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ARTICLE

A New Approach to Vague Soft Bi-Topological Spaces

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ABSTRACT

Fuzzy soft topology considers only membership value. It has nothing to do with the non-membership value. So an extension was needed in this direction. Vague soft topology addresses both membership and non-membership values simultaneously. Sometimes vague soft topology (single structure) is unable to address some complex structures. So an extension to vague soft bi-topology (double structure) was needed in this direction. To make this situation more meaningful, a new concept of vague soft bi-topological space is introduced and its structural characteristics are attempted with a new definition. In this article, new concept of vague soft bi-topological space (VSBTS) is initiated and its structural behaviors are attempted. This approach is based on generalized vague soft open sets, known as vague soft β open sets. An ample of examples are given to understand the structures. For the non-validity of some results, counter examples are provided to pay the price. Pair-wise vague soft β open and pair-wise vague soft β close sets are also addressed with examples in (VSBTS). Vague soft separation axioms are initiated in (VSBTS) concerning soft points of the space. Other separation axioms are also addressed relative to soft points of the space. Vague soft bi-topological properties are studied with the application of vague soft β open sets with respect to soft points of the spaces. The characterization of vague soft β close as well as vague soft β open sets, characteristics of Bolzano Weirstrass property, vague soft compactness and its marriage with sequences, interconnection between sequential compactness and countable compactness in (VSBTS) with respect to soft β open sets are addressed.

KEYWORDS

Vague soft set; vague soft point; vague soft bi-topological space; vague soft β-open set and vague soft β-separation axioms

1 Introduction

During the study of the potential applications in traditional and non-classical logic, it is essential to have fuzzy soft sets, vague soft sets and neutrosophical soft sets. Researchers now deal with the complications of modeling uncertain information in the economic, engineering, environmental, sociology, medical, and numerous other fields. Classical methods do not always succeed because there may be different types of uncertainties in those fields. Zadeh [1] has created a new approach to the fuzzy set theory that was the most appropriate schedule to address uncertainty. Pawlak [2] pioneered the concept of a rough set. The approximate operations on sets were investigated. Each theory has its own inherent challenges, as Molodtsov has pointed them out [3]. Molodtsov [3] proposed an entirely new and advanced approach to modeling vagueness and uncertainty free of the complications of existing procedures. In soft set theory, among other related issues, the problem of setting the member function simply does not arise. Soft sets are regarded as neighborhood systems and they are a special case of contextual fuzzy sets. Soft set theory has potential applications in many different fields, counting the smoothness of functions, game theory, operations research, Riemann integration, Perron integration, probability theory, and measurement theory.

Maji et al. [4] functionalized soft sets in multicriteria decision making problems by applying the technique of knowledge reduction to the information table induced by soft set. Maji et al. [5] discussed different fundamentals of soft set theory. Pei et al. [6] discussed the relationship between soft sets and information systems. Soft set is a kind of particular information system. The more general results show that soft sets and information systems of partitions have the same formally designed structures and that those soft sets and fuzzy information systems are equivalent after soft sets are extended to several classes of general cases.

Chen et al. [7] pointed out some drawbacks in [4]. They improved the work of Maji et al. [4]. Smarandache [8] generalized the soft set to the hyper-soft set by transforming the function F into a multi-attribute function. Further the author introduced the hybrids of crisp, fuzzy, intuitionistic fuzzy, neutrosophic, and plithogenic hyper-soft set.

Cagman et al. [9] defined and presented the concept of soft topology on a soft set. The authors also discussed the basis of soft topological spaces theory. Shabir et al. [10] introduced soft topological areas and examined some basic notions. Bayramov et al. [11] investigated some basic notions of soft topological spaces by using soft point approach. Khattak et al. [12] ushered the notion of soft (α, β)-open sets and their characterization in soft single point topology. Atanassov [13] initiated the idea of "intuitionistic fuzzy set" (IFS) which is an extension of the concept 'fuzzy set'. The authors addressed various characteristics comprising of operations and relations over sets. Bayramov et al. [14] introduced some important properties of intuitionistic fuzzy soft topological spaces and defined the intuitionistic fuzzy soft closure and interior of an intuitionistic fuzzy soft set. Chen [15] addressed similarity measures between vague sets and between elements.

Hong et al. [16] discovered new functions to discuss the degree of accuracy in the grades of membership of each substitute relative to a set of conditions embodied by the values of vague. Ye [17] discovered that an improved precision function for a vague set is recommended by considering the effect on the fitting to which every alternative catches the choice maker's necessities of a vague level, unknown degree. The author added that the precise function is more judicious than the current precise function which is in some cases is not favorable. Alhazaymeh et al. [18] introduced the conception of interval-valued vague soft sets which are an extension of the soft set. Alhazaymeh et al. [19] generalized vague soft set and its operations. Al-Quran et al. [20] extended notion of classical soft sets to neutrosophic vague soft sets by applying the theory of soft sets to neutrosophic vague sets to be more effective and beneficial.

Selvachandran et al. [21] studied vague entropy measure for complex vague soft sets. Wei et al. [22] introduced five elements associated with vague soft sets that enable GML to represent fuzziness and implement vague soft set GML modeling, which solves the problem of lack of fuzzy information expression in GML. Tahat et al. [23] introduced the concept of vague soft set ordering and addressed certain relevant properties. Xu et al. [24] introduced a vague soft set, an extension of the soft set. They presented and deliberated the fundamental features of vague soft sets. With a standard approach, Wang et al. [25] examined a few basic characteristics of vague soft topological areas.

This reference [25] has become a motivating source and it leads me to my research work excellently. Inthumathi et al. [26] introduced some generalization of vague soft open sets in vague soft topological spaces and obtained a decomposition of vague α-soft open sets by using them. In our study, the intersection, union and difference operations are re-defined on the vague soft sets in contrast to the studies [24] and the properties related to these operations are presented. Then, considering these newly defined processes, contrasting [25] vague soft topology is remodeled and further the study is extended to vague soft bitopological spaces with respect to soft points of the spaces under generalized vague soft open sets, known as vague soft β-open sets. An ample of examples are given to understand the structures.

In this article, the concept of vague soft topology is initiated and its structural characteristics are attempted with new definitions. The rest of the article is pictured as follows. Originality begins from this Section 2 because in this section some new operations are defined in vague soft set theory. Main results are addressed in Section 3. In this Section 3, some new definitions of vague soft open sets are given. With the support of this new definition, soft separations axioms and soft other separation axioms are defined in VSBTS with respect to softs. These separation axioms are verified through suitable examples. The engagement of these separation axioms with other results are also addressed.

In Section 4, some results are discussed in two vague soft bi-topological space with respect to vague soft β open sets. Vague soft product spaces are discussed with respect to soft points. The characterization of vague soft β closed as well as soft β open sets, characteristics of Bolzano Weirstrass property, vague soft compactness and its marriage with sequences, interconnection between sequentially compactness and countably compactness in vague soft bi-topology with respect to soft β open sets are addressed. In the final Section 5, some concluding remarks and future work are given.

2 Preliminaries

In this section, some basic definitions, which are soft sets, soft sub-space, soft equal space, soft difference, soft null set, soft absolute set, soft point, soft union, soft intersection, vague soft topology and vague soft neighborhood are addressed.

Definition 2.1. Let $\mathcal{X}$ be an initial universe set and θ be a set of parameters. Then pair $\langle \hbar, \theta\rangle$ is called as vague soft set (VSS) over $\mathcal{X}$, where $\hbar$ is a mapping from θ to $\mathrm{V}(\mathcal{X})$.

The set of all NSS over $\mathcal{X}$ is denoted by $\mathrm{VSS}(\mathcal{X})$. A vague set $\langle \hbar, \theta\rangle$ can be written as

$$\langle \hbar, \theta\rangle = \left\{(\imath\,, < x, \mathrm{M}_{\hbar(\imath)}\,(x)\,, \mathrm{N}_{\hbar(\imath)}\,(x) > : x \in \mathcal{X}, \imath \in \theta\right\}$$

Definition 2.2. Let $\mathcal{X}$ be an initial universe set and θ be a set of parameters. Then the vague soft set $x^{\imath}{}_{(\alpha,\gamma)}$ defined as

$$x^{\imath}{}_{(\alpha,\gamma)}\left(\imath'\right)(\mathrm{y}) = \begin{cases}(\alpha,\gamma) & \text{if } \imath = \imath' \text{ and } x = \mathrm{y}\\ (0,1) & \text{if } \imath \neq \imath' \text{ and } x \neq \mathrm{y}\end{cases}$$

for all $x \in \mathcal{X}, 0 < \alpha, \gamma \leq 1, i \in \theta$, is called a vague soft point.

Definition 2.3. Let $\langle \tilde{f}_1, \theta \rangle, \langle \tilde{f}_2, \theta \rangle \in VSS(\mathcal{X})$. Then for all $x \in \mathcal{X}$

1. Vague Soft Subset: $\langle \tilde{f}_1, \theta \rangle \subset \langle \tilde{f}_2, \theta \rangle$ if $M_{\tilde{f}_1(i)}(x) \leq M_{\tilde{f}_2(i)}(x)$ and $N_{\tilde{f}_1(i)}(x) \geq N_{\tilde{f}_2(i)}(x)$ for all $i \in \theta$.

2. Vague Soft Equality: $\langle \tilde{f}_1, \theta \rangle = \langle \tilde{f}_2, \theta \rangle$ if $\langle \tilde{f}_1, \theta \rangle \subset \langle \tilde{f}_2, \theta \rangle$ and $\langle \tilde{f}_2, \theta \rangle \subset \langle \tilde{f}_1, \theta \rangle$.

3. Vague Soft Intersection:

$\langle \tilde{f}_1, \theta \rangle \cap \langle \tilde{f}_2, \theta \rangle = \langle \tilde{f}_3, \theta \rangle$ and is defined as

$\langle \tilde{f}_3, \theta \rangle = \{(i, \{< x, \min\{M_{\tilde{f}_1(i)}(x), M_{\tilde{f}_2(i)}(x)\}, \max\{N_{\tilde{f}_1(i)}(x), N_{\tilde{f}_2(i)}(x)\} >\} : i \in \theta\}$.

4. Vague Soft Union:

$\langle \tilde{f}_1, \theta \rangle \cup \langle \tilde{f}_2, \theta \rangle = \{(i, \{< x, \max\{M_{\tilde{f}_1(i)}(x), M_{\tilde{f}_2(i)}(x)\}, \min\{N_{\tilde{f}_1(i)}(x), N_{\tilde{f}_2(i)}(x)\} >\} : i \in \theta\}$

More generally, the Vague Soft intersection and the Vague Soft union of a collection of $\{\langle f_i, \theta \rangle\} \subset VSS(\mathcal{X})$ are defined by

$$\bigcap_{i \in I} \left\langle \tilde{f}_i, \theta \right\rangle = \left\{ (i, \left\{ < x, \inf \left\{ M_{f_{i(i)}}(x) \right\}, \sup \left\{ N_{f_{i(i)}}(x) \right\} > \right\} : i \in \theta \right\}$$

$$\bigcup_{i \in I} \left\langle \tilde{f}_i, \theta \right\rangle = \left\{ (i, \left\{ < x, \sup \left\{ M_{f_{i(i)}}(x) \right\}, \inf \left\{ N_{f_{i(i)}}(x) \right\} > \right\} : i \in \theta \right\}$$

5. The VSS defined as $M_{\tilde{f}_1(i)}(x) = 1$ and $N_{\tilde{f}_1(i)}(x) = 0$, for all $i \in \theta$ and $x \in \mathcal{X}$ is called the universal VSS denoted by $1_{(\mathcal{X},\theta)}$. Also the vague set defined as $M_{\tilde{f}_1(i)}(x) = 0$ and $N_{\tilde{f}_1(i)}(x) = 1$ for all $i \in \theta$ and $x \in \mathcal{X}$ is called the empty VSS denoted by $0_{(\mathcal{X},\theta)}$.

6. Vague Soft Complement: $\langle \tilde{f}_1, \theta \rangle^c = 1_{(\mathcal{X},\theta)} \backslash \langle \tilde{f}_1, \theta \rangle = \{(i < e, N_{\tilde{f}_1(i)}(x), M_{\tilde{f}_1(i)}(x) > : i \in \theta\}$

Clearly, the complements of $1_{(\mathcal{X},\theta)}$ and $0_{(\mathcal{X},\theta)}$ are defined

$$\left(1_{(\mathcal{X},\theta)}\right)^c = 1_{(\mathcal{X},\theta)} \backslash 1_{(\mathcal{X},\theta)} = \{(i, \{< x, 1 >\} : i \in \theta\} = 0_{(\mathcal{X},\theta)}$$

$$\left(0_{(\mathcal{X},\theta)}\right)^c = 1_{(\mathcal{X},\theta)} \backslash 0_{(\mathcal{X},\theta)} = \{(i, \{< x, 0 >\} : i \in \theta\} = 1_{(\mathcal{X},\theta)}$$

Definition 2.4. Let $\tau \subset VSS(\mathcal{X})$ then τ is named a vague soft topology on $\mathcal{X}$ if the following conditions hold:

(1) $0_{(\mathcal{X},\theta)}, 1_{(\mathcal{X},\theta)} \in \tau$.

(2) τ is closed under union of VSSs.

(3) τ is closed under finite intersection of VSSs.

Then the order triple $\langle \mathcal{X}, \tau, \theta \rangle$ is called vague soft topology on $\mathcal{X}$.

3 Main Results

In this section, the notion of vague soft bi-topological space is leaked out. Examples are also reflected to understand the structures. Vague soft β separation axioms are inaugurated in vague soft bi-topological spaces concerning soft points of the space. Other β separation axioms are also addressed relative to soft points of the space in vague soft bi-topological spaces. An ample of examples are provided to secure the results.

Definition 3.1. If $\langle \mathcal{X}, \tau_1, \theta \rangle$ and $\langle \mathcal{X}, \tau_2, \theta \rangle$ are two VSTS, then $\langle \mathcal{X}, \tau_1, \tau_2, \theta \rangle$ is called VSBTS. If $\langle \mathcal{X}, \tau_1, \tau_2, \theta \rangle$ be VSBTS. A vague soft sub-set (f, θ) is said to be vague soft β open in $\langle \mathcal{X}, \tau_1, \tau_2, \theta \rangle$ if $(\mathrm{f}, \theta) \subseteq VScl(VSint(VScl(\mathrm{f}, \theta)))$. A vague soft sub-set (f, θ) is said to be vague soft β close in $\langle \mathcal{X}, \tau_1, \tau_2, \theta \rangle$ if $(\mathrm{f}, \theta) \supseteq VSint(VScl(VSint(\mathrm{f}, \theta)))$. A vague soft sub-set (f, θ) is said to be vague soft β open in $\langle \mathcal{X}, \tau_1, \tau_2, \theta \rangle$ pair-wisly if there exists a vague soft β open set (f_1, θ) in τ_1 and vague soft β open set (f_2, θ) in τ_2 such that $(\mathrm{f}, \theta) = (\mathrm{f}_1, \theta) \cup (\mathrm{f}_2, \theta)$.

Example 3.2. Let $\mathcal{X} = \{x_1, x_2, x_3\}$, $\theta = \{i_1, i_2\}$, $\tau_1 = \{0_{(\mathcal{X},\theta)}, 1_{(\mathcal{X},\theta)}, (\mathrm{f}_1, \theta), (\mathrm{f}_2, \theta)\}$,

$\tau_2 = \{0_{(\mathcal{X},\theta)}, 1_{(\mathcal{X},\theta)}, (\mathrm{g}_1, \theta), (\mathrm{g}_2, \theta)\}$, where $(\mathrm{f}_1, \theta), (\mathrm{f}_2, \theta), (\mathrm{g}_1, \theta)$ and (g_2, θ) being VSSs are as following:

$$\begin{bmatrix}
(\mathrm{f}_1,\theta)(i_1) = \left\{ < x_1, \frac{8}{10}, \frac{2}{10} >, < x_2, \frac{4}{10}, \frac{4}{10} >, < x_3, \frac{3}{10}, \frac{2}{10} > \right\} \\
(\mathrm{f}_1,\theta)(i_2) = \left\{ < x_1, \frac{6}{10}, \frac{3}{10} >, < x_2, \frac{5}{10}, \frac{1}{10} >, < x_3, \frac{5}{10}, \frac{4}{10} > \right\} \\
(\mathrm{f}_2,\theta)(i_1) = \left\{ < x_1, \frac{6}{10}, \frac{4}{10} >, < x_2, \frac{3}{10}, \frac{4}{10} >, < x_3, \frac{2}{10}, \frac{3}{10} > \right\} \\
(\mathrm{f}_2,\theta)(i_2) = \left\{ < x_1, \frac{5}{10}, \frac{3}{10} >, < x_2, \frac{4}{10}, \frac{2}{10} >, < x_3, \frac{3}{10}, \frac{4}{10} > \right\} \\
(\mathrm{g}_1,\theta)(i_1) = \left\{ < x_1, \frac{4}{10}, \frac{5}{10} >, < x_2, \frac{2}{10}, \frac{6}{10} >, < x_3, \frac{1}{10}, \frac{4}{10} > \right\} \\
(\mathrm{g}_1,\theta)(i_2) = \left\{ < x_1, \frac{3}{10}, \frac{4}{10} >, < x_2, \frac{3}{10}, \frac{3}{10} >, < x_3, \frac{1}{10}, \frac{5}{10} > \right\} \\
(\mathrm{g}_2,\theta)(i_1) = \left\{ < x_1, \frac{7}{10}, \frac{1}{10} >, < x_2, \frac{3}{10}, \frac{3}{10} >, < x_3, \frac{2}{10}, \frac{1}{10} > \right\} \\
(\mathrm{g}_2,\theta)(i_2) = \left\{ < x_1, \frac{7}{10}, \frac{1}{10} >, < x_2, \frac{3}{10}, \frac{3}{10} >, < x_3, \frac{2}{10}, \frac{1}{10} > \right\}
\end{bmatrix}$$

Then $(\mathrm{f}_1, \theta) \cap (\mathrm{f}_2, \theta) = ((\mathrm{f}_2, \theta), (\mathrm{f}_1, \theta) \cap (\mathrm{g}_1, \theta) = (\mathrm{g}_1, \theta), (\mathrm{f}_1, \theta) \cap (\mathrm{g}_2, \theta) = (\mathrm{f}_2, \theta), (\mathrm{g}_1, \theta) \cap (\mathrm{g}_2, \theta) = (\mathrm{g}_1, \theta), (\mathrm{f}_2, \theta) \cap (\mathrm{g}_2, \theta) = (\mathrm{f}_2, \theta)$ and $(\mathrm{f}_1, \theta) \cup (\mathrm{f}_2, \theta) = (\mathrm{f}_2, \theta), (\mathrm{f}_1, \theta) \cup (\mathrm{g}_1, \theta) = (\mathrm{g}_1, \theta), (\mathrm{f}_1, \theta) \cup (\mathrm{g}_2, \theta) = (\mathrm{f}_2, \theta), (\mathrm{g}_1, \theta) \cup (\mathrm{g}_2, \theta) = (\mathrm{g}_1, \theta), (\mathrm{f}_2, \theta) \cup (\mathrm{g}_2, \theta) = (\mathrm{g}_2, \theta)$.

Therefore τ_1 and τ_2 are vague soft topologies on $\mathcal{X}$ and so $\langle \mathcal{X}, \tau_1, \tau_2, \theta \rangle$ is a vague soft bitopological space.

Remark 3.3. Let $\langle \mathcal{X}, \tau_1, \tau_2, \theta \rangle$ be a vague soft bitopological space then $\tau_1 \cup \tau_2$ need not necessarily be a vague soft topological space on $\mathcal{X}$.

Example 3.4. Let $\mathcal{X} = \{x_1, x_2, x_3\}$, $\theta = \{i_1, i_2\}$, $\tau_1 = \{0_{(\mathcal{X},\theta)}, 1_{(\mathcal{X},\theta)}, (f_1,\theta), (f_2,\theta)\}$, $\tau_2 = \{0_{(\mathcal{X},\theta)}, 1_{(\mathcal{X},\theta)}, (g_1,\theta), (g_2,\theta)\}$, where $(f_1,\theta), (f_2,\theta), (g_1,\theta)$ and (g_2,θ) being VSSs are as following:

$$\begin{bmatrix} (f_1,\theta)(i_1) = \left\{ < x_1, \frac{8}{10}, \frac{2}{10} >, < x_2, \frac{4}{10}, \frac{4}{10} >, < x_3, \frac{3}{10}, \frac{2}{10} > \right\} \\ (f_1,\theta)(i_2) = \left\{ < x_1, \frac{6}{10}, \frac{3}{10} >, < x_2, \frac{5}{10}, \frac{1}{10} >, < x_3, \frac{5}{10}, \frac{4}{10} > \right\} \\ (f_2,\theta)(i_1) = \left\{ < x_1, \frac{6}{10}, \frac{4}{10} >, < x_2, \frac{3}{10}, \frac{4}{10} >, < x_3, \frac{2}{10}, \frac{3}{10} > \right\} \\ (f_2,\theta)(i_2) = \left\{ < x_1, \frac{5}{10}, \frac{3}{10} >, < x_2, \frac{4}{10}, \frac{2}{10} >, < x_3, \frac{3}{10}, \frac{4}{10} > \right\} \\ (g_1,\theta)(i_1) = \left\{ < x_1, \frac{4}{10}, \frac{5}{10} >, < x_2, \frac{2}{10}, \frac{6}{10} >, < x_3, \frac{1}{10}, \frac{4}{10} > \right\} \\ (g_1,\theta)(i_2) = \left\{ < x_1, \frac{3}{10}, \frac{4}{10} >, < x_2, \frac{3}{10}, \frac{3}{10} >, < x_3, \frac{1}{10}, \frac{5}{10} > \right\} \\ (g_2,\theta)(i_1) = \left\{ < x_1, \frac{7}{10}, \frac{1}{10} >, < x_2, \frac{3}{10}, \frac{3}{10} >, < x_3, \frac{2}{10}, \frac{1}{10} > \right\} \\ (g_2,\theta)(i_2) = \left\{ < x_1, \frac{7}{10}, \frac{1}{10} >, < x_2, \frac{3}{10}, \frac{3}{10} >, < x_3, \frac{2}{10}, \frac{1}{10} > \right\}. \end{bmatrix}$$

Here $\tau_1 \cup \tau_2 = \{0_{(\mathcal{X},\theta)}, 1_{(\mathcal{X},\theta)}, (f_1,\theta), (f_2,\theta), (f_3,\theta), (g_1,\theta), (g_2,\theta)\}$ is not a vague soft topology on $\mathcal{X}$.

Definition 3.5. Let $\langle \mathcal{X}, \tau_1, \tau_2, \theta \rangle$ be a vague soft bitopological space. Then a VSS

$\langle f, \theta \rangle = \left\{ (i, \left\{ < x, M_{f(i)}(x), N_{f(i)}(x) > \right\} : x \in \mathcal{X}, i \in \theta \right\}$

is called as a pairwise vague soft β open set if there exist a vague soft open $\langle f_1, \theta \rangle$ in τ_1 and a vague soft β open $\langle f_2, \theta \rangle$ in τ_2 such that for all $x \in \mathcal{X}$.

$\langle f, \theta \rangle = \langle f_1, \theta \rangle \cup \langle f_2, \theta \rangle = \left\{ \left(i, \left\{ < x, \max \left\{ M_{f_1 i}(x), M_{f_2(i)}(x) \right\}, \min \left\{ N_{f_1(i)}(x \right\}, N_{f_2(i)}(x) \right\} > \right\} \right) : i \in \theta \right\}$

Definition 3.6. Let $\langle \mathcal{X}, \tau_1, \tau_2, \theta \rangle$ be a vague soft bitopological space. Then a VSS

$\langle f, \theta \rangle = \left\{ \left(i, \left\{ < x, M_{f(i)}(x), N_{f(i)}(x) > \right\} \right) : x \in \mathcal{X}, i \in \theta \right\}$

is called as a pairwise vague soft β open set if there exist a vague soft open set $\langle f_1, \theta \rangle$ in τ_1 and a vague soft β open set $\langle f_2, \theta \rangle$ in τ_2 such that for all $x \in \mathcal{X}$.

$\langle f, \theta \rangle = \langle f_1, \theta \rangle \cup \langle f_2, \theta \rangle = \left\{ \left(i, \left\{ < x, \max \left\{ M_{f_1(i)}(x), M_{f_2(i)}(x) \right\}, \min \left\{ N_{f_1(i)}(x), N_{f_2(i)}(x) \right\} > \right\} \right) : i \in \theta \right\}$

The set of all pairwise vague open sets in $\langle \mathcal{X}, \tau_1, \tau_2, \theta \rangle$ is denoted by PVSO $\langle \mathcal{X}, \tau_1, \tau_2, \theta \rangle$.

Definition 3.7. Let $\langle \mathcal{X}, \tau_1, \tau_2, \theta \rangle$ be a vague soft bitopological space. Then a NSS

$\langle f, \theta \rangle = \left\{ \left(i, \left\{ < x, M_{f(i)}(\mathrm{x}), N_{f(i)}(x) > \right\} \right) : x \in \mathcal{X}, i \in \theta \right\}$

is called as a pairwise vague soft β closed set if $\langle f, \theta \rangle^c$ is a pairwise vague soft β open set. It is clear that $\langle f, \theta \rangle$ is a pairwise neutrosophic soft β closed set if there exist a vague soft β closed set $\langle f_1, \theta \rangle$ in τ_1 and a vague soft β closed set $\langle f_2, \theta \rangle$ in τ_2 such that for all $x \in \mathcal{X}$.

$\langle f, \theta \rangle = \langle f_1, \theta \rangle \cap \langle f_2, \theta \rangle = \left\{ \left(i, \left\{ < x, \min \left\{ M_{f_1(i)}(x), M_{f_2(i)}(x) \right\}, \max \left\{ N_{f_1(i)}(x), N_{f_2(i)}(x) \right\} > \right\} \right) : i \in \theta \right\}$

The set of all pairwise vague β closed sets in $\langle \mathcal{X}, \tau_1, \tau_2, \theta \rangle$ is denoted by PVS β $C\langle \mathcal{X}, \tau_1, \tau_2, \theta \rangle$.

Example 3.8. Let $\mathcal{X} = \{x_1, x_2, x_3\}$, $\theta = \{i_1, i_2\}$, $\tau_1 = \{0_{(\mathcal{X},\theta)}, 1_{(\mathcal{X},\theta)}, (f_1, \theta)\}$, $\tau_2 = \{0_{(\mathcal{X},\theta)}, 1_{(\mathcal{X},\theta)}, (f_2, \theta)\}$ where (f_1, θ) and (f_2, θ) are defined as

$$\left[\begin{array}{l} (f_1,\theta)(i_1) = \left\{ < x_1, \frac{8}{10}, \frac{2}{10} >, < x_2, \frac{4}{10}, \frac{4}{10} >, < x_3, \frac{3}{10}, \frac{2}{10} > \right\} \\ (f_1,\theta)(i_2) = \left\{ < x_1, \frac{6}{10}, \frac{3}{10} >, < x_2, \frac{5}{10}, \frac{1}{10} >, < x_3, \frac{5}{10}, \frac{4}{10} > \right\} \\ (f_2,\theta)(i_1) = \left\{ < x_1, \frac{6}{10}, \frac{4}{10} >, < x_2, \frac{5}{10}, \frac{1}{10} >, < x_3, \frac{5}{10}, \frac{4}{10} > \right\} \\ (f_2,\theta)(i_2) = \left\{ < x_1, \frac{6}{10}, \frac{4}{10} >, < x_2, \frac{3}{10}, \frac{4}{10} >, < x_3, \frac{2}{10}, \frac{3}{10} > \right\}. \\ (f_1,\theta) \cup (f_2,\theta) = \left\{ \left(i_1, \left\{ \left\langle x_1, \frac{8}{10}, \frac{2}{10} \right\rangle, \left\langle x_2, \frac{5}{10}, \frac{1}{10} \right\rangle, \left\langle x_3, \frac{5}{10}, \frac{2}{10} \right\rangle \right\} \right), \right. \\ \qquad \left. \left(i_2, \left\{ \left\langle x_1, \frac{6}{10}, \frac{3}{10} \right\rangle, \left\langle x_2, \frac{5}{10}, \frac{1}{10} \right\rangle, \left\langle x_3, \frac{5}{10}, \frac{3}{10} \right\rangle \right\} \right) \right\} \end{array}\right]$$

is a pairwise vague soft β open set. Also

$$\left[\begin{array}{l} (f_1,\theta)^c(i_1) = \left\{ < x_1, \frac{2}{10}, \frac{8}{10} >, < x_2, \frac{6}{10}, \frac{6}{10} >, < x_3, \frac{2}{10}, \frac{3}{10} > \right\} \\ (f_1,\theta)^c(i_2) = \left\{ < x_1, \frac{3}{10}, \frac{6}{10} >, < x_2, \frac{1}{10}, \frac{5}{10} >, < x_3, \frac{4}{10}, \frac{5}{10} > \right\} \\ (f_2,\theta)^c(i_2) = \left\{ < x_1, \frac{3}{10}, \frac{6}{10} >, < x_2, \frac{1}{10}, \frac{5}{10} >, < x_3, \frac{4}{10}, \frac{3}{10} > \right\} \\ (f_2,\theta)^c(i_2) = \left\{ < x_1, \frac{4}{10}, \frac{6}{10} >, < x_2, \frac{4}{10}, \frac{3}{10} >, < x_3, \frac{3}{10}, \frac{2}{10} > \right\}. \\ \Longrightarrow (f_1,\theta)^c \cap (f_2,\theta)^c = \left\{ \left(i_1, \left\{ \left\langle x_1, \frac{2}{10}, \frac{8}{10} \right\rangle, \left\langle x_2, \frac{6}{10}, \frac{6}{10} \right\rangle, \left\langle x_3, \frac{7}{10}, \frac{8}{10} \right\rangle \right\} \right), \right. \\ \qquad \left. \left(i_2, \left\{ \left\langle x_1, \frac{4}{10}, \frac{7}{10} \right\rangle, \left\langle x_2, \frac{5}{10}, \frac{9}{10} \right\rangle, \left\langle x_3, \frac{1}{10}, \frac{6}{10} \right\rangle \right\} \right) \right\} \end{array}\right]$$

is a pairwise vague soft β close set.

Theorem 3.9. Let $\langle \mathcal{X}, \tau_1, \tau_2, \theta \rangle$ be a vague soft bitopological space. In this case

(1) $0_{(\mathcal{X},\theta)}, 1_{(\mathcal{X},\theta)} \in \mathrm{PVSO}\langle \mathcal{X}, \tau_1, \tau_2, \theta \rangle$.
(2) If $\{(f_i, \theta) \mid i \in \Delta\} \subseteq \mathrm{PVSO}\langle \mathcal{X}, \tau_1, \tau_2, \theta \rangle$ then $\bigcup_{i \in I} (f_i, \theta) \in \mathrm{PVSO}\langle \mathcal{X}, \tau_1, \tau_2, \theta \rangle$.
(3) If $\{(g_i, \theta) \mid i \in \Delta\} \subseteq \mathrm{PVSC}\langle \mathcal{X}, \tau_1, \tau_2, \theta \rangle$ then $\bigcap_{i \in I} (g_i, \theta) \in \mathrm{PVSC}\langle \mathcal{X}, \tau_1, \tau_2, \theta \rangle$.

Proof.

1. Since $0_{(\mathcal{X},\theta)} \cup 0_{(\mathcal{X},\theta)} = 0_{(\mathcal{X},\theta)}$ and $1_{(\mathcal{X},\theta)} \cup 1_{(\mathcal{X},\theta)} = 1_{(\mathcal{X},\theta)}$ then $0_{(\mathcal{X},\theta)}$ and $1_{(\mathcal{X},\theta)}$ are pairwise vague soft β closed sets.

2. Since $(f_i,\theta) \in PVSO\langle \mathcal{X},\tau_1,\tau_2,\theta\rangle$, there exist $(f_i^1,\theta) \in \tau_1$ and $(f_i^1,\theta) \in \tau_2$ such that $(f_i^1,\theta) = (f_i^1,\theta) \cup (f_i^1,\theta)$ for all $i \in I$. Then

$$\bigcup_{i\in I}(f_i,\theta) = \bigcup_{i\in I}\left((f_i^1,\theta)\cup(f_i^1,\theta)\right) = \left(\bigcup_{i\in I}(f_i^1,\theta)\right)\cup\left(\bigcup_{i\in I}(f_i^2,\theta)\right).$$

As τ_1 and τ_2 are vague soft topologies on X, $\bigcup_{i\in I}(f_i^1,\theta)) \in \tau_1$, $\bigcup_{i\in I}(f_i^2,\theta) \in \tau_1$.

Therefore $\bigcup_{i\in I}(f_i^1,\theta) \in PVSO(\langle \mathcal{X},\tau_1,\tau_2,\theta\rangle$.

3. Since $(g_i,\theta) \in PVSC\langle \mathcal{X},\tau_1,\tau_2,\theta\rangle$, there exist $(g_i^1,\theta)^c \in \tau_1$ and $(g_i^2,\theta)^c \in \tau_2$ such that $(g_i,\theta) = (g_i^1,\theta)\cap(g_i^1,\theta)$ for all $i \in \Delta$. Then

$$\bigcap_{i\in I}(g_i,\theta) = \bigcap_{i\in I}\left((g_i^1,\theta)\cap(g_i^2,\theta)\right) = \left(\bigcap_{i\in I}(g_i^1,\theta)\right)\cap\left(\bigcap_{i\in I}(g_i^2,\theta)\right).$$

Then $\bigcap_{i\in I}(g_i,\theta) \in PVSC\langle \mathcal{X},\tau_1,\tau_2,\theta\rangle$ as $\left(\bigcap_{i\in I}(g_i^1,\theta)\right)^c \in \tau_1$ and $\left(\bigcap_{i\in I}(g_i^2,\theta)\right)^c \in \tau_1$.

Definition 3.10. Let $\langle \mathcal{X},\tau_1,\tau_2,\theta\rangle$ be a vague soft bitopological space and $(f,\theta) \in VSS(\mathcal{X})$. The pairwise vague soft closure of (f,θ), denoted by $cl_P^{VSS}(f,\theta)$, is the intersection of all pairwise vague soft β closed sets containing (f,θ), i.e.,

$$cl_P^{VSS}(f,\theta) = \cap\{(g,\theta) \in PVSC(X) \mid (f,\theta) \subset (g,\theta)\}$$

It is clear that $cl_P^{VSS}(f,\theta)$ is the smallest pairwise vague soft β closed set containing (f,θ).

Example 3.11. Let $\langle \mathcal{X},\tau_1,\tau_2,\theta\rangle$ be the same as in Example 3.4 and

$$\begin{bmatrix} (g,\theta) = \left\{\left(i_1,\left\{<x_1,\frac{3}{10},\frac{7}{10}>,<x_2,\frac{3}{10},\frac{4}{10}>,<x_3,\frac{2}{10},\frac{4}{10}>\right\}\right)\right. \\ \left.\left(i_2,\left\{<x_1,\frac{2}{10},\frac{6}{10}>,<x_2,\frac{1}{10},\frac{7}{10}>,<x_3,\frac{3}{10},\frac{4}{10}\right\rangle\right\}\right)\right\}. \end{bmatrix}$$

be a vague soft set over $\mathcal{X}$. Now, we need to determine pairwise vague soft β closed sets in $\langle \mathcal{X},\tau_1,\tau_2,\theta\rangle$ to find $cl_P^{VSS}(g,\theta)$ then,

$$\begin{bmatrix} (f_2,\theta)(i_1) = \left\{<x_1,\frac{6}{10},\frac{4}{10}>,<x_2,\frac{3}{10},\frac{4}{10}>,<x_3,\frac{2}{10},\frac{3}{10}>\right\} \\ (f_2,\theta)(i_2) = \left\{<x_1,\frac{5}{10},\frac{3}{10}>,<x_2,\frac{4}{10},\frac{2}{10}>,<x_3,\frac{3}{10},\frac{4}{10}>\right\}, \\ (f_2,\theta)^c = \left\{\left(i_1,\left\{<x_1,\frac{4}{10},\frac{6}{10}>,<x_2,\frac{4}{10},\frac{3}{10}>,<x_3,\frac{3}{10},\frac{2}{10}>\right\}\right),\right. \\ \left.\left(i_2,\left\{<x_1,\frac{3}{10},\frac{5}{10}>,<x_2,\frac{2}{10},\frac{6}{10}>,<x_3,\frac{4}{10},\frac{3}{10}>\right\}\right)\right\}. \end{bmatrix}$$

The pairwise vague soft β closed sets which contains (g,θ) are $(f_2,\theta)^c$, $1_{(\mathcal{X},\theta)}$. Therefore $cl_P^{VSS}(g,\theta) = (f_2,\theta)^c \cap 1_{(\mathcal{X},\theta)} = (f_2,\theta)^c$.

Definition 3.12. Let $\langle\langle\tilde{\mathcal{X}}\rangle,\tau_1,\tau_2,\theta\rangle$ be vague soft bitopological space, $x^{i}{}_{(i,j)} \neq y^{i'}{}_{(i',j')}$ are VS points. If there exist $VS\beta$ open sets $(\tilde{f},\theta) \in \tau_1 \cup \tau_2$ and $(\tilde{g},\theta) \in \tau_1 \cup \tau_2$ such that $x^{i}{}_{(i,j)} \in (\tilde{f},\theta), x^{i}{}_{(i,j)} \sqcap (\tilde{g},\theta) = 0_{(\langle\tilde{\mathcal{X}}\rangle,\theta)}$ *or* $y^{i'}{}_{(i',j')} \in (\tilde{g},\theta),\ y^{i'}{}_{(i',j')} \sqcap (\tilde{f},\theta) = 0_{(\langle\tilde{\mathcal{X}}\rangle,\theta)}$, Then $\langle\langle\tilde{\mathcal{X}}\rangle,\tau_1,\tau_2,\theta\rangle$ is called a $VS\beta_0$.

Definition 3.13. Let $\langle\langle\tilde{\mathcal{X}}\rangle,\tau_1,\tau_2,\theta\rangle$ be a $VSTS$ over $\langle\tilde{\mathcal{X}}\rangle$, $x^{i}{}_{(i,j)} \neq y^{i'}{}_{(i',j')}$ are VS points. If there exists $VS\beta$ open sets $(\tilde{f},\theta) \in \tau_1 \cup \tau_2$ & $(\tilde{g},\theta) \in \tau_1 \cup \tau_2\, s.t. x^{i}{}_{(i,j)} \in (\tilde{f},\theta)$, $x^{i}{}_{(i,j)} \sqcap (\tilde{g},\theta) = 0_{(\langle\tilde{\mathcal{X}}\rangle,\theta)}$ and $y^{i'}{}_{(i',j')} \in (\tilde{g},\theta), y^{i'}{}_{(i',j')} \sqcap (\tilde{f},\theta) = 0_{(\langle\tilde{\mathcal{X}}\rangle,\theta)}$, Then $\langle\langle\tilde{\mathcal{X}}\rangle,\tau_1,\tau_2,\theta\rangle$ is called a $VS\beta_1$.

Definition 3.14. Let $\langle\langle\tilde{\mathcal{X}}\rangle,\tau_1,\tau_2,\theta\rangle$ be a $VSTS$ over $\langle\tilde{\mathcal{X}}\rangle$, $x^{i}{}_{(i,j)} \neq y^{i'}{}_{(i',j')}$ are VS points. If $\exists VS\beta$ open sets $(\tilde{f},\theta) \in \tau_1 \cup \tau_2$ and $(\tilde{g},\theta) \in \tau_1 \cup \tau_2$ such that $x^{i}{}_{(a,c)} \in (\tilde{f},\theta)$ & $y^{i'}{}_{(i',j')} \in (\tilde{g},\theta)$ & $(\tilde{f},\theta) \sqcap (\tilde{g},\theta) = 0_{(\langle\tilde{\mathcal{X}}\rangle,\theta)}$, Then $\langle\langle\tilde{\mathcal{X}}\rangle,\tau_1,\tau_2,\theta\rangle$ is called a $VS\beta_2$.

Theorem 3.15. Let $(\langle\tilde{\mathcal{X}}\rangle,\ \tau_1,\tau_2,\ \theta)$ be a $VSBTS$. Then $(\langle\tilde{\mathcal{X}}\rangle,\ \tau_1,\tau_2,\ \theta)$ be a $VS\beta_1$ structure if and only if each VS point is a $VS\beta$-close.

Proof. Let $(\langle\tilde{\mathcal{X}}\rangle,\ \tau_1,\tau_2,\ \theta)$ be a $VSBTS$ over $\langle\widetilde{\mathcal{X}}\rangle.(x^{i}{}_{(i,j)},\theta)$ be an arbitrary VS point. We establish $(x^{i}{}_{(i,j)},\theta)$ is a soft β-open set. Let $(y^{i'}{}_{(i',j')},\theta) \neq (x^{i}{}_{(i,j)},\theta)$. This means that $(y^{i'}{}_{(i',j')},\theta)$ & $(x^{i}{}_{(i,j)},\theta)$ are two are distinct VS points. Since $(\langle\tilde{\mathcal{X}}\rangle,\ \tau_1,\tau_2,\ \theta)$ be a $VS\beta_1$ structure, there exists a $VS\beta$-open set $(\tilde{g},\theta)$ so that $(y^{i'}{}_{(i',j')},\theta) \in (\tilde{g},\theta)$ *and* $(x^{i}{}_{(i,j)},\theta) \sqcap (\tilde{g},\theta) = 0_{(\langle\tilde{\mathcal{X}}\rangle,\theta)}$. Since, $(x^{i}{}_{(i,j)},\theta) \sqcap (\tilde{g},\theta) = 0_{(\langle\tilde{\mathcal{X}}\rangle,\theta).}$ So $(y^{i'}{}_{(i',j')},\theta) \in (\tilde{g},\theta) \subset (x^{i}{}_{(i,j)},\theta)$. Thus $(x^{i}{}_{(i,j)},\theta)$ is a $NS\beta$-open set, i.e., $(x^{i}{}_{(i,j)},\theta)$ is a $VS\beta$-close set. Suppose that each VS point $(x^{i}{}_{(i,j)},\theta)$ is a $VS\beta$-close. Then $(x^{i}{}_{(i,j)},\theta)^c$ is a $VS\beta$-open set. Let $(x^{i}{}_{(i,j)},\theta) \sqcap (y^{i'}{}_{(i',j')},\theta) = 0_{(\langle\tilde{\mathcal{X}}\rangle,\theta).}$ Thus $(y^{i'}{}_{(i',j')},\theta) \in (x^{i}{}_{(i,j)},\theta)^c$ and $(x^{e}{}_{(i,j)},\theta) \sqcap (x^{i}{}_{(i,j)},\theta)^c = 0_{(\langle\tilde{\mathcal{X}}\rangle,\theta).}$ So $(\langle\tilde{\mathcal{X}}\rangle,\ \tau_1,\tau_2,\ \theta)$ be a VS-β_1 space.

Theorem 3.16. Let $(\langle\tilde{\mathcal{X}}\rangle,\ \tau_1,\tau_2,\ \theta)$ be a $VSBTS$ over universal $set\langle\widetilde{\mathcal{X}}\rangle$. Then $(\langle\tilde{\mathcal{X}}\rangle,\ \tau_1,\tau_2,\ \theta)$ is VS-β_2 space *iff* for distinct VS points $(x^{i}{}_{(i,j)},\theta)$ & $(y^{i'}{}_{(i',j')},\theta)$, there exists a $VS\beta$-open set $(\tilde{f},\theta)$ containing there exists but not $(y^{i'}{}_{(i',j')},\theta)$ such that $(y^{i'}{}_{(i',j')},\theta) \notin \overline{(\tilde{f},\theta).}$

Proof. Let $(x^{i}{}_{(i,j)},\theta) \neq (y^{i'}{}_{(i',j')},\theta)$ be two VS points in $VS\beta_2$ space. Then there exists disjoint $VS\beta$ open sets $(\tilde{f},\theta)$ and $(\tilde{g},\theta)$ such that $(x^{i}{}_{(i,j)},\theta) \in (\tilde{f},\theta)$ and $(y^{i'}{}_{(i',j')},\theta) \in (\tilde{g},\theta)$. *Since* $(x^{i}{}_{(i,j)},\theta) \sqcap (y^{i'}{}_{(i',j')},\theta) = 0_{(\langle\tilde{\mathcal{X}}\rangle,\theta)}$ and $(\tilde{f},\theta) \sqcap (\tilde{g},\theta) = 0_{(\langle\tilde{\mathcal{X}}\rangle,\theta)}.(y^{i'}{}_{(i',j')},\theta) \notin (\tilde{f},\theta) \Longrightarrow (y^{i'}{}_{(i',j')},\theta) \notin \overline{(\tilde{f},\theta).}$ Next suppose that, $(x^{i}{}_{(i,j)},\theta) \succ (y^{i'}{}_{(i',j')},\theta)$, there exists a $VS\beta$ open set $(\tilde{f},\theta)$ containing $(x^{i}{}_{(i,j)},\theta)$ but not $(y^{i'}{}_{(i',j')},\theta)$ such that $(y^{i'}{}_{(i',j')},\theta) \notin \overline{(\tilde{f},\theta)}^c$ that is $(\tilde{f},\theta)$ and $\overline{(\tilde{f},\theta)}^c$ are mutually exclusive $VS\beta$ open sets supposing $(x^{i}{}_{(i,j)},\theta)$ and $(y^{i}{}_{(i',j')},\theta)$, respectively.

Theorem 3.17. Let $(\langle\tilde{\mathcal{X}}\rangle,\ \tau_1,\tau_2,\ \theta)$ be a $VSBTS$. Then $(\langle\tilde{\mathcal{X}}\rangle,\ \tau_1,\tau_2,\ \theta)$ is $VS\beta_1$ space if every VS point $(x^{i}{}_{(i,j)},\theta) \in (\tilde{f},\theta) \in (\langle\tilde{\mathcal{X}}\rangle,\tau,\theta)$. If there exists a $VS\beta$ open set $(\tilde{g},\theta)$ such that $(x^{e}{}_{(i,j)},\theta) \in (\tilde{g},\theta) \subset \overline{(\tilde{g},\theta)} \subset (\tilde{f},\theta)$, Then $(\langle\tilde{\mathcal{X}}\rangle,\ \tau,\ \theta)$ a $VS\beta_2$ space.

Proof. Suppose $(x^{i}{}_{(i,j)}, \theta) \sqcap (y^{i/}{}_{(i/,j/)}, \theta) = 0_{(\langle\tilde{\mathcal{X}}\rangle,\theta)}$. Since $(\langle\tilde{\mathcal{X}}\rangle, \tau_1, \tau_2, \theta)$ is $NS\beta_1$ space. $(x^{i}{}_{(i,j)}, \theta)$ & $(y^{i/}{}_{(i/,j/)}, \theta)$ *are* $VS\beta$ close sets in $(\langle\tilde{\mathcal{X}}\rangle, \tau, \theta)$. Then $(x^{i}{}_{(i,j)}, \theta) \in ((y^{i/}{}_{(i/,j/)}, \theta))^c \in (\langle\tilde{\mathcal{X}}\rangle, \tau, \theta)$. Thus there exists a $VS\beta$ open set $(\tilde{g}, \theta) \in (\langle\tilde{\mathcal{X}}\rangle, \tau, \theta)$ such that $(x^{i}{}_{(i,j)}, \theta) \in (\tilde{g}, \theta) \subset \overline{(\tilde{g}, \theta)} \subset ((y^{e/}{}_{(i/,j/)}, \theta))^c$. So $(y^{i/}{}_{(i/,j/)}, \theta) \in (\tilde{g}, \theta)$ and $(\tilde{g}, \theta) \sqcap ((\tilde{g}, \theta))^c = 0_{(\langle\tilde{\mathcal{X}}\rangle,\theta)}$ that is $(\langle\tilde{\mathcal{X}}\rangle, \tau_1, \tau_2, \theta)$ is a VS soft β_2 *space*.

Definition 3.18. Let $(\langle\tilde{\mathcal{X}}\rangle, \tau_1, \tau_2, \theta)$ be a $VSBTS$. $(\tilde{f}, \theta)$ be a $VS\beta$ closed set and $(x^{i}{}_{(i,j)}, \theta) \sqcap (\tilde{f}, \theta) = 0_{(\langle\tilde{\mathcal{X}}\rangle,\theta)}$. If there exists $VS\beta$-open sets $(\tilde{g}_1, \theta)$ & $(\tilde{g}_2, \theta)$ such that $(x^{i}{}_{(i,j)}, \theta) \in (\tilde{g}_1, \theta), (\tilde{f}, \theta) \subset (\tilde{g}_2, \theta)$ and $(x^{e}{}_{(i,j)}, \theta) \sqcap (\tilde{g}_1, \theta) = 0_{(\langle\tilde{\mathcal{X}}\rangle,\theta)}$, then $(\langle\tilde{\mathcal{X}}\rangle, \tau_1, \tau_2, \theta)$ is called a $VS\beta$-regular space. $(\langle\tilde{\mathcal{X}}\rangle, \tau_1, \tau_2, \theta)$ is said to be $VS\beta_3$ *space*, if is both a VS regular and $VS\beta_1$*space*.

Theorem 3.19. Let $(\langle\tilde{\mathcal{X}}\rangle, \tau_1, \tau_2, \theta)$ be a $VSBTS$. $(\langle\tilde{\mathcal{X}}\rangle, \tau_1, \tau_2, \theta)$ is VS β_3 *space iff* for every $(x^{i}{}_{(i,j)}, \theta) \in (\tilde{f}, \theta)$ that is $(\tilde{g}, \theta) \in ((\langle\tilde{\mathcal{X}}\rangle, \tau_1, \tau_2, \theta)$ such that $(x^{i}{}_{(i,j)}, \theta) \in (\tilde{g}, \theta) \subset \overline{(\tilde{g}, \theta)} \subset (\tilde{f}, \theta)$.

Proof. Let $(\langle\tilde{\mathcal{X}}\rangle, \tau_1, \tau_2, \theta)$ is $VS\beta_3$ space and $(x^{i}{}_{(i,j)}, \theta) \in (\tilde{f}, \theta) \in (\langle\tilde{\mathcal{X}}\rangle, \tau, \theta)$. Since $(\langle\tilde{\mathcal{X}}\rangle, \tau_1, \tau_2, \theta)$ is $VS\beta_3$ space for the VS point $(x^{i}{}_{(i,j)}, \theta)$ and β closed set $(\tilde{f}, \theta)^c$, there exists $(\tilde{g}_1, \theta)$ and $(\tilde{g}_2, \theta)$ such that $(x^{i}{}_{(i,j)}, \theta) \in (\tilde{g}_1, \theta), (\tilde{f}, \theta)^c \subset (\tilde{g}_2, \theta)$ & $(\tilde{g}_1, \theta) \cap (\tilde{g}_2, \theta) = 0_{(\langle\tilde{\mathcal{X}}\rangle,\theta)}$. Then we have $(x^{i}{}_{(i,j)}, \theta) \in (\tilde{g}_1, \theta) \subset (\tilde{g}_2, \theta)^c \subset (\tilde{f}, \theta)$, since $(\tilde{g}_2, \theta)^c VS\beta$ close set $\overline{(\tilde{g}_1, \theta)} \subset (\tilde{g}_2, \theta)^c$. Conversely, let $(x^{i}{}_{(i,j)}, \theta) \cap (\tilde{h}, \theta) = 0_{(\langle\tilde{\mathcal{X}}\rangle,\theta)}$ and $(\tilde{h}, \theta)$ be a $VS\beta$ close set. $(x^{i}{}_{(i,j)}, \theta) \propto (\tilde{h}, \theta)^c$ and from the condition of the theorem, we have $(x^{i}{}_{(i,j)}, \theta) \in (\tilde{g}, \theta) \subset \overline{(\tilde{g}, \theta)} \subset (\tilde{h}, \theta)^c$. Thus $(x^{i}{}_{(i,j)}, \theta) \in (\tilde{g}, \theta), (\tilde{h}, \theta) \subset$ *and* $(\tilde{g}, \theta) \cap \overline{(\tilde{g}, \theta)}^c = 0_{(\langle\tilde{\mathcal{X}}\rangle,\theta)}$. So $(\langle\tilde{\mathcal{X}}\rangle, \tau, \theta)$ is $VS\beta_3$ space.

Definition 3.20. Let $(\langle\tilde{\mathcal{X}}\rangle, \tau_1, \tau_2, \theta)$ be a $VSBTS$. This space is a $VS\beta$ normal space, if for every pair of disjoint $VS\beta$ close sets $(\tilde{f}_1, \theta)$ and $(\tilde{f}_2, \theta)$, there exists disjoint $VS\beta$ open sets $(\tilde{g}_1, \theta)$ and $(\tilde{g}_2, \theta)$ such that $(\tilde{f}_1, \theta) \subset (\tilde{g}_1, \theta)$ and $(\tilde{f}_2, \theta) \subset (\tilde{g}_2, \theta)$.

$(\langle\tilde{\mathcal{X}}\rangle, \tau, \theta)$ is said to be a $VS\beta_4$ *space* if it is both a $VS\beta$ normal and $VS\beta_1$ space.

Theorem 3.21. Let $(\langle\tilde{\mathcal{X}}\rangle, \tau_1, \tau_2, \theta)$ be a $VSBTS$ over universal set $\langle\widetilde{\mathcal{X}}\rangle$. This space is a $VS\beta_4$ space if and only if for each $VS\beta$ closed set $(\tilde{f}, \theta)$ and $VS\beta$ open set $(\tilde{g}, \theta)$ with $(\tilde{f}, \theta) \subset (\tilde{g}, \theta)$, there exists a $VS\beta$ open set $(\tilde{\pi}, \theta)$ such that $(\tilde{f}, \theta) \subset (\tilde{\pi}, \theta) \subset \overline{(\tilde{\pi}, \theta)} \subset (\tilde{g}, \theta)$.

Proof. Let $(\langle\tilde{\mathcal{X}}\rangle, \tau_1, \tau_2, \theta)$ be a $NS\beta_4$ over universal set $\langle\tilde{\mathcal{X}}\rangle$ and let $(\tilde{f}, \theta) \subset (\tilde{g}, \theta)$. Then $(\tilde{g}, \theta)^c$ is a $VS\beta$ close set and $(\tilde{f}, \theta) \cap (\tilde{g}, \theta) = 0_{(\langle\tilde{\mathcal{X}}\rangle,\theta)}$. Since $(\langle\tilde{\mathcal{X}}\rangle, \tau_1, \tau_2, \theta)$ be a $VS\beta_4$ space, there exists $VS\beta$-open sets $(\tilde{\pi}_1, \theta)$ *and* $(\tilde{\pi}_2, \theta)$ such that $(\tilde{f}, \theta) \subset (\tilde{\pi}_1, \theta), (\tilde{g}, \theta)^c \subset (\tilde{\pi}_2, \theta)$ *and* $(\tilde{\pi}_1, \theta) \cap (\tilde{\pi}_2, \theta) = 0_{(\langle\tilde{\mathcal{X}}\rangle,\theta)}$. Thus $(\tilde{f}, \theta) \subset (\tilde{\pi}_1, \theta) \subset (\tilde{\pi}_2, \theta)^c \subset (\tilde{g}, \theta), (\tilde{\pi}_2, \theta)^c$ is a $VS\beta$ close set and $\overline{(\tilde{\pi}_1, \theta)} \subset (\tilde{\pi}_2, \theta)^c$. So $(\tilde{f}, \theta) \subset (\tilde{\pi}_1, \theta) \subset \overline{(\tilde{\pi}_1, \theta)} \subset (\tilde{g}, \theta)$.

Conversely, $(\tilde{f}_1, \theta)$ and $(\tilde{f}_2, \theta)$ be two disjoint $VS\beta$ close sets. Then $(\tilde{f}_1, \theta) \subset (\tilde{f}_2, \theta)^c$ implies there exists $VS\beta$ open set (π, θ) such that $(\tilde{f}_1, \theta) \subset (\tilde{\pi}, \theta) \subset \overline{(\tilde{\pi}_1, \theta)} \subset (\tilde{f}_2, \theta)^c$. Thus $(\tilde{\pi}, \theta)$ and $(\tilde{\pi}, \theta)^c$ are $VS\beta$ open sets and $(\tilde{f}_1, \theta) \subset (\tilde{f}_2, \theta)^c$, $(\tilde{f}_2, \theta) \subset \overline{(\tilde{\pi}, \theta)}^c$ *and* $(\tilde{\pi}, \theta)$*and*$\overline{(\tilde{\pi}, \theta)}^c = 0_{(\langle\tilde{\mathcal{X}}\rangle,\theta)}$. $(\langle\tilde{\mathcal{X}}\rangle, \tau, \theta)$ be a $VS\beta_4$ space.

Theorem 3.22. A VBS countable space in which every VS convergent sequence has a unique soft limit is a $VSB\beta$ Hausdorff space.

Proof. Let $(\langle\tilde{\mathcal{X}}\rangle, \tau_1,\tau_2, \theta)$ be *VBS* Hausdorff space and let $\langle(\widetilde{x^{i}_{(i,j)},\theta})_n\rangle$ be a soft convergent sequence in $(\langle\tilde{\mathcal{X}}\rangle, \tau_1,\tau_2, \theta)$. We prove that the limit of this sequence is unique. We prove this result by contradiction. Suppose $\langle(\widetilde{x^{i}_{(i,j)},\theta})_n\rangle$ converges to two soft points $\tilde{l}$ and $\tilde{m}$ such that $\tilde{l}\neq\tilde{m}$. Then by trichotomy law either $\tilde{l}<\tilde{m}$ *or* $\tilde{l}>\tilde{m}$. Since the possess the *VSBβ* Hausdorff characterstics, there must happen two *VSβ* open sets $\langle f,\theta\rangle$ and $\langle\rho,\theta\rangle$ such that $\langle f,\theta\rangle\tilde{\cap}\langle\rho,\theta\rangle = 0_{(\langle\tilde{\mathcal{X}}\rangle,\theta)}$. Now, $\langle(\widetilde{x^{i}_{(i,j)},\theta})_n\rangle$ converges to $\tilde{l}$ so there exists an integer n_1 such that $(\widetilde{x^{i}_{(i,j)},\theta})_n\in\langle f,\theta\rangle\forall n\geq n_1$. Also, $\langle(\widetilde{x^{i}_{(i,j)},\theta})_n\rangle$ converges to $\tilde{m}$ so there exists an integer n_2 such that $(\widetilde{x^{i}_{(i,j)},\theta})_n\in\langle\rho,\theta\rangle\forall n\geq n_2$. We are interested to discuss the maximum possibility, for that we must suppose maximum of both the integers which will enable us to discuss the soft sequence for single soft number now $max(n_1,n_2)=n_0$. Which leads to the situation $(\widetilde{x^{i}_{(i,j)},\theta})_n\in\langle f,\theta\rangle\forall n\geq n_0$ and $(\widetilde{x^{i}_{(i,j)},\theta})_n\in\langle\rho,\theta\rangle\forall n\geq n_0$. This implies that $(\widetilde{x^{i}_{(i,j)},\theta})_n\in\langle f,\theta\rangle$ and $(\widetilde{x^{i}_{(i,j)},\theta})_n\in\langle\rho,\theta\rangle\forall n\geq n_0$. Implies that $(\widetilde{x^{i}_{(i,j)},\theta})_n\in(\langle f,\partial\rangle\tilde{\cap}\langle\rho,\theta\rangle)\forall n\geq n_0$. Which beautifully contradict the fact that $\langle f,\theta\rangle\tilde{\cap}\langle\rho,\theta\rangle=\widetilde{0_{(\langle\tilde{\mathcal{X}}\rangle,\theta)}}$. Hence, the limit of the *VS* sequence must be unique.

4 More Results in Vague Soft Bitopological Spaces

In section, some results are discussed in two vague soft bi-topological space with respect to vague soft β open sets. Vague soft product spaces are discussed with respect to soft points. The characterization of vague soft β closed as well as vague soft β open sets, characteristics of Bolzano Weirstrass property, vague soft compactness and its marriage with sequences, interconnection between sequentially compactness and countably compactness in vague soft bi-topology with respect to soft β open sets are addressed.

Theorem 4.1. Let $(\langle\tilde{\mathcal{X}}\rangle, \tau_1,\tau_2, \theta)$ be *VSBTS* such that it is *VSβ* Hausdorff space and $(\langle\tilde{Y}\rangle,\mathfrak{F}_1,\mathfrak{F}_2,\theta)$ be any *VSBST*. Let $\langle f,\theta\rangle$: $(\langle\tilde{\mathcal{X}}\rangle,\tau_1,\tau_2,\theta)\longrightarrow(\langle\tilde{Y}\rangle,\mathfrak{F}_1,\mathfrak{F}_2,\theta)$ be a soft function such that it is soft monotone and continuous. Then $(\langle\tilde{Y}\rangle,\mathfrak{F}_1,\mathfrak{F}_2,\theta)$ is also of characteristics of *VBSβ* Hausdorffness.

Proof. Suppose $(x^{i}_{(i,j)},\theta)_1,(x^{i}_{(i,j)},\theta)_2\in\langle\tilde{\mathcal{X}}\rangle$ such that either $(x^{i}_{(i,j)},\theta)_1\neq(x^{i}_{(i,j)},\theta)_2$. Since $\langle f,\theta\rangle$ is soft monotone. Let us suppose the monotonically increasing case. Suppose $(y^{i/}_{(i/,j/)},\theta)_1,(y^{i/}_{(i/,j/)},\theta)_2\in\langle\tilde{Y}\rangle$ such that $(y^{i/}_{(i/,j/)},\theta)_1\neq(y^{i/}_{(i/,j/)},\theta)_2$ respectively such that $(y^{i/}_{(i/,j/)},\theta)=f_{(x^{i}_{(i,j)},\theta)_1},(y^{i/}_{(i/,j/)},\theta)_2=f_{(x^{i}_{(i,j)},\theta)_2}$. Since, $(\langle\tilde{\mathcal{X}}\rangle, \tau_1,\tau_2, \theta)$ is *VSBβ* Hausdorff space so there exists mutually disjoint *VSβ* open sets $\langle k_1,\theta\rangle$ and $\langle k_2,\theta\rangle\in(\langle\tilde{\mathcal{X}}\rangle,\tau_1,\tau_2,\theta)\Longrightarrow f(\langle k_1,\theta\rangle)$ and $f(\langle k_2,\theta\rangle)\in(\langle\tilde{Y}\rangle,\mathfrak{F}_1,\mathfrak{F}_2,\theta)$. We claim that $f(\langle k_1,\theta\rangle)\tilde{\cap}f(\langle k_2,\theta\rangle)=\widetilde{0_{(\langle\tilde{\mathcal{X}}\rangle,\theta}}$. Otherwise $f(\langle k_1,\theta\rangle)\tilde{\cap}f(\langle k_2,\theta\rangle)\neq\widetilde{0_{(\langle\tilde{\mathcal{X}}\rangle,\theta)}}$. Suppose there exists $(t^{i//}_{(i//,j//)},\theta)_1\in f(\langle k_1,\partial\rangle)\tilde{\cap}f(\langle k_2,\theta\rangle)\Longrightarrow(t^{i//}_{(i//,j//)},\theta)_1\in f(\langle k_1,\theta\rangle)$ and $(t^{i//}_{(i//,j//)},\theta)_1\in f(\langle k_2,\theta\rangle),(t^{i//}_{(i//,j//)},\theta)_1\in f(\langle k_1,\theta\rangle), f$ is soft one-one there exists $(t^{i//}_{(i//,j//)},\theta)_2\in\langle k_1,\theta\rangle$ such that $(t^{i//}_{(i//,j//)},\theta)_1=f((t^{i//}_{(i//,j//)},\theta)_2),(t^{i//}_{(i//,j//)},\theta)_1\in f(\langle k_2,\theta\rangle)$ implies there exists $(t^{i//}_{(i//,j//)},\theta)_3\in\langle k_2,\theta\rangle$ such that $f((t^{i//}_{(i//,j//)},\theta)_3)\Longrightarrow f((t^{i//}_{(i//,j//)},\theta)_2)=f((t^{i//}_{(i//,j//)},\theta)_3)$. Since, f is soft one-one $\Longrightarrow(t^{i//}_{(i//,j//)},\theta)_2=(t^{i//}_{(i//,j//)},\theta)_3\Longrightarrow(t^{i//}_{(i//,j//)},\theta)_2\in f(\langle k_1,\theta\rangle),(t^{i//}_{(i//,j//)},\theta)_2\in f(\langle k_2,\theta\rangle)\Longrightarrow(t^{e//}_{(i//,j//)},\theta)_2\in f(\langle k_1,\theta\rangle)\tilde{\cap}f(\langle k_2,\theta\rangle)$. This is contradiction because $\langle k_1,\theta\rangle\tilde{\cap}\langle k_2,\theta\rangle=0_{(\langle\tilde{\mathcal{X}}\rangle,\theta)}$. Therefore $f(\langle k_1,\theta\rangle)\tilde{\cap}f(\langle k_2,\theta\rangle)=0_{(\langle\tilde{\mathcal{X}}\rangle,\theta)}$.

Finally, $(x^{i}{}_{(i,j)},\theta)_1 \succ (x^{i}{}_{(i,j)},\theta)_2$ *or* $(x^{i}{}_{(i,j)},\theta)_1 \prec (x^{i}{}_{(i,j)},\theta)_2 \Longrightarrow (x^{i}{}_{(i,j)},\theta)_1 \neq (x^{i}{}_{(i,j)},\theta)_2 \Longrightarrow f((x^{i}{}_{(i,j)},\theta)_1) \succ f((x^{i}{}_{(i,j)},\theta)_2)$ *or* $f((x^{i}{}_{(i,j)},\theta)_1) \prec f((x^{i}{}_{(i,j)},\theta)_2) \Longrightarrow f((x^{i}{}_{(i,j)},\theta)_1) \neq f((x^{i}{}_{(i,j)},\theta)_2)$. Given a pair of points $(y^{i'}{}_{(i',j')},\theta)_1, (y^{i'}{}_{(i',j')},\theta)_2 \in \langle \tilde{Y} \rangle$ such that $(y^{i'}{}_{(i',j')},\theta)_1 \neq (y^{i'}{}_{(i',j')},\theta)_2$. We are able to find out mutually exclusive $VS\beta$ open sets $f(\langle k_1,\theta\rangle), f(\langle k_2,\theta\rangle) \in (\langle \tilde{Y} \rangle, \mathfrak{F}_1, \mathfrak{F}_2, \theta)$ such that $(y^{i'}{}_{(i',j')},\theta)_1 \in f(\langle k_1,\theta\rangle), (y^{i'}{}_{(i',j')},\theta)_2 \in f(\langle k_2,\theta\rangle)$. This proves that $(\langle \tilde{Y} \rangle, \mathfrak{F}_1, \mathfrak{F}_2, \theta)$ is $VSB\beta$ Husdorff space.

Theorem 4.2. Let $(\langle \tilde{\mathcal{X}} \rangle, \tau_1, \tau_2, \theta)$ be $VSBTS$ and $(\langle \tilde{Y} \rangle, \mathfrak{F}_1, \mathfrak{F}_2, \theta)$ be an-other $VSBTS$ which satisfies one more condition of $VSB\beta$ Hausdorffness. Let $\langle f,\theta\rangle$: $(\langle \tilde{\mathcal{X}} \rangle, \tau_1, \tau_2, \theta) \longrightarrow (\langle \tilde{Y} \rangle, \mathfrak{F}_1, \mathfrak{F}_2, \theta)$ be a soft fuction such that it is soft monotone and continuous. Then $(\langle \tilde{\mathcal{X}} \rangle, \tau_1, \tau_2, \theta)$ is also of characteristics of VSβ Hausdorfness.

Proof. Suppose $(x^{i}{}_{(i,j)},\theta)_1, (x^{i}{}_{(i,j)},\theta)_2 \in \langle \tilde{\mathcal{X}} \rangle$ such that either $(x^{i}{}_{(i,j)},\theta)_1 \neq (x^{i}{}_{(i,j)},\theta)_2$. Let us suppose the VS monotonically increasing case. So, $(x^{i}{}_{(i,j)},\theta)_1 \neq (x^{i}{}_{(i,j)},\theta)_2$ implies that $f((x^{i}{}_{(i,j)},\theta)_1) \neq f((x^{i}{}_{(i,j)},\theta)_2)$ respectively. Suppose $(y^{i'}{}_{(i',j')},\theta)_1, (y^{i'}{}_{(i',j')},\theta)_2 \in \langle \tilde{Y} \rangle$ such that $(y^{i'}{}_{(i',j')},\theta)_1 \neq (y^{i'}{}_{(i',j')},\theta)_2$. So, $(y^{i'}{}_{(i',j')},\theta)_1 \neq (y^{i'}{}_{(i',j')},\theta)_2$ respectively such that $(y^{i'}{}_{(i',j')},\theta)_1 = f((x^{i}{}_{(i,j)},\theta)_1), (y^{i'}{}_{(i',j')},\theta)_2 = f((x^{i}{}_{(i,j)},\theta)_2)$ such that $(x^{i}{}_{(i,j)},\theta)_1 = f^{-1}(y_1)$ and $(x^{i}{}_{(i,j)},\theta)_2 = f^{-1}((y^{i'}{}_{(i',j')},\theta)_2)$. Since $(y^{i'}{}_{(i',j')},\theta)_1, (y^{i'}{}_{(i',j')},\theta)_2 \in \langle \tilde{Y} \rangle$ but $(\langle \tilde{Y} \rangle, \mathfrak{F}_1, \mathfrak{F}_2, \theta)$ is $VS\beta$ Hausdorff space. So according to definition $(y^{i'}{}_{(i',j')},\theta)_1 \neq (y^{i'}{}_{(i',j')},\theta)_2$. There definitely exists $VS\beta$ open sets $\langle k_1,\theta\rangle$ and $\langle k_2,\theta\rangle \in (\langle \tilde{Y} \rangle, \mathfrak{F}_1, \mathfrak{F}_2, \theta)$ such that $(y^{i'}{}_{(i',j')},\theta)_1 \in \langle k_1,\theta\rangle$ and $(y^{i'}{}_{(i',j')},\theta)_2 \in \langle k_2,\theta\rangle$ and these two $VS\beta$ open sets are guaranteedly mutually exclusive because the possibility of one rules out the possibility of other. Since $f^{-1}(\langle k_1,\theta\rangle)$ and $f^{-1}(\langle k_2,\theta\rangle)$ are $VS\beta$ open in $(\langle \tilde{\mathcal{X}} \rangle, \tau_1, \tau_2, \theta)$. Now, $f^{-1}(\langle k_1,\theta\rangle) \tilde{\cap} f^{-1}(\langle k_1,\theta\rangle) = f^{-1}(\langle k_1,\theta\rangle \tilde{\cap} \langle k_2,\theta\rangle) = f^{-1}(\tilde{\varnothing}) = \widetilde{0_{(\langle \tilde{\mathcal{X}} \rangle,\theta)}}$ and $(y^{i'}{}_{(a',c')},\theta)_1 \in \langle k_1,\theta\rangle \Rightarrow f^{-1}((y^{i'}{}_{(i',j')},\theta)_1) \in f^{-1}(\langle k_1,\theta\rangle) \Rightarrow (x^{i}{}_{(i,j)},\theta)_1 \in (\langle k_1,\theta\rangle)$, $(y^{i'}{}_{(i',j')},\theta)_2 \in \langle k_2,\theta\rangle \Rightarrow f^{-1}((y^{e'}{}_{(i',j')},\theta)_2) \in f^{-1}(\langle k_2,\theta\rangle)$ implies that $(x^{i}{}_{(i,j)},\theta)_2 \in (\langle k_2,\theta\rangle)$. We see that it has been shown for every pair of distinct points of $\langle \tilde{\mathcal{X}} \rangle$, there suppose disjoint $NS\beta$ open sets namely, $f^{-1}(\langle k_1,\theta\rangle)$ and $f^{-1}(\langle k_2,\theta\rangle)$ belong to $(\langle \tilde{\mathcal{X}} \rangle, \tau_1, \tau_2, \theta)$ such that $(x^{i}{}_{(i,j)},\theta)_1 \in f^{-1}(\langle k_1,\theta\rangle)$ and $(x^{i}{}_{(i,j)},\theta)_2 \in f^{-1}(\langle k_2,\theta\rangle)$. Accordingly, $VSBTS$ is $VSB\beta$ Hausdorff space.

Theorem 4.3. Let $(\langle \tilde{\mathcal{X}} \rangle, \tau_1, \tau_2, \theta)$ be $VSBTS$ and $(\langle \tilde{Y} \rangle, \mathfrak{F}_1, \mathfrak{F}_2, \theta)$ be an-other $VSBTS$. Let $\langle f,\theta\rangle$: $(\langle \tilde{\mathcal{X}} \rangle, \tau_1, \tau_2, \theta) \longrightarrow (\langle \tilde{Y} \rangle, \mathfrak{F}_1, \mathfrak{F}_2, \theta)$ be a soft mapping such that it is continuous mapping. Let $(\langle \tilde{Y} \rangle, \mathfrak{F}_1, \mathfrak{F}_2, \theta)$ is $VS\beta$ Hausdorff space then it is guaranteed that $\{((i,j),(y^{i'}{}_{(i',j')},\theta)): f((x^{i}{}_{(i,j)},\theta)) = f((y^{i'}{}_{(i',j')},\theta))\}$ is a $VS\beta$ close sub-set of $(\langle \tilde{\mathcal{X}} \rangle, \tau_1, \tau_2, \theta) \times (\langle \tilde{Y} \rangle, \mathfrak{F}_1, \mathfrak{F}_2, \theta)$.

Proof. Given that $(\langle \tilde{\mathcal{X}} \rangle, \tau_1, \tau_2, \theta)$ be $VSBTS$ and $(\langle \tilde{Y} \rangle, \mathfrak{F}_1, \mathfrak{F}_2, \theta)$ be an-other $VSBTS$. Let $\langle f,\theta\rangle$: $(\langle \tilde{\mathcal{X}} \rangle, \tau_1, \tau_2, \theta) \longrightarrow (\langle \tilde{Y} \rangle, \mathfrak{F}_1, \mathfrak{F}_2, \theta)$ be a soft mapping such that it is continuous mapping. $(\langle \tilde{Y} \rangle, \mathfrak{F}_1, \mathfrak{F}_2, \theta)$ is $VSB\beta$ Hausdorff space Then we will prove that $\{((x^{i}{}_{(i,j)},\theta'),(y^{i'}{}_{(i',j')},\theta)): f((x^{i}{}_{(i,j)},\theta)) = f((y^{i'}{}_{(i',j')},\theta))\}$ is a $VS\beta$ closed sub-set of $(\langle \tilde{\mathcal{X}} \rangle, \tau_1, \tau_2, \theta) \times (\langle \tilde{Y} \rangle, \mathfrak{F}_1, \mathfrak{F}_2, \theta)$. Equvalently, we will prove that $\{((x^{i}{}_{(i,j)},\theta),(y^{i'}{}_{(i',j')},\theta)): f((x^{i}{}_{(i,j)},\theta)) = (y^{i'}{}_{(i',j')},\theta)\}^c$ is $VS\beta$ open sub-set of $(\langle \tilde{\mathcal{X}} \rangle, \tau_1, \tau_2, \theta) \times (\langle \tilde{Y} \rangle, \mathfrak{F}_1, \mathfrak{F}_2, \theta)$. Let $((x^{i}{}_{(i,j)},\theta),(y^{i'}{}_{(i',j')},\theta)) \in \{((x^{i}{}_{(i,j)},\theta),(y^{i'}{}_{(i',j')},\theta))$ *with* $(x^{i}{}_{(i,j)},\theta)$

$\succ (\mathcal{y}^{i/}{}_{(i/,j/)}, \theta)$: $\mathcal{f}((x^{i}{}_{(i,j)}, \theta)) \succ \mathcal{f}((\mathcal{y}^{i/}{}_{(i/,j/)}, \theta))\}^{c} or((x^{i}{}_{(i,j)}, \theta), (\mathcal{y}^{i/}{}_{(i/,j/)}, \theta)) \in \{((x^{i}{}_{(i,j)}, \theta), (\mathcal{y}^{i/}{}_{(i/,j/)}, \theta))$ *with* $(x^{i}{}_{(i,j)}, \theta) \prec (\mathcal{y}^{i/}{}_{(i/,j/)}, \theta)$: $\mathcal{f}((x^{i}{}_{(i,j)}, \theta)) \prec \mathcal{f}((\mathcal{y}^{i/}{}_{(i/,j/)}, \theta))\}^{c}$. Then, $\mathcal{f}((x^{i}{}_{(i,j)}, \theta)) \succ \mathcal{f}((\mathcal{y}^{i/}{}_{(i/,j/)}, \theta))$ or $\prec \mathcal{f}((x^{i}{}_{(i,j)}, \theta)) \prec \mathcal{f}((\mathcal{y}^{i/}{}_{(i/,j/)}, \theta))$ accordingly. Since, $(\langle \tilde{Y}\rangle, \mathfrak{F}_1, \mathfrak{F}_2, \theta)$ is $VSB\beta$-Hausdorff space. Certainly, $\mathcal{f}((x^{i}{}_{(i,j)}, \theta)), \mathcal{f}((\mathcal{y}^{i/}{}_{(i/,j/)}, \theta))$ are points of $(\langle \tilde{Y}\rangle, \mathfrak{F}_1, \mathfrak{F}_2, \theta)$, there exists $VS\beta$ open sets $\langle \mathcal{G}, \theta\rangle, \langle \mathcal{k}, \theta\rangle \in (\langle \tilde{Y}\rangle, \widetilde{\mathfrak{F}_1, \mathfrak{F}_2}, \theta)$ such that $\mathcal{f}((x^{i}{}_{(i,j)}, \theta)) \in \langle \mathcal{G}, \theta\rangle$ & $\mathcal{f}((x^{i}{}_{(i,j)}, \theta)) \in \langle \mathcal{k}, \theta\rangle$ provided $\langle \mathcal{G}, \theta\rangle \tilde{\cap} \langle \mathcal{k}, \partial\rangle = 0_{\widetilde{(\langle \tilde{\mathcal{X}}\rangle, \theta)}_{\langle \tilde{Y}\rangle}}$. Since, $\langle \mathcal{f}, \theta\rangle$ is soft continuous, $\mathcal{f}^{-1}(\langle \mathcal{G}, \theta\rangle$ & $\mathcal{f}^{-1}(\langle \mathcal{k}, \theta\rangle$ are $VS\beta$ open sets in $(\langle \tilde{\mathcal{X}}\rangle, \tau_1, \tau_2, \theta)$ supposing $(x^{i}{}_{(i,j)}, \theta)$ and $(\mathcal{y}^{i/}{}_{(i/,j/)}, \theta)$ respectively and so $\mathcal{f}^{-1}(\langle \mathcal{G}, \theta\rangle \times \mathcal{f}^{-1}(\langle h, \theta\rangle$ is basic $VS\beta$ open set in $\widetilde{(\langle \tilde{\mathcal{X}}\rangle}, \tau_1, \tau_2, \theta) \times (\langle \tilde{Y}\rangle, \mathfrak{F}_1, \mathfrak{F}_2, \theta)$ containing $((x^{i}{}_{(i,j)}, \theta), (\mathcal{y}^{i/}{}_{(i/,j/)}, \theta))$. Since $\langle \mathcal{G}, \theta\rangle \tilde{\cap} \langle \mathcal{k}, \theta\rangle = 0_{\widetilde{(\langle \tilde{\mathcal{X}}\rangle, \theta)}_{Y}}$, it is clear by the definition of $\{((x^{i}{}_{(i,j)}, \theta), (\mathcal{y}^{i/}{}_{(i/,j/)}, \theta))$: $\mathcal{f}((x^{i}{}_{(i,j)}, \theta)) = \mathcal{f}((\mathcal{y}^{i/}{}_{(i/,j/)}, \theta))\}$ that $\{\mathcal{f}^{-1}(\langle \mathcal{G}, \theta\rangle$ & $\mathcal{f}^{-1}(\langle \mathcal{k}, \theta\rangle\} \tilde{\cap} \{((x^{i}{}_{((i,j))}, \theta), (\mathcal{y}^{i/}{}_{(i/,j/)}, \theta))$: $\mathcal{f}(x) = \mathcal{f}((\mathcal{y}^{i/}{}_{(i/,j/)}, \theta))\} = 0_{(\langle \tilde{\mathcal{X}}\rangle, \theta)}$, that is $\mathcal{f}^{-1}(\langle \mathcal{G}, \theta\rangle \times \mathcal{f}^{-1}(\langle \mathcal{k}, \theta\rangle \Subset \{((x^{i}{}_{(i,j)}, \theta), (\mathcal{y}^{i/}{}_{(i/,j/)}, \theta))$: $\mathcal{f}((x^{i}{}_{(i,j)}, \theta)) = \mathcal{f}((\mathcal{y}^{i/}{}_{(i/,j/)}, \theta))\}^{c}$. Hence $\{((x^{i}{}_{(i,j)}, \theta), (\mathcal{y}^{i/}{}_{(i/,j/)}, \theta))$: $\mathcal{f}((x^{i}{}_{(\mathfrak{a},\mathfrak{c})}, \theta)) = \mathcal{f}((\mathcal{y}^{i/}{}_{(i/,j/)}, \theta))\}^{c}$ implies that $\{((x^{i}{}_{(\mathfrak{a},\mathfrak{c})}, \theta), (\mathcal{y}^{i/}{}_{(i/,j/)}, \theta))$: $\mathcal{f}((x^{i}{}_{(i,j)}, \theta)) = \mathcal{f}((\mathcal{y}^{i/}{}_{(i/,j/)}, \theta))\}$ is $VS\beta$ close.

Theorem 4.4. Let $(\langle \tilde{\mathcal{X}}\rangle, \tau_1, \tau_2, \theta)$ a *VSB* second countable space and let $\langle \mathcal{f}, \theta\rangle$ be *VSB* uncountable sub set of $(\langle \tilde{\mathcal{X}}\rangle, \tau_1, \tau_2, \theta)$. Then, at least one point of $\langle \mathcal{f}, \theta\rangle$ is a soft limit point of $\langle \mathcal{f}, \theta\rangle$.

Proof. Let $\mathfrak{W} = \langle \mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4, \ldots \mathcal{B}_n : n \in \mathbb{N}\rangle$ *for* $(\langle \tilde{\mathcal{X}}\rangle, \tau_1, \tau_2, \theta)$.

Let, if possible, no point of $\langle \mathcal{f}, \theta\rangle$ be a soft limit point of $\langle \mathcal{f}, \theta\rangle$. Then, for each $(x^{i}{}_{(i,j)}, \theta) \in \langle \mathcal{f}, \partial\rangle$ there exists $VS\beta$ open set $\langle \rho, \theta\rangle_{(x^{i}{}_{(i,j)}, \theta)}$ such that $(x^{i}{}_{(i,j)}, \theta) \in \langle \rho, \theta\rangle_{(x^{i}{}_{(i,j)}, \theta)}$, $\langle \rho, \theta\rangle_{(x^{i}{}_{(i,j)}, \theta)} \tilde{\cap} \langle \mathcal{f}, \theta\rangle = \{(x^{i}{}_{(i,j)}, \theta)\}$. Since $\mathfrak{W}$ is soft base there exists $\mathcal{B}_{n(x^{i}{}_{(i,j)}, \theta)} \in \mathfrak{W}$ such that $(x^{i}{}_{(i,j)}, \theta) \in \mathcal{B}_{n(x^{i}{}_{(i,j)}, \theta)} \Subset \langle \rho, \partial\rangle_{(x^{i}{}_{(i,j)}, \theta)}$. Therefore, $\mathcal{B}_{n(x^{i}{}_{(i,j)}, \theta)} \tilde{\cap} \langle \mathcal{f}, \theta\rangle \Subset \langle \rho, \theta\rangle_{(x^{i}{}_{(i,j)}, \theta)} \tilde{\cap} \langle \mathcal{f}, \theta\rangle = \{(x^{i}{}_{(i,j)}, \theta)\}$. More-over, if $(x^{i}{}_{(i,j)}, \theta)_1$ and $(x^{e}{}_{(i,j)}, \theta)_2$ be any two VS points such that $(x^{v}{}_{(i,j)}, \theta)_1 \neq (x^{i}{}_{(i,j)}, \theta)_2$ which means either $(x^{i}{}_{(i,j)}, \theta)_1 \succ (x^{e}{}_{(i,j)}, \theta)_2$ or $(x^{i}{}_{(i,j)}, \theta)_1 \prec (i,j)_2$ then there exists $\mathcal{B}_{n(x^{i}{}_{(i,j)}, \theta)_1}$ and $\mathcal{B}_{n(x^{i}{}_{(i,j)}, \theta)_2}$ in $\mathfrak{W}$ such that $\mathcal{B}_{n(x^{i}{}_{(i,j)}, \theta)_1} \tilde{\cap} \langle \mathcal{f}, \theta\rangle = \{(x^{i}{}_{(i,j)}, \theta)_1\}$ and $\mathcal{B}_{n(x^{i}{}_{(i,j)}, \theta)_2} \tilde{\cap} \langle \mathcal{f}, \theta\rangle = \{(x^{i}{}_{(i,j)}, \theta)_2\}$. Now, $(x^{i}{}_{(i,j)}, \theta)_1 \neq (x^{i}{}_{(i,j)}, \theta)_2$ which guarantees that $\{(x^{i}{}_{(i,j)}, \theta)_1\} \neq \{(x^{i}{}_{(i,j)}, \theta)_2\}$ which implies that $\mathcal{B}_{n(x^{i}{}_{(i,j)}, \theta)_1} \tilde{\cap} \langle \mathcal{f}, \theta\rangle \neq \mathcal{B}_{n(x^{i}{}_{(i,j)}, \theta)_2} \tilde{\cap} \langle \mathcal{f}, \theta\rangle$ which implies $\mathcal{B}_{n(x^{i}{}_{(i,j)}, \theta)_1} \neq \mathcal{B}_{n(x^{i}{}_{(i,j)}, \theta)_2}$. Thus, there exists a one to one soft correspondence of $\langle \mathcal{f}, \partial\rangle$ on to $\{\mathcal{B}_{n(x^{i}{}_{(i,j)}, \theta)} : (x^{e}{}_{(i,j)}, \theta) \in \langle \mathcal{f}, \theta\rangle\}$. Now, $\langle \mathcal{f}, \theta\rangle$ being VS uncountable, it follows that $\{\mathcal{B}_{n(x^{i}{}_{(i,j)}, \theta)} : (x^{i}{}_{(i,j)}, \theta) \in \langle \mathcal{f}, \theta\rangle\}$ is VS uncountable. But, this is purely a contradiction, since $\{\mathcal{B}_{n(x^{i}{}_{(i,j)}, \theta)} : (x^{i}{}_{(i,j)}, \theta) \in \langle \mathcal{f}, \theta\rangle\}$ being a VS sub-family of the NS countable collection $\mathfrak{W}$. This contradiction is taking birth that on point of $\langle \mathcal{f}, \theta\rangle$ is a soft limit point of $\langle \mathcal{f}, \theta\rangle$, so at least one point of $\langle \mathcal{f}, \theta\rangle$ is a soft limit point of $\langle \mathcal{f}, \theta\rangle$.

Theorem 4.5. Let $(\langle \tilde{\mathcal{X}}\rangle, \tau_1, \tau_2, \theta)$ *VSBTS* such that is is *VSB* countably compact then this space has the characteristics of Bolzano Weirstrass property.

Proof. Let $(\langle \tilde{\mathcal{X}}\rangle, \tau_1, \tau_2, \theta)$ be a *VSB* countably compact space and suppose, if possible, that it negates the Bolzano Weierstrass Property (*BWP*). Then there must exists an infinite $VS\beta$ set $\langle \mathcal{f}, \theta\rangle$ supposing no soft limit point. Further suppose $\langle \rho, \theta\rangle$ be soft countability infinite soft subset $\langle \mathcal{f}, \theta\rangle$ that is $\langle \rho, \theta\rangle \Subset \langle \mathcal{f}, \theta\rangle$. Then the guarantee $\langle \rho, \theta\rangle$ has no soft limit point. It follows that $\langle\rho,\theta\rangle$ is *VS* soft β closed set. Also for each $\widetilde{(x^{i}{}_{(i,j)},\theta)}_n \in \langle\rho,\theta\rangle, \widetilde{(x^{i}{}_{(i,j)},\theta)}_n$ is not a soft limit point of $\langle\rho,\theta\rangle$. Hence there exists $VS\beta$ open set $\langle\mathcal{G}_n,\theta\rangle$, such that $\widetilde{(x^{i}{}_{(i,j)},\theta)}_n \in \langle\mathcal{G}_n,\theta\rangle$, $\langle\mathcal{G}_n,\theta\rangle\tilde{\cap}\langle\rho,\theta\rangle = \{\widetilde{(x^{i}{}_{(i,j)},\theta)}_n\}$. The the collection $\{\langle\mathcal{G}_n,\theta\rangle: n\in N\}\tilde{\cap}\langle\rho,\theta\rangle^c$ is countable $VS\beta$ open cover of $(\langle\tilde{\mathcal{X}}\rangle,\tau_1,\tau_2,\theta)$. this soft cover has no finite sub-cover. For this we exhaust a single $\langle\mathcal{G}_n,\theta\rangle$, it would not be a soft cover of $(\langle\tilde{\mathcal{X}}\rangle,\ \tau_1,\tau_2,\ \theta)$ since then $\langle\widetilde{(x^{i}{}_{(i,j)},\theta)}_n\rangle$ would be covered. Result in $(\langle\tilde{\mathcal{X}}\rangle,\ \tau_1,\tau_2,\ \theta)$ is not *VS* countably compact. But this contradicts the given. Hence, we are compelled to accept $(\langle\tilde{\mathcal{X}}\rangle,\ \tau_1,\tau_2,\ \theta)$ must have Bolzano Weirstrass Property.

Theorem 4.6. Let $(\langle\tilde{\mathcal{X}}\rangle,\ \tau_1,\tau_2,\ \theta)$ be a *VSTS* and $(\langle\tilde{Y}\rangle,\mathfrak{F}_1,\mathfrak{F}_2,\theta)$ be *VS* sub-space of $(\langle\tilde{\mathcal{X}}\rangle,\ \tau_1,\tau_2,\ \theta)$. The necessary and sufficient condition for $(\langle\tilde{Y}\rangle,\mathfrak{F}_1,\mathfrak{F}_2,\theta)$ to be $VS\beta$ compact relative to $(\langle\tilde{Y}\rangle,\mathfrak{F}_1,\mathfrak{F}_2,\theta)$ is that $\langle\mathfrak{f},\theta\rangle$ is $VS\beta$ compact relative to $(\langle\tilde{\mathcal{X}}\rangle,\ \tau_1,\tau_2,\ \theta)$.

Proof. First we prove that $\langle\mathfrak{f},\theta\rangle$ relative to $(\langle\tilde{\mathcal{X}}\rangle,\ \tau_1,\tau_2,\ \theta)$ Let $\{\langle\mathfrak{k},\theta\rangle_i: i\in I\}$ that is $\{\langle\mathfrak{k},\theta\rangle_1,\langle\mathfrak{k},\theta\rangle_2,\langle\mathfrak{k},\theta\rangle_3,\langle\mathfrak{k},\theta\rangle_4,\ldots\}$ be $(\langle\tilde{\mathcal{X}}\rangle,\tau_1,\tau_2,\theta) - VS\beta$ open cover of $\langle\mathfrak{f},\theta\rangle$, then $\langle\mathfrak{f},\theta\rangle \Subset \tilde{\bigcup}_{\mathfrak{z}}\langle\mathfrak{k},\theta\rangle_{\mathfrak{z}}.\langle\mathfrak{k},\theta\rangle_{\mathfrak{z}} \in (\langle\tilde{\mathcal{X}}\rangle,\tau_1,\tau_2,\theta) \Longrightarrow \exists\langle g,\theta\rangle_{\mathfrak{z}} \in (\langle\tilde{\mathcal{X}}\rangle,\tau_1,\tau_2,\theta)$ such that $\langle\mathfrak{k},\theta\rangle_{\mathfrak{z}} = \langle g,\theta\rangle_{\mathfrak{z}}\tilde{\cap}\langle\mathfrak{f},\theta\rangle \Subset \langle g,\theta\rangle_{\mathfrak{z}} \Longrightarrow \exists\langle g,\theta\rangle_{\mathfrak{z}} \in (\langle\tilde{\mathcal{X}}\rangle,\tau_1,\tau_2,\theta)$ such that $\langle\mathfrak{k},\theta\rangle_{\mathfrak{z}} \Subset \langle g,\theta\rangle_{\mathfrak{z}} \Longrightarrow \tilde{\bigcup}_{\mathfrak{z}}\langle\mathfrak{k},\theta\rangle_{\mathfrak{z}} \Subset \mathfrak{z}$ but $\langle\mathfrak{f},\theta\rangle \Subset \langle\mathfrak{k},\theta\rangle_{\mathfrak{z}}$. So that $\langle\mathfrak{f},\theta\rangle \Subset \tilde{\bigcup}_i\langle\mathfrak{k},\theta\rangle_{\mathfrak{z}}$. This guarantees that $\{\langle g,\theta\rangle_{\mathfrak{z}}: \mathfrak{z}\in I\}$ is a $(\langle\tilde{\mathcal{X}}\rangle,\tau_1,\tau_2,\theta) - VS\beta$ open cover of $\langle\mathfrak{k},\theta\rangle$ which is known to be $VS\beta$ compact relative $(\langle\tilde{\mathcal{X}}\rangle,\ \tau_1,\tau_2,\ \theta)$ and hence the soft cover $\{\mathfrak{z}: \mathfrak{z}\in I\}$ must be freezable to a finite soft cub cover, say, $\{\langle g,\theta\rangle_{\mathfrak{z}r}: r=1,2,3,4,\ldots,n\}$, Then $\langle\mathfrak{f},\theta\rangle \Subset \widetilde{\bigcup_{r=1}^{n}\langle G,\theta\rangle}_{\mathfrak{z}r} \Longrightarrow$

$$\langle\mathfrak{f},\theta\rangle\tilde{\cap}\langle\mathfrak{f},\theta\rangle \Subset \langle\mathfrak{f},\theta\rangle\tilde{\cap}\left[\widetilde{\bigcup_{r=1}^{n}}\langle G,\theta\rangle_{\mathfrak{z}r}\right]$$

$= \widetilde{\bigcup_{r=1}^{n}}(\langle\mathfrak{f},\theta\rangle\tilde{\cap}\langle g,\theta\rangle_{\mathfrak{z}r} = \widetilde{\bigcup_{r=1}^{n}\langle\mathfrak{f},\theta\rangle}_{\mathfrak{z}r}$ *or* $\langle\mathfrak{f},\theta\rangle \Subset \widetilde{\bigcup_{r=1}^{n}\langle\mathfrak{f},\theta\rangle}_{ir} \Longrightarrow \{\langle\mathfrak{k},\partial\rangle_{\mathfrak{z}r}: 1\le r\le n\}$ is a $(\langle\tilde{\mathcal{X}}\rangle,\tau_1,\tau_2,\theta) - VS\beta$ open cover of $\langle\mathfrak{f},\theta\rangle$. Steping from an arbitrary $(\langle\tilde{\mathcal{X}}\rangle,\tau_1,\tau_2,\theta) - \beta$ open cover of $(\langle\tilde{Y}\rangle,\mathfrak{F}_1,\mathfrak{F}_2,\theta)$, we are able to show that the $VS\beta$ cover is freezable to a finite soft sub cover $\{\langle\mathfrak{k},\partial\rangle_{\mathfrak{z}r}: 1\le r\le n\}$ of $\langle\mathfrak{f},\theta\rangle$, meaning there by $\langle\mathfrak{f},\theta\rangle$ is $(\langle\tilde{\mathcal{X}}\rangle,\tau_1,\tau_2,\theta) - VS\beta$ compact. The condition is sufficient: Suppose $(\langle\tilde{Y}\rangle,\mathfrak{F}_1,\mathfrak{F}_2,\theta)$ be soft sub-space of $(\langle\tilde{\mathcal{X}}\rangle,\ \tau_1,\tau_2,\ \theta)$ and also $\langle\mathfrak{f},\theta\rangle$ is $(\langle\tilde{\mathcal{X}}\rangle,\tau_1,\tau_2,\theta) - VS$ compact. We have to prove that $\langle\mathfrak{f},\theta\rangle$ is $(\langle\tilde{\mathcal{X}}\rangle,\tau_1,\tau_2,\theta) - VS\beta$ compact. Let $\{\langle\mathfrak{k},\theta\rangle_1,\langle\mathfrak{k},\theta\rangle_2,\langle\mathfrak{k},\theta\rangle_3,\langle\mathfrak{k},\theta\rangle_4,\ldots\}$ be soft $(\langle\tilde{\mathcal{X}}\rangle,\tau_1,\tau_2,\theta) - VS\beta$ open cover of $\langle\mathfrak{f},\theta\rangle$, so that $\langle\mathfrak{f},\theta\rangle \Subset \tilde{\bigcup}_i\langle g,\theta\rangle_i$ from which $\langle\mathfrak{f},\theta\rangle\tilde{\cap}\langle\mathfrak{f},\theta\rangle \Subset \langle\mathfrak{f},\theta\rangle\tilde{\cap}(\tilde{\bigcup}_i\langle g,\theta\rangle_{\mathfrak{z}})$ *or*, $\langle\mathfrak{f},\theta\rangle \Subset \tilde{\bigcup}_i(\langle\mathfrak{f},\theta\rangle\tilde{\cap}\langle g,\theta\rangle_{\mathfrak{z}})$. On taking $\langle\mathfrak{k},\theta\rangle_{\mathfrak{z}} = \langle g,\theta\rangle_i\tilde{\cap}\langle\mathfrak{f},\theta\rangle$, we get $\langle\mathfrak{f},\theta\rangle \Subset \tilde{\bigcup}\langle\mathfrak{k},\theta\rangle_i, \langle g,\theta\rangle_i \in (\langle\tilde{\mathcal{X}}\rangle,\tau_1,\tau_2,\theta) \Longrightarrow \langle\mathfrak{k},\theta\rangle_{\mathfrak{z}} = \langle g,\theta\rangle_i\tilde{\cap}\langle\mathfrak{f},\theta\rangle \in (\langle\tilde{Y}\rangle,\mathfrak{F}_1,\mathfrak{F}_2,\theta)\ldots.(1)$. Now from (1) it is clear that $\{\langle\mathfrak{k},\theta\rangle_1,\langle\mathfrak{k},\theta\rangle_2,\langle\mathfrak{k},\theta\rangle_3,\langle\mathfrak{k},\theta\rangle_4,\ldots\}$ is $(\langle\tilde{Y}\rangle,\mathfrak{F}_1,\mathfrak{F}_2,\theta) - VS\beta$open soft cover of $\langle\mathfrak{f},\theta\rangle$ which is known to be $(\langle\tilde{Y}\rangle,\mathfrak{F}_1,\mathfrak{F}_2,\theta) - VS\beta$ compact hence this soft cover must be reducible to a finite soft sub-cover. say, $\{\langle\mathfrak{k},\theta\rangle_{\mathfrak{z}r}: 1\le r\le n\}$. *This* $\Longrightarrow \langle\mathfrak{f},\theta\rangle \Subset \widetilde{\bigcup_{r=1}^{n}\langle\mathfrak{k},\theta\rangle}_{ir} = \bigcup_{r=1}^{n}(((\langle g,\theta\rangle_{\mathfrak{z}r})\tilde{\cap}\langle\mathfrak{f},\theta\rangle \in (\langle\tilde{\mathcal{X}}\rangle,\tau_1,\tau_2,\theta)$, *or*

$$\langle\mathfrak{f},\theta\rangle \Subset \left(\bigcup_{r=1}^{n}(((\langle g,\theta\rangle_{ir})\tilde{\cap}\langle\mathfrak{f},\theta\rangle) \Subset \widetilde{\bigcup_{r=1}^{n}}\langle g,\theta\rangle_{ir},\ or\right.$$

$\langle \mathfrak{f}, \theta\rangle \bigcup_{r=1}^{n} \langle g, \theta\rangle_{\mathfrak{z}r}$. This proves that $\{\langle g, \theta\rangle_{\mathfrak{z}r} : 1 \le r \le n\}$ is a finite soft sub-cover to the soft cover $\langle g, \theta\rangle_{\mathfrak{z}}$. Starting from an arbitrary $(\langle\tilde{\mathcal{X}}\rangle, \tau_1, \tau_2, \theta) - VS\beta$ open soft cover of $\langle \mathfrak{f}, \theta\rangle$, we are able to show that this soft vague open cover is freezable to a finite soft sub-cover, showing there by $\langle \mathfrak{f}, \theta\rangle$ is $(\langle\tilde{\mathcal{X}}\rangle, \tau_1, \tau_2, \theta) - V\beta$ compact.

Theorem 4.7. Let $(\langle\tilde{\mathcal{X}}\rangle, \tau_1, \tau_2, \theta)\, VSBTS$ and let $\langle \widetilde{(x^i{}_{(i,j)}, \theta)_n}\rangle$ be a VS sequence in $(\langle\tilde{\mathcal{X}}\rangle$, τ_1, τ_2, $\theta)$ such that it converges to a point $(x^i{}_{(i,j)}, \theta)_{n_0}$ then the soft set $\langle g, \theta\rangle$ consisting of the points $(x^i{}_{(i,j)}, \theta)_{n_0}$ and $(x^i{}_{(i,j)}, \theta)_n (n = 1, 2, 3, \ldots)$ is soft $VS\beta$ compact.

Proof. Given $(\langle\tilde{\mathcal{X}}\rangle, \tau_1, \tau_2, \theta)\, VSBTS$ and let $\langle \widetilde{(x^i{}_{(i,j)}, \theta)_n}\rangle$ be a VS sequence in $(\langle\tilde{\mathcal{X}}\rangle$, τ_1, τ_2, $\theta)$ such that it converges to a point $(x^i{}_{(i,j)}, \theta)_{n_0}$ that is $\widetilde{(x^i{}_{(i,j)}, \theta)}_n \longrightarrow (x^i{}_{(i,j)}, \theta)_{n_0} \in \langle\tilde{\mathcal{X}}\rangle$. Let $\langle g, \theta\rangle = \left\langle \begin{matrix} \widetilde{(x^i{}_{(i,j)}, \theta)}_1, \widetilde{(x^i{}_{(i,j)}, \theta)}_2, \widetilde{(x^i{}_{(i,j)}, \theta)}_3, \\ (x^i{}_{(i,j)}, \theta)_4, (x^i{}_{(i,j)}, \theta)_5, (x^i{}_{(i,j)}, \theta)_7, \ldots \end{matrix} \right\rangle$. Let $\{\langle \mathfrak{S}, \partial\rangle_\alpha : \alpha \in \Delta\}$ be $VS\beta$ open covering of $\langle g, \theta\rangle$ so that $\langle g, \theta\rangle \Subset \tilde{\cup}\{\langle \mathfrak{S}, \theta\rangle_\alpha : \alpha \in \Delta\}, (x^i{}_{(i,j)}, \theta)_{n_0} \in \langle g, \theta\rangle$ implies that there exists $\alpha_0 \in \Delta s.t. (x^i{}_{(i,j)}, \theta)_{n_0} \in \langle \mathfrak{S}, \theta\rangle_{\alpha_0}$. According to the definition of soft convergence, $(x^i{}_{(i,j)}, \theta)_{n_0} \in \langle \mathfrak{S}, \theta\rangle_{\alpha_0} \in (\langle\tilde{\mathcal{X}}\rangle, \tau_1, \tau_2, \theta)$ implies there exists $n_0 \in V$ such that $n \ge n_0$ and $(x^i{}_{(i,j)}, \theta)_n \in \langle \mathfrak{S}, \theta\rangle_{\alpha_0}$. Evidently, $\langle \mathfrak{S}, \partial\rangle_{\alpha_0}$ contains the points $(x^i{}_{(i,j)}, \theta)_{n_0}, (x^i{}_{(i,j)}, \theta)_{n_0+1}, (x^i{}_{(i,j)}, \theta)_{n_0+2}, (x^i{}_{(i,j)}, \theta)_{n_0+3}, (x^i{}_{(i,j)}, \theta), \ldots . (x^i{}_{(i,j)}, \theta)_{n_0+n}, \ldots$ Look carefully at the points and train them in a way as, $(x^i{}_{(i,j)}, \theta)_1, (x^i{}_{(i,j)}, \theta)_2, (x^i{}_{(i,j)}, \theta)_3, (x^i{}_{(i,j)}$, $\theta)_4, \ldots . (x^i{}_{(i,j)}, \theta), \ldots$ generating a finite soft set. Let $1 \le n_{0-1}$. Then $(x^i{}_{(i,j)}, \theta)_i \in \langle g, \theta\rangle$. For this $i, (x^i{}_{(i,j)}, \theta)_i \in \langle g, \theta\rangle$. Hence there exists $\alpha_i \in \Delta$ such that $(x^i{}_{(i,j)}, \theta)_i \in \langle \mathfrak{S}, \theta\rangle_{\alpha_i}$. Evidently $\langle g, \partial\rangle \Subset \widetilde{\bigcup_{r=0}^{n_{0-1}}} \langle \mathfrak{S}, \theta\rangle_{\alpha_i}$. This shows that $\{\langle \mathfrak{S}, \theta\rangle_{\alpha_i} : 0 \le n_{0-1}\}$ is $VS\beta$ open cover of $\langle g, \theta\rangle$. Thus an arbitrary VS β open cover $\{\langle \mathfrak{S}, \theta\rangle_\alpha : \alpha \in \Delta\}$ of $\langle g, \theta\rangle$ is reducible to a finite VS cub-cover $\{\langle \mathfrak{S}, \theta\rangle_{\alpha i} : i = 0, 1, 2, 3, \ldots n_{0-1}\}$, it follows that $\langle g, \theta\rangle$ is soft $VS\beta$ compact.

Theorem 4.8. Let $(\langle\tilde{\mathcal{X}}\rangle, \tau_1, \tau_2, \theta)\, VSBTS$ and $(\langle\tilde{Y}\rangle, \mathfrak{F}_1, \mathfrak{F}_2, \theta)$ be another $VSBTS$. Let $\langle f, \theta\rangle$ be a soft continuous mapping of a soft vague sequentially compact $VS\beta$ space $(\langle\tilde{\mathcal{X}}\rangle, \tau_1, \tau_2, \theta)$ into $(\langle\tilde{Y}\rangle, \mathfrak{F}_1, \mathfrak{F}_2, \theta)$.Then, $\langle f, \theta\rangle(\langle\tilde{\mathcal{X}}\rangle, \tau_1, \tau_2, \theta)$ is $VS\beta$ sequentially compact.

Proof. Given $(\langle\tilde{\mathcal{X}}\rangle, \tau_1, \tau_2, \theta)\, VSBTS$ and $(\langle\tilde{Y}\rangle, \mathfrak{F}_1, \mathfrak{F}_2, \theta)$ be another $VSBTS$. Let $\langle f, \theta\rangle$ be a soft continuous mapping of a VS sequentially compact space $(\langle\tilde{\mathcal{X}}\rangle, \tau_1, \tau_2, \theta)$ into $(\langle\tilde{Y}\rangle, \mathfrak{F}_1, \mathfrak{F}_2, \theta)$ then we have to prove $\langle f, \theta\rangle(\langle\tilde{\mathcal{X}}\rangle, \tau_1, \tau_2, \theta) VS$ sequentially. For this we proceed as. Let $\left\langle \begin{matrix} \widetilde{(y^{i'}{}_{(i',j')}, \theta)}_1, \widetilde{(y^{i'}{}_{(i',j')}, \theta)}_2, \\ , \widetilde{(y^{i'}{}_{(i',j')}, \theta)}_5, \widetilde{(y^{i'}{}_{(i',j')}, \theta)}_6, \\ (y^{i'}{}_{(i',j')}, \theta)_7, \ldots (y^{i'}{}_{(i',j')}, \theta)_n, \ldots \end{matrix} \right\rangle$ be a soft sequence of VS points in $\langle f, \theta\rangle(\langle\tilde{\mathcal{X}}\rangle, \tau_1, \tau_2, \theta)$, Then for each $n \in N$ there exists $\left\langle \begin{array}{c} \widetilde{(x^i{}_{(i,j)},\theta)}_1, \widetilde{(x^i{}_{(i,j)},\theta)}_2, \\ \widetilde{(x^i{}_{(i,j)},\theta)}_4, \widetilde{(x^i{}_{(i,j)},\theta)}_5, \\ \widetilde{(x^i{}_{(i,j)},\theta)}_7, \ldots \widetilde{(x^i{}_{(i,j)},\theta)}_n, \ldots \in (\langle \tilde{\mathcal{X}} \rangle, \tau_1, \tau_2, \theta) \end{array} \right\rangle$ such that

$$\langle f,\theta\rangle \left(\left\langle \begin{array}{c} \widetilde{(x^i{}_{(i,j)},\theta)}_1, \widetilde{(x^i{}_{(i,j)},\theta)}_2, \\ \widetilde{(x^i{}_{(i,j)},\theta)}_3, \\ \widetilde{(x^i{}_{(i,j)},\theta)}_7, \ldots \widetilde{(x^i{}_{(i,j)},\theta)}_n, \ldots \end{array} \right\rangle \right) = \left\langle \begin{array}{c} \left(\widetilde{y^{i'}{}_{(i',j')},\theta}\right)_1, \left(\widetilde{y^{i'}{}_{(i',j')},\theta}\right)_2, \\ \left(\widetilde{y^{i'}{}_{(i',j')},\theta}\right)_3, \left(\widetilde{y^{i'}{}_{(i',j')},\theta}\right)_4, \\ \left(\widetilde{y^{i'}{}_{(i',j')},\theta}\right)_6, \left(\widetilde{y^{i'}{}_{(i',j')},\theta}\right)_7, \\ \ldots \left(\widetilde{y^{i'}{}_{(i',j')},\theta}\right)_n, \ldots \end{array} \right\rangle$$

. Thus we obtain a soft sequence $\left\langle \begin{array}{c} \widetilde{(x^i{}_{(i,j)},\theta)}_1, \widetilde{(x^i{}_{(i,j)},\theta)}_2, \\ \widetilde{(x^i{}_{(i,j)},\theta)}_3, \widetilde{(x^i{}_{(i,j)},\theta)}_4, \\ \widetilde{(x^i{}_{(i,j)},\theta)}, \widetilde{(x^i{}_{(i,j)},\theta)}_6, \\ \widetilde{(x^i{}_{(i,j)},\theta)}_7, \ldots \widetilde{(x^i{}_{(i,j)},\theta)}_n, \ldots \end{array} \right\rangle$ in $(\langle \tilde{\mathcal{X}} \rangle, \tau_1, \tau_2, \theta)$. But $(\langle \tilde{\mathcal{X}} \rangle, \tau_1, \tau_2, \theta)$ being soft sequentially $VSB\beta$ compact, there is a VSB sub-sequence $\langle \widetilde{(x^i{}_{(i,j)},\theta)}_{n_i} \rangle$ of $\langle \widetilde{(x^i{}_{(i,j)},\theta)}_n \rangle$ such that $\langle \widetilde{(x^i{}_{(i,j)},\theta)}_{n_i} \rangle \longrightarrow \widetilde{(x^i{}_{(i,j)},\theta)} \in (\langle \tilde{\mathcal{X}} \rangle, \tau_1, \tau_2, \theta)$. So, by $VS\beta$ continuity of $\langle f,\theta\rangle, \langle \widetilde{(x^i{}_{(i,j)},\theta)}_{n_i} \rangle \longrightarrow \widetilde{(x^i{}_{(i,j)},\theta)}$ implies that $\langle f,\theta\rangle(\langle \widetilde{(x^i{}_{(i,j)},\theta)}_{n_i} \rangle) \longrightarrow \langle f,\theta\rangle(\langle \widetilde{(x^i{}_{(i,j)},\theta)}_n \rangle) \in \langle f,\partial\rangle(\langle \tilde{\mathcal{X}} \rangle, \tau_1, \tau_2, \theta)$. Thus, $\langle f,\partial\rangle(\langle \widetilde{(x^i{}_{(i,j)},\theta)}_{n_i} \rangle)$ is a soft sub-sequence of $\left\langle \begin{array}{c} \left(\widetilde{y^{i'}{}_{(i',j')},\theta}\right)_1, \left(\widetilde{y^{i'}{}_{(i',j')},\theta}\right)_2, \\ \left(\widetilde{y^{i'}{}_{(i',j')},\theta}\right), \widetilde{(x^i{}_{(i,j)},\theta)}_4, \\ \left(\widetilde{y^{i'}{}_{(i',j')},\theta}\right)_5, \left(\widetilde{y^{i'}{}_{(i',j')},\theta}\right)_6, \\ \left(y^{i'}{}_{(i',j')},\theta\right)_7, \ldots \left(y^{i'}{}_{(i',j')},\theta\right)_n, \ldots \end{array} \right\rangle$ converges to $(\langle f,\theta\rangle)(\widetilde{(x^i{}_{(i,j)},\theta)})$ in $\langle f,\theta\rangle(\langle \tilde{\mathcal{X}} \rangle, \tau_1, \tau_2, \theta)$. Hence, $\langle f,\theta\rangle(\langle \tilde{\mathcal{X}} \rangle, \tau_1, \tau_2, \theta)$ is $VS\beta$ sequentially compact.

5 Conclusion

Fuzzy soft topology considers only membership value. It has nothing to do with non-membership value. So extension was needed in this direction. The concept of vague soft topology was introduced to address the issue with fuzzy soft topology. Vague soft topology addresses both membership and non-membership values simultaneously. To make this object more meaningful, the conception of vague soft bi-topological structure is ushered in and its structural characteristics are attempted with new definitions. An ample of examples are also given to understand the structures. For the non-validity of some results, counter examples are provided. Pair-wise vague open and pair-wise vague soft closed sets are also addressed with examples in vague soft bi-topological spaces. Vague soft β separation axioms are inaugurated in vague soft bitopological spaces concerning soft points of the space. Other β separation axioms are also addressed relative to soft points of the space. Vague soft bi-topological properties from one space to another and then from other space to the first space with respect to vague soft β open sets are addressed. Vague soft product spaces are discussed with respect to soft points. In future, we will try to address [27,28] in soft setting with respect to soft points of the spaces. Also, we will try to convert the work of this manuscript to hyper soft and plithogenic hyper soft set based on reference [29].

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