voronovskaya type asymptotic formula for the most popular generalizations of the q-Bernstein polynomials. Based on the studies [2,3], many approximation properties of the well-known linear positive operators via q-calculus were investigated in details by several researchers. We refer readers for some notable results [4–13].

In 2007, a Kantorovich type Bernstein polynomials via q-calculus are constructed by Dalmanoğlu [14] as follows:

$$B_{m,q}(\mu;y)=[m+1]_q\sum_{j=0}^{m}b_{m,j}(y;q)q^{-j}\int_{[j]_q/[m+1]_q}^{[j+1]_q/[m+1]_q}\mu(t)d_qt,\ \ y\in[0,1], \tag{1}$$

$$b_{m,j}(y;q):=\begin{bmatrix}m\\ j\end{bmatrix}_q y^j\prod_{r=0}^{m-j-1}(1-q^ry),$$

where $B_{m,q}$: $C[0,1]\to C[0,1]$ are defined for any $m\in\mathbb{N}$ and any function $\mu\in C[0,1]$.

She obtained several approximation properties of the constructed polynomials (1) and estimated the order of approximation in terms of the modulus of continuity. Now, before proceeding further, we recall several basic notations based on q-calculus see details in [15]. Let $0<q\leq 1$, for all integer $j>0$, the q-integer $[j]_q$ is defined by

$$[j]_q:=\begin{cases}\dfrac{1-q^j}{1-q}, & q\neq 1\\ j, & q=1\end{cases}.$$

The q-factorial $[j]_q!$ and q-binomial $\begin{bmatrix}j\\ l\end{bmatrix}_q$ are defined respectively, as below:

$$[j]_q!:=\begin{cases}[j]_q[j-1]_q..[1]_q, & j=1,2,..\\ 1, & j=0\end{cases}$$

and

$$\begin{bmatrix}j\\ l\end{bmatrix}_q:=\frac{[j]_q!}{[l]_q!\,[j-l]_q!},\ \ (j\geq l\geq 0).$$

The q-analogue of the integration on the interval $[0,B]$ is defined as

$$\int_0^B\mu(t)d_qt:=B(1-q)\sum_{j=0}^{\infty}\mu(Bq^j)q^j,\qquad q\in(0,1).$$

In 1962's, French engineer Bézier handled the Bernstein basis functions, since they have a simple construction to utilization, to develop the shape design of surface and curve of cars. Moreover, these magnificent polynomials of Bernstein led to many application fields of Mathematics, such as computer graphics, computer-aided geometric design (CAGD), numerical solution of partial differential equations and, etc. Some applications in CAGD, one can refer to [16–19].

In 2019, Cai et al. [20] proposed and studied some statistical approximation properties of a new generalization of (λ,q)-Bernstein polynomials via Bézier bases with shape parameter $\lambda\in[-1,1]$ as follows:

$$\widetilde{B}_{m,q,\lambda}(\mu;y)=\sum_{j=0}^{m}\widetilde{b}_{m,j}(y;q,\lambda)\mu\left(\frac{[j]_q}{[m]_q}\right), \tag{2}$$

where

$$\widetilde{b}_{m,0}(y;q,\lambda) = b_{m,0}(y;q) - \frac{\lambda}{[m]_q+1} b_{m+1,1}(y;q),$$

$$\widetilde{b}_{m,j}(y;q,\lambda) = b_{m,j}(y;q) \ + \lambda \left(\frac{[m]_q - 2[j]_q + 1}{[m]_q^2 - 1} b_{m+1,j}(y;q) \right.$$

$$\left. - \frac{[m]_q - 2[j]_q - 1}{[m]_q^2 - 1} b_{m+1,j+1}(y;q) \right) \ (j = 1,2,\ldots,m-1), \tag{3}$$

$$\widetilde{b}_{m,m}(y;q,\lambda) = b_{m,m}(y;q) - \frac{\lambda}{[m]_q+1} b_{m+1,m}(y;q),$$

$m \geq 2$, $y \in [0,1]$, $0 < q \leq 1$, $\lambda \in [-1,1]$ and $b_{m,j}(y;q)$ are defined in (1).

In the year 2020, Mursaleen et al. [21] considered Chlodowsky kind (λ,q)-Bernstein-Stancu polynomials and derived Korovkin-type convergence, and Voronovskaya-type asymptotic theorems. For some recent relevant works we refer to [22–34].

Motivated by all write above-mentioned works, we will introduce and examine the following Kantorovich version of (λ,q)-Bernstein operators:

$$R_{m,q,\lambda}(\mu;y) = [m+1]_q \sum_{j=0}^{m} \widetilde{b}_{m,j}(y;q,\lambda) q^{-j} \int_{[j]_q/[m+1]_q}^{[j+1]_q/[m+1]_q} \mu(t) d_q t. \tag{4}$$

Remark 1.1. Let the operators $R_{m,q,\lambda}(\mu;y)$ be defined by (4). Then,

- ▷ For $\lambda = 0$, operators (4) reduce to the Kantorovich type q-Bernstein polynomials constructed by Dalmanoğlu [14].
- ▷ For $\lambda = 0$ and $q = 1$, operators (4) reduce to the classical Kantorovich polynomials [35].
- ▷ For $q = 1$, operators (4) reduce to the Kantorovich type λ-Bernstein polynomials proposed by Cai [36].

The present research is organized as follows: In Section 2, we compute several preliminary outcomes such as moments and central moments. In Section 3, we give a Korovkin-type convergence theorem and estimate the degree of convergence in terms of the ordinary modulus of continuity, the functions belong to Lipschitz-type class and Peetre's K-functional, respectively. In the final section, we demonstrate the comparison of the convergence of operators (4) to a function with some illustrations and error estimation table using Maple software.

2 Preliminaries

Lemma 2.1. (See [20]) Let $e_u(t) = t^u$, $u \in \mathbb{N} \cup \{0\}$, $y \in [0,1]$, $0 < q \leq 1$, $\lambda \in [-1,1]$ and $m > 1$. Then, for the operators given by (2) we have

$$\widetilde{B}_{m,q,\lambda}(e_0;y) = 1. \tag{5}$$

$$
\begin{aligned}
\widetilde{B}_{m,q,\lambda}(e_1;y) = y &+ \frac{[m+1]_q\lambda y\,(1-y^m)}{[m]_q([m]_q-1)} \\
&- \frac{2[m+1]_q\lambda y}{[m]_q^2-1}\left(\frac{1-y^m}{[m]_q}+qy(1-y^{m-1})\right) \\
&+ \frac{\lambda}{q[m]_q([m]_q+1)}\left(1-\prod_{r=0}^{m}\left(1-q^r y\right)-y^{m+1}-[m+1]_q y(1-y^m)\right) \\
&+ \frac{\lambda}{[m]_q^2-1}\left\{2[m+1]_q y^2(1-y^{m-1})-\frac{2[m+1]_q y(1-y^m)}{q[m]_q}\right. \\
&\left.+\frac{2}{q[m]_q}\left(1-\prod_{r=0}^{m}\left(1-q^r y\right)-y^{m+1}\right)\right\}. \qquad (6)
\end{aligned}
$$

$$
\begin{aligned}
&\widetilde{B}_{m,q,\lambda}(e_2;y) \\
&= y^2+\frac{y(1-y)}{[m]_q}+\frac{[m+1]_q\lambda y}{[m]_q\left([m]_q-1\right)}\left(qy(1-y^{m-1})+\frac{1-y^m}{[m]_q}\right) \\
&-\frac{2[m+1]_q\lambda}{[m]_q\left([m]_q^2-1\right)}\left\{\frac{y(1-y^m)}{[m]_q}+q(q+2)y^2(1-y^{m-1})\right. \\
&\left.+q^3[m-1]_q y^3(1-y^{m-2})\right\}-\frac{\lambda}{q[m]_q\left([m]_q+1\right)}\left\{[m+1]_q y^2(1-y^{m-1})\right. \\
&\left.-\frac{[m+1]_q y(1-y^m)}{q[m]_q}+\frac{1-\prod_{r=0}^{m}(1-q^r y)-y^{m+1}}{q[m]_q}\right\}+\frac{2\lambda}{[m]_q\left([m]_q^2-1\right)} \\
&\times\left\{q[m-1]_q[m+1]_q y^3(1-y^{m-2})-\frac{(1-q)[m+1]_q y^2(1-y^{m-1})}{q}\right. \\
&\left.+\frac{[m+1]_q y(1-y^m)}{q^2[m]_q}-\frac{1-\prod_{r=0}^{m}(1-q^r y)-y^{m+1}}{q^2[m]_q}\right\}. \qquad (7)
\end{aligned}
$$

Lemma 2.2. Let $y\in[0,1]$, $0<q\leq 1$, $\lambda\in[-1,1]$ and $m>1$. Then, we have $R_{m,q,\lambda}(1;y)=1$. (8)

Proof. Taking into consideration the definition of q-integral, it follows:

$$
\begin{aligned}
R_{m,q,\lambda}(1;y) &= [m+1]_q\sum_{j=0}^{m}\widetilde{b}_{m,j}(y;q,\lambda)q^{-j}\int_{[j]_q/[m+1]_q}^{[j+1]_q/[m+1]_q} d_q t \\
&= [m+1]_q\sum_{j=0}^{m}\widetilde{b}_{m,j}(y;q,\lambda)q^{-j}\frac{q^j}{[m+1]_q}
\end{aligned}
$$

$$= \sum_{j=0}^{m} \widetilde{b}_{m,j}(y;q,\lambda)$$

$$= \widetilde{B}_{m,q,\lambda}(1;y) = 1.$$

Hence, we get (8).

Lemma 2.3 Let $y \in [0,1]$, $0 < q \leq 1$, $\lambda \in [-1,1]$ and $m > 1$. Then, we have

$$\begin{aligned}
R_{m,q,\lambda}(t;y) &= y - \frac{q^m y}{[m+1]_q} + \frac{\lambda y(1-y^m)}{[m]_q - 1} \\
&- \frac{2[m]_q \lambda y}{[m]_q^2 - 1}\left(\frac{1-y^m}{[m]_q} + qy(1-y^{m-1})\right) \\
&+ \frac{\lambda}{q[m+1]_q([m]_q+1)}\left(1 - \prod_{r=0}^{m}(1-q^r y) - y^{m+1} - [m+1]_q y(1-y^m)\right) \\
&+ \frac{\lambda[m]_q}{[m+1]_q([m]_q^2-1)}\left\{2[m+1]_q y^2(1-y^{m-1}) - \frac{2[m+1]_q y(1-y^m)}{q[m]_q}\right. \\
&\left. + \frac{2}{q[m]_q}\left(1 - \prod_{r=0}^{m}(1-q^r y) - y^{m+1}\right)\right\} + \frac{1}{(1+q)[m+1]_q}.
\end{aligned} \tag{9}$$

Proof. By simple computing, one has

$$\int_{[j]_q/[m+1]_q}^{[j+1]_q/[m+1]_q} t d_q t = \frac{q^j}{(1+q)[m+1]_q^2}\left((1+q)[j]_q + 1\right).$$

Then,

$$\begin{aligned}
R_{m,q,\lambda}(t;y) &= [m+1]_q \sum_{j=0}^{m} \widetilde{b}_{m,j}(y;q,\lambda) q^{-j} \int_{[j]_q/[m+1]_q}^{[j+1]_q/[m+1]_q} t d_q t \\
&= [m+1]_q \sum_{j=0}^{m} \widetilde{b}_{m,j}(y;q,\lambda) q^{-j} \frac{q^j}{[m+1]_q^2}\left([j]_q + \frac{1}{1+q}\right) \\
&= \sum_{j=0}^{m} \widetilde{b}_{m,j}(y;q,\lambda) \frac{[j]_q}{[m+1]_q} + \sum_{j=0}^{m} \widetilde{b}_{m,j}(y;q,\lambda) \frac{1}{(1+q)[m+1]_q} \\
&= \frac{[m]_q}{[m+1]_q} \sum_{j=0}^{m} \widetilde{b}_{m,j}(y;q,\lambda) \frac{[j]_q}{[m]_q} + \frac{1}{(1+q)[m+1]_q} \sum_{j=0}^{m} \widetilde{b}_{m,j}(y;q,\lambda) \\
&= \frac{[m]_q}{[m+1]_q} \widetilde{B}_{m,q,\lambda}(t;y) + \frac{1}{(1+q)[m+1]_q} \widetilde{B}_{m,q,\lambda}(1;y).
\end{aligned}$$

In view of (5) and (6), we obtain desired sequel (9).

Lemma 2.4. Let $y \in [0,1]$, $0 < q \leq 1$, $\lambda \in [-1,1]$ and $m > 1$. Then, we have

$$\begin{aligned}
R_{m,q,\lambda}(t^2;y) &= y^2 - \frac{2q^m y^2}{[m+1]_q} + \frac{q^{2m}y^2}{[m+1]_q^2} \\
&+ \frac{[m]_q y}{[m+1]_q^2}\left(\frac{2q+1+(q^2+q+1)(1-y)}{q^2+q+1}\right) \\
&+ \frac{\lambda y}{[m+1]_q\left([m]_q-1\right)}\left(q[m]_q y(1-y^{m-1}) + \frac{(q^2+3q+2)(1-y^m)}{q^2+q+1}\right) \\
&- \frac{2[m]_q\lambda}{[m+1]_q\left([m]_q^2-1\right)}\left\{\frac{(q^2+3q+2)y(1-y^m)}{(q^2+q+1)[m]_q} + q^3y^3[m-1]_q(1-y^{m-2})\right. \\
&\left. + \frac{q(q^3+3q^2+5q+1)y^2(1-y^{m-1})}{q^2+q+1}\right\} \\
&- \frac{\lambda}{q[m+1]_q^2\left([m]_q^2-1\right)}\left\{[m]_q[m+1]_q y^2(1-y^{m-1})\right. \\
&\left. - \frac{(3q^2+2q+1)[m+1]_q y(1-y^m)}{q(q^2+q+1)} + \frac{3q^2+2q+1}{q(q^2+q+1)}\left(1-\prod_{r=0}^{m}\left(1-q^r y\right) - y^{m+1}\right)\right\} \\
&+ \frac{2\lambda[m]_q}{[m+1]_q^2\left([m]_q^2-1\right)}\left\{q[m-1]_q[m+1]_q y^3(1-y^{m-2})\right. \\
&+ \frac{(q^3+q^2+2q-1)[m+1]_q y^2(1-y^{m-1})}{q(q^2+q+1)} - \frac{(3q^2+q-1)[m+1]_q y(1-y^m)}{q^2(q^2+q+1)} \\
&\left. + \frac{3q^2+q-1}{q^2(q^2+q+1)}\frac{\left(1-\prod_{r=0}^{m}(1-q^r y)-y^{m+1}\right)}{[m]_q}\right\} + \frac{1}{(q^2+q+1)[m+1]_q^2}. \qquad (10)
\end{aligned}$$

Proof. As in previous Lemmas, again using the definition of q-integral, we get

$$\int_{[j]_q/[m+1]_q}^{[j+1]_q/[m+1]_q} t^2 d_q t = \frac{q^j([j+1]_q^2 + [j+1]_q[j]_q + [j]_q^2)}{(q^2+q+1)[m+1]_q^3}.$$

By the fact that $[j+1]_q = 1 + q[j]_q$, one can has the following easily:

$$\begin{aligned}
R_{m,q,\lambda}(t^2;y) &= \frac{1}{[m+1]_q^2}\sum_{j=0}^{m}\widetilde{b}_{m,q,\lambda}(y)\left([j]_q^2 + \frac{(2q+1)}{q^2+q+1}[j]_q + \frac{1}{q^2+q+1}\right) \\
&= \frac{[m]_q^2}{[m+1]_q^2}\widetilde{B}_{m,q,\lambda}(t^2;y) + \frac{(2q+1)[m]_q}{\left(q^2+q+1\right)[m+1]_q^2}\widetilde{B}_{m,q,\lambda}(t;y) \\
&+ \frac{1}{\left(q^2+q+1\right)}\frac{1}{[m+1]_q^2}\widetilde{B}_{m,q,\lambda}(1;y).
\end{aligned}$$

Then, by (5), (6) and (7), we arrive at (10) so the proof is done.

Corollary 2.1. Let $y \in [0,1]$, $0 < q \leq 1$, $\lambda \in [-1,1]$ and $m > 1$. As a consequence of Lemmas 2.2, 2.3 and 2.4, we arrive the following central moments:

$$\text{(i)} \quad R_{m,q,\lambda}(t-y;y) \leq \frac{q^m y}{[m+1]_q} + \frac{y\,(1-y^m)}{[m]_q - 1}$$

$$+\frac{2[m]_q y}{[m]_q^2-1}\left(\frac{1+y^m}{[m]_q} + qy(1-y^{m-1})\right)$$

$$+\frac{1}{q[m+1]_q([m]_q+1)}\left(1+\prod_{r=0}^{m}\left(1-q^r y\right) + y^{m+1} + [m+1]_q y(1-y^m)\right)$$

$$+\frac{[m]_q}{[m+1]_q([m]_q^2-1)}\left\{2[m+1]_q y^2(1-y^{m-1}) + \frac{2[m+1]_q y(1-y^m)}{q[m]_q}\right.$$

$$\left.+\frac{2}{q[m]_q}\left(1+\prod_{r=0}^{m}\left(1-q^r y\right)+y^{m+1}\right)\right\} + \frac{1}{(1+q)[m+1]_q} =: \alpha_m(y,q).$$

$$\text{(ii)} \quad R_{m,q,\lambda}((t-y)^2;y) \leq \frac{q^{2m}y^2}{[m+1]_q^2} + \frac{[m]_q y}{[m+1]_q^2}\left(\frac{2q+1+(q^2+q+1)(1-y)}{q^2+q+1}\right)$$

$$+\frac{2y^2\,(1-y^m)}{[m]_q-1} + \frac{4y^2[m]_q}{\left([m]_q^2-1\right)}\left(\frac{1+y^m}{[m]_q} + qy(1-y^{m-1})\right)$$

$$+\frac{y}{[m+1]_q\left([m]_q-1\right)}\left(q[m]_q y(1-y^{m-1}) + \frac{(q^2+3q+2)(1-y^m)}{q^2+q+1}\right)$$

$$+\frac{2y}{q[m+1]_q\left([m]_q+1\right)}\left(1+\prod_{r=0}^{m}\left(1-q^r y\right) + y^{m+1} + [m+1]_q y(1-y^m)\right)$$

$$+\frac{1}{q[m+1]_q^2\left([m]_q^2-1\right)}\left\{\frac{(3q^2+2q+1)[m+1]_q y(1-y^m)}{q(q^2+q+1)} + [m]_q[m+1]_q y^2(1-y^{m-1})\right.$$

$$\left.+\frac{3q^2+2q+1}{q(q^2+q+1)}\left(1+\prod_{r=0}^{m}\left(1-q^r y\right)+y^{m+1}\right)\right\}$$

$$+\frac{2[m]_q y}{[m+1]_q\left([m]_q^2-1\right)}\left\{\frac{(q^3+3q^2+2q+2(q^2+q+1)[m+1]_q y)(1-y^m)}{q^2(q^2+q+1)[m]_q}\right.$$

$$+q[m-1]_q[m+1]_q y^3(1-y^{m-2})$$

$$\left.+\frac{(2(q^2+q+1)[m+1]_q y + q(q^3+3q^2+5q+1))y(1-y^{m-1})}{q^2+q+1}\right\}$$

$$+\frac{2[m]_q}{[m+1]_q^2\left([m]_q^2-1\right)}\left\{\frac{(q^3+q^2+2q+1)[m+1]_q y^2(1-y^{m-1})}{q(q^2+q+1)}\right.$$
$$+\frac{3q^2+2q+1}{q^2(q^2+q+1)}\left(\frac{1+\prod_{r=0}^{m}(1-q^r y)+y^{m+1}}{[m]_q}\right)+q[m-1]_q[m+1]_q y^3(1-y^{m-2})$$
$$\left.+\frac{(3q^2+q+1)[m+1]_q y(1-y^m)}{q^2(q^2+q+1)}\right\}$$
$$+\frac{2y}{(q+1)[m+1]_q}+\frac{1}{(q^2+q+1)[m+1]_q^2}=:\beta_m(y,q).$$

Remark 2.1. It can be seen that for a fixed $q\in(0,1)$, $\lim_{m\to\infty}[m]_q=\frac{1}{1-q}$. In order to provide the convergence results, we get the sequence $q:=(q_m)$ such that $0<q_m<1, q_m\to 1, \frac{1}{[m]_{q_m}}\to 0$ as $m\to\infty$.

3 Convergence Results of $R_{m,q,\lambda}(\mu;y)$

In this section, we first introduce a Korovkin-type approximation theorem for $R_{m,q,\lambda}(\mu;y)$. It is a known fact that, the space $C[0,1]$ denotes the real-valued continuous function on $[0,1]$ and equipped with the norm for the function μ as $\|\mu\|_{C[0,1]}=\sup_{y\in[0,1]}|\mu(y)|$.

Theorem 3.1. Assume that $q:=\{q_m\}$ satisfies the conditions as in Remark. Then, for all $\mu\in C[0,1]$, $\lambda\in[-1,1]$, $y\in[0,1]$ and $m>1$, we have

$$\lim_{m\to\infty}\left\|R_{m,q_m,\lambda}(\mu;.)-\mu\right\|=0. \tag{11}$$

Proof. Consider the sequence of functions $e_s(y)=y^s$, where $s\in\{0,1,2\}$ and $y\in[0,1]$. According to the Bohman-Korovkin theorem [37], we have to show that

$$\lim_{m\to\infty}\left\|R_{m,q_m,\lambda}(e_s;.)-e_s\right\|=0,\ \text{ for } s=0,1,2. \tag{12}$$

The proof of (12) follows easily, using (8), (9) and (10). Hence, we get the required sequel.

Now, we will estimate the order of convergence in terms of the usual modulus of continuity, for the function belong to the Lipschitz type class and the Peetre's K-functional. By first, we symbolize the usual modulus of continuity of $\mu\in C[0,1]$ as follows:

$$\omega(\mu;\eta):=\sup_{0<a\leq\eta}\ \sup_{y\in[0,1]}|\mu(y+a)-\mu(y)|.$$

Since $\eta>0$, $\omega(\mu;\eta)$ has some useful properties see details by [38]. Also, we present an element of Lipschitz continuous function with $Lip_L(\zeta)$, where $L>0$ and $0<\zeta\leq 1$. If the expression below:

$$|\mu(t)-\mu(y)|\leq L\,|t-y|^{\zeta},\quad (t,y\in\mathbb{R}),$$

holds, then one can say μ is belong to $Lip_L(\zeta)$.

Also, the Peetre's K-functional is defined by

$$K_2(\mu,\eta)=\inf_{\nu\in C^2[0,1]}\left\{\|\mu-\nu\|+\eta\left\|\nu''\right\|\right\},$$

where $\eta > 0$ and $C^2[0,1] = \left\{\nu \in C[0,1]\colon \nu', \nu'' \in C[0,1]\right\}$.

Taking into account [39], there exists an absolute constant $C > 0$ such that

$$K_2(\mu;\eta) \le C\omega_2(\mu;\sqrt{\eta}), \quad \eta > 0 \tag{13}$$

where $\omega_2(\mu;\eta) = \sup\limits_{0<z\le\eta} \sup\limits_{y\in[0,1]} |\mu(y+2z) - 2\mu(y+z) + \mu(y)|$,

is the second order modulus of smoothness of $\mu \in C[0,1]$.

Theorem 3.2. Assume that $q := \{q_m\}$ satisfies the conditions as in Remark 2.1. Then, for all $\mu \in C[0,1]$, $\lambda \in [-1,1]$, $y \in [0,1]$ and $m > 1$, we obtain

$$\left|R_{m,q_m,\lambda}(\mu;y) - \mu(y)\right| \le 2\omega(\mu;\sqrt{\beta_m(y,q_m)}),$$

where $\beta_m(y,q_m) = R_{m,q_m,\lambda}((t-y)^2;y)$ same as in Corollary 2.1.

Proof. Taking $|\mu(t) - \mu(y)| \le \left(1 + \frac{|t-y|}{\delta}\right)\omega(\mu;\delta)$ into account and after operating $R_{m,q_m,\lambda}(.;y)$, we get

$$\left|R_{m,q_m,\lambda}(\mu;y) - \mu(y)\right| \le \left(1 + \frac{1}{\delta} R_{m,q_m,\lambda}(|t-y|;y)\right)\omega(\mu;\delta).$$

Utilizing the Cauchy-Bunyakovsky-Schwarz inequality, it follows:

$$\begin{aligned}\left|R_{m,q_m,\lambda}(\mu;y) - \mu(y)\right| &\le \left(1 + \frac{1}{\delta}\sqrt{R_{m,q_m,\lambda}((t-y)^2;y)}\right)\omega(\mu;\delta)\\ &\le \left(1 + \frac{1}{\delta}\sqrt{\beta_m(y,q_m)}\right)\omega(\mu;\delta).\end{aligned}$$

Choosing $\delta = \sqrt{\beta_m(y,q_m)}$, which completes the proof.

Theorem 3.3. Assume that $q\colon = \{q_m\}$ satisfies the conditions as in Remark 2.1. Then, for all $\mu \in Lip_L(\zeta)$, $0 < \zeta \le 1$, $y \in [0,1]$, $\lambda \in [-1,1]$ and $m > 1$, we obtain

$$\left|R_{m,q_m,\lambda}(\mu;y) - \mu(y)\right| \le L(\beta_m(y,q_m))^{\frac{\zeta}{2}}.$$

Proof. Let $\mu \in Lip_L(\zeta)$. Using the linearity and monotonicity properties of the operators (4), therefore

$$\begin{aligned}&\left|R_{m,q_m,\lambda}(\mu;y) - \mu(y)\right| \le R_{m,q_m,\lambda}(|\mu(t) - \mu(y)|;y)\\ &= [m+1]_{q_m} \sum_{j=0}^{m} \widetilde{b}_{m,q_m,\lambda}(y) q_m^{-j} \int_{[j]_{q_m}/[m+1]_{q_m}}^{[j+1]_{q_m}/[m+1]_{q_m}} |\mu(t) - \mu(y)|\, d_{q_m}t\\ &\le L\left([m+1]_{q_m} \sum_{j=0}^{m} \widetilde{b}_{m,q_m,\lambda}(y) q^{-j} \int_{[j]_{q_m}/[m+1]_{q_m}}^{[j+1]_{q_m}/[m+1]_{q_m}} |t-y|^{\zeta}\, d_{q_m}t\right).\end{aligned}$$

Using the Hölder's inequality and with $p_1 = \frac{2}{\zeta}$ and $p_2 = \frac{2}{2-\zeta}$, one has $\frac{1}{p_1} + \frac{1}{p_2} = 1$. Hence, we may write

$$\left|R_{m,q_m,\lambda}(\mu;y) - \mu(y)\right|$$

$$\leq L\left([m+1]_{q_m}\sum_{j=0}^{m}\widetilde{b}_{m,q_m,\lambda}(y)q^{-j}\int_{[j]_{q_m}/[m+1]_{q_m}}^{[j+1]_{q_m}/[m+1]_{q_m}}|t-y|^2\,d_{q_m}t\right)^{\frac{\zeta}{2}}$$

$$\leq L\left(R_{m,q_m,\lambda}((t-y)^2;y)\right)^{\frac{\zeta}{2}} \leq L(\beta_m(y,q_m))^{\frac{\zeta}{2}}.$$

Hence, Theorem 3.3 is proved.

Theorem 3.4. For all $\mu \in C[0,1]$, $y \in [0,1]$ and $\lambda \in [-1,1]$, the following inequality holds true

$$\left|R_{m,q,\lambda}(\mu;y) - \mu(y)\right| \leq C\omega_2(\mu;\frac{1}{2}\sqrt{\beta_m(y,q) + (\alpha_m(y,q))^2}) + \omega(\mu;\alpha_m(y,q)),$$

where $C > 0$ is a positive constant, $\alpha_m(y,q)$ and $\beta_m(y,q)$ are same as in Corollary 2.1.

Proof. Firstly, we define the following auxiliary operators:

$$\widehat{R}_{m,q,\lambda}(\mu;y) = R_{m,q,\lambda}(\mu;y) - \mu(R_{m,q,\lambda}(t;y)) + \mu(y). \tag{14}$$

In view of (8) and (9), we find

$$\widehat{R}_{m,q,\lambda}(t-y;y) = 0.$$

Using Taylor's expansion, one has

$$\xi(t) = \xi(y) + (t-y)\xi'(y) + \int_t^y (t-u)\xi''(u)du, \quad (\xi \in C^2[0,1]). \tag{15}$$

After operating $\widehat{R}_{m,q,\lambda}(.;y)$ to (15), yields

$$\begin{aligned}
\widehat{R}_{m,q,\lambda}(\xi;y) - \xi(y) &= \widehat{R}_{m,q,\lambda}((t-y)\xi'(y);y) + \widehat{R}_{m,q,\lambda}\left(\int_t^y (t-u)\xi''(u)du;y\right)\\
&= \xi'(y)\widehat{R}_{m,q,\lambda}(t-y;y) + R_{m,q,\lambda}\left(\int_t^y (t-u)\xi''(u)du;y\right)\\
&\quad - \int_{R_{m,q,\lambda}(t;y)}^{y} (R_{m,q,\lambda}(t;y) - u)\xi''(u)du\\
&= R_{m,q,\lambda}\left(\int_t^y (t-u)\xi''(u)du;y\right) - \int_{R_{m,q,\lambda}(t;y)}^{y} (R_{m,q,\lambda}(t;y) - u)\xi''(u)du.
\end{aligned}$$

Considering (14), we get

$$\left|\widehat{R}_{m,q,\lambda}(\xi;y)-\xi(y)\right| \le \left|R_{m,q,\lambda}\left(\int_t^y (t-u)\xi''(u)du;y\right)\right| + \left|\int_y^{R_{m,q,\lambda}(t;y)} (R_{m,q,\lambda}(t;y)-u)\xi''(u)du\right|$$

$$\le R_{m,q,\lambda}\left(\left|\int_y^t (t-u)\right| \left|\xi''(u)\right| |du|;y\right)$$

$$+\int_y R_{m,q,\lambda}(t;y)\left|R_{m,q,\lambda}(t;y)-u\right| \left|\xi''(u)\right| |du|$$

$$\le \|\xi''\| \left\{R_{m,q,\lambda}((t-y)^2;y)+\left(R_{m,q,\lambda}(t;y)-y\right)^2\right\}$$

$$\le \left\{\beta_m(y,q)+(\alpha_m(y,q))^2\right\} \|\xi''\|.$$

On the other hand using (8), (9) and (14), it deduce the following:

$$\left|\widehat{R}_{m,q,\lambda}(\mu;y)\right| \le \left|R_{m,q,\lambda}(\mu;y)\right| + 2\|\mu\| \le \|\mu\| R_{m,q,\lambda}(1;y)+2\|\mu\| \le 3\|\mu\|. \tag{16}$$

With the help of (15) and (16), we get

$$\left|R_{m,q,\lambda}(\mu;y)-\mu(y)\right| \le \left|\widehat{R}_{m,q,\lambda}(\mu-\xi;y)-(\mu-\xi)(y)\right|$$

$$+\left|\widehat{R}_{m,q,\lambda}(\xi;y)-\xi(y)\right| + \left|\mu(y)-\mu(R_{m,q,\lambda}(t;y))\right|$$

$$\le 4\|\mu-\xi\| + \left\{\beta_m(y,q)+(R_{m,q,\lambda}(t;y))^2\right\} \|\xi''\| + \omega(\mu;\alpha_m(y,q)).$$

On account of this, if we take the infimum on the right hand side over all $\xi \in C^2[0,1]$ and in view of (13), we arrive

$$\left|R_{m,q,\lambda}(\mu;y)-\mu(y)\right| \le 4K_2\left(\mu;\frac{\{\beta_m(y,q)+(\alpha_m(y,q))^2\}}{4}\right)+\omega(\mu;\alpha_m(y,q))$$

$$\le C\omega_2(\mu;\frac{1}{2}\sqrt{\beta_m(y,q)+(\alpha_m(y,q))^2})+\omega(\mu;\alpha_m(y,q)).$$

Hence, we obtain the proof of this theorem.

4 Graphical and Numerical Examples

In this section, we present some graphics and error estimation table to see the convergence behaviour of $R_{m,q,\lambda}(\mu;y)$ operators to a function.

Example. Consider the function

$$\mu(y) = 1-\sin(\pi y)$$

on the interval $[0,1]$.

In Fig. 1, we demonstrate the approximations of $R_{m,q,1}(\mu;y)$ and $R_{m,q,-1}(\mu;y)$ operators to $\mu(y)$ for different values of m and q.

In Fig. 2, we show the approximations of $R_{m,q,0}(\mu;y)$ and $R_{m,q,1}(\mu;y)$ operators to $\mu(y)$ for $m=10$ and $q=0.9$.

Also, in Table 1, we calculate the maximum error of approximation of $R_{m,q,\lambda}(\mu; y)$ operators for some values of m, q and λ in view of $\mu(y)$.

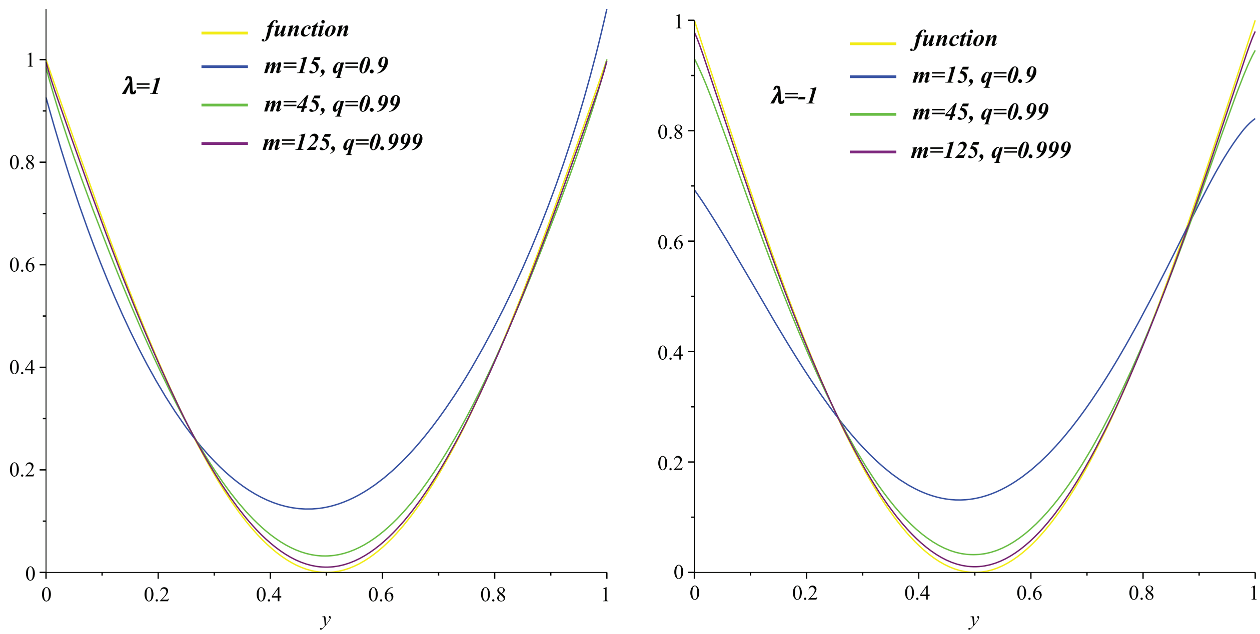

Figure 1: The convergence of $R_{m,q,\lambda}(\mu; y)$ operators to $\mu(y) = 1 - \sin(\pi y)$ for $\lambda = 1$, $\lambda = -1$ and different values of m and q

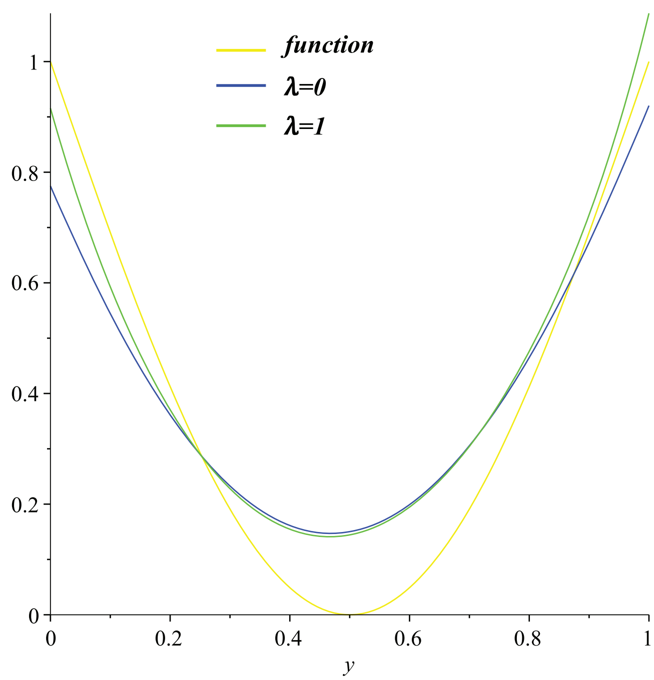

Figure 2: The convergence of $R_{m,q,\lambda}(\mu; y)$ operators to $\mu(y) = 1 - \sin(\pi y)$ for $\lambda = 0$, $\lambda = 1$, $m = 10$ and $q = 0.9$

Table 1: Error of approximation $R_{m,q,\lambda}(\mu;y)$ operators to $\mu(y) = 1 - \sin(\pi y)$ for different values of m, q and λ

λ	q	$\lvert\mu(y) - R_{m,q,\lambda}(\mu;y)\rvert$		
		$m = 15$	$m = 45$	$m = 125$
1	0.9	0.1272102421	0.1091438514	0.1084604470
	0.99	0.0747917138	0.0320712502	0.0168560799
	0.999	0.0705273105	0.0266032428	0.0102987106
0	0.9	0.1305031803	0.1106365735	0.1099013419
	0.99	0.0761998009	0.0321428950	0.0168623572
	0.999	0.0718123066	0.0266565471	0.0103014327
-1	0.9	0.1337961186	0.1121292958	0.1113422366
	0.99	0.0776078880	0.0322145398	0.0168686344
	0.999	0.0730973026	0.0267098514	0.0103041547

It is obvious from Table 1 that, in case $\lambda = 1$ and as the value m increases than the absolute error bounds of $\lvert\mu(y) - R_{m,q,1}(\mu;y)\rvert$ are smaller than $\lvert\mu(y) - R_{m,q,0}(\mu;y)\rvert$. Namely, our newly defined operators (4), in case $\lambda > 0$ has better approximation than Kantorovich type q-Bernstein polynomials.

5 Conclusion

In this work, we introduced a Kantorovich type of q-Bernstein operators based on the Bézier basis functions with shape parameter $\lambda \in [-1, 1]$. Due to the parameter λ, we have more flexibility in modeling. Some approximation properties of these operators such as a Korovkin type convergence theorem and as well as the order of convergence concerning with the usual modulus of continuity, Lipschitz-type functions and Peetre's K-functional are studied. Moreover, we made our work more intuitive by using some graphs and error estimation table. As future works, we may consider Durrmeyer and Stancu kinds of q-Bernstein operators with shape parameter λ.

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References

1. Bernstein, S. N. (1912). Démonstration du théorème de Weierstrass fondée sur le calcul des probabilités. *Communications of the Kharkov Mathematical Society, 13(1),* 1–2.
2. Lupas, A. (1987). A q-analogue of the Bernstein operator. *Seminar on Numerical and Statistical calculus,* vol. 9, pp. 85–92. Cluj-Napoca.
3. Phillips, G. M. (1997). Bernstein polynomials based on the q-integers. *Annals of Numerical Mathematics, 4,* 511–518.
4. Oruç, H., Phillips, G. M. (2003). q-Bernstein polynomials and Bézier curves. *Journal of Computational and Applied Mathematics, 151(1),* 1–12. DOI 10.1016/S0377-0427(02)00733-1.
5. Nowak, G. (2009). Approximation properties for generalized q-Bernstein polynomials. *Journal of Mathematical Analysis and Applications, 350(1),* 50–55. DOI 10.1016/j.jmaa.2008.09.003.
6. Aslan, R., Izgi, A. (2020). Ağırlıklı Uzaylarda q-Szász-Kantorovich-Chlodowsky Operatörlerinin Yaklaşımları. *Erciyes Üniversitesi Fen Bilimleri Enstitüsü Fen Bilimleri Dergisi, 36(1),* 137–149.
7. Mursaleen, M., Khan, A. (2013). Generalized q-Bernstein-Schurer operators and some approximation theorems. *Journal of Function Spaces,* 2013. DOI 10.1155/2013/719834.
8. Dalmanoğlu, O., Dogru, O. (2010). On statistical approximation properties of Kantorovich type q-Bernstein operators. *Mathematical and Computer Modelling, 52,* 760–771. DOI 10.1016/j.mcm.2010.05.005.
9. Acar, T., Aral, A. (2015). On pointwise convergence of q-Bernstein operators and their q-derivatives. *Numerical Functional Analysis and Optimization, 36,* 287–304. DOI 10.1080/01630563.2014.970646.
10. Mahmudov, N. I., Sabancigil, P. (2013). Approximation theorems for q-Bernstein-Kantorovich operators. *Filomat, 27,* 721–730. DOI 10.2298/FIL1304721M.
11. Neer, T., Agrawal, P. N., Araci, S. (2017). Stancu-Durrmeyer type operators based on q-integers. *Applied Mathematics and Information Sciences, 11(3),* 1–9. DOI 10.18576/amis/110317.
12. Gupta, V. (2008). Some approximation properties of q-Durrmeyer operators. *Applied Mathematics and Computation, 197,* 172–178. DOI 10.1016/j.amc.2007.07.056.
13. Karsli, H., Gupta, V. (2008). Some approximation properties of q-Chlodowsky operators. *Applied Mathematics and Computation, 195,* 220–229. DOI 10.1016/j.amc.2007.04.085.
14. Dalmanoğlu, O. (2007). Approximation by Kantorovich type q-Bernstein operators. *Proceedings of the 12th WSEAS International Conference on Applied Mathematics*, pp. 113–117. Cairo, Egypt.
15. Kac, V., Cheung, P. (2001). *Quantum calculus*. Berlin, Germany: Springer Science & Business Media.
16. Khan, K., Lobiyal, D. K., Kilicman, A. (2018). A de Casteljau Algorithm for Bernstein type Polynomials based on (p, q)-integers. *Applications and Applied Mathematics, 13(2),* 997–1017.
17. Khan, K., Lobiyal, D. K. (2017). Bézier curves based on Lupas (p, q)-analogue of Bernstein functions in CAGD. *Journal of Computational and Applied Mathematics, 317,* 458–477. DOI 10.1016/j.cam.2016.12.016.
18. Farin, G. (1993). *Curves and surfaces for computer-aided geometric design: A practical guide*. Amsterdam, Netherlands: Elsevier.
19. Sederberg, T. W. (2007). *Computer aided geometric design course notes*. Provo, Utah: Brigham Young University.
20. Cai, Q. B., Zhou, G., Li, J. (2019). Statistical approximation properties of λ-Bernstein operators based on q-integers. *Open Mathematics, 17,* 487–498. DOI 10.1515/math-2019-0039.
21. Mursaleen, M., Al-Abied, A. A. H., Salman, M. A. (2020). Chlodowsky type (λ, q)-Bernstein-Stancu operators. *Azerbaijan Journal of Mathematics, 10,* 75–101.
22. Rahman, S., Mursaleen, M., Acu, A. M. (2019). Approximation properties of λ-Bernstein-Kantorovich operators with shifted knots. *Mathematical Methods in the Applied Sciences, 42(11),* 4042–4053. DOI 10.1002/mma.5632.
23. Cai, Q. B., Lian, B. Y., Zhou, G. (2018). Approximation properties of λ-Bernstein operators. *Journal of Inequalities and Applications, 2018(1),* 61. DOI 10.1186/s13660-018-1653-7.
24. Özger, F. (2019). Weighted statistical approximation properties of univariate and bivariate λ-Kantorovich operators. *Filomat, 33(11),* 3473–3486. DOI 10.2298/FIL1911473O.

25. Özger, F. (2020). On new Bézier bases with Schurer polynomials and corresponding results in approximation theory. *Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics, 69,* 376–393. DOI 10.31801/cfsuasmas.510382.
26. Özger, F., Demirci, K., Yildiz, S. (2020). Approximation by Kantorovich variant of λ-Schurer operators and related numerical results. In: *Topics in contemporary mathematical analysis and applications*, pp. 77–94, Boca Raton, USA: CRC Press.
27. Özger, F. (2020). Applications of generalized weighted statistical convergence to approximation theorems for functions of one and two variables. *Numerical Functional Analysis and Optimization, 41(16),* 1990–2006. DOI 10.1080/01630563.2020.1868503.
28. Özger, F., Srivastava, H. M., Mohiuddine, S. A. (2020). Approximation of functions by a new class of generalized Bernstein-Schurer operators. *Revista de la Real Academia de Ciencias Exactas, Fsicas y Naturales. Serie A. Matemáticas, 114(4),* 1–21. DOI 10.1007/s13398-020-00903-6.
29. Mohiuddine, S. A., Özger, F. (2020). Approximation of functions by Stancu variant of Bernstein-Kantorovich operators based on shape parameter α. *Revista de la Real Academia de Ciencias Exactas, Fsicas y Naturales. Serie A. Matemáticas, 114(2),* 1–17. DOI 10.1007/s13398-020-00802-w.
30. Srivastava, H. M., Ansari, K. J., Özger, F., Ödemiş Özger, Z. (2021). A link between approximation theory and summability methods via four-dimensional infinite matrices. *Mathematics, 9(16),* 1895. DOI 10.3390/math9161895.
31. Kumar, A. (2020). Approximation properties of generalized λ-Bernstein-Kantorovich type operators. *Rendiconti del Circolo Matematico di Palermo Series, 2(70),* 505–520. DOI 10.1007/s12215-020-00509-2.
32. Acu, A. M., Manav, N., Sofonea, D. F. (2018). Approximation properties of λ-Kantorovich operators. *Journal of Inequalities and Applications, 2018(1),* 202. DOI 10.1186/s13660-018-1795-7 2018.
33. Cai, Q. B., Torun, G., Dinlemez Kantar, Ü. (2021). Approximation properties of generalized λ-Bernstein-Stancu-type operators. *Journal of Mathematics, 2021(2),* 1–17. DOI 10.1155/2021/5590439.
34. Srivastava, H. M., Özger, F., Mohiuddine, S. A. (2019). Construction of Stancu-type Bernstein operators based on Bézier bases with shape parameter λ. *Symmetry, 11(3),* 316. DOI 10.3390/sym11030316.
35. Kantorovich, L. V. (1930). *Sur certain dé veloppements suivant les polynômes de la forme de S. Bernstein I, II*, pp. 563–568. CR Acad, URSS.
36. Cai, Q. B. (2018). The Bézier variant of Kantorovich type λ-Bernstein operators. *Journal of Inequalities and Applications, 2018(1),* 90. DOI 10.1186/s13660-018-1688-9.
37. Korovkin, P. P. (1953). On convergence of linear positive operators in the space of continuous functions. *Doklady Akademii Nauk SSSR, 90,* 961–964.
38. Altomare, F., Campiti, M. (2011). *Korovkin-type approximation theory and its applications*. Berlin: de Gruyter.
39. DeVore, R. A., Lorentz, G. G. (1993). *Constructive approximation*. Berlin, Heidelberg: Springer.