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ARTICLE

Bounds on Fractional-Based Metric Dimension of Petersen Networks

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ABSTRACT

The problem of investigating the minimum set of landmarks consisting of auto-machines (Robots) in a connected network is studied with the concept of location number or metric dimension of this network. In this paper, we study the latest type of metric dimension called as local fractional metric dimension (LFMD) and find its upper bounds for generalized Petersen networks $\mathbb{G}P(n, 3)$, where $n \geq 7$. For $n \geq 9$. The limiting values of LFMD for $\mathbb{G}P(n, 3)$ are also obtained as 1 (bounded) if n approaches to infinity.

KEYWORDS

Metric dimension; local fractional metric dimension; Petersen network; local resolving neighborhoods

1 Introduction

The idea of metric dimension (MD) was firstly introduced by Melter et al. [1]. It has various applications in different areas such as the navigation system, image processing and drug discoveries. A network consists of nodes that are represented by vertices and connections between different vertices are denoted by edges. With the help of edges, an agent can change its position from one vertex to another. Some vertices are referred to as landmarks from which an agent can easily find its location in the network. The set with the minimum number of landmarks is known as the metric basis, and the cardinality of the aforesaid set is known as MD [2,3].

Moreover, the concept of MD in integer programming problem (IPP) was studied by Chartrand et al. [4]. Later on, Oellermann et al. [5,6] produced more refined results of (IPP) through MD. Fehr et al. [7] also derived various results for different graphs which are used to solve relaxation problems by using MD. For more results on MD, see [8,9].

The idea of fractional metric dimension (FMD) for different networks flourished through the work of Arumugam et al. [10]. Moreover, different networks are studied with the help of FMD such as hierarchical, Cartesian, corona, comb and lexicographic products [11–14]. Yi [15] and Liu et al. [16] calculated FMD for permutation and generalized Jahangir networks. Moreover, the sharps bounds of FMD for all the connected networks are studied in [17]. The idea of local fractional metric dimension (LFMD) came through the work of Aisyah et al. in which they computed the LFMD for the corona product of networks [18]. Later on, Liu et al. [19] discussed the LFMD for a particular class of planar networks called by circular ladders and rotationally symmetric networks. Recently, Javaid et al. calculated the bounds of LFMD of connected and prism related networks in [20,21].

In this paper, we find local resolving neighborhood sets (LRNs) of generalized Petersen network $\mathbb{G}P(n,3)$ for $n \geq 5$. After that, we calculated the sharp bounds of the local fractional metric dimension with the help of LRNs. The organization of paper is: Section 1 describes the introduction, Section 2 presents the preliminaries, Section 3 includes the local fractional metric dimensions of the Generalized Petersen network and Section 4 presents the discussion and conclusion.

2 Preliminaries

Mathematically, a network $\mathcal{N}$ consists of vertices set $V(\mathcal{N})$and edges set E($\mathcal{N}$) with property $E(\mathcal{N}) \subseteq V(\mathcal{N}) \times V(\mathcal{N})$. In the present study, only the simple networks without any loop or parallel edges are considered. The distance between two vertices is considered as the length (number of edges) of the shortest path existing between them. For more basic notions, we refer to [22,23].

For any connected graph, $y \in V(\mathcal{N})$ can resolve pair $\{v,w\} \in V(\mathcal{N}$, if $d(y,v) \neq d(y,w)$. Let T be a set which is subset of $V(\mathcal{N})$ known as resolving set of $\mathcal{N}$ if all pair of vertices in $\mathcal{N}$ are resolved by some vertices of T. The cardinality of resolving set is denoted by $|T|$. The set having minimum cardinality among all the resolving sets of $\mathcal{N}$ is called as metric dimension (MD).

For $vw \in E(\mathcal{N})$, the local resolving neighborhood (LRN) set $LR(vw)$ of vw is defined as $LR(vw) = \{x \in V(\mathcal{N}) : d(x,v) \neq d(x,w)\}$. A local resolving function (LRF) is a real valued function $\phi : V(\mathcal{N}) \rightarrow [0,1]$ such that $\phi(LR(vw)) \geq 1$ for each $LR(vw)$ of $\mathcal{N}$, where $\phi\left(LR\left(vw\right)\right) = \sum_{z \in LR(vw)} \phi\left(z\right)$. An LRF g is called minimal if there exists an other function $\phi : V(\mathcal{N}) \rightarrow [0,1]$ such that $\phi \leq g$ and $\phi(u) \neq g(u)$ for at least one $u \in V$, that is not a LRF of $\mathcal{N}$. If $|g| = \sum_{y \in V(\mathcal{N})} \phi\left(x\right)$, then LFMD of $\mathcal{N}$ is defined as

$dim_{lf}(\mathcal{N}) = \min\{|g| : g \text{ is a minimal LRF of } \mathcal{N}\}$. For more detail, see [1,10,19].

For $n \geq 7$, let $\mathbb{G}P(n,3)$ be a generalized Petersen network with vertex set $V(\mathbb{G}P(n,3)) = \{x_i, y_j : 1 \leq i,j \leq n\}$ and edge set $E(\mathbb{G}P(n,3)) = \{x_i x_{i+1} : 1 \leq i \leq n-1\} \cup \{x_i y_i : 1 \leq i \leq n\} \cup \{x_n x_1\} \cup \{y_i y_{i+3} : 1 \leq i \leq n-3\} \cup \{y_{n-2}y_1, y_{n-1}y_2, y_n y_3\}$, where $|V(\mathbb{G}P(n,3))| = 2n$, $|E(\mathbb{G}P(n,3))| = 3n$ (see Fig. 1).

Now we present the following important result which will be frequently used in the main results.

Theorem 1: (*see* [15]) Let $\mathcal{N}(V\mathcal{N}, E(\mathcal{N}))$ be a connected network and $LR(c)$ be a local resolving neighborhood for some $c \in E(\mathcal{N})$. If $|LR(c) \cap Y| \geq \alpha$ for all $c \in E(\mathcal{N})$, then, $dim_{lf}\left(\mathcal{N}\right) \leq \frac{|Y|}{\alpha}$, where, $\alpha = min\{|LR(c)| : c \in E(\mathcal{N})\}$, $Y = \cup\{LR(c) : |LR(c)| = \alpha\}$.

Theorem 2: (*see* [15]) For a connected network $\mathbb{N}$, $dim_{lf}(\mathbb{N}) = 1$ if $\mathbb{N}$ is bipartite.

Theorem 3: (*see* [20]) Let $\mathcal{N}(V\mathcal{N}, E(\mathcal{N}))$ be a connected network and $LR(c)$ be a local resolving neighborhood set. Then, $\frac{|V(\mathcal{N})|}{\gamma} \leq dim_{lf}\left(\mathcal{N}\right)$, where, $\gamma = max\{|LR(c)| : c \in E(\mathcal{N})\}$ and $2 \leq \gamma \leq |V(\mathcal{N})|$.

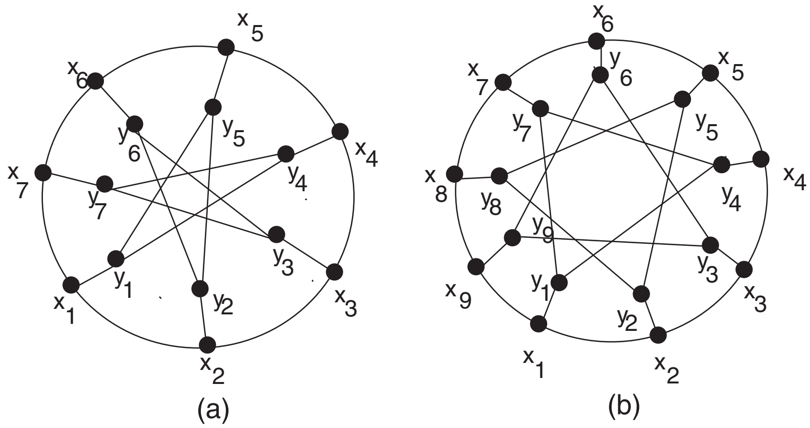

Figure 1: Petersen graphs (a) $\mathbb{G}P(6, 3)$ and (b) $\mathbb{G}P(9, 3)$

3 Local Fractional Metric Dimension Generalized Petersen Network

This section deals with the main findings of the present studies.

Theorem 4. The LFMD of generalized Petersen network $\mathbb{G}P(n, 3)$ for $n = 7$ is $\frac{14}{13} \leq dim_{lf}(\mathbb{G}P(7,3)) \leq \frac{7}{5}$.

Proof: The LRNs of $\mathbb{G}P(n, 3)$ for $n = 7$ are given by:

$LR_1 = LR(x_1x_2) = V(\mathbb{G}P(7,3)) - \{x_5, y_5\}$, $LR_2 = LR(x_2x_3) = V(\mathbb{G}P(7,3)) - \{x_6, y_6\}$,

$LR_3 = LR(x_3x_4) = V(\mathbb{G}P(7,3)) - \{x_7, y_7\}$, $LR_4 = LR(x_4x_5) = V(\mathbb{G}P(7,3)) - \{x_1, y_1\}$,

$LR_5 = LR(x_5x_6) = V(\mathbb{G}P(7,3)) - \{x_2, y_2\}$, $LR_6 = LR(x_6x_7) = V(\mathbb{G}P(7,3)) - \{x_3, y_3\}$,

$LR_7 = LR(x_7x_1) = V(\mathbb{G}P(7,3)) - \{x_4, y_4\}$, $LR_8 = LR(y_1y_4) = V(\mathbb{G}P(7,3)) - \{y_6\}$,

$LR_9 = LR(y_2y_5) = V(\mathbb{G}P(7,3)) - \{y_7\}$, $LR_{10} = LR(y_3y_6) = V(\mathbb{G}P(7,3)) - \{y_1\}$,

$LR_{11} = LR(y_4y_7) = V(\mathbb{G}P(7,3)) - \{y_2\}$, $LR_{12} = LR(y_5y_1) = V(\mathbb{G}P(7,3)) - \{y_3\}$,

$LR_{13} = LR(y_6y_2) = V(\mathbb{G}P(7,3)) - \{y_4\}$, $LR_{14} = LR(y_7y_3) = V(\mathbb{G}P(7,3)) - \{y_5\}$,

$LR(x_1y_1) = V(\mathbb{G}P(7,3)) - \{y_2, y_3, y_6, y_7\} = LR(e_1)$,

$LR(x_2y_2) = V(\mathbb{G}P(7,3)) - \{y_3, y_4, y_7, y_1\} = LR(e_2)$,

$LR(x_3y_3) = V(\mathbb{G}P(7,3)) - \{y_4, y_5, y_1, y_2\} = LR(e_3)$,

$LR(x_4y_4) = V(\mathbb{G}P(7,3)) - \{y_5, y_6, y_2, y_3\} = LR(e_4)$,

$LR(x_5y_5) = V(\mathbb{G}P(7,3)) - \{y_6, y_7, y_3, y_4\} = LR(e_5)$,

$LR(x_6y_6) = V(\mathbb{G}P(7,3)) - \{y_7, y_1, y_4, y_5\} = LR(e_6)$,

$LR(x_7y_7) = V(\mathbb{G}P(7,3)) - \{y_1, y_2, y_5, y_6\} = LR(e_7)$.

For $1 \le m \le 14$ and $1 \le j \le 7$ LRN are $|LR(e_j)| = 10 < |LR_m|$. Furthermore, $\overset{7}{\underset{j=1}{\cup}} LR(e_j) = V(\mathbb{G}P(7,3))$, $|\overset{7}{\underset{j=1}{\cup}} LR(e_j)| = 14$ and $|LR_m \cap \overset{7}{\underset{j=1}{\cup}} LR(e_j)| \ge |LR(e_j)| = 10$. Moreover, $1 \le j \le 7$, $LR(e_j)$ are pairwise nonempty. There exist a minimal LRF $\psi : V(\mathbb{G}P(7,3)) \to [0,1]$ is defined as $\psi(y) = \frac{1}{10}$ for each $y \in \overset{7}{\underset{j=1}{\cup}} LR(e_j)$ and $\psi(y) = 0$ for the vertices of $\mathbb{G}P(7,3)$ which are not in $\overset{7}{\underset{j=1}{\cup}} LR(e_j)$. Therefore, by theorem 1, $dim_{lf}(\mathbb{G}P(7,3)) \le \sum_{j=1}^{14} \frac{1}{10} = \frac{14}{10}$. Since $|V(\mathbb{G}P(7,3))| = \gamma = 13$, then by Theorem 3 we have $\frac{14}{13} \le dim_{lf}(\mathbb{G}P(7,3))$ (as $\mathbb{G}P(7,3)$ is not bipartite network). Therefore, $\frac{14}{13} \le dim_{lf}(\mathbb{G}P(n,3)) \le \frac{7}{5}$.

Lemma 1: Let $\mathbb{G}P(n,3)$ be Generalized Petersen network for, $n \equiv 3(mod\, 6)$ and $n \ge 9$. Then, for $1 \le i \le n-3$, $1 \le j \le n |LR(e_j)| = |LR(e_j = y_i y_{i+3})| = 2n-6 = |LR(y_{n-2}y_1)| = |LR(y_{n-1}y_2)| = |LR(y_n y_3)|$. Moreover, $\overset{n}{\underset{j=1}{\cup}} LR(e_j) = \{x_p : 1 \le p \le n\} \cup \{y_q : 1 \le q \le n\}$ and $|\overset{n}{\underset{j=1}{\cup}} LR(e_j)| = \alpha = 2n$.

Proof: For, $n \ge 9$ and $n \equiv 3(mod\, 6)$ the local resolving neighborhood of generalized Petersen network $\mathbb{G}P(n,3)$, for $1 \le i \le n-3$, $1 \le j \le n$, $p, q \ne \frac{n+3}{2}, \frac{n+5}{2}, \frac{n+7}{2}$

$$LR(y_i y_{i+3}) = \begin{cases} x_p : & 1 \le p \le n \\ \\ y_q : & 1 \le q \le n \end{cases}$$

with $|LR(e_j)| = 2n-6$ and $\overset{n}{\underset{j=1}{\cup}} LR(e_j) = \{x_p : 1 \le p \le n\} \cup \{y_q : 1 \le q \le n\}$ and we have $|\overset{n}{\underset{j=1}{\cup}} LR(e_j)| = 2n$.

Lemma 2: Let $\mathbb{G}P(n,3)$ be generalized Petersen network with $n \equiv 3(mod\, 6)$ and $n \ge 9$, then, for $1 \le i \le n$, $1 \le j \le n$. (a) $|LR(e_j)| < |LR(x_i x_{i+1})|$ and $|LR(x_i x_{i+1}) \cap \left(\overset{n}{\underset{j=1}{\cup}} LR(e_j)\right| \ge |LR(e_j)|$,

(b) $|LR(e_j)| < |LR(x_i y_i)|$ and $|LR(x_i y_i) \cap \left(\overset{n}{\underset{j=1}{\cup}} LR(e_j)\right| \ge |LR(e_j)|$,

Proof: (a) The local resolving neighborhood for $1 \le i \le n$, $1 \le j \le n$, $p, q \ne \frac{n+3}{2}$

$$LR(x_i x_{i+1}) = \begin{cases} x_p : & 1 \le p \le n \\ \\ y_q : & 1 \le q \le n \end{cases}$$

with $|LR(x_i x_{i+1})| = 2n-2 > 2n-6 = |LR(e_j)|$, Therefore, $|LR(x_i x_{i+1}) \cap \left(\overset{n}{\underset{j=1}{\cup}} LRe_j\right)| = 2n-2 > |LR(e_j)|$.

(b) The local resolving neighborhood for $1 \le i \le n$, $1 \le j \le n$,

$$LR(x_i y_i) = \begin{cases} x_p : & 1 \le p \le n \\ \\ y_q : & 1 \le q \le n \end{cases}$$

with $|LR(x_iy_i)| = 2n > 2n - 6 = |LR(e_j)|$, Therefore, $|LR\,(x_iy_i) \cap \left(\bigcup_{j=1}^{n} LRe_j\right)| = 2n > |LR\left(e_j\right)|$.

Theorem 5: Let $\mathbb{G}P(n,3)$ with $n \equiv 3(mod\,6)$ be a generalized Petersen network, where $|V(\mathbb{G}P(n,3))| = 2n$ and $n \geq 9$. Then, $1 \leq dim_{lf}\,(\mathbb{G}P\,(n,3)) \leq \frac{2n}{2n-6}$.

Proof:

Case 1: The LRNs of $\mathbb{G}P(n,3)$ for $n = 9$ are given by:

$LR_1 = LR(x_1x_2) = V(\mathbb{G}P(9,3)) - \{x_6, y_6\}$, $LR_2 = LR(x_2x_3) = V(\mathbb{G}P(9,3)) - \{x_7, y_7\}$,

$LR_3 = LR(x_3x_4) = V(\mathbb{G}P(9,3)) - \{x_8, y_8\}$, $LR_4 = LR(x_4x_5) = V(\mathbb{G}P(9,3)) - \{x_9, y_9\}$,

$LR_5 = LR(x_5x_6) = V(\mathbb{G}P(9,3)) - \{x_1, y_1\}$, $LR_6 = LR(x_6x_7) = V(\mathbb{G}P(9,3)) - \{x_2, y_2\}$,

$LR_7 = LR(x_7x_8) = V(\mathbb{G}P(9,3)) - \{x_3, y_3\}$, $LR_8 = LR(x_8x_9) = V(\mathbb{G}P(9,3)) - \{x_4, y_4\}$,

$LR_9 = LR(x_9x_1) = V(\mathbb{G}P(9,3)) - \{x_5, y_5\}$, $LR_{10} = LR(x_1y_1) = V(\mathbb{G}P(9,3))$,

$LR_{11} = LR(x_2y_2) = V(\mathbb{G}P(9,3))$, $LR_{12} = LR(x_3y_3) = V(\mathbb{G}P(9,3))$,

$LR_{13} = LR(x_4y_4) = V(\mathbb{G}P(9,3))$, $LR_{14} = LR(x_5y_5) = V(\mathbb{G}P(9,3))$,

$LR_{15} = LR(x_6y_6) = V(\mathbb{G}P(9,3))$, $LR_{16} = LR(x_7y_7) = V(\mathbb{G}P(9,3))$,

$LR_{17} = LR(x_8y_8) = V(\mathbb{G}P(9,3))$, $LR_{18} = LR(x_9y_9) = V(\mathbb{G}P(9,3))$,

$LR(y_1y_4) = V(\mathbb{G}P(9,3)) - \{x_6, x_7, x_8, y_6, y_7, y_8\} = LR(e_1)$,

$LR(y_2y_5) = V(\mathbb{G}P(9,3)) - \{x_7, x_8, x_9, y_7, y_8, y_9\} = LR(e_2)$,

$LR(y_3y_6) = V(\mathbb{G}P(9,3)) - \{x_8, x_9, x_1, y_8, y_9, y_1\} = LR(e_3)$,

$LR(y_4y_7) = V(\mathbb{G}P(9,3)) - \{x_9, x_1, x_2, y_9, y_1, y_2\} = LR(e_4)$,

$LR(y_5y_8) = V(\mathbb{G}P(9,3)) - \{x_1, x_2, x_3, y_1, y_2, y_3\} = LR(e_5)$,

$LR(y_6y_9) = V(\mathbb{G}P(9,3)) - \{x_2, x_3, x_4, y_2, y_3, y_4\} = LR(e_6)$,

$LR(y_7y_1) = V(\mathbb{G}P(9,3)) - \{x_3, x_4, x_5, y_3, y_4, y_5\} = LR(e_7)$,

$LR(y_8y_2) = V(\mathbb{G}P(9,3)) - \{x_4, x_5, x_6, y_4, y_5, y_6\} = LR(e_8)$,

$LR(y_9y_3) = V(\mathbb{G}P(9,3)) - \{x_5, x_6, x_7, y_5, y_6, y_7\} = LR(e_9)$.

For $1 \leq m \leq 18$ and $1 \leq j \leq 9$ LRN are $|LR(e_j)| = 12 < |LR_m|$. Furthermore, $\bigcup_{j=1}^{9} LR\left(e_j\right) = V\,(\mathbb{G}P\,(9,3))$, $|\bigcup_{j=1}^{9} LR\left(e_j\right)| = 18$ and $|LR_m \cap \bigcup_{j=1}^{9} LR\left(e_j\right)| \geq |LR\left(e_j\right)| = 12$. Moreover, $1 \leq j \leq 9$, $LR(e_j)$ are pairwise nonempty. There exist a minimal LRF $\psi : V(\mathbb{G}P(9,3)) \rightarrow [0,1]$ is defined as $\psi\,(y) = \frac{1}{12}$ for each $y \in \bigcup_{j=1}^{9} LR\left(e_j\right)$ and $\psi(y) = 0$ for the vertices of $\mathbb{G}P(9,3)$ which are not in $\bigcup_{j=1}^{9} LR\left(e_j\right)$. Therefore, by Theorem 1, $dim_{lf}\,(\mathbb{G}P\,(9,3)) \leq \sum_{j=1}^{18} \frac{18}{12} = \frac{3}{2}$. Since $|V(\mathbb{G}P(9,3))| = \gamma = 18$, then by Theorem 3

$\frac{18}{18} \leq dim_{lf}\left(\mathbb{G}P\left(9,3\right)\right)$ implies $1 \leq dim_{lf}(\mathbb{G}P(9,3))$ (as $\mathbb{G}P(9,3)$ is not bipartite network). Therefore, $1 \leq dim_{lf}\left(\mathbb{G}P\left(9,3\right)\right) \leq \frac{3}{2}$.

Case 2: The LRNs of $\mathbb{G}P(n,3)$ for $n = 15$ are given by:

$LR_1 = LR(x_1x_2) = V(\mathbb{G}P(15,3)) - \{x_9, y_9\}$, $LR_2 = LR(x_2x_3) = V(\mathbb{G}P(15,3)) - \{x_{10}, y_{10}\}$,

$LR_3 = LR(x_3x_4) = V(\mathbb{G}P(15,3)) - \{x_{11}, y_{11}\}$, $LR_4 = LR(x_4x_5) = V(\mathbb{G}P(15,3)) - \{x_{12}, y_{12}\}$,

$LR_5 = LR(x_5x_6) = V(\mathbb{G}P(15,3)) - \{x_{13}, y_{13}\}$, $LR_6 = LR(x_6x_7) = V(\mathbb{G}P(15,3)) - \{x_{14}, y_{14}\}$,

$LR_7 = LR(x_7x_8) = V(\mathbb{G}P(15,3)) - \{x_{15}, y_{15}\}$, $LR_8 = LR(x_8x_9) = V(\mathbb{G}P(15,3)) - \{x_1, y_1\}$,

$LR_9 = LR(x_9x_{10}) = V(\mathbb{G}P(15,3)) - \{x_2, y_2\}$, $LR_{10} = LR(x_{10}x_{11}) = V(\mathbb{G}P(15,3)) - \{x_3, y_3\}$,

$LR_{11} = LR(x_{11}x_{12}) = V(\mathbb{G}P(15,3)) - \{x_4, y_4\}$, $LR_{12} = LR(x_{12}x_{13}) = V(\mathbb{G}P(15,3)) - \{x_5, y_5\}$,

$LR_{13} = LR(x_{13}x_{14}) = V(\mathbb{G}P(15,3)) - \{x_6, y_6\}$, $LR_{14} = LR(x_{14}x_{15}) = V(\mathbb{G}P(15,3)) - \{x_7, y_7\}$,

$LR_{15} = LR(x_{15}x_1) = V(\mathbb{G}P(15,3)) - \{x_8, y_8\}$, $LR_{16} = LR(x_1y_1) = V(\mathbb{G}P(15,3))$,

$LR_{17} = LR(x_2y_2) = V(\mathbb{G}P(15,3))$, $LR_{18} = LR(x_3y_3) = V(\mathbb{G}P(15,3))$,

$LR_{19} = LR(x_4y_4) = V(\mathbb{G}P(15,3))$, $LR_{20} = LR(x_5y_5) = V(\mathbb{G}P(15,3))$,

$LR_{21} = LR(x_6y_6) = V(\mathbb{G}P(15,3))$, $LR_{22} = LR(x_7y_7) = V(\mathbb{G}P(15,3))$,

$LR_{23} = LR(x_8y_8) = V(\mathbb{G}P(15,3))$, $LR_{24} = LR(x_9y_9) = V(\mathbb{G}P(15,3))$,

$LR_{25} = LR(x_{10}y_{10}) = V(\mathbb{G}P(15,3))$, $LR_{26} = LR(x_{11}y_{11}) = V(\mathbb{G}P(15,3))$,

$LR_{27} = LR(x_{12}y_{12}) = V(\mathbb{G}P(15,3))$, $LR_{28} = LR(x_{13}y_{13}) = V(\mathbb{G}P(15,3))$,

$LR_{29} = LR(x_{14}y_{14}) = V(\mathbb{G}P(15,3))$, $LR_{30} = LR(x_{15}y_{15}) = V(\mathbb{G}P(15,3))$,

$LR(y_1y_4) = V(\mathbb{G}P(15,3)) - \{x_9, x_{10}, x_{11}, y_9, y_{10}, y_{11}\} = LR(e_1)$,

$LR(y_2y_5) = V(\mathbb{G}P(15,3)) - \{x_{10}, x_{11}, x_{12}, y_{10}, y_{11}, y_{12}\} = LR(e_2)$,

$LR(y_3y_6) = V(\mathbb{G}P(15,3)) - \{x_{11}, x_{12}, x_{13}, y_{11}, y_{12}, y_{13}\} = LR(e_3)$,

$LR(y_4y_7) = V(\mathbb{G}P(15,3)) - \{x_{12}, x_{13}, x_{14}, y_{12}, y_{13}, y_{14}\} = LR(e_4)$,

$LR(y_5y_8) = V(\mathbb{G}P(15,3)) - \{x_{13}, x_{14}, x_{15}, y_{13}, y_{14}, y_{15}\} = LR(e_5)$,

$LR(y_6y_9) = V(\mathbb{G}P(15,3)) - \{x_{14}, x_{15}, x_1, y_{14}, y_{15}, y_1\} = LR(e_6)$,

$LR(y_7y_{10}) = V(\mathbb{G}P(15,3)) - \{x_{15}, x_1, x_2, y_{15}, y_1, y_2\} = LR(e_7)$,

$LR(y_8y_{11}) = V(\mathbb{G}P(15,3)) - \{x_1, x_2, x_3, y_1, y_2, y_3\} = LR(e_8)$,

$LR(y_9y_{12}) = V(\mathbb{G}P(15,3)) - \{x_2, x_3, x_4, y_2, y_3, y_4\} = LR(e_9)$,

$LR(y_{10}y_{13}) = V(\mathbb{G}P(15,3)) - \{x_3, x_4, x_5, y_3, y_4, y_5\} = LR(e_{10})$,

$LR(y_{11}y_{14}) = V(\mathbb{G}P(15,3)) - \{x_4, x_5, x_6, y_4, y_5, y_6\} = LR(e_{11})$,

$LR(y_{12}y_{15}) = V(\mathbb{G}P(15,3)) - \{x_5, x_6, x_7, y_5, y_6, y_7\} = LR(e_{12})$,

$LR(y_{13}y_1) = V(\mathbb{G}P(15,3)) - \{x_6, x_7, x_8, y_6, y_7, y_8\} = LR(e_{13})$,

$LR(y_{14}y_2) = V(\mathbb{G}P(15,3)) - \{x_7, x_8, x_9, y_7, y_8, y_9\} = LR(e_{14})$,

$LR(y_{15}y_3) = V(\mathbb{G}P(15,3)) - \{x_8, x_9, x_{10}, y_8, y_9, y_{10}\} = LR(e_{15})$.

For $1 \le m \le 30$ and $1 \le j \le 15$ LRN are $|LR(e_j)| = 24 < |LR_m|$. Furthermore, $\overset{15}{\underset{j=1}{\cup}} LR\left(e_j\right) = V\left(\mathbb{G}P\left(15,3\right)\right)$, $|\overset{15}{\underset{j=1}{\cup}} LR\left(e_j\right)| = 30$ and $|LR_m \cap \overset{15}{\underset{j=1}{\cup}} LR\left(e_j\right)| \ge |LR\left(e_j\right)| = 24$. Moreover, $1 \le j \le 15$, $LR(e_j)$ are pairwise nonempty. There exist a minimal LRF $\psi : V(\mathbb{G}P(15,3)) \to [0,1]$ is defined as $\psi\left(y\right) = \frac{1}{24}$ for each $y \in \overset{15}{\underset{j=1}{\cup}} LR\left(e_j\right)$ and $\psi(y) = 0$ for the vertices of $\mathbb{G}P(15,3)$ which are not in $\overset{15}{\underset{j=1}{\cup}} LR\left(e_j\right)$. Therefore, by Theorem 1, $dim_{lf}\left(\mathbb{G}P\left(15,3\right)\right) \le \sum_{j=1}^{30} \frac{30}{20} = \frac{5}{4}$. Since $|V(\mathbb{G}P(15,3))| = \gamma = 30$, then by Theorem 3 we have $\frac{30}{30} \le dim_{lf}\left(\mathbb{G}P\left(15,3\right)\right)$ implies $1 \le dim_{lf}(\mathbb{G}P(15,3))$. As $\mathbb{G}P(15,3)$ is not bipartite network therefore, $1 \le dim_{lf}\left(\mathbb{G}P\left(15,3\right)\right) \le \frac{5}{4}$.

Case 3: For $1 \le i \le n$, $1 \le j \le n$ and $n \ge 19$, $LR(e_j) = LR(y_iy_{i+3})$, $LR(x_ix_{i+1})$, $LR(x_iy_i)$. By Lemmas 1, 2, we have (i) $|LR(x_ix_{i+1})|$, $LR\left(x_iy_i\right) \ge |LR\left(e_j\right)| = 2n-6 = \alpha$, (ii) $|LR\left(x_ix_{i+1}\right) \cap \overset{n}{\underset{j=1}{\cup}} LR\left(e_j\right)|$, $|LR\left(x_iy_i\right) \cap \overset{n}{\underset{j=1}{\cup}} LR\left(e_j\right)| \ge |LR\left(e_j\right)|$ and $\overset{n}{\underset{j=1}{\cup}} LR\left(e_j\right) = 2n = \beta$. The intersection of LRS having minimum cardinality is not empty. Therefore, there exist a mimimal local resolving $\psi' : V(\mathbb{G}P(n,3)) \to [0,1]$ such that $|\psi'| < |\psi|$, where the minimal LRF $\psi : V(\mathbb{G}P(n,3)) \to [0,1]$ is defined as $\phi\left(v\right) = \left\{\frac{1}{\alpha}\ for\ v \in \overset{n}{\underset{j=1}{\cup}} LR\left(e_j\right)\right\}$.

Therefore, by Theorem 1, $dim_{lf}\left(\mathbb{G}P\left(n,3\right)\right) \le \sum_{t=1}^{\beta} \frac{1}{\alpha} = \frac{2n}{2n-6}$. Since $|V(\mathbb{G}P(n,3))| = \gamma = 2n$, then by Theorem 3 we have $\frac{2n}{2n} \le dim_{lf}\left(\mathbb{G}P\left(n,3\right)\right)$ implies $1 \le dim_{lf}(\mathbb{G}P(n,3))$. As $\mathbb{G}P(n,3)$ is not bipartite network therefore, $1 \le dim_{lf}\left(\mathbb{G}P\left(n,3\right)\right) \le \frac{2n}{2n-6}$.

Lemma 3: Let $\mathbb{G}P(n,3)$ be Generalized Petersen network for, $n \equiv 3(mod\,6)$ and $n \ge 11$. Then, for $1 \le i \le n-3$, $1 \le j \le n|LR(e_j)| = |LR(e_j = y_iy_{i+3})| = 2n-6 = |LR(y_{n-2}y_1)| = |LR(y_{n-1}y_2)| = |LR(y_ny_3)|$. Moreover, $\overset{n}{\underset{j=1}{\cup}} LR\left(e_j\right) = \left\{x_p : 1 \le p \le n\right\} \cup \left\{y_q : 1 \le q \le n\right\}$ and $|\overset{n}{\underset{j=1}{\cup}} LR\left(e_j\right)| = \alpha = 2n$.

Proof: For, $n \ge 11$ and $n \equiv 3(mod\,6)$ the local resolving neighborhood of generalized Petersen network $\mathbb{G}P(n,3)$, for $1 \le i \le n-3$, $1 \le j \le n$, $p \ne \frac{n+1}{2}, \frac{n+5}{2}, \frac{n+9}{2}$, $q \ne \frac{n-5}{2}, \frac{n+5}{2}, \frac{n+15}{2}$,

$$LR\left(y_iy_{i+3}\right) = \begin{cases} x_p : & 1 \le p \le n \\ y_q : & 1 \le q \le n \end{cases}$$

with $|LR(e_j)| = 2n-6$ and $\overset{n}{\underset{j=1}{\cup}} LR\left(e_j\right) = \left\{x_p : 1 \le p \le n\right\} \cup \left\{y_q : 1 \le q \le n\right\}$ and we have $|\overset{n}{\underset{j=1}{\cup}} LR\left(e_j\right)| = 2n$.

Lemma 4: Let $\mathbb{G}P(n,3)$ be generalized Petersen network with $n \equiv 3(mod\,6)$ and $n \ge 11$, then, for $1 \le i \le n$, $1 \le j \le n$.

(a) $|LR(e_j)| < |LR(x_ix_{i+1})|$ and $|LR\left(x_ix_{i+1}\right) \cap \left(\overset{n}{\underset{j=1}{\cup}} LR\left(e_j\right)\right| \ge |LR\left(e_j\right)|$,

(b) $|LR(e_j)| < |LR(x_iy_i)|$ and $|LR(x_iy_i) \cap \left(\bigcup_{j=1}^{n} LR(e_j)\right| \geq |LR(e_j)|$,

Proof: (a) The local resolving neighborhood for $1 \leq i \leq n$, $1 \leq j \leq n$, $p, q \neq \frac{n+3}{2}$

$$LR(x_ix_{i+1}) = \begin{cases} x_p: & 1 \leq p \leq n \\ y_q: & 1 \leq q \leq n \end{cases}$$

with $|LR(x_ix_{i+1})| = 2n - 2 > 2n - 6 = |LR(e_j)|$, Therefore, $|LR(x_ix_{i+1}) \cap \left(\bigcup_{j=1}^{n} LRe_j\right)| = 2n - 2 > |LR(e_j)|$.

(b) The local resolving neighborhood for $1 \leq i \leq n$, $1 \leq j \leq n$,

$$LR(x_iy_i) = \begin{cases} x_p: & 1 \leq p \leq n \\ y_q: & 1 \leq q \leq n \end{cases}$$

with $|LR(x_iy_i)| = 2n > 2n - 6 = |LR(e_j)|$, Therefore, $|LR(x_iy_i) \cap \left(\bigcup_{j=1}^{n} LRe_j\right)| = 2n > |LR(e_j)|$.

Theorem 6: Let $\mathbb{G}P(n,3)n \equiv 5(mod\,6)$ be a generalized Petersen network, where $|V(\mathbb{G}P(n,2))| = 2n$ and $n \geq 11$. Then, $1 \leq dim_{lf}(\mathbb{G}P(n,3)) \leq \frac{2n}{2n-6}$.

Proof:

The LRNs of $\mathbb{G}P(n,3)$ for $n = 11$ are given by:

$LR_1 = LR(x_1x_2) = V(\mathbb{G}P(11,3)) - \{x_7, y_7\}$, $LR_2 = LR(x_2x_3) = V(\mathbb{G}P(11,3)) - \{x_8, y_8\}$,

$LR_3 = LR(x_3x_4) = V(\mathbb{G}P(11,3)) - \{x_9, y_9\}$, $LR_4 = LR(x_4x_5) = V(\mathbb{G}P(11,3)) - \{x_{10}, y_{10}\}$,

$LR_5 = LR(x_5x_6) = V(\mathbb{G}P(11,3)) - \{x_{11}, y_{11}\}$, $LR_6 = LR(x_6x_7) = V(\mathbb{G}P(11,3)) - \{x_1, y_1\}$,

$LR_7 = LR(x_7x_8) = V(\mathbb{G}P(11,3)) - \{x_2, y_2\}$, $LR_8 = LR(x_8x_9) = V(\mathbb{G}P(11,3)) - \{x_3, y_3\}$,

$LR_9 = LR(x_9x_{10}) = V(\mathbb{G}P(11,3)) - \{x_4, y_4\}$, $LR_{10} = LR(x_{10}x_{11}) = V(\mathbb{G}P(11,3)) - \{x_5, y_5\}$,

$LR_{11} = LR(x_{11}x_1) = V(\mathbb{G}P(11,3)) - \{x_6, y_6\}$, $LR_{12} = LR(x_1y_1) = V(\mathbb{G}P(11,3))$,

$LR_{13} = LR(x_2y_2) = V(\mathbb{G}P(11,3))$, $LR_{14} = LR(x_3y_3) = V(\mathbb{G}P(11,3))$,

$LR_{15} = LR(x_4y_4) = V(\mathbb{G}P(11,3))$, $LR_{16} = LR(x_5y_5) = V(\mathbb{G}P(11,3))$,

$LR_{17} = LR(x_6y_6) = V(\mathbb{G}P(11,3))$, $LR_{18} = LR(x_7y_7) = V(\mathbb{G}P(11,3))$,

$LR_{19} = LR(x_8y_8) = V(\mathbb{G}P(11,3))$, $LR_{20} = LR(x_9y_9) = V(\mathbb{G}P(11,3))$,

$LR_{21} = LR(x_{10}y_{10}) = V(\mathbb{G}P(11,3))$, $LR_{22} = LR(x_{11}y_{11}) = V(\mathbb{G}P(11,3))$,

$LR(y_1y_4) = V(\mathbb{G}P(11,3)) - \{x_6, x_8, x_{10}, y_2, y_3, y_8\} = LR(e_1)$,

$LR(y_2y_5) = V(\mathbb{G}P(11,3)) - \{x_7, x_9, x_{11}, y_3, y_4, y_9\} = LR(e_2)$,

$LR(y_3y_6) = V(\mathbb{G}P(11,3)) - \{x_8, x_{10}, x_1, y_4, y_5, y_{10}\} = LR(e_3)$,

$LR(y_4y_7) = V(\mathbb{G}P(11,3)) - \{x_9, x_{11}, x_2, y_5, y_6, y_{11}\} = LR(e_4)$,

$LR(y_5y_8) = V(\mathbb{G}P(11,3)) - \{x_{10}, x_1, x_3, y_6, y_7, y_1\} = LR(e_5)$,

$LR(y_6y_9) = V(\mathbb{G}P(11,3)) - \{x_{11}, x_2, x_4, y_7, y_8, y_2\} = LR(e_6)$,

$LR(y_7y_{10}) = V(\mathbb{G}P(11,3)) - \{x_1, x_3, x_5, y_8, y_9, y_3\} = LR(e_7)$,

$LR(y_8y_{11}) = V(\mathbb{G}P(11,3)) - \{x_2, x_4, x_6, y_9, y_{10}, y_4\} = LR(e_8)$,

$LR(y_9y_1) = V(\mathbb{G}P(11,3)) - \{x_3, x_5, x_7, y_{10}, y_{11}, y_5\} = LR(e_9)$,

$LR(y_{10}y_2) = V(\mathbb{G}P(11,3)) - \{x_4, x_6, x_8, y_{11}, y_1, y_6\} = LR(e_{10})$,

$LR(y_{11}y_3) = V(\mathbb{G}P(11,3)) - \{x_5, x_7, x_9, y_1, y_2, y_7\} = LR(e_{11})$.

For $1 \le m \le 22$ and $1 \le j \le 11$ LRN are $|LR(e_j)| = 16 < |LR_m|$. Furthermore, $\bigcup_{j=1}^{11} LR(e_j) = V(\mathbb{G}P(11,2))$, $|\bigcup_{j=1}^{11} LR(e_j)| = 22$ and $|LR_m \cap \bigcup_{j=1}^{11} LR(e_j)| \ge |LR(e_j)| = 16$. Moreover, $1 \le j \le 11$, $LR(e_j)$ are pairwise nonempty. There exist a minimal LRF $\psi : V(\mathbb{G}P(14,2)) \to [0,1]$ is defined as $\psi(y) = \frac{1}{16}$ for each $y \in \bigcup_{j=1}^{11} LR(e_j)$ and $\psi(y) = 0$ for the vertices of $\mathbb{G}P(11,3)$ which are not in $\bigcup_{j=1}^{14} LR(e_j)$. Therefore, by Theorem 1, $dim_{lf}(\mathbb{G}P(11,3)) \le \sum_{j=1}^{22} \frac{1}{16} = \frac{11}{8}$. Since $|V(\mathbb{G}P(11,3))| = \gamma = 22$, then by Theorem 3 we have $\frac{22}{22} \le dim_{lf}(\mathbb{G}P(11,3))$ implies $1 \le dim_{lf}(\mathbb{G}P(11,3))$. As $\mathbb{G}P(11,3)$ is not bipartite network therefore, $1 \le dim_{lf}(\mathbb{G}P(11,3)) \le \frac{11}{8}$.

Case 2: The LRNs of $\mathbb{G}P(n,3)$ for $n = 17$ are given by:

$LR_1 = LR(x_1x_2) = V(\mathbb{G}P(17,3)) - \{x_{10}, y_{10}\}$, $LR_2 = LR(x_2x_3) = V(\mathbb{G}P(17,3)) - \{x_{11}, y_{11}\}$,

$LR_3 = LR(x_3x_4) = V(\mathbb{G}P(17,3)) - \{x_{12}, y_{12}\}$, $LR_4 = LR(x_4x_5) = V(\mathbb{G}P(17,3)) - \{x_{13}, y_{13}\}$,

$LR_5 = LR(x_5x_6) = V(\mathbb{G}P(17,3)) - \{x_{14}, y_{14}\}$, $LR_6 = LR(x_6x_7) = V(\mathbb{G}P(17,3)) - \{x_{15}, y_{15}\}$,

$LR_7 = LR(x_7x_8) = V(\mathbb{G}P(17,3)) - \{x_{16}, y_{16}\}$, $LR_8 = LR(x_8x_9) = V(\mathbb{G}P(17,3)) - \{x_{17}, y_{17}\}$,

$LR_9 = LR(x_9x_{10}) = V(\mathbb{G}P(17,3)) - \{x_1, y_1\}$, $LR_{10} = LR(x_{10}x_{11}) = V(\mathbb{G}P(17,3)) - \{x_2, y_2\}$,

$LR_{11} = LR(x_{11}x_{12}) = V(\mathbb{G}P(17,3)) - \{x_3, y_3\}$, $LR_{12} = LR(x_{12}x_{13}) = V(\mathbb{G}P(17,3)) - \{x_4, y_4\}$,

$LR_{13} = LR(x_{13}x_{14}) = V(\mathbb{G}P(17,3)) - \{x_5, y_5\}$, $LR_{14} = LR(x_{14}x_{15}) = V(\mathbb{G}P(17,3)) - \{x_6, y_6\}$,

$LR_{15} = LR(x_{15}x_{16}) = V(\mathbb{G}P(17,3)) - \{x_7, y_7\}$, $LR_{16} = LR(x_{16}x_{17}) = V(\mathbb{G}P(17,3)) - \{x_8, y_8\}$,

$LR_{17} = LR(x_{17}x_1) = V(\mathbb{G}P(17,3)) - \{x_9, y_9\}$, $LR_{18} = LR(x_1y_1) = V(\mathbb{G}P(17,3))$,

$LR_{19} = LR(x_2y_2) = V(\mathbb{G}P(17,3))$, $LR_{20} = LR(x_3y_3) = V(\mathbb{G}P(17,3))$,

$LR_{21} = LR(x_4y_4) = V(\mathbb{G}P(17,3))$, $LR_{22} = LR(x_5y_5) = V(\mathbb{G}P(17,3))$,

$LR_{23} = LR(x_6y_6) = V(\mathbb{G}P(17,3))$, $LR_{24} = LR(x_7y_7) = V(\mathbb{G}P(17,3))$,

$LR_{25} = LR(x_8y_8) = V(\mathbb{G}P(17,3))$, $LR_{26} = LR(x_9y_9) = V(\mathbb{G}P(17,3))$,

$LR_{27} = LR(x_{10}y_{10}) = V(\mathbb{G}P(17,3))$, $LR_{28} = LR(x_{11}y_{11}) = V(\mathbb{G}P(17,3))$,

$LR_{29} = LR(x_{12}y_{12}) = V(\mathbb{G}P(17,3))$, $LR_{30} = LR(x_{13}y_{13}) = V(\mathbb{G}P(17,3))$,

$LR_{31} = LR(x_{14}y_{14}) = V(\mathbb{G}P(17,3))$, $LR_{32} = LR(x_{15}y_{15}) = V(\mathbb{G}P(17,3))$,
$LR_{33} = LR(x_{16}y_{16}) = V(\mathbb{G}P(17,3))$, $LR_{34} = LR(x_{17}y_{17}) = V(\mathbb{G}P(17,3))$,
$LR(y_1y_4) = V(\mathbb{G}P(17,3)) - \{x_9, x_{11}, x_{13}, y_6, y_{11}, y_{16}\} = LR(e_1)$,
$LR(y_2y_5) = V(\mathbb{G}P(17,3)) - \{x_{10}, x_{12}, x_{14}, y_7, y_{12}, y_{17}\} = LR(e_2)$,
$LR(y_3y_6) = V(\mathbb{G}P(17,3)) - \{x_{11}, x_{13}, x_{15}, y_8, y_{13}, y_1\} = LR(e_3)$,
$LR(y_4y_7) = V(\mathbb{G}P(17,3)) - \{x_{12}, x_{14}, x_{16}, y_9, y_{14}, y_2\} = LR(e_4)$,
$LR(y_5y_8) = V(\mathbb{G}P(17,3)) - \{x_{13}, x_{15}, x_{17}, y_{10}, y_{15}, y_3\} = LR(e_5)$,
$LR(y_6y_9) = V(\mathbb{G}P(17,3)) - \{x_{14}, x_{16}, x_1, y_{11}, y_{16}, y_4\} = LR(e_6)$,
$LR(y_7y_{10}) = V(\mathbb{G}P(17,3)) - \{x_{15}, x_{17}, x_2, y_{12}, y_{17}, y_5\} = LR(e_7)$,
$LR(y_8y_{11}) = V(\mathbb{G}P(17,3)) - \{x_{16}, x_1, x_3, y_{13}, y_1, y_6\} = LR(e_8)$,
$LR(y_9y_{12}) = V(\mathbb{G}P(17,3)) - \{x_{17}, x_2, x_4, y_{14}, y_2, y_7\} = LR(e_9)$,
$LR(y_{10}y_{13}) = V(\mathbb{G}P(17,3)) - \{x_1, x_3, x_5, y_{15}, y_3, y_8\} = LR(e_{10})$,
$LR(y_{11}y_{14}) = V(\mathbb{G}P(17,3)) - \{x_2, x_4, x_6, y_{16}, y_4, y_9\} = LR(e_{11})$,
$LR(y_{12}y_{15}) = V(\mathbb{G}P(17,3)) - \{x_3, x_5, x_7, y_{17}, y_5, y_{10}\} = LR(e_{12})$,
$LR(y_{13}y_{16}) = V(\mathbb{G}P(17,3)) - \{x_4, x_6, x_8, y_1, y_6, y_{11}\} = LR(e_{13})$,
$LR(y_{14}y_{17}) = V(\mathbb{G}P(17,3)) - \{x_5, x_7, x_9, y_2, y_7, y_{12}\} = LR(e_{14})$,
$LR(y_{15}y_1) = V(\mathbb{G}P(17,3)) - \{x_6, x_8, x_{10}, y_3, y_8, y_{13}\} = LR(e_{15})$,
$LR(y_{16}y_2) = V(\mathbb{G}P(17,3)) - \{x_7, x_9, x_{11}, y_4, y_9, y_{14}\} = LR(e_{16})$,
$LR(y_{17}y_3) = V(\mathbb{G}P(17,3)) - \{x_8, x_{10}, x_{12}, y_5, y_{10}, y_{15}\} = LR(e_{17})$.

For $1 \le m \le 34$ and $1 \le j \le 17$ LRN are $|LR(e_j)| = 28 < |LR_m|$. Furthermore, $\overset{17}{\underset{j=1}{\cup}} LR\left(e_j\right) = V\left(\mathbb{G}P\left(17,3\right)\right)$, $|\overset{17}{\underset{j=1}{\cup}} LR\left(e_j\right)| = 34$ and $|LR_m \cap \overset{17}{\underset{j=1}{\cup}} LR\left(e_j\right)| \ge |LR\left(e_j\right)| = 28$. Moreover, $1 \le j \le 17$, $LR(e_j)$ are pairwise nonempty. There exist a minimal LRF $\psi : V(\mathbb{G}P(17,3)) \to [0,1]$ is defined as $\psi\left(y\right) = \frac{1}{28}$ for each $y \in \overset{17}{\underset{j=1}{\cup}} LR\left(e_j\right)$ and $\psi(y) = 0$ for the vertices of $\mathbb{G}P(17,3)$ which are not in $\overset{17}{\underset{j=1}{\cup}} LR\left(e_j\right)$. Therefore, by Theorem 1, $dim_{lf}\left(\mathbb{G}P\left(17,3\right)\right) \le \sum_{j=1}^{34} \frac{1}{24} = \frac{17}{14}$.

$$dim_{lf}\left(\mathbb{G}P\left(17,3\right)\right) \le \frac{17}{14}.$$

Case 3: For $1 \le i \le n$, $1 \le j \le n$ and $n \ge 23$, $LR(e_j) = LR(y_iy_{i+3})$, $LR(x_ix_{i+1})$, $LR(x_iy_i)$. By Lemmas 3, 4, we have (i) $|LR(x_ix_{i+1})|$, $LR(x_iy_i) \ge |LR(e_j)| = 2n - 6 = \alpha$, (ii) $|LR\left(x_ix_{i+1}\right) \cap \overset{n}{\underset{j=1}{\cup}} LR\left(e_j\right)|$, $|LR\left(x_iy_i\right) \cap \overset{n}{\underset{j=1}{\cup}} LR\left(e_j\right)| \ge |LR\left(e_j\right)|$ and $\overset{n}{\underset{j=1}{\cup}} LR\left(e_j\right) = 2n = \beta$. The intersection of LRS having minimum cardinality is not empty. Therefore, there exist a mimimal local resolving $\psi' : V(\mathbb{G}P(n,3)) \rightarrow [0,1]$ such that $|\psi'| < |\psi|$, where the minimal LRF $\psi : V(\mathbb{G}P(n,3)) \rightarrow [0,1]$ is defined as $\phi(v) = \left\{\frac{1}{\alpha}\ for\ v \in \overset{n}{\underset{j=1}{\cup}} LR\left(e_j\right)\right\}$.

Therefore, by Theorem 1, $dim_{lf}(\mathbb{G}P(n,3)) \leq \sum_{t=1}^{\beta} \frac{1}{\alpha} = \frac{2n}{2n-6}$. Since $|V(\mathbb{G}P(n,3))| = \gamma = 2n$, then by Theorem 3 we have $\frac{2n}{2n} \leq dim_{lf}(\mathbb{G}P(n,3))$ implies $1 \leq dim_{lf}(\mathbb{G}P(n,3))$. As $\mathbb{G}P(n,3)$ is not bipartite network therefore, $1 \leq dim_{lf}(\mathbb{G}P(n,3)) \leq \frac{2n}{2n-6}$.

Lemma 5: Let $\mathbb{G}P(n,3)$ be Generalized Petersen network for, $n \equiv 1(mod\,6)$ and $n \geq 13$. Then, for $1 \leq i \leq n-3$, $1 \leq j \leq n\,|LR(e_j)| = |LR(e_j = y_iy_{i+3})| = 2n-6 = |LR(y_{n-2}y_1)| = |LR(y_{n-1}y_2)| = |LR(y_ny_3)|$. Moreover, $\overset{n}{\underset{j=1}{\cup}} LR\left(e_j\right) = \left\{x_p : 1 \leq p \leq n\right\} \cup \left\{y_q : 1 \leq q \leq n\right\}$ and $|\overset{n}{\underset{j=1}{\cup}} LR\left(e_j\right)| = \alpha = 2n$.

Proof: For, $n \geq 13$ and $n \equiv 3(mod\,6)$ the local resolving neighborhood of generalized Petersen network $\mathbb{G}P(n,3)$, for $1 \leq i \leq n-3$, $1 \leq j \leq n$, $p \neq \frac{n+3}{2}, \frac{n+5}{2}, \frac{n+7}{2}$, $q \neq \frac{n-3}{2}, \frac{n+5}{2}, \frac{n+13}{2}$,

$$LR\left(y_iy_{i+3}\right) = \begin{cases} x_p: & 1 \leq p \leq n \\ y_q: & 1 \leq q \leq n \end{cases}$$

with $|LR(e_j)| = 2n-6$ and $\overset{n}{\underset{j=1}{\cup}} LR\left(e_j\right) = \left\{x_p : 1 \leq p \leq n\right\} \cup \left\{y_q : 1 \leq q \leq n\right\}$ and we have $|\overset{n}{\underset{j=1}{\cup}} LR\left(e_j\right)| = 2n$.

Lemma 6: Let $\mathbb{G}P(n,3)$ be generalized Petersen network with $n \equiv 3(mod\,6)$ and $n \geq 13$, then, for $1 \leq i \leq n$, $1 \leq j \leq n$. (a) $|LR(e_j)| < |LR(x_ix_{i+1})|$ and $|LR\left(x_ix_{i+1}\right) \cap \left(\overset{n}{\underset{j=1}{\cup}} LR\left(e_j\right) | \geq |LR\left(e_j\right)|\right.$,

(b) $|LR(e_j)| < |LR(x_iy_i)|$ and $|LR\left(x_iy_i\right) \cap \left(\overset{n}{\underset{j=1}{\cup}} LR\left(e_j\right) | \geq |LR\left(e_j\right)|\right.$,

Proof: (a) The local resolving neighborhood for $1 \leq i \leq n$, $1 \leq j \leq n$, $p, q \neq \frac{n+3}{2}$

$$LR\left(x_ix_{i+1}\right) = \begin{cases} x_p: & 1 \leq p \leq n \\ y_q: & 1 \leq q \leq n \end{cases}$$

with $|LR(x_ix_{i+1})| = 2n-2 > 2n-6 = |LR(e_j)|$, Therefore, $|LR\left(x_ix_{i+1}\right) \cap \left(\overset{n}{\underset{j=1}{\cup}} LRe_j\right)| = 2n-2 > |LR\left(e_j\right)|$.

(b) The local resolving neighborhood for $1 \leq i \leq n$, $1 \leq j \leq n$,

$$LR\left(x_iy_i\right) = \begin{cases} x_p: & 1 \leq p \leq n \\ y_q: & 1 \leq q \leq n \end{cases}$$

with $|LR(x_iy_i)| = 2n > 2n-6 = |LR(e_j)|$, Therefore, $|LR\left(x_iy_i\right) \cap \left(\overset{n}{\underset{j=1}{\cup}} LRe_j\right)| = 2n > |LR\left(e_j\right)|$.

Theorem 6: Let $\mathbb{G}P(n,3)$ with $n \equiv 3(mod\,6)$ be a generalized Petersen network, where $|V(\mathbb{G}P(n,3))| = 2n$ and $n \geq 13$. Then, $1 \leq dim_{lf}(\mathbb{G}P(n,3)) \leq \frac{2n}{2n-6}$.

Proof:

The LRNs of $\mathbb{G}P(n,3)$ for $n = 13$ are given by:

$LR_1 = LR(x_1x_2) = V(\mathbb{G}P(13,3)) - \{x_8, y_8\}$, $LR_2 = LR(x_2x_3) = V(\mathbb{G}P(13,3)) - \{x_9, y_9\}$,

$LR_3 = LR(x_3x_4) = V(\mathbb{G}P(13,3)) - \{x_{10}, y_{10}\}$, $LR_4 = LR(x_4x_5) = V(\mathbb{G}P(13,3)) - \{x_{11}, y_{11}\}$,

$LR_5 = LR(x_5x_6) = V(\mathbb{G}P(13,3)) - \{x_{12}, y_{12}\}$, $LR_6 = LR(x_6x_7) = V(\mathbb{G}P(13,3)) - \{x_{13}, y_{13}\}$,

$LR_7 = LR(x_7x_8) = V(\mathbb{G}P(13,3)) - \{x_1, y_1\}$, $LR_8 = LR(x_8x_9) = V(\mathbb{G}P(13,3)) - \{x_2, y_2\}$,

$LR_9 = LR(x_9x_{10}) = V(\mathbb{G}P(13,3)) - \{x_3, y_3\}$, $LR_{10} = LR(x_{10}x_{11}) = V(\mathbb{G}P(13,3)) - \{x_4, y_4\}$,

$LR_{11} = LR(x_{11}x_{12}) = V(\mathbb{G}P(13,3)) - \{x_5, y_5\}$, $LR_{12} = LR(x_{12}x_{13}) = V(\mathbb{G}P(13,3)) - \{x_6, y_6\}$,

$LR_{13} = LR(x_{13}x_1) = V(\mathbb{G}P(13,3)) - \{x_7, y_7\}$, $LR_{14} = LR(x_1y_1) = V(\mathbb{G}P(13,3))$,

$LR_{15} = LR(x_2y_2) = V(\mathbb{G}P(13,3))$, $LR_{16} = LR(x_3y_3) = V(\mathbb{G}P(13,3))$,

$LR_{17} = LR(x_4y_4) = V(\mathbb{G}P(13,3))$, $LR_{18} = LR(x_5y_5) = V(\mathbb{G}P(13,3))$,

$LR_{19} = LR(x_6y_6) = V(\mathbb{G}P(13,3))$, $LR_{20} = LR(x_7y_7) = V(\mathbb{G}P(13,3))$,

$LR_{21} = LR(x_8y_8) = V(\mathbb{G}P(13,3))$, $LR_{22} = LR(x_9y_9) = V(\mathbb{G}P(13,3))$,

$LR_{23} = LR(x_{10}y_{10}) = V(\mathbb{G}P(13,3))$, $LR_{24} = LR(x_{11}y_{11}) = V(\mathbb{G}P(13,3))$,

$LR_{25} = LR(x_{12}y_{12}) = V(\mathbb{G}P(13,3))$, $LR_{26} = LR(x_{13}y_{13}) = V(\mathbb{G}P(13,3))$,

$LR(y_1y_4) = V(\mathbb{G}P(13,3)) - \{x_8, x_9, x_{10}, y_5, y_9, y_{13}\} = LR(e_1)$,

$LR(y_2y_5) = V(\mathbb{G}P(13,3)) - \{x_9, x_{10}, x_{11}, y_6, y_{10}, y_1\} = LR(e_2)$,

$LR(y_3y_6) = V(\mathbb{G}P(13,3)) - \{x_{10}, x_{11}, x_{12}, y_7, y_{11}, y_2\} = LR(e_3)$,

$LR(y_4y_7) = V(\mathbb{G}P(13,3)) - \{x_{11}, x_{12}, x_{13}, y_8, y_{12}, y_3\} = LR(e_4)$,

$LR(y_5y_8) = V(\mathbb{G}P(13,3)) - \{x_{12}, x_{13}, x_1, y_9, y_{13}, y_4\} = LR(e_5)$,

$LR(y_6y_9) = V(\mathbb{G}P(13,3)) - \{x_{13}, x_1, x_2, y_{10}, y_1, y_5\} = LR(e_6)$,

$LR(y_7y_{10}) = V(\mathbb{G}P(13,3)) - \{x_1, x_2, x_3, y_{11}, y_2, y_6\} = LR(e_7)$,

$LR(y_8y_{11}) = V(\mathbb{G}P(13,3)) - \{x_2, x_3, x_4, y_{12}, y_3, y_7\} = LR(e_8)$,

$LR(y_9y_{12}) = V(\mathbb{G}P(13,3)) - \{x_3, x_4, x_5, y_{13}, y_4, y_8\} = LR(e_9)$,

$LR(y_{10}y_{13}) = V(\mathbb{G}P(13,3)) - \{x_4, x_5, x_6, y_1, y_5, y_9\} = LR(e_{10})$,

$LR(y_{11}y_1) = V(\mathbb{G}P(13,3)) - \{x_5, x_6, x_7, y_2, y_6, y_{10}\} = LR(e_{11})$,

$LR(y_{12}y_2) = V(\mathbb{G}P(13,3)) - \{x_6, x_7, x_8, y_3, y_7, y_{11}\} = LR(e_{12})$,

$LR(y_{13}y_3) = V(\mathbb{G}P(13,3)) - \{x_7, x_8, x_9, y_4, y_8, y_{12}\} = LR(e_{13})$.

For $1 \le m \le 26$ and $1 \le j \le 13$ LRN are $|LR(e_j)| = 20 < |LR_m|$. Furthermore, $\bigcup_{j=1}^{13} LR\left(e_j\right) = V\left(\mathbb{G}P\left(13,3\right)\right)$, $|\bigcup_{j=1}^{13} LR\left(e_j\right)| = 26$ and $|LR_m \cap \bigcup_{j=1}^{13} LR\left(e_j\right)| \ge |LR\left(e_j\right)| = 20$. Moreover, $1 \le j \le 13$,

$LR(e_j)$ are pairwise nonempty. There exist a minimal LRF $\psi : V(\mathbb{G}P(13,2)) \rightarrow [0,1]$ is defined as $\psi\,(y) = \frac{1}{20}$ for each $y \in \bigcup_{j=1}^{13} LR\left(e_j\right)$ and $\psi(y) = 0$ for the vertices of $\mathbb{G}P(13,3)$ which are not in $\bigcup_{j=1}^{13} LR\left(e_j\right)$. Therefore, by Theorem 1, $dim_{lf}\,(\mathbb{G}P\,(13,3)) \leq \sum_{j=1}^{26} \frac{1}{20} = \frac{13}{10}$. Since $|V(\mathbb{G}P(13,3))| = \gamma = 26$, then by Theorem 3 we have $\frac{26}{26} \leq dim_{lf}\,(\mathbb{G}P\,(13,3))$ implies $1 \leq dim_{lf}(\mathbb{G}P(13,3))$. As $\mathbb{G}P(13,3)$ is not bipartite network, therefore, $1 \leq dim_{lf}\,(\mathbb{G}P\,(13,3)) \leq \frac{13}{10}$.

Case 2: The LRNs of $\mathbb{G}P(n,3)$ for $n = 19$ are given by:

$LR_1 = LR(x_1x_2) = V(\mathbb{G}P(19,3)) - \{x_{11}, y_{11}\}$, $LR_2 = LR(x_2x_3) = V(\mathbb{G}P(19,3)) - \{x_{12}, y_{12}\}$,

$LR_3 = LR(x_3x_4) = V(\mathbb{G}P(19,3)) - \{x_{13}, y_{13}\}$, $LR_4 = LR(x_4x_5) = V(\mathbb{G}P(19,3)) - \{x_{14}, y_{14}\}$,

$LR_5 = LR(x_5x_6) = V(\mathbb{G}P(19,3)) - \{x_{15}, y_{15}\}$, $LR_6 = LR(x_6x_7) = V(\mathbb{G}P(19,3)) - \{x_{16}, y_{16}\}$,

$LR_7 = LR(x_7x_8) = V(\mathbb{G}P(19,3)) - \{x_{17}, y_{17}\}$, $LR_8 = LR(x_8x_9) = V(\mathbb{G}P(19,3)) - \{x_{18}, y_{18}\}$,

$LR_9 = LR(x_9x_{10}) = V(\mathbb{G}P(19,3)) - \{x_{19}, y_{19}\}$, $LR_{10} = LR(x_{10}x_{11}) = V(\mathbb{G}P(19,3)) - \{x_1, y_1\}$,

$LR_{11} = LR(x_{11}x_{12}) = V(\mathbb{G}P(19,3)) - \{x_2, y_2\}$, $LR_{12} = LR(x_{12}x_{13}) = V(\mathbb{G}P(19,3)) - \{x_3, y_3\}$,

$LR_{13} = LR(x_{13}x_{14}) = V(\mathbb{G}P(19,3)) - \{x_4, y_4\}$, $LR_{14} = LR(x_{14}x_{15}) = V(\mathbb{G}P(19,3)) - \{x_5, y_5\}$,

$LR_{15} = LR(x_{15}x_{16}) = V(\mathbb{G}P(19,3)) - \{x_6, y_6\}$, $LR_{16} = LR(x_{16}x_{17}) = V(\mathbb{G}P(19,3)) - \{x_7, y_7\}$,

$LR_{17} = LR(x_{17}x_{18}) = V(\mathbb{G}P(19,3)) - \{x_8, y_8\}$, $LR_{18} = LR(x_{18}x_{19}) = V(\mathbb{G}P(19,3)) - \{x_9, y_9\}$,

$LR_{19} = LR(x_{19}x_1) = V(\mathbb{G}P(19,3)) - \{x_{10}, y_{10}\}$,

$LR_{20} = LR(x_1y_1) = V(\mathbb{G}P(19,3))$,

$LR_{21} = LR(x_2y_2) = V(\mathbb{G}P(19,3))$, $LR_{22} = LR(x_3y_3) = V(\mathbb{G}P(19,3))$,

$LR_{23} = LR(x_4y_4) = V(\mathbb{G}P(19,3))$, $LR_{24} = LR(x_5y_5) = V(\mathbb{G}P(19,3))$,

$LR_{25} = LR(x_6y_6) = V(\mathbb{G}P(19,3))$, $LR_{26} = LR(x_7y_7) = V(\mathbb{G}P(19,3))$,

$LR_{27} = LR(x_8y_8) = V(\mathbb{G}P(19,3))$, $LR_{28} = LR(x_9y_9) = V(\mathbb{G}P(19,3))$,

$LR_{29} = LR(x_{10}y_{10}) = V(\mathbb{G}P(19,3))$, $LR_{30} = LR(x_{11}y_{11}) = V(\mathbb{G}P(19,3))$,

$LR_{31} = LR(x_{12}y_{12}) = V(\mathbb{G}P(19,3))$, $LR_{32} = LR(x_{13}y_{13}) = V(\mathbb{G}P(19,3))$,

$LR_{33} = LR(x_{14}y_{14}) = V(\mathbb{G}P(19,3))$, $LR_{34} = LR(x_{15}y_{15}) = V(\mathbb{G}P(19,3))$,

$LR_{35} = LR(x_{16}y_{16}) = V(\mathbb{G}P(19,3))$, $LR_{36} = LR(x_{17}y_{17}) = V(\mathbb{G}P(19,3))$,

$LR_{37} = LR(x_{18}y_{18}) = V(\mathbb{G}P(19,3))$, $LR_{38} = LR(x_{19}y_{19}) = V(\mathbb{G}P(19,3))$,

$LR(y_1y_4) = V(\mathbb{G}P(19,3)) - \{x_{11}, x_{12}, x_{13}, y_8, y_{12}, y_{16}\} = LR(e_1)$,

$LR(y_2y_5) = V(\mathbb{G}P(19,3)) - \{x_{12}, x_{13}, x_{14}, y_9, y_{13}, y_{17}\} = LR(e_2)$,

$LR(y_3y_6) = V(\mathbb{G}P(19,3)) - \{x_{13}, x_{14}, x_{15}, y_{10}, y_{14}, y_{18}\} = LR(e_3)$,

$LR(y_4y_7) = V(\mathbb{G}P(19,3)) - \{x_{14}, x_{15}, x_{16}, y_{11}, y_{15}, y_{19}\} = LR(e_4)$,

$LR(y_5y_8) = V(\mathbb{G}P(19,3)) - \{x_{15}, x_{16}, x_{17}, y_{12}, y_{16}, y_1\} = LR(e_5)$,

$LR(y_6y_9) = V(\mathbb{G}P(19,3)) - \{x_{16}, x_{17}, x_{18}, y_{13}, y_{17}, y_2\} = LR(e_6)$,

$LR(y_7y_{10}) = V(\mathbb{G}P(19,3)) - \{x_{17}, x_{18}, x_{19}, y_{14}, y_{18}, y_3\} = LR(e_7)$,

$LR(y_8y_{11}) = V(\mathbb{G}P(19,3)) - \{x_{18}, x_{19}, x_1, y_{15}, y_{19}, y_4\} = LR(e_8)$,

$LR(y_9y_{12}) = V(\mathbb{G}P(19,3)) - \{x_{19}, x_1, x_2, y_{16}, y_1, y_5\} = LR(e_9)$,

$LR(y_{10}y_{13}) = V(\mathbb{G}P(19,3)) - \{x_1, x_2, x_3, y_{17}, y_2, y_6\} = LR(e_{10})$,

$LR(y_{11}y_{14}) = V(\mathbb{G}P(19,3)) - \{x_2, x_3, x_4, y_{18}, y_3, y_7\} = LR(e_{11})$,

$LR(y_{12}y_{15}) = V(\mathbb{G}P(19,3)) - \{x_3, x_4, x_5, y_{19}, y_4, y_8\} = LR(e_{12})$,

$LR(y_{13}y_{16}) = V(\mathbb{G}P(19,3)) - \{x_4, x_5, x_6, y_1, y_5, y_9\} = LR(e_{13})$,

$LR(y_{14}y_{17}) = V(\mathbb{G}P(19,3)) - \{x_5, x_6, x_7, y_2, y_6, y_{10}\} = LR(e_{14})$,

$LR(y_{15}y_{18}) = V(\mathbb{G}P(19,3)) - \{x_6, x_7, x_8, y_3, y_7, y_{11}\} = LR(e_{15})$,

$LR(y_{16}y_{19}) = V(\mathbb{G}P(19,3)) - \{x_7, x_8, x_9, y_4, y_8, y_{12}\} = LR(e_{16})$,

$LR(y_{17}y_1) = V(\mathbb{G}P(19,3)) - \{x_8, x_9, x_{10}, y_5, y_9, y_{13}\} = LR(e_{17})$,

$LR(y_{18}y_2) = V(\mathbb{G}P(19,3)) - \{x_9, x_{10}, x_{11}, y_6, y_{10}, y_{14}\} = LR(e_{18})$,

$LR(y_{19}y_3) = V(\mathbb{G}P(19,3)) - \{x_{10}, x_{11}, x_{12}, y_7, y_{11}, y_{15}\} = LR(e_{19})$.

For $1 \le m \le 38$ and $1 \le j \le 19$ LRN are $|LR(e_j)| = 32 < |LR_m|$. Furthermore, $\overset{19}{\underset{j=1}{\cup}} LR\left(e_j\right) = V\left(\mathbb{G}P\left(19,3\right)\right)$, $|\overset{19}{\underset{j=1}{\cup}} LR\left(e_j\right)| = 38$ and $|LR_m \cap \overset{19}{\underset{j=1}{\cup}} LR\left(e_j\right)| \ge |LR\left(e_j\right)| = 32$. Moreover, $1 \le j \le 19$, $LR(e_j)$ are pairwise nonempty. There exist a minimal LRF $\psi : V(\mathbb{G}P(19,3)) \to [0,1]$ is defined as $\psi\left(y\right) = \frac{1}{32}$ for each $y \in \overset{19}{\underset{j=1}{\cup}} LR\left(e_j\right)$ and $\psi(y) = 0$ for the vertices of $\mathbb{G}P(19,3)$ which are not in $\overset{19}{\underset{j=1}{\cup}} LR\left(e_j\right)$. Therefore, by Theorem 1, $dim_{lf}\left(\mathbb{G}P\left(19,3\right)\right) \le \sum_{j=1}^{38} \frac{1}{20} = \frac{19}{16}$. Since $|V(\mathbb{G}P(19,3))| = \gamma = 38$, then by Theorem 3 we have $\frac{38}{38} \le dim_{lf}\left(\mathbb{G}P\left(19,3\right)\right)$ implies $1 \le dim_{lf}(\mathbb{G}P(19,3))$. As $\mathbb{G}P(19,3)$ is not bipartite network therefore, $1 \le dim_{lf}\left(\mathbb{G}P\left(19,3\right)\right) \le \frac{19}{16}$.

Case 3: For $1 \le i \le n$, $1 \le j \le n$ and $n \ge 25$, $LR(e_j) = LR(y_iy_{i+3})$, $LR(x_ix_{i+1})$, $LR(x_iy_i)$. By Lemma 5, 6, we have (i) $|LR(x_ix_{i+1})|$, $LR(x_iy_i) \ge |LR(e_j)| = 2n - 6 = \alpha$, (ii) $|LR\left(x_ix_{i+1}\right) \cap \overset{n}{\underset{j=1}{\cup}} LR\left(e_j\right)|$, $|LR\left(x_iy_i\right) \cap \overset{n}{\underset{j=1}{\cup}} LR\left(e_j\right)| \ge |LR\left(e_j\right)|$ and $\overset{n}{\underset{j=1}{\cup}} LR\left(e_j\right) = 2n = \beta$. The intersection of LRS having minimum cardinality is not empty. Therefore, there exist a mimimal local resolving $\psi' : V(\mathbb{G}P(n,3)) \to [0,1]$ such that $|\psi'| < |\psi|$, where the minimal LRF $\psi : V(\mathbb{G}P(n,3)) \to [0,1]$ is defined as $\phi\left(v\right) = \left\{\frac{1}{\alpha}\ for\ v \in \overset{n}{\underset{j=1}{\cup}} LR\left(e_j\right)\right\}$.

Therefore, by Theorem 1, $dim_{lf}\left(\mathbb{G}P\left(n,3\right)\right) \leq \sum_{t=1}^{\beta} \frac{1}{\alpha} = \frac{2n}{2n-6}$. Since $|V(\mathbb{G}P(n,3))| = \gamma = 2n$, then by Theorem 3 we have $\frac{2n}{2n} \leq dim_{lf}\left(\mathbb{G}P\left(n,3\right)\right)$ implies $1 \leq dim_{lf}(\mathbb{G}P(n,3))$. As $\mathbb{G}P(n,3)$ is not bipartite network therefore, $1 \leq dim_{lf}\left(\mathbb{G}P\left(n,3\right)\right) \leq \frac{2n}{2n-6}$.

Theorem 7. The LFMD of generalized Petersen network $\mathbb{G}P(n,3)$ for $n \geq 8$ and $n \equiv 0(mod\,2)$ is 1.

Proof: As generalized Petersen network $\mathbb{G}P(n,3)$ for $n \geq 8$ and $n \equiv 0(mod\,2)$ is bipartite network. Therefore, by Theorem 2 we have $dim_{lf}(\mathbb{G}P(n,3)) = 1$.

Table 1: Upper and lower bounds of LFMD of generalized petersen $\mathbb{G}P(n,3)$ network for, $n = 7, n \geq 9$ for $n \equiv 3(mod\,6)$, $n \geq 11$ for $n \equiv 5(mod\,6)$, and $n \geq 13$ for $n \equiv 1(mod\,6)$

$\mathbb{G}P(n,3)$	Upper bounds of LFMD	Lower bounds of LFMD
$n = 7$	$\frac{7}{5}$	$\frac{14}{13}$
$n \geq 9$	$\frac{2n}{2n-6}$	1
$n \geq 11$	$\frac{2n}{2n-6}$	1
$n \geq 13$	$\frac{2n}{2n-6}$	1

Table 2: Limiting values of LFMD of generalized petersen $\mathbb{G}P(n,3)$ network for, $n \geq 9$ for $n \equiv 3(mod\,6)$, $n \geq 11$ for $n \equiv 5(mod\,6)$, and $n \geq 13$ for $n \equiv 1(mod\,6)$

$\mathbb{G}P(n,3)$	Limiting values of upper bound of LFMD	Comment
$n \geq 9$	$\lim_{n\to\infty} \frac{2n}{2n-6} = 1$	Bounded
$n \geq 11$	$\lim_{n\to\infty} \frac{2n}{2n-6} = 1$	Bounded
$n \geq 13$	$\lim_{n\to\infty} \frac{2n}{2n-6} = 1$	Bounded

4 Discussion and Conclusion

In this paper, we have investigated the LFMD generalized Petersen network $\mathbb{G}P(n,3)$ for $n \geq 7$ with the exact value of lower and upper bounds. We have also checked the bounded and unbounded behavior of networks and found that for $n \geq 8$ and $n \equiv 0(mod\,2)\mathbb{G}P(n,3)$ is bipartite Network having LFMD is 1. The details of computed values of LFMD are given in Tables 1 and 2. Even before the aforementioned tables, we illustrate Theorem 5 with the help of a example finding LFMD for the generalized Petersen graph with $n = 9$. By Fig. 1b and Theorem 5 (Case A), it can be observed that the LRN sets of thee edges $y_1y_4, y_2y_5, y_3y_6, y_4y_7, y_5y_8, y_7y_8, y_8y_2, y_9y_3$ have the cardinality of 12 which is minimum among all the other LRNs. Moreover, the union of these LRNs is equal to the order of $\mathbb{G}P(n,3)$. The cardinality of the other LRNs with the intersection of this union is larger or equal to 12. By Theorem 1, $dim_{lf}\mathbb{G}P\left(9,3\right) \leq \frac{18}{12} = \frac{3}{2}$. In addition, the cardinality of LRNs with maximum cardinality is 18 consequently by Theorem 3, $dim_{lf}\mathbb{G}P\left(9,3\right) \geq \frac{18}{18} = 1$. Therefore, $1 \leq dim_{lf}\mathbb{G}P\left(9,3\right) \leq \frac{3}{2}$.

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