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Solving Fractional Differential Equations via Fixed Points of Chatterjea Maps

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ABSTRACT

In this paper, we present the existence and uniqueness of fixed points and common fixed points for Reich and Chatterjea pairs of self-maps in complete metric spaces. Furthermore, we study fixed point theorems for Reich and Chatterjea nonexpansive mappings in a Banach space using the Krasnoselskii-Ishikawa iteration method associated with $\mathscr{S}_\lambda$ and consider some applications of our results to prove the existence of solutions for nonlinear integral and nonlinear fractional differential equations. We also establish certain interesting examples to illustrate the usability of our results.

KEYWORDS

Common fixed points; Reich and Chatterjea mappings; Krasnoselskii-Ishikawa iteration; complete metric space; Banach space; integral equation; nonlinear fractional differential equation

1 Introduction

Fixed point theory plays an important role in various branches of mathematics as well as in nonlinear functional analysis, and is very useful for solving many existence problems in nonlinear differential and integral equations with applications in engineering and behavioural sciences. Recently, many authors have provided the extended fixed point theorems for the different classes of contraction type mappings, such as Kannan, Reich, Chatterjea and Ćirić-Reich-Rus mappings (see [1–10]).

Let (Λ, d) be a metric space. A mapping $\mathscr{S}$ is said to be a contraction if there exists $\alpha \in [0, 1)$ such that

$$d(\mathscr{S}\mu, \mathscr{S}\omega) \leq \alpha d(\mu, \omega), \tag{1}$$

for each $\mu, \omega \in \Lambda$. A self-mapping $\mathscr{S}$ on Λ is nonexpansive if $\alpha = 1$. A point $\nu \in \Lambda$ is said to be a fixed point of $\mathscr{S}$ if $\mathscr{S}(\nu) = \nu$. We denote the set of all fixed points of $\mathscr{S}$ as $Fix(\mathscr{S})$.

Kannan [11] established a fixed point theorem for mapping satisfying:

$$d(\mathscr{S}\mu, \mathscr{S}\omega) \leq \alpha\{d(\mu, \mathscr{S}\mu) + d(\omega, \mathscr{S}\omega)\}, \tag{2}$$

for each $\mu, \omega \in \Lambda$ where $\alpha \in \left[0, \frac{1}{2}\right)$. We know that if Λ is complete, then every contraction and every Kannan mapping has a unique fixed point. A mapping $\mathscr{S}$ is called Kannan nonexpansive if $\alpha = 1/2$ in (2). Nonexpansive mappings are always continuous but Kannan nonexpansive mappings are discontinuous (see [12]).

In 1980, Gregus [13] combined nonexpansive and Kannan nonexpansive mappings as follows:

$$d(\mathscr{S}\mu, \mathscr{S}\omega) \leq \alpha d(\mu, \omega) + \beta d(\mu, \mathscr{S}\mu) + \gamma d(\omega, \mathscr{S}\omega), \ \ \mu, \omega \in \Lambda, \tag{3}$$

where α, β, γ are non-negative numbers. If $\alpha + \beta + \gamma < 1$, then the mapping $\mathscr{S}$ is known as a Reich contraction. A mapping satisfying (3) is said to be a Reich type nonexpansive mapping, if $\alpha+\beta+\gamma = 1$ (see [14,15]).

In [15], the authors considered the Rhoades mapping satisfying the following condition:

$$d(\mathscr{S}\mu, \mathscr{S}\omega) \leq \alpha d(\mu, \omega) + \beta d(\omega, \mathscr{S}\mu) + \gamma d(\mu, \mathscr{S}\omega), \tag{4}$$

for each $\mu, \omega \in \Lambda$ where α, β, γ are non-negative numbers such that $\alpha + \beta + \gamma < 1$. A mapping satisfying (4) is said to be Chatterjea type nonexpansive mapping if $\alpha + \beta + \gamma = 1$. Reich [16] showed the generalized Banach's theorem and observed that Kannan's theorem is a particular case of it with a suitable selection of the constant. Reich type mappings and generalized nonexpansive mappings have been important research area on their own for many authors which has been applied in various spaces such as metric space, Banach space, and partially ordered Banach spaces (see [5,8,9,17–19]).

In 1971, Ćirić [20] introduced the notion of orbital continuity. Sastry et al. [21] defined the notion of orbital continuity for a pair of mappings. We now recall some relevant definitions.

Definition 1.1. [20] If $\mathscr{S}$ is a self-mapping on metric space (Λ, d), then the set

$$O(\mathscr{S}, \mu, n) = \{\mu, \mathscr{S}\mu, \ldots, \mathscr{S}^n\mu\}, \ n \geq 0,$$

is said to be an orbit of $\mathscr{S}$ at μ. A metric space Λ is said to be $\mathscr{S}$-orbital complete if every Cauchy sequence contained in the set

$$O(\mathscr{S}, \mu, \infty) = \{\mu, \mathscr{S}\mu, \mathscr{S}^2\mu, \ldots\},$$

for some $\mu \in \Lambda$ converges in Λ.

In addition, $\mathscr{S}$ is said to be orbital continuous at a point $\nu \in \Lambda$, if for any sequence $\{\mu_n\} \subset O(\mathscr{S}, \mu, n)$, then, $\lim_{n\to\infty} \mu_n = \nu$ implies $\lim_{n\to\infty} \mathscr{S}\mu_n = \mathscr{S}\nu$. Every continuous mapping $\mathscr{S}$ is orbital continuity, but the converse is not true, see [20].

Definition 1.2. [21] Let $\mathscr{S}$ and $\mathscr{T}$ be two self-mappings of a metric space (Λ, d), and $\{\mu_n\}$ be a sequence in Λ such that $\mu_{2n+1} = \mathscr{T}\mu_{2n}, \ \mu_{2n+2} = \mathscr{S}\mu_{2n+1}, n \geq 0$. Then, the set

$$O(\mathscr{S}, \mathscr{T}, \mu_0, n) = \{\mu_n, \ \ n = 1, 2, \ldots\},$$

is called the $(\mathscr{S}, \mathscr{T})$-orbit at μ_0. The mapping $\mathscr{T}$ (or $\mathscr{S}$) is called $(\mathscr{S}, \mathscr{T})$-orbital continuous if $\lim_{n\to\infty} \mathscr{S}\mu_n = \nu$ implies $\lim_{n\to\infty} \mathscr{T}\mathscr{S}\mu_n = \mathscr{T}\nu$ or ($\lim_{n\to\infty} \mathscr{S}\mu_n = \nu$ implies $\lim_{n\to\infty} \mathscr{S}\mathscr{S}\mu_n = \mathscr{S}\nu$). The mappings $\mathscr{S}$ and $\mathscr{T}$ are said to be orbital continuous if $\mathscr{S}$ is $(\mathscr{S}, \mathscr{T})$-orbital continuous and $\mathscr{T}$ is $(\mathscr{S}, \mathscr{T})$-orbital continuous.

Ćirić in [22] proved that continuity of $\mathscr{S}$ implies orbital continuity but the converse is not true.

Definition 1.3. [23] A mapping $\mathscr{S} : \Lambda \to \Lambda$ of metric space Λ is said to be κ-continuous, if $\lim_{n\to\infty} \mathscr{S}^{\kappa-1}\mu_n = \nu$, then $\lim_{n\to\infty} \mathscr{S}^{\kappa}\mu_n = \mathscr{S}\nu$ such that $\kappa > 1$.

Note that, 1-continuity is equivalent to continuity and for any $\kappa = 1, 2, \ldots$, κ-continuity implies $\kappa+1$-continuity while the converse is not true. Further, continuity of the mapping $\mathscr{S}^{\kappa}$ and κ-continuity of $\mathscr{S}$ are independent conditions when $\kappa > 1$, for more detail and examples (see [23]).

On the other hand, the concept of asymptotic regularity has been introduced by Browder et al. [24] in connection with the study of fixed points of nonexpansive mappings. Asymptotic regularity is a fundamentally important concept in metric fixed point theory. A self-mapping $\mathscr{S}$ of a metric space (Λ, d) is called asymptotically regular if $\lim_{n\to\infty} d\left(\mathscr{S}^{n}\mu, \mathscr{S}^{n+1}\mu\right) = 0$ for all $\mu \in \Lambda$. A mapping $\mathscr{S}$ is called asymptotically regular with respect to $\mathscr{T}$ at $\mu_0 \in \Lambda$ if there exists a sequence $\mu_n \in \Lambda$ such that $\mathscr{T}\mu_{n+1} = \mathscr{S}\mu_n$, $n \geq 0$ and $\lim_{n\to\infty} d\left(\mathscr{T}\mu_{n+1}, \mathscr{T}\mu_{n+2}\right) = 0$. The mapping $\mathscr{S}$ has an approximate fixed point sequence if there exists a sequence $\mu_n \subset \Lambda$, such that $d(\mu_n, \mathscr{S}\mu_n) \to 0$ as $n \to \infty$. The self-mappings $\mathscr{S}$ and $\mathscr{T}$ are called compatible [6] if $\lim_{n\to\infty} d\left(\mathscr{S}\mathscr{T}\mu_n, \mathscr{T}\mathscr{S}\mu_n\right) = 0$, whenever $\{\mu_n\}$ is a sequence in Λ such that $\lim_{n\to\infty} \mathscr{S}\mu_n = \lim_{n\to\infty} \mathscr{T}\mu_n = \nu$ for some $\nu \in \Lambda$. $C(\mathscr{S}, \mathscr{T}) = \{\mu \in \Lambda : \mathscr{S}\mu = \mathscr{T}\mu\}$ denotes the set of coincidence points of $\mathscr{S}$ and $\mathscr{T}$.

In [25], Gòrnicki proved the following fixed point theorem:

Theorem 1.1. Let (Λ, d) be a complete metric space and $\mathscr{S} : \Lambda \to \Lambda$ be a continuous asymptotically regular mapping satisfying

$$d(\mathscr{S}\mu, \mathscr{S}\omega) \leq \alpha d(\mu, \omega) + \beta\{d(\mu, \mathscr{S}\mu) + d(\omega, \mathscr{S}\omega)\}, \tag{5}$$

for all $\mu, \omega \in \Lambda$ where $\alpha \in [0, 1)$ and $\beta \in [0, \infty)$. Then $\mathscr{S}$ has a unique fixed point $\nu \in \Lambda$ and $\mathscr{S}^{n}\mu \to \nu$ for any $\mu \in \Lambda$.

Recently, Bisht [26] showed that the continuity assumption considered in Theorem 1.1 can be weakened by the notion of orbital continuity or κ-continuity.

Theorem 1.2. Let (Λ, d) be a complete metric space and $\mathscr{S} : \Lambda \to \Lambda$ be an asymptotically regular mapping. Assume that there exist $\alpha \in [0, 1)$ and $\beta \in [0, \infty)$ satisfying (5) for all $\mu, \omega \in \Lambda$. Then $\mathscr{S}$ has a unique fixed point $\nu \in \Lambda$, provided that $\mathscr{S}$ is either κ-continuous for some $\kappa \geq 1$ or orbitally continuous. Moreover, $\mathscr{S}^{n}\mu \to \nu$ for any $\mu \in \Lambda$.

This paper is organised as follows: First, we establish some fixed point theorems for Reich and Chatterjea nonexpansive mappings to include asymptotically regular or continuous mappings in complete metric spaces. After that, we prove some fixed point theorems and common fixed points for Reich and Chatterjea type nonexpansive mappings in Banach space using the Krasnoselskii-Ishikawa method associated with $\mathscr{S}_{\lambda}$. In addition, several examples are provided to illustrate our results. Further, we study the existence of solutions for nonlinear integral equations and nonlinear fractional differential equations. Our work generalizes and complements the comparable results in the current literature.

2 Asymptotic Behaviour of Mappings in Complete Metric Spaces

In this section, we study fixed point and common fixed point theorems for Reich and Chatterjea type nonexpansive mappings in complete metric space.

To start with the following lemma, which is useful to prove the results of this section:

Lemma 2.1. [27] Let $\{h_n\}$ be a sequence of non-negative real numbers satisfying

$$h_{n+1} \leq (1 - \psi_n)h_n + \psi_n\vartheta_n, \quad n \geq 0,$$

where $\{\psi_n\}$,$\{\vartheta_n\}$ are sequences of real numbers such that:

(i) $\psi_n \subset [0,1]$ and $\sum_{n=0}^{\infty}\psi_n = \infty$,
(ii) $\limsup_{n\to\infty}\vartheta_n \le 0$, or
(iii) $\sum_{n=0}^{\infty}|\psi_n\vartheta_n|$ is convergent.

Then, $\lim_{n\to\infty} h_n = 0$.

Theorem 2.1. Let (Λ, d) be a complete metric space and $\mathscr{S}, \mathscr{T} : \Lambda \to \Lambda$ be asymptotically regular. Assume that there exist non-negative numbers α, β, γ where $\alpha + \beta + \gamma = 1$, satisfying

$$d(\mathscr{S}\mu, \mathscr{T}\omega) \le \alpha d(\mu, \omega) + \beta d(\mu, \mathscr{S}\mu) + \gamma d(\omega, \mathscr{T}\omega), \tag{6}$$

for all $\mu, \omega \in \Lambda$ and $\alpha \in [0, 1)$. Further, $\mathscr{S}$ and $\mathscr{T}$ are either κ-continuous for some $\kappa \ge 1$ or orbitally continuous. Then $\mathscr{S}$ and $\mathscr{T}$ have a unique common fixed point ν. Moreover, for any $\mu \in \Lambda$, $\lim_{n\to\infty}\mathscr{S}^n\mu = \nu = \lim_{n\to\infty}\mathscr{T}^n\mu$.

Proof. The proof of the theorem is organized in three steps:

Step 1: We shall prove that $\lim_{n\to\infty} d(\mathscr{S}^n\mu, \mathscr{T}^n\mu) = 0$, for any $\mu \in \Lambda$. The result is trivial if $\mathscr{S} = \mathscr{T}$. Suppose that $\mathscr{S} \ne \mathscr{T}$ and $\alpha = 0$. Then (6) becomes

$$d(\mathscr{S}\mu, \mathscr{T}\omega) \le \beta d(\mu, \mathscr{S}\mu) + \gamma d(\omega, \mathscr{T}\omega),$$

for all $\mu, \omega \in \Lambda$. Defining $\mu_1 = \mathscr{S}^n\mu$ and $\omega_1 = \mathscr{T}^n\mu$, for any $\mu \in \Lambda$, we get

$$d(\mathscr{S}^{n+1}\mu, \mathscr{T}^{n+1}\mu) \le \beta d(\mathscr{S}^n\mu, \mathscr{S}^{n+1}\mu) + \gamma d(\mathscr{T}^n\mu, \mathscr{T}^{n+1}\mu).$$

As $n \to \infty$, the asymptotic regularity of $\mathscr{S}$ and $\mathscr{T}$, implies that

$$\lim_{n\to\infty} d\left(\mathscr{S}^{n+1}\mu, \mathscr{T}^{n+1}\mu\right) = 0.$$

Using triangle inequality and asymptotic regularity of $\mathscr{S}$ and $\mathscr{T}$, obtain

$$d(\mathscr{S}^n\mu, \mathscr{T}^n\mu) \le d(\mathscr{S}^n\mu, \mathscr{S}^{n+1}\mu) + d(\mathscr{S}^{n+1}\mu, \mathscr{T}^{n+1}\mu) + d(\mathscr{T}^{n+1}\mu, \mathscr{T}^n\mu) \to 0. \text{ as } n \to \infty.$$

Thereafter, suppose that $\mathscr{S} \ne \mathscr{T}$ and $\alpha \ne 0$. Define $\mu_1 = \mathscr{S}^n\mu$ and $\omega_1 = \mathscr{T}^n\mu$. Then, (6) becomes

$$d(\mathscr{S}^{n+1}\mu, \mathscr{T}^{n+1}\mu) \le \alpha d(\mathscr{S}^n\mu, \mathscr{T}^n\mu) + \beta d(\mathscr{S}^n\mu, \mathscr{S}^{n+1}\mu) + \gamma d(\mathscr{T}^n\mu, \mathscr{T}^{n+1}\mu).$$

Let $h_n = d(\mathscr{S}^n\mu, \mathscr{T}^n\mu)$, $\psi_n = 1 - \alpha$ and $\vartheta_n = \dfrac{\beta}{1-\alpha} d\left(\mathscr{S}^n\mu, \mathscr{S}^{n+1}\mu\right) + \dfrac{\gamma}{1-\alpha} d\left(\mathscr{T}^n\mu, \mathscr{T}^{n+1}\mu\right)$.By asymptotically regularity of $\mathscr{S}$ and $\mathscr{T}$, we have $\lim_{n\to\infty}\vartheta_n = 0$. Furthermore, $\sum_{n=1}^{\infty}\psi_n = \infty$. Hence, by Lemma 2.1, we get that $\lim_{n\to\infty} d(\mathscr{S}^n\mu, \mathscr{T}^n\mu) = 0$ for any $\mu \in \Lambda$.

Step 2: Let $\mu_n = \mathscr{S}^n\mu$ for any $\mu \in \Lambda$. Now, we show that $\{\mu_n\}$ is a Cauchy sequence converging to $\nu \in \Lambda$. Moreover, $\{\mathscr{T}^n\mu\} \to \nu \in \Lambda$. Suppose on contrary that $\{\mu_n\}$ is not a Cauchy sequence. Then there exists an $\varepsilon > 0$ and two subsequences of integers $\{m_\kappa\}$ and $\{n_\kappa\}$ such that for every $m_\kappa > n_\kappa \ge \kappa$, we have

$$d(\mathscr{S}^{m_\kappa}, \mathscr{S}^{n_\kappa}) \ge \varepsilon, \tag{7}$$

where $\kappa = 1, 2, \ldots$.

Choosing m_κ, the smallest number exceeding n_κ for which (7) holds. In addition, we assume that

$$d(\mathscr{S}^{m_{(\kappa-1)}}\mu, \mathscr{S}^{n_\kappa}\mu) < \varepsilon.$$

Thus, we have

$$\varepsilon \le d(\mathscr{S}^{m_\kappa}\mu, \mathscr{S}^{n_\kappa}\mu) \le d(\mathscr{S}^{m_\kappa}\mu, \mathscr{S}^{m_{(\kappa-1)}}\mu) + d(\mathscr{S}^{m_{(\kappa-1)}}\mu, \mathscr{S}^{n_\kappa}\mu)$$
$$< d(\mathscr{S}^{m_\kappa}\mu, \mathscr{S}^{m_{(\kappa-1)}}\mu) + \varepsilon.$$

As $\kappa \to \infty$, it follows by asymptotic regularity of $\mathscr{S}$ that

$$\lim_{\kappa\to\infty} d\left(\mathscr{S}^{m_\kappa}\mu, \mathscr{S}^{n_\kappa}\mu\right) = \varepsilon. \tag{8}$$

Further, by asymptotic regularity of $\mathscr{S}$ and the following inequality

$$d(\mathscr{S}^{m_{(\kappa-1)}}\mu, \mathscr{S}^{n_{(\kappa-1)}}\mu) \le d(\mathscr{S}^{m_{(\kappa-1)}}\mu, \mathscr{S}^{m_\kappa}\mu) + d(\mathscr{S}^{m_\kappa}\mu, \mathscr{S}^{n_\kappa}\mu) + d(\mathscr{S}^{n_\kappa}\mu, \mathscr{S}^{n_{(\kappa-1)}}\mu),$$

the implication is that

$$\lim_{k\to\infty} d\left(\mathscr{S}^{m_{(\kappa-1)}}\mu, \mathscr{S}^{n_{(\kappa-1)}}\mu\right) = \varepsilon. \tag{9}$$

Now, using (6) we get

$$d(\mathscr{S}^{m_\kappa}\mu, \mathscr{S}^{n_\kappa}\mu) \le d(\mathscr{S}^{m_\kappa}\mu, \mathscr{T}^{n_\kappa}\mu) + d(\mathscr{T}^{n_\kappa}\mu, \mathscr{S}^{n_\kappa}\mu)$$
$$\le d(\mathscr{T}^{n_\kappa}\mu, \mathscr{S}^{n_\kappa}\mu) + \alpha[d(\mathscr{S}^{m_{(\kappa-1)}}\mu, \mathscr{S}^{n_{(\kappa-1)}}\mu) + d(\mathscr{S}^{n_{(\kappa-1)}}\mu, \mathscr{T}^{n_{(\kappa-1)}}\mu)]$$
$$+ \beta d(\mathscr{S}^{m_{(\kappa-1)}}\mu, \mathscr{S}^{m_\kappa}\mu) + \gamma d(\mathscr{T}^{n_{(\kappa-1)}}\mu, \mathscr{T}^{n_\kappa}\mu).$$

Taking limit as $\kappa \to \infty$, on the both sides of the above inequality, and using (8), (9) and asymptotic regularity of $\mathscr{S}$ and $\mathscr{T}$, we obtain that, $\varepsilon \le \alpha\varepsilon$, which is a contradiction. Hence, $\{\mu_n\}$ is a Cauchy sequence in complete space Λ, there exists a point ν in Λ such that $\mu_n \to \nu$. Moreover,

$$d(\mathscr{T}^n\mu, \nu) \le d(\mathscr{T}^n\mu, \mathscr{S}^n\mu) + d(\mathscr{S}^n\mu, \nu),$$

from Step 1 and $\mu_n \to \nu$, we get that $\mathscr{T}^n\mu$ converges to $\nu \in \Lambda$.

Step 3: We will prove that μ is the unique common fixed point of $\mathscr{S}$ and $\mathscr{T}$. Assume that $\mathscr{S}$ is κ-continuous. Since $\lim_{n\to\infty} \mathscr{S}^{\kappa-1}\mu_n = \nu$, $\kappa-$continuity of $\mathscr{S}$ implies that

$$\lim_{n\to\infty} \mathscr{S}^\kappa\mu_n = \mathscr{S}\nu.$$

For the uniqueness of the limit, we get $\mathscr{S}\nu = \nu$.

Similarly, let $\mathscr{S}$ be orbitally continuous. Since $\lim_{n\to\infty} \mu_n = \nu$, orbital continuity of $\mathscr{S}$ implies

$$\lim_{n\to\infty} \mathscr{S}\mu_n = \mathscr{S}\nu,$$

we obtain $\mathscr{S}\nu = \nu$.

In addition, since $\lim_{n\to\infty} \mathscr{T}^n\mu = \nu$, this gives $\mathscr{T}\nu = \nu$ whenever $\mathscr{T}$ is κ-continuous or orbitally continuous. Hence, $\nu \in Fix(\mathscr{S}) \cap Fix(\mathscr{T})$. Now, we prove the uniqueness of the common fixed point, suppose that there is $\nu \ne \nu^*$ in $Fix(\mathscr{S}) \cap Fix(\mathscr{T})$. Let $\mu = \nu$ and $\omega = \nu^*$. Then, (6) implies $d(\nu, \nu^*) \le \alpha d(\nu, \nu^*)$, which is a contradiction. We have $\nu = \nu^*$. In the other words, the point ν is the unique common fixed point of $\mathscr{S}$ and $\mathscr{T}$.

Example 2.2. Consider $\Lambda = [0, 1]$, equipped with the metric d defined by $d(\mu, \omega) = |\mu - \omega|$. Let $\mathscr{S}$ and $\mathscr{T}$ be such that

$$\mathscr{S}\mu = \begin{cases} \frac{\mu}{7} & \text{if } \mu \in [0,1)\,, \\ 0 & \text{if } \mu = 1, \end{cases} \quad \textit{and} \quad \mathscr{T}\mu = \begin{cases} \frac{3\mu}{7} & \text{if } \mu \in [0,1)\,, \\ \frac{1}{7} & \text{if } \mu = 1. \end{cases}$$

Clearly, the two mappings $\mathscr{S}$ and $\mathscr{T}$ are asymptotically regular. For $\mu = \frac{1}{2} \in \Lambda$ we have $\lim_{n\to\infty} d\left(\mathscr{S}^n\mu, S^{n+1}\mu\right) = 0$. Similarly, we can show that $\mathscr{T}$ is asymptotically regular. The mappings $\mathscr{S}$ and $\mathscr{T}$ are orbitally continuous at 0 and discontinuous at 1. Now, we show that $\mathscr{S}$ and $\mathscr{T}$ satisfy the condition (6) with $\alpha = \frac{1}{12}$, $\beta = \frac{1}{6}$ and $\gamma = \frac{3}{4}$. In fact, we have the following four cases:

Case 1: Let $\mu, \omega \in [0, 1)$, such that $\omega \le \mu$. We get

$$\begin{aligned} \alpha d\left(\mu, \omega\right) + \beta d\left(\mu, \mathscr{S}\mu\right) + \gamma d\left(\omega, \mathscr{T}\omega\right) &= \frac{1}{12}|\mu - \omega| + \frac{1}{6}\left|\mu - \frac{\mu}{7}\right| + \frac{3}{4}\left|\omega - \frac{3}{7}\omega\right| \\ &= \frac{1}{12}|\mu - \omega| + \frac{1}{7}|\mu| + \frac{3}{7}|\omega| \\ &\ge \frac{1}{12}|\mu - \omega| + \frac{1}{7}|\mu + 3\omega| \\ &\ge \frac{1}{7}|\mu - 3\omega| = d\left(\mathscr{S}\mu, \mathscr{T}\omega\right). \end{aligned}$$

Case 2: Let $\mu \in [0, 1)$ and $\omega = 1$. We have

$$\begin{aligned} \alpha d\left(\mu, \omega\right) + \beta d\left(\mu, \mathscr{S}\mu\right) + \gamma d\left(\omega, \mathscr{T}\omega\right) &= \frac{1}{12}|\mu - 1| + \frac{1}{6}|\mu - \frac{\mu}{7}| + \frac{3}{4}|1 - \frac{1}{7}| \\ &= \frac{1}{12}\left(1 - \mu\right) + \frac{\mu}{7} + \frac{9}{14} \\ &\ge \frac{61}{84} + \frac{5}{84}\mu > \frac{1}{7}|\mu - 1| = d\left(\mathscr{S}\mu, \mathscr{T}\omega\right). \end{aligned}$$

Case 3: Let $\omega \in [0, 1)$ and $\mu = 1$. We obtain

$$\begin{aligned} \alpha d\left(\mu, \omega\right) + \beta d\left(\mu, \mathscr{S}\mu\right) + \gamma d\left(\omega, \mathscr{T}\omega\right) &= \frac{1}{12}|1 - \omega| + \frac{1}{6} + \frac{3}{4}|\omega - \frac{3}{7}\omega| \\ &= \frac{1}{12}\left(1 - \omega\right) + \frac{1}{6} + \frac{3}{7}\omega \\ &> \frac{3}{7}\omega = d\left(\mathscr{S}\mu, \mathscr{T}\omega\right). \end{aligned}$$

Case 4: If $\mu = \omega = 1$. We get

$$\begin{aligned} \alpha d(\mu,\omega) + \beta d(\mu,\mathscr{S}\mu) + \gamma d(\omega,\mathscr{T}\omega) &= \frac{1}{6} + \frac{3}{4}|1 - \frac{1}{7}| \\ &= \frac{1}{6} + \frac{9}{14} \\ &> \frac{2}{14} = \frac{1}{7} = d(\mathscr{S}\mu,\mathscr{T}\omega). \end{aligned}$$

Therefore, in all the cases, $\mathscr{S}$ and $\mathscr{T}$ satisfy the condition (6) for all $\mu, \omega \in \Lambda$. Moreover, all the assumptions of Theorem 2.1 hold, hence, the mappings $\mathscr{S}$ and $\mathscr{T}$ have a unique common fixed point at 0. Further, $\lim_{n\to\infty} \mathscr{S}^n\mu = 0 = \lim_{n\to\infty} \mathscr{T}^n\mu$ for any $\mu \in \Lambda$.

In the special case of our result, we can generate the Theorem 1.4 of Gòrnicki [28].

Corollary 2.1. Let (Λ, d) be a complete metric space and $\mathscr{S}$ and $\mathscr{T}$ are self-mappings on Λ which $\mathscr{S}^p$ and $\mathscr{T}^q$ are asymptotically regular for some positive integers p and q, respectively. Assume that there exist non-negative numbers α, β, γ where $\alpha + \beta + \gamma = 1$, such that, the following condition holds

$$d(\mathscr{S}^p\mu, \mathscr{T}^q\omega) \le \alpha d(\mu,\omega) + \beta d(\mu, \mathscr{S}^p\mu) + \gamma d(\omega, \mathscr{T}^q\omega), \tag{10}$$

for all $\mu, \omega \in \Lambda$. Then, $\mathscr{S}$ and $\mathscr{T}$ have a unique common fixed point $\nu \in \Lambda$, provided that both $\mathscr{S}^p$ and $\mathscr{T}^q$ are either κ-continuous for some $\kappa \ge 1$ or orbitally continuous.

Proof. Take $f = \mathscr{S}^p$ and $g = \mathscr{T}^q$. From (10) obtain

$$d(f\mu, g\omega) \le \alpha d(\mu,\omega) + \beta d(\mu, f\mu) + \gamma d(\omega, g\omega), \tag{11}$$

for all $\mu, \omega \in \Lambda$. By Theorem 2.1, we obtain f and g have a unique common fixed point ν. Then,

$$f(\mathscr{S}\nu) = \mathscr{S}^p(\mathscr{S}\nu) = \mathscr{S}^{p+1}\nu = \mathscr{S}(\mathscr{S}^p\nu) = \mathscr{S}(f\nu) = \mathscr{S}\nu,$$

which implies that $\mathscr{S}\nu$ is a fixed point of f. Similarly, we derive that $\mathscr{T}\nu$ is a fixed point of g. By (11), we obtain

$$\begin{aligned} d(f\mathscr{S}\mu, g\mathscr{T}\omega) = d(\mathscr{S}\mu, \mathscr{T}\omega) &\le \alpha d(\mathscr{S}\mu, \mathscr{T}\omega) + \beta d(\mathscr{S}\mu, f\mathscr{S}\mu) + \gamma d(\mathscr{T}\omega, g\mathscr{T}\omega) \\ &= \alpha d(\mathscr{S}\mu, \mathscr{T}\omega) < d(\mathscr{S}\mu, \mathscr{T}\omega), \end{aligned}$$

which implies that $\mathscr{T}\nu = \mathscr{S}\nu$. From the uniqueness of the common fixed point of f and g, it follows that $\mathscr{S}\nu = \mathscr{T}\nu = \nu$. Assuming that $\nu^* \ne \nu$ are two common fixed points of $\mathscr{S}$ and $\mathscr{T}$ such that $f\nu^* = g\nu^* = \nu^*$. The uniqueness of the common fixed point of f and g implies that $\nu = \nu^*$. Hence, the common fixed point of $\mathscr{S}$ and $\mathscr{T}$ is unique.

Example 2.3. Let $\Lambda = c_0 = \{\nu = (\nu_n)_{n\in\mathbb{N}} : \lim_{n\to\infty} \nu_n = 0\}$ be the space of all real sequences convergent to zero, endowed with the usual metric d_∞ defined by $d_\infty(\mu,\omega) = \sup_n|\mu_n - \omega_n|$ for all $\mu = (\mu_n)_n$ and $\omega = (\omega_n)_n \in \Lambda$. Then (Λ, d_∞) is a complete metric space. Let $\mathscr{S}$ and $\mathscr{T}$ be such that

$$\mathscr{S}(\mu) = \mathscr{S}(\mu_n) \begin{cases} \left(\frac{\mu_1}{2}, \frac{\mu_2}{2}, \frac{\mu_3}{2}, \ldots\right) & \text{if there is atleast one } \mu_n \text{with } |\mu_n| \ge 1, \\ \left(\frac{\mu_n}{2n+1}, 0, 0, \ldots\right) & \text{otherwise,} \end{cases}$$

and

$$\mathscr{T}(\mu) = \mathscr{T}(\mu_n) = \begin{cases} \left(\frac{\mu_1}{4}, \frac{\mu_2}{4}, \frac{\mu_3}{4}, \ldots\right) & \text{if there is atleast one } \mu_{\text{n}} \text{ with } |\mu_{\text{n}}| \geq 1, \\ \left(\frac{\mu_n}{4n+1}, 0, 0, \ldots\right) & \text{otherwise.} \end{cases}$$

Choosing $\mu = (1, 1, 1, \ldots)$ we have $\lim_{n\to\infty} d\left(\mathscr{S}^n\mu, \mathscr{S}^{n+1}\mu\right) = 0$ and $\lim_{n\to\infty} d\left(\mathscr{T}^n\mu, \mathscr{T}^{n+1}\mu\right) = 0$. Clearly, $\mathscr{S}$ and $\mathscr{T}$ are asymptotically regular, and orbitally continuous. There are non-negative numbers α, β, γ such that $\alpha + \beta + \gamma = 1$, we get

$$d(\mathscr{S}^p(\mu), \mathscr{T}^q(\omega)) \leq \alpha d(\mu, \omega) + \beta d(\mu, \mathscr{S}^p(\mu)) + \gamma d(\omega, \mathscr{T}^q(\omega)),$$

for some $p, q \in \mathbb{N}$.

Obviously, as all the assumptions of Corollary 2.1 hold, $\mathscr{S}$ and $\mathscr{T}$ have $\mu = (0, 0, 0, \ldots) \in \Lambda$, as their unique common fixed point. Also, $\lim_{n\to\infty} \mathscr{S}^p\mu = 0 = \lim_{n\to\infty} \mathscr{T}^q\mu$ for any $\mu \in \Lambda$.

Theorem 2.4. Let (Λ, d) be a complete metric space and $\mathscr{S} : \Lambda \to \Lambda$ be an asymptotically regular mapping. Assume that there exist non-negative numbers α, β, γ such that $\alpha + \beta + \gamma = 1$, and $\alpha < 1$, satisfying

$$d(\mathscr{S}\mu, \mathscr{S}\omega) \leq \alpha d(\mu, \omega) + \beta d(\mu, \mathscr{S}\mu) + \gamma d(\omega, \mathscr{S}\omega), \tag{12}$$

for all $\mu, \omega \in \Lambda$. The one of the following conditions hold:

(i) The mapping $\mathscr{S}$ is continuous. Further, $\mathscr{S}^n\mu \to \nu$ for each $\mu \in \Lambda$, as $n \to \infty$.

(ii) For $\kappa \geq 1$, $\mathscr{S}$ is κ-continuous or orbitally continuous.

Then, $\mathscr{S}$ has a unique fixed point $\nu \in \Lambda$.

Proof. First, we shall prove condition (i). Let $\mu_0 \in \Lambda$ be arbitrary and define a sequence $\{\mu_n\}$ by $\mu_{n+1} = \mathscr{S}\mu_n$ for all $n \geq 0$.

Using the triangle inequality and asymptotic regularity in (12) we get for any n and $\kappa > 0$,

$$\begin{aligned} d(\mu_{n+\kappa}, \mu_n) &\leq d(\mu_{n+\kappa}, \mu_{n+\kappa+1}) + d(\mu_{n+\kappa+1}, \mu_{n+1}) + d(\mu_{n+1}, \mu_n) \\ &\leq d(\mu_{n+\kappa}, \mu_{n+\kappa+1}) + \alpha d(\mu_{n+\kappa}, \mu_n) + \beta d(\mu_{n+\kappa}, \mu_{n+\kappa+1}) \\ &\quad + \gamma d(\mu_n, \mu_{n+1}) + d(\mu_{n+1}, \mu_n). \end{aligned}$$

Thus,

$$(1 - \alpha)d(\mu_{n+\kappa}, \mu_n) \leq (1 + \beta)d(\mu_{n+\kappa}, \mu_{n+\kappa+1}) + (1 + \gamma)d(\mu_n, \mu_{n+1}) \to 0,$$

as $n \to \infty$ and $\alpha < 1$. This shows that $\{\mu_n\}$ is a Cauchy sequence in complete metric space Λ. There exists $\nu \in \Lambda$ such that $\mu_n \to \nu$. The continuity of $\mathscr{S}$ and $\mu_{n+1} = \mathscr{S}\mu_n$, implies that $\nu = \mathscr{S}\nu$. Let $\nu^* \neq \nu$ be another fixed point of $\mathscr{S}$. Then

$$\begin{aligned} 0 < d(\nu, \nu^*) = d(\mathscr{S}\nu, \mathscr{S}\nu^*) &\leq \alpha d(\nu, \nu^*) + \beta d(\nu, \mathscr{S}\nu) + \gamma d(\nu^*, \mathscr{S}\nu^*) \\ &= \alpha d(\nu, \nu^*) < d(\nu, \nu^*), \end{aligned}$$

which is a contradiction. Hence, $\mathscr{S}$ has a unique fixed point $\nu \in \Lambda$. Now, we show that $\mathscr{S}^n\mu \to \nu$. From (12) we have

$$d(\mathscr{S}^n\mu, \nu) = d(\mathscr{S}^n\mu, \mathscr{S}^{n+1}\nu) \leq d(\mathscr{S}^n\mu, \mathscr{S}^{n+1}\mu) + d(\mathscr{S}^{n+1}\mu, \mathscr{S}^{n+1}\nu)$$
$$\leq d(\mathscr{S}^n\mu, \mathscr{S}^{n+1}\mu) + \alpha d(\mathscr{S}^n\mu, \mathscr{S}^n\nu) + \beta d(\mathscr{S}^n\mu, \mathscr{S}^{n+1}\mu)$$
$$+ \gamma d(\mathscr{S}^n\nu, \mathscr{S}^{n+1}\nu).$$

Hence,

$$(1-\alpha)d(\mathscr{S}^n\mu, \nu) \leq (1+\beta)d(\mathscr{S}^n\mu, \mathscr{S}^{n+1}\mu) \to 0, as\ n \to \infty.$$

This shows that $\mathscr{S}^n\mu \to \nu$ for any $\mu \in \Lambda$ as $\alpha < 1$.

Next, we will consider condition (ii). Choose μ_0 as an arbitrary point in Λ. We consider a sequence $\{\mu_n\} \in \Lambda$ given by $\mu_{n+1} = \mathscr{S}\mu_n$ for any $n \geq 0$. Then, from (i) we have proven that $\{\mu_n\}$ is a Cauchy sequence in a complete metric space. There exists a point $\nu \in \Lambda$ such that $\mu_n \to \nu$ as $n \to \infty$. and $\mathscr{S}\mu_n \to \nu$. Moreover, for all $\kappa \geq 1$ we have $\mathscr{S}^\kappa\mu_n \to \nu$ as $n \to \infty$. Suppose that $\mathscr{S}$ is κ-continuous. Since $\mathscr{S}^{\kappa-1}\mu_n \to \nu$, we get $\lim_{n\to\infty}\mathscr{S}^\kappa\mu_n = \mathscr{S}\nu$. This implies $\nu = \mathscr{S}\nu$, that is ν is a fixed point of $\mathscr{S}$. Finally, we assume that $\mathscr{S}$ is orbitally continuous. Since $\mu_n \to \nu$, orbital continuity implies that $\lim_{n\to\infty}\mathscr{S}\mu_n = \mathscr{S}\nu$. This yields $\mathscr{S}\nu = \nu$, that is $\mathscr{S}$ has a fixed point at ν.

Theorem 2.5. Let (Λ, d) be a complete metric space and $\mathscr{S} : \Lambda \to \Lambda$ be a continuous mapping satisfying (12). Assume that $\mathscr{S}$ has an approximate fixed point sequence. Then, $\mathscr{S}$ has a unique fixed point ν. In particular, $\mu_n \to \nu$ as $n \to \infty$.

Proof. Suppose that there exist $m, n \in \mathbb{N}$ such that $m > n$. Then, by triangle inequality and (12), we obtain

$$d(\mu_n, \mu_m) \leq d(\mu_n, \mathscr{S}\mu_n) + d(\mathscr{S}\mu_n, \mathscr{S}\mu_m) + d(\mathscr{S}\mu_m, \mu_m)$$
$$\leq d(\mu_n, \mathscr{S}\mu_n) + \alpha d(\mu_n, \mu_m) + \beta d(\mu_n, \mathscr{S}\mu_n) + (\gamma + 1)d(\mu_m, \mathscr{S}\mu_m),$$

which implies

$$(1-\alpha)d(\mu_n, \mu_m) \leq (1+\beta)d(\mu_n, \mathscr{S}\mu_n) + (1+\gamma)d(\mu_m, \mathscr{S}\mu_m).$$

As $n, m \to \infty$, we have $\lim_{n,m\to\infty} d(\mu_n, \mu_m) = 0$. Since Λ is a complete metric space, then $\{\mu_n\}$ is a Cauchy sequence. Hence, the sequence $\{\mu_n\}$ converges to $\nu \in \Lambda$. Since $\lim_{n\to\infty} d(\mu_n, \mathscr{S}\mu_n) = 0$, from the continuity of $\mathscr{S}$ we get that ν is a fixed point of $\mathscr{S}$. The uniqueness of the fixed point follows from (12).

Theorem 2.6. Let (Λ, d) be a complete metric space and $\mathscr{S}, \mathscr{T} : \Lambda \to \Lambda$. Suppose that $\mathscr{S}$ is asymptotically regular with respect to $\mathscr{T}$. Assume that there exist non-negative numbers α, β, γ such that $\alpha + \beta + \gamma = 1$, as $\alpha < 1$, satisfying

$$d(\mathscr{S}\mu, \mathscr{S}\omega) \leq \alpha d(\mathscr{T}\mu, \mathscr{T}\omega) + \beta d(\mathscr{T}\mu, \mathscr{S}\mu) + \gamma d(\mathscr{T}\omega, \mathscr{S}\omega), \tag{13}$$

for each $\mu, \omega \in \Lambda$. Further, suppose that $\mathscr{S}$ and $\mathscr{T}$ are $(\mathscr{S}, \mathscr{T})$-orbitally continuous and compatible. Then $C(\mathscr{S}, \mathscr{T}) \neq \phi$ and $\mathscr{S}$ and $\mathscr{T}$ have a unique common fixed point.

Proof. Since $\mathscr{S}$ is asymptotically regular with respect to $\mathscr{T}$ at $\mu_0 \in \Lambda$, so, there exists a sequence $\{\omega_n\} \in \Lambda$ such that $\omega_n = \mathscr{S}\mu_n = \mathscr{T}\mu_{n+1}$ for each $n \geq 0$, and $\lim_{n\to\infty} d(\mathscr{T}\mu_{n+1}, \mathscr{T}\mu_{n+2}) = \lim_{n\to\infty} d(\omega_n, \omega_{n+1}) = 0$. We show that $\{\omega_n\}$ is a Cauchy sequence. From (13) and triangle inequality, for any n and any $\kappa > 0$, we have

$$\begin{aligned} d(\mathscr{S}\mu_{n+\kappa}, \mathscr{S}\mu_n) = d(\omega_{n+\kappa}, \omega_n) &\le d(\omega_{n+\kappa}, \omega_{n+\kappa+1}) + d(\omega_{n+\kappa+1}, \omega_{n+1}) + d(\omega_{n+1}, \omega_n) \\ &\le d(\omega_{n+\kappa}, \omega_{n+\kappa+1}) + \alpha d(\omega_{n+\kappa}, \omega_n) + \beta d(\omega_{n+\kappa}, \omega_{n+\kappa+1}) \\ &\quad + \gamma d(\omega_n, \omega_{n+1}) + d(\omega_{n+1}, \omega_n) \\ &= (1+\beta) d(\omega_{n+\kappa}, \omega_{n+\kappa+1}) + \alpha d(\omega_{n+\kappa}, \omega_n) + (\gamma+1) d(\omega_n, \omega_{n+1}). \end{aligned}$$

Thus,

$$(1-\alpha) d(\omega_{n+\kappa}, \omega_n) \le (1+\beta) d(\omega_{n+\kappa}, \omega_{n+\kappa+1}) + (1+\gamma) d(\omega_{n+1}, \omega_n).$$

Since $\mathscr{S}$ is asymptotically regular with respect to $\mathscr{T}$, then $\lim_{n\to\infty} d(\omega_{n+\kappa}, \omega_n) = 0$. Therefore, $\{\omega_n\}$ is a Cauchy sequence in a complete metric space. There exists a point $\nu \in \Lambda$ such that $\omega_n \to \nu$ as $n \to \infty$. Moreover, $\omega_n = \mathscr{S}\mu_n = \mathscr{T}\mu_{n+1} \to \nu$.

Suppose that $\mathscr{S}$ and $\mathscr{T}$ are compatible mappings. By the orbital continuity of $\mathscr{S}$ and $\mathscr{T}$,

$$\lim_{n\to\infty} \mathscr{S}\mathscr{S}\mu_n = \lim_{n\to\infty} \mathscr{S}\mathscr{T}\mu_n = \mathscr{S}\nu,$$

further

$$\lim_{n\to\infty} \mathscr{T}\mathscr{S}\mu_n = \lim_{n\to\infty} \mathscr{T}\mathscr{T}\mu_n = \mathscr{T}\nu.$$

The compatibility of $\mathscr{S}$ and $\mathscr{T}$ implies $\lim_{n\to\infty} d(\mathscr{S}\mathscr{T}\mu_n, \mathscr{T}\mathscr{S}\mu_n) = 0$. Taking limit as $n \to \infty$ we have $\mathscr{S}\nu = \mathscr{T}\nu$, which means, $C(\mathscr{S}, \mathscr{T}) \neq \phi$. Since ν is a coincidence point, the compatibility of $\mathscr{S}$ and $\mathscr{T}$ implies the commutativity of ν. Hence, $\mathscr{T}\mathscr{S}\nu = \mathscr{S}\mathscr{S}\nu = \mathscr{T}\mathscr{T}\nu$. Using (13), we obtain

$$d(\mathscr{S}\nu, \mathscr{S}\mathscr{S}\nu) \le \alpha d(\mathscr{T}\nu, \mathscr{T}\mathscr{S}\nu) + \beta d(\mathscr{T}\nu, \mathscr{S}\nu) + \gamma d(\mathscr{T}\mathscr{S}\nu, \mathscr{S}\mathscr{S}\nu) = \alpha d(\mathscr{S}\nu, \mathscr{S}\mathscr{S}\nu),$$

which is a contradiction, thence, $\mathscr{S}\nu = \mathscr{S}\mathscr{S}\nu$. Hence $\mathscr{S}\nu = \mathscr{S}\mathscr{S}\nu = \mathscr{T}\mathscr{S}\nu$ and $\mathscr{S}\nu$ is a common fixed point of $\mathscr{S}$ and $\mathscr{T}$. The uniqueness of the common fixed point follows from (13).

In the next theorem, we establish a common fixed point result on Chatterjea nonexpansive mapping.

Theorem 2.7. Let (Λ, d) be a complete metric space and $\mathscr{S}$, $\mathscr{T}$ be asymptotically regular self-mapping on Λ. Assume that there exist non-negative numbers α, β, γ such that $2\alpha + \beta + 2\gamma = 1$, satisfying

$$d(\mathscr{S}\mu, \mathscr{T}\omega) \le \alpha d(\mu, \omega) + \beta d(\omega, \mathscr{S}\mu) + \gamma d(\mu, \mathscr{T}\omega), \tag{14}$$

for each $\mu, \omega \in \Lambda$. Suppose further that $\mathscr{S}$ and $\mathscr{T}$ are either κ-continuous for some $\kappa \ge 1$ or orbitally continuous. Then, $\mathscr{S}$ and $\mathscr{T}$ have a unique common fixed point ν. Moreover, for any $\mu \in \Lambda$, $\lim_{n\to\infty} \mathscr{S}^n\mu = \nu = \lim_{n\to\infty} \mathscr{T}^n\mu$.

Proof. We follow the lines of Theorem 2.1 to prove this theorem. The proof is divided into three steps as follow:

Step 1: We shall prove that $\lim_{n\to\infty} d(\mathscr{S}^n\mu, \mathscr{T}^n\mu) = 0$, for any $\mu \in \Lambda$. The result is trivial if $\mathscr{S} = \mathscr{T}$. Suppose $\mathscr{S} \neq \mathscr{T}$ and $\alpha = 0$, from (14) we obtain

$$d(\mathscr{S}\mu, \mathscr{T}\omega) \le \beta d(\omega, \mathscr{S}\mu) + \gamma d(\mu, \mathscr{T}\omega),$$

for each $\mu, \omega \in \Lambda$. Define $\mu_1 = \mathscr{S}^n\mu$ and $\omega_1 = \mathscr{T}^n\mu$, for any $\mu \in \Lambda$. Then

$$\begin{aligned} d(\mathscr{S}^{n+1}\mu, \mathscr{T}^{n+1}\mu) &\leq \beta d(\mathscr{T}^n\mu, \mathscr{S}^{n+1}\mu) + \gamma d(\mathscr{S}^n\mu, \mathscr{T}^{n+1}\mu) \\ &\leq \beta\{d(\mathscr{T}^n\mu, \mathscr{T}^{n+1}\mu) + d(\mathscr{T}^{n+1}\mu, \mathscr{S}^{n+1}\mu)\} \\ &\quad + \gamma\{d(\mathscr{S}^n\mu, \mathscr{S}^{n+1}\mu) + d(\mathscr{S}^{n+1}\mu, \mathscr{T}^{n+1}\mu)\}, \end{aligned}$$

which implies

$$d\left(\mathscr{S}^{n+1}\mu, \mathscr{T}^{n+1}\mu\right) \leq \frac{\beta}{2\alpha+\gamma} d\left(\mathscr{T}^n\mu, \mathscr{T}^{n+1}\mu\right) + \frac{\gamma}{2\alpha+\gamma} d\left(\mathscr{S}^n\mu, \mathscr{S}^{n+1}\mu\right).$$

As $n \to \infty$, since $\mathscr{S}$ and $\mathscr{T}$ are asymptotic regularity, we have

$$\lim_{n\to\infty} d\left(\mathscr{S}^{n+1}\mu, \mathscr{T}^{n+1}\mu\right) = 0. \tag{15}$$

By the asymptotic regularity of $\mathscr{S}$ and $\mathscr{T}$ and triangle inequality, we have

$$d(\mathscr{S}^n\mu, \mathscr{T}^n\mu) \leq d(\mathscr{S}^n\mu, \mathscr{S}^{n+1}\mu) + d(\mathscr{S}^{n+1}\mu, \mathscr{T}^{n+1}\mu) + d(\mathscr{T}^{n+1}\mu, \mathscr{T}^n\mu) \to 0, \text{ as } n \to \infty.$$

Next, suppose that $\mathscr{S} \neq \mathscr{T}$ and $\alpha \neq 0$. Let $\mu_1 = \mathscr{S}^n\mu$ and $\omega_1 = \mathscr{T}^n\mu$, for any $\mu \in \Lambda$. Then, by (14) we obtain

$$\begin{aligned} d(\mathscr{S}^{n+1}\mu, \mathscr{T}^{n+1}\mu) &\leq \alpha d(\mathscr{S}^n\mu, \mathscr{T}^n\mu) + \beta d(\mathscr{T}^n\mu, \mathscr{S}^{n+1}\mu) + \gamma d(\mathscr{S}^n\mu, \mathscr{T}^{n+1}\mu) \\ &\leq \alpha d(\mathscr{S}^n\mu, \mathscr{T}^n\mu) + \beta\{d(\mathscr{T}^n\mu, \mathscr{T}^{n+1}\mu) + d(\mathscr{T}^{n+1}\mu, \mathscr{S}^{n+1}\mu)\} \\ &\quad + \gamma\{d(\mathscr{S}^n\mu, \mathscr{S}^{n+1}\mu) + d(\mathscr{S}^{n+1}\mu, \mathscr{T}^{n+1}\mu)\}, \end{aligned}$$

obtain

$$(1-\beta-\gamma)d(\mathscr{S}^{n+1}\mu, \mathscr{T}^{n+1}\mu) \leq \alpha d(\mathscr{S}^n\mu, \mathscr{T}^n\mu) + \beta d(\mathscr{T}^n\mu, \mathscr{T}^{n+1}\mu) + \gamma d(\mathscr{S}^n\mu, \mathscr{S}^{n+1}\mu)$$

which implies

$$d\left(\mathscr{S}^{n+1}\mu, \mathscr{T}^{n+1}\mu\right) \leq \frac{\alpha}{2\alpha+\gamma} d\left(\mathscr{S}^n\mu, \mathscr{T}^n\mu\right) + \frac{\beta}{2\alpha+\gamma} d\left(\mathscr{T}^n\mu, \mathscr{T}^{n+1}\mu\right) + \frac{\gamma}{2\alpha+\gamma} d\left(\mathscr{S}^n\mu, \mathscr{S}^{n+1}\mu\right).$$

Let $h_n = d(\mathscr{S}^n\mu, \mathscr{T}^n\mu)$, $\psi_n = 1 - \dfrac{\alpha}{2\alpha+\gamma}$ and $\vartheta_n = \dfrac{\beta}{\alpha} d\left(\mathscr{T}^n\mu, \mathscr{T}^{n+1}\mu\right) + \dfrac{\gamma}{\alpha} d\left(\mathscr{S}^n\mu, \mathscr{S}^{n+1}\mu\right)$. By the asymptotically regularity of $\mathscr{S}$ and $\mathscr{T}$, we have $\lim_{n\to\infty} \vartheta_n = 0$. Furthermore, $\sum_{n=1}^{\infty} \psi_n = \infty$. Hence, by Lemma 2.1, we get that $\lim_{n\to\infty} d\left(\mathscr{S}^n\mu, \mathscr{T}^n\mu\right) = 0$ for any $\mu \in \Lambda$.

Step 2: Let $\mu_n = \mathscr{S}^n\mu$ for any μ in Λ and $n \geq 0$. Then, we show that $\{\mu_n\}$ is a Cauchy sequence which is convergent to $\nu \in \Lambda$. Moreover, $\{\mathscr{T}^n\mu\} \to \nu \in \Lambda$.

Assume that $\{\mathscr{S}^n\mu\}$ is not a Cauchy sequence. Then, there exists an $\varepsilon > 0$ and sequences of integers $\{m_\kappa\}$ and $\{n_\kappa\}$ such that $m_\kappa > n_\kappa \geq \kappa$, for $\kappa = 1, 2, \ldots$, we have

$$d(\mathscr{S}^{m_\kappa}, \mathscr{S}^{n_\kappa}) \geq \varepsilon. \tag{16}$$

Choosing m_κ, the smallest number exceeding n_κ for which (16) holds, we also assume that

$$d(\mathscr{S}^{m_{(\kappa-1)}}\mu, \mathscr{S}^{n_\kappa}\mu) < \varepsilon.$$

Now, we have that

$$\varepsilon \leq d(\mathscr{S}^{m_\kappa}\mu, \mathscr{S}^{n_\kappa}\mu) \leq d(\mathscr{S}^{m_\kappa}\mu, \mathscr{S}^{m_{(\kappa-1)}}\mu) + d(\mathscr{S}^{m_{(\kappa-1)}}\mu, \mathscr{S}^{n_\kappa}\mu)$$
$$< d(\mathscr{S}^{m_\kappa}\mu, \mathscr{S}^{m_{(\kappa-1)}}\mu) + \varepsilon.$$

As $\kappa \to \infty$, it follows by asymptotic regularity of $\mathscr{S}$ that

$$\lim_{\kappa\to\infty} d\,(\mathscr{S}^{m_\kappa}\mu, \mathscr{S}^{n_\kappa}\mu) = \varepsilon. \tag{17}$$

Furthermore, by asymptotic regularity of $\mathscr{S}$ and the above inequality, we obtain

$$d(\mathscr{S}^{m_{(\kappa-1)}}\mu, \mathscr{S}^{n_{(\kappa-1)}}\mu) \leq d(\mathscr{S}^{m_{(\kappa-1)}}\mu, \mathscr{S}^{m_\kappa}\mu) + d(\mathscr{S}^{m_\kappa}\mu, \mathscr{S}^{n_\kappa}\mu) + d(\mathscr{S}^{n_\kappa}\mu, \mathscr{S}^{n_{(\kappa-1)}}\mu),$$

implying that

$$\lim_{\kappa\to\infty} d\,(\mathscr{S}^{m_{(\kappa-1)}}\mu, \mathscr{S}^{n_{(\kappa-1)}}\mu) = \varepsilon. \tag{18}$$

Then, using (14), we have

$$\begin{aligned} d(\mathscr{S}^{m_\kappa}\mu, \mathscr{S}^{n_\kappa}\mu) &\leq d(\mathscr{S}^{m_\kappa}\mu, \mathscr{T}^{n_\kappa}\mu) + d(\mathscr{T}^{n_\kappa}\mu, \mathscr{S}^{n_\kappa}\mu) \\ &\leq d(\mathscr{T}^{n_\kappa}\mu, \mathscr{S}^{n_\kappa}\mu) + \alpha d(\mathscr{S}^{m_{(\kappa-1)}}\mu, \mathscr{T}^{n_{(\kappa-1)}}\mu) + \beta d(\mathscr{T}^{n_{(\kappa-1)}}\mu, \mathscr{S}^{m_\kappa}\mu) \\ &\quad + \gamma d(\mathscr{S}^{m_{(\kappa-1)}}\mu, \mathscr{T}^{n_\kappa}\mu) \\ &\leq d(\mathscr{T}^{n_\kappa}\mu, \mathscr{S}^{n_\kappa}\mu) + \alpha\{d(\mathscr{S}^{m_{(\kappa-1)}}\mu, \mathscr{S}^{n_{(\kappa-1)}}\mu) + d(\mathscr{S}^{n_{(\kappa-1)}}\mu, \mathscr{T}^{n_{(\kappa-1)}}\mu)\} \\ &\quad + \beta\{d(\mathscr{T}^{n_{(\kappa-1)}}\mu, \mathscr{S}^{n_{\kappa-1}}\mu) + d(\mathscr{S}^{n_{(\kappa-1)}}\mu, \mathscr{S}^{m_{(\kappa-1)}}\mu) + d(\mathscr{S}^{m_{(\kappa-1)}}\mu, \mathscr{S}^{m_\kappa}\mu)\} \\ &\quad + \gamma\{d(\mathscr{S}^{m_{(\kappa-1)}}\mu, \mathscr{S}^{n_{(\kappa-1)}}\mu) + d(\mathscr{S}^{n_{(\kappa-1)}}\mu, \mathscr{S}^{n_\kappa}\mu) + d(\mathscr{S}^{n_\kappa}\mu, \mathscr{T}^{n_\kappa}\mu)\}, \end{aligned}$$

as $\kappa \to \infty$, From (17), (18) and the asymptotic regularity of $\mathscr{S}$ and $\mathscr{T}$, we have, $\varepsilon \leq (\alpha + \beta + \gamma)\varepsilon$. This is a contradiction. Hence, $\{\mu_n\}$ is a Cauchy sequence in complete space Λ. There exists ν in Λ such that $\mu_n \to \nu$. Moreover

$$d(\mathscr{T}^n\mu, \nu) \leq d(\mathscr{T}^n\mu, \mathscr{S}^n\mu) + d(\mathscr{S}^n\mu, \nu),$$

from (15) and $\mu_n \to \nu$ implies $\mathscr{T}^n\mu$ converges to $\nu \in \Lambda$.

Step 3: We prove the uniqueness of the common fixed point of $\mathscr{S}$ and $\mathscr{T}$.

Let $\mathscr{S}$ be κ-continuous. Since $\lim_{n\to\infty} \mathscr{S}^{\kappa-1}\mu_n = \nu$, we have that

$$\lim_{n\to\infty} \mathscr{S}^\kappa\mu_n = \mathscr{S}\nu.$$

Since the limit is unique, it implies $\mathscr{S}\nu = \nu$.

Similarly, suppose that $\mathscr{S}$ is orbitally continuous. Since $\lim_{n\to\infty} \mu_n = \nu$, we have that

$$\lim_{n\to\infty} \mathscr{S}\mu_n = \mathscr{S}\mu,$$

implies that $\mathscr{S}\nu = \nu$. Furthermore, since $\lim_{n\to\infty} \mathscr{T}^n\mu = \nu$, we have that $\mathscr{T}\nu = \nu$ whenever $\mathscr{T}$ is κ-continuous or orbitally continuous. Hence, $\nu \in Fix(\mathscr{S}) \cap Fix(\mathscr{T})$. For the uniqueness of fixed point, suppose that $\nu \neq \nu^*$ are two common fixed points of $\mathscr{S}$ and $\mathscr{T}$. Let $\mu = \nu$ and $\omega = \nu^*$. Then, (14) implies $d(\nu, \nu^*) \leq (\alpha + \beta + \gamma)d(\nu, \nu^*)$. Hence, this is a contradiction. We must have, ν as the unique common fixed point of $\mathscr{S}$ and $\mathscr{T}$.

Example 2.8. Let $\Lambda = C^0([0,1]\times[0,1])$ be the space of all continuous functions on [0, 1]. Defined $d : \Lambda \times \Lambda \to \mathbb{R}^+$ by

$d((f_1,f_2),(g_1,g_2)) = \|f_1 - g_1\|_\infty + \|f_2F_1 - g_2G_1\|_\infty.$

where $F_1(t) = 1 + \int_0^t f_1(u)du$ and $G_1(t) = 1 + \int_0^t g_1(u)du$ for each $t \in [0,1]$. Let $\mathscr{S}$ and $\mathscr{T}$ be such that

$$\mathscr{S}(f_1,f_2) = \begin{cases} \left(\frac{1}{5},\frac{1}{5}\right) & \text{if } (f_1,f_2) \neq \left(\frac{1}{5},0\right), \\ \left(\frac{1}{5},\frac{2}{15}\right) & \text{if } (f_1,f_2) = \left(\frac{1}{5},0\right), \end{cases}$$

and

$$\mathscr{T}(f_1,f_2) = \begin{cases} \left(\frac{1}{5},\frac{1}{5}\right) & \text{if } (f_1,f_2) \neq \left(\frac{1}{5},0\right), \\ \left(\frac{1}{5},\frac{3}{15}\right) & \text{if } (f_1,f_2) = \left(\frac{1}{5},0\right). \end{cases}$$

If we choose $f = (f_1,f_2) = \left(\frac{1}{2},\frac{1}{2}\right)$ we have $\lim_{n\to\infty} d\left(\mathscr{S}^n f, \mathscr{S}^{n+1} f\right) = \lim_{n\to\infty} d\left(\mathscr{T}^n f, \mathscr{T}^{n+1} f\right) = 0.$ Then, $\mathscr{S}$ and $\mathscr{T}$ are asymptotically regular, orbitally continuous at $\left(\frac{1}{5},\frac{1}{5}\right)$ and discontinuous at $\left(\frac{1}{5},0\right)$. Now, we show that $\mathscr{S}$ and $\mathscr{T}$ satisfy the condition (14). We choose $\alpha = \frac{1}{15}$, $\beta = \frac{1}{5}$ and $\gamma = \frac{1}{3}$.

In fact, we have the following four cases:

Case 1: Let $(f_1(t), f_2(t)) \neq \left(\frac{1}{5},0\right)$, and $(g_1(t), g_2(t)) \neq \left(\frac{1}{5},0\right)$, we have

$$\begin{aligned} \alpha d(f,g) + \beta d(g,\mathscr{S}(f)) + \gamma d(f,\mathscr{T}(g)) &= \frac{1}{15} d(f,g) + \frac{1}{5} d\left((g_1,g_2),\left(\frac{1}{5},\frac{1}{5}\right)\right) + \frac{1}{3} d\left((f_1,f_2),\left(\frac{1}{5},\frac{1}{5}\right)\right) \\ &\geq 0 = d(\mathscr{S}(f),\mathscr{T}(g)). \end{aligned}$$

Case 2: If $(f_1(t), f_2(t)) \neq \left(\frac{1}{5},0\right)$, and $(g_1(t), g_2(t)) = \left(\frac{1}{5},0\right)$, implies

$$\begin{aligned} d(\mathscr{S}(f),\mathscr{T}(g)) &= d\left(\left(\frac{1}{5},\frac{1}{5}\right),\left(\frac{1}{5},\frac{3}{15}\right)\right) \\ &= \left\|\frac{1}{5} - \frac{1}{5}\right\|_\infty + \left\|\frac{1}{5} + \int_0^1 \frac{1}{25} du - \frac{3}{15} - \frac{3}{15}\int_0^1 \frac{1}{5} du\right\|_\infty \\ &= \left\|\frac{1}{25} - \frac{3}{75}\right\|_\infty = 0. \end{aligned}$$

Therefore,

$$\begin{aligned} \alpha d(f,g) + \beta d(g,\mathscr{S}(f)) + \gamma d(f,\mathscr{T}(g)) &= \frac{1}{15} d(f,g) + \frac{1}{5} d(g,\mathscr{S}(f)) + \frac{1}{3} d(f,\mathscr{T}(g)) \\ &\geq 0 = d(\mathscr{S}(f),\mathscr{T}(g)). \end{aligned}$$

Case 3: If $(f_1(t), f_2(t)) = \left(\frac{1}{5}, 0\right)$, and $(g_1(t), g_2(t)) \neq \left(\frac{1}{5}, 0\right)$, we obtain

$$\begin{aligned}
d(\mathscr{S}(f), \mathscr{T}(g)) &= d\left(\left(\frac{1}{5}, \frac{2}{15}\right), \left(\frac{1}{5}, \frac{1}{5}\right)\right) \\
&= \left\|\frac{1}{5} - \frac{1}{5}\right\|_\infty + \left\|\frac{2}{15} + \frac{2}{15}\int_0^1 \frac{1}{5}du - \frac{1}{5} - \frac{1}{5}\int_0^1 \frac{1}{5}du\right\|_\infty \\
&= \left\|-\frac{6}{75}\right\|_\infty = \frac{6}{75}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\alpha d(f,g) + \beta d(g, \mathscr{S}(f)) + \gamma d(f, \mathscr{T}(g)) &= \frac{1}{15}d(f,g) + \frac{1}{5}d(g, \mathscr{S}(f)) + \frac{1}{3}d(f, \mathscr{T}(g)) \\
&= \frac{1}{15}d(f,g) + \frac{1}{5}d(g, \mathscr{S}(f)) + \frac{1}{3}d\left(\left(\frac{1}{5}, 0\right), \left(\frac{1}{5}, \frac{1}{5}\right)\right) \\
&= \frac{1}{15}d(f,g) + \frac{1}{5}d(g, \mathscr{S}(f)) + \frac{1}{3}\left\|\frac{6}{25}\right\|_\infty \\
&= \frac{1}{15}d(f,g) + \frac{1}{5}d(g, \mathscr{S}(f)) + \frac{6}{75} \geq d(\mathscr{S}(f), \mathscr{T}(g)).
\end{aligned}$$

Case 4: If $(f_1(t), f_2(t)) = \left(\frac{1}{5}, 0\right)$, and $(g_1(t), g_2(t)) = \left(\frac{1}{5}, 0\right)$, we obtain

$$\begin{aligned}
d(\mathscr{S}(f), \mathscr{T}(g)) &= d\left(\left(\frac{1}{5}, \frac{2}{15}\right), \left(\frac{1}{5}, \frac{3}{15}\right)\right) \\
&= \left\|\frac{1}{5} - \frac{1}{5}\right\|_\infty + \left\|\frac{2}{15} + \frac{2}{15}\int_0^1 \frac{1}{5}du - \frac{3}{15} - \frac{3}{15}\int_0^1 \frac{1}{5}du\right\|_\infty \\
&= \left\|-\frac{4}{75}\right\|_\infty = \frac{4}{75}.
\end{aligned}$$

However,

$$\begin{aligned}
\alpha d(f,g) + \beta d(g, \mathscr{S}(f)) + \gamma d(f, \mathscr{T}(g)) &= \frac{1}{15}d\left(\left(\frac{1}{5}, 0\right), \left(\frac{1}{5}, 0\right)\right) + \frac{1}{5}d\left(\left(\frac{1}{5}, 0\right), \left(\frac{1}{5}, \frac{2}{15}\right)\right) \\
&\quad + \frac{1}{3}d\left(\left(\frac{1}{5}, 0\right), \left(\frac{1}{5}, \frac{3}{15}\right)\right) \\
&= \frac{1}{5}\left\|\frac{12}{75}\right\|_\infty + \frac{1}{3}\left\|\frac{18}{75}\right\|_\infty \\
&= \frac{4}{125} + \frac{6}{75} \\
&= \frac{6}{75} > \frac{4}{75} = d(\mathscr{S}(f), \mathscr{T}(g)).
\end{aligned}$$

Therefore, in all cases, $\mathscr{S}$ and $\mathscr{T}$ satisfy the condition (14) for all $\mu, \omega \in \Lambda$. Moreover, all the assumptions of Theorem 2.7 hold. Point $\left(\frac{1}{5}, \frac{1}{5}\right)$ is the unique common fixed point of $\mathscr{S}$ and $\mathscr{T}$. Moreover, $\lim_{n\to\infty} \mathscr{S}^n\mu = 0 = \lim_{n\to\infty} \mathscr{T}^n\mu$ for any $\mu \in \Lambda$.

Corollary 2.2. Let (Λ, d) be a complete metric space and $\mathscr{S}$ and $\mathscr{T}$ be self-mappings on Λ where $\mathscr{S}^p$ and $\mathscr{T}^q$ are asymptotically regular for some positive integers p and q, respectively. Assume that there exist non-negative numbers α, β, γ with $2\alpha+\beta+2\gamma = 1$ such that the following condition holds

$$d(\mathscr{S}^p\mu, \mathscr{T}^q\omega) \leq \alpha d(\mu, \omega) + \beta d(\omega, \mathscr{S}^p\mu) + \gamma d(\mu, \mathscr{T}^q\omega), \tag{19}$$

for all $\mu, \omega \in \Lambda$. Then, $\mathscr{S}$ and $\mathscr{T}$ have a unique common fixed point $\nu \in \Lambda$, provided that both $\mathscr{S}^p$ and $\mathscr{T}^q$ are either κ-continuous for some $\kappa \geq 1$ or orbitally continuous.

Proof. Take $f = \mathscr{S}^p$ and $g = \mathscr{T}^q$. Then, for each $\mu, \omega \in \Lambda$, the (19) becomes

$$\begin{aligned} d(f\mu, g\omega) &\leq \alpha d(\mu, \omega) + \beta d(\omega, f\mu) + \gamma d(\mu, g\omega) \\ &\leq \alpha d(\mu, \omega) + \beta\{d(\omega, \mu) + d(\mu, f\mu)\} + \gamma\{d(\mu, \omega) + (\omega, g\omega)\} \\ &\leq (\alpha + \beta + \gamma) d(\omega, \mu) + \beta d(\mu, f\mu) + \gamma d(\omega, g\omega) \\ &\leq d(\omega, \mu) + \beta d(\mu, f\mu) + \gamma d(\omega, g\omega). \end{aligned} \tag{20}$$

According to Theorem 2.7, f and g have a unique common fixed point ν. Now, we show that $\mathscr{S}\nu$ is a fixed point of f, obtain

$$f(\mathscr{S}\nu) = \mathscr{S}^p(\mathscr{S}\nu) = \mathscr{S}^{p+1}\nu = \mathscr{S}(f\nu) = \mathscr{S}\nu,$$

implies that $\mathscr{S}\nu$ is a fixed point of f. Similarly, we can prove that $\mathscr{T}\nu$ is also a fixed point of g. Using (20), given that $\mathscr{S}\nu = \mathscr{T}\nu$, since f and g have a unique common fixed point, it follows that $\mathscr{S}\nu = \mathscr{T}\nu = \nu$. Suppose that ν^* is another common fixed point of $\mathscr{S}$ and $\mathscr{T}$ such that $\nu^* \neq \nu$. we have, $f\nu^* = g\nu^* = \nu^*$. The uniqueness of the common fixed point of f and g implies $\nu^* = \nu$. Then ν is a unique common fixed point of $\mathscr{S}$ and $\mathscr{T}$.

Theorem 2.9. Let (Λ, d) be a complete metric space. The mapping $\mathscr{S} : \Lambda \to \Lambda$ is asymptotically regular. Assume that there exist non-negative numbers α, β and γ with $2\alpha + \beta + 2\gamma = 1$ such that

$$d(\mathscr{S}\mu, \mathscr{S}\omega) \leq \alpha d(\mu, \omega) + \beta d(\omega, \mathscr{S}\mu) + \gamma d(\mu, \mathscr{S}\omega), \tag{21}$$

for all $\mu, \omega \in \Lambda$, provided that one of the following conditions hold:

(i) The mapping $\mathscr{S}$ is a continuous. Further $\mathscr{S}^n\mu \to \nu$ for each $\mu \in \Lambda$, as $n \to 0$.

(ii) For $\kappa \geq 1$, $\mathscr{S}$ is κ-continuous or orbitally continuous.

Then $\mathscr{S}$ has a unique fixed point $\mu \in \Lambda$

Proof. First, we will proof the condition (i) holds. Let $\mu_0 \in \Lambda$ be arbitrary. Then define a sequence $\{\mu_n\}$ by $\mu_{n+1} = \mathscr{S}\mu_n$ for all $n \geq 0$.

Using the triangle inequality and asymptotic regularity in (21) we obtain for any n and $\kappa > 0$,

$$\begin{aligned}
d(\mu_{n+\kappa},\mu_n) &\le d(\mu_{n+\kappa},\mu_{n+\kappa+1}) + d(\mu_{n+\kappa+1},\mu_{n+1}) + d(\mu_{n+1},\mu_n)\\
&\le d(\mu_{n+\kappa},\mu_{n+\kappa+1}) + \alpha d(\mu_{n+\kappa},\mu_n) + \beta d(\mu_n,\mu_{n+\kappa+1})\\
&\quad + \gamma d(\mu_{n+\kappa},\mu_{n+1}) + d(\mu_{n+1},\mu_n)\\
&\le d(\mu_{n+\kappa},\mu_{n+\kappa+1}) + \alpha d(\mu_{n+\kappa},\mu_n) + \beta\{d(\mu_n,\mu_{n+\kappa}) + d(\mu_{n+\kappa},\mu_{n+\kappa+1})\}\\
&\quad + \gamma\{d(\mu_{n+\kappa},\mu_n) + d(\mu_n,\mu_{n+1})\} + d(\mu_{n+1},\mu_n)\\
&= (1+\beta)d(\mu_{n+\kappa},\mu_{n+\kappa+1}) + (\alpha+\beta+\gamma)d(\mu_{n+\kappa},\mu_n) + (1+\gamma)d(\mu_{n+1},\mu_n).
\end{aligned}$$

Then,

$$d(\mu_{n+\kappa},\mu_n) \le \frac{1+\beta}{\alpha+\gamma} d(\mu_{n+\kappa},\mu_{n+\kappa+1}) + \frac{1+\gamma}{\alpha+\gamma} d(\mu_n,\mu_{n+1}) \to 0,$$

as $n \to \infty$. This shows that $\{\mu_n\}$ is a Cauchy sequence in a complete metric space Λ, there exists $\nu \in \Lambda$ such that $\mu_n \to \nu$. As $\mathscr{S}$ is continuous and $\mu_{n+1} = \mathscr{S}\mu_n$, we obtain that $\nu = \mathscr{S}\nu$. Suppose that $\nu^* \neq \nu$ is another fixed point of $\mathscr{S}$. Then, we have

$$\begin{aligned}
0 < d(\nu,\nu^*) = d(\mathscr{S}\nu,\mathscr{S}\nu^*) &\le \alpha d(\nu,\nu^*) + \beta d(\nu^*,\mathscr{S}\nu) + \gamma d(\nu,\mathscr{S}\nu^*)\\
&= (\alpha+\beta+\gamma)d(\nu,\nu^*) < d(\nu,\nu^*),
\end{aligned}$$

which is a contradiction. Hence, $\mathscr{S}$ has a unique fixed point $\nu \in \Lambda$. Now, we show that $\mathscr{S}^n \to \nu$. From (21), we have

$$\begin{aligned}
d(\mathscr{S}^n\mu,\nu) = d(\mathscr{S}^n\mu,\mathscr{S}^n\nu) &\le d(\mathscr{S}^n\mu,\mathscr{S}^{n+1}\mu) + d(\mathscr{S}^{n+1}\mu,\mathscr{S}^{n+1}\nu)\\
&\le d(\mathscr{S}^n\mu,\mathscr{S}^{n+1}\mu) + \alpha d(\mathscr{S}^n\mu,\mathscr{S}^n\nu) + \beta d(\mathscr{S}^n\nu,\mathscr{S}^{n+1}\mu)\\
&\quad + \gamma d(\mathscr{S}^n\mu,\mathscr{S}^{n+1}\nu)\\
&\le d(\mathscr{S}^n\mu,\mathscr{S}^{n+1}\mu) + \alpha d(\mathscr{S}^n\mu,\nu) + \beta\{d(\nu,\mathscr{S}^{n+1}\mu) + d(\mathscr{S}^n\mu,\mathscr{S}^{n+1}\mu)\}\\
&\quad + \gamma d(\mathscr{S}^n\mu,\nu)\\
&= (1+\beta)d(\mathscr{S}^n\mu,\mathscr{S}^{n+1}\mu) + (\alpha+\beta+\gamma)d(\mathscr{S}^n\mu,\nu).
\end{aligned}$$

Hence,

$$d(\mathscr{S}^n\mu,\nu) \le \frac{1+\beta}{\alpha+\gamma} d\left(\mathscr{S}^n\mu,\mathscr{S}^{n+1}\mu\right) \to 0, \textit{as } n \to \infty.$$

This shows that $\mathscr{S}^n \to \nu$ for any $\mu \in \Lambda$.

We next prove the condition (ii) holds. Suppose that μ_0 is any point in Λ. The sequence $\{\mu_n\} \in \Lambda$ is given by $\mu_{n+1} = \mathscr{S}\mu_n = \mathscr{S}^n\mu$, for each $n \ge 0$. Then, in the proof condition (i) we have shown that $\{\mu_n\}$ is a Cauchy sequence in complete space Λ. There exists a point $\nu \in \Lambda$ such that $\mu_n \to \nu$ as $n \to \infty$, in addition, $\mathscr{S}\mu_n \to \nu$. Moreover, for each $\kappa \ge 1$ we have $\mathscr{S}^\kappa\mu_n \to \nu$ as $n \to \infty$. Suppose that $\mathscr{S}$ is κ-continuous, and $\mathscr{S}^{\kappa-1}\mu_n \to \nu$, implies $\lim_{n\to\infty}\mathscr{S}^\kappa\mu_n = \mathscr{S}\nu$. Hence, ν is a fixed point of $\mathscr{S}$.

Finally, we show that ν as a fixed point of $\mathscr{S}$. Assume that $\mathscr{S}$ is orbitally continuous. Since $\mu_n \to \nu$, orbital continuity implies that $\lim_{n\to\infty}\mathscr{S}\mu_n = \mathscr{S}\nu$. Then ν is a fixed point of $\mathscr{S}$.

Theorem 2.10. Let (Λ, d) be a complete metric space and $\mathscr{S}:\Lambda \to \Lambda$ be a continuous mapping satisfying (21). Suppose that $\mathscr{S}$ has an approximate fixed point sequence. Then, $\mathscr{S}$ has a unique fixed point ν. In particular, $\mu_n \to \nu$ as $n \to \infty$.

Proof. Let $m > n$ for all $n, m \in \mathbb{N}$. Then, using triangle inequality and (21), we obtain

$$\begin{aligned} d(\mu_n, \mu_m) &\le d(\mu_n, \mathscr{S}\mu_n) + d(\mathscr{S}\mu_n, \mathscr{S}\mu_m) + d(\mathscr{S}\mu_m, \mu_m) \\ &\le d(\mu_n, \mathscr{S}\mu_n) + \alpha d(\mu_n, \mu_m) + \beta d(\mu_m, \mathscr{S}\mu_n) + \gamma d(\mu_n, \mathscr{S}\mu_m) + d(\mathscr{S}\mu_m, \mu_m) \\ &\le d(\mu_n, \mathscr{S}\mu_n) + \alpha d(\mu_n, \mu_m) + \beta\{d(\mu_m, \mu_n) + d(\mu_n, \mathscr{S}\mu_n)\} \\ &\quad + \gamma\{d(\mu_n, \mu_m) + d(\mu_m, \mathscr{S}\mu_m)\} + d(\mathscr{S}\mu_m, \mu_m) \\ &\le (1+\beta)d(\mu_n, \mathscr{S}\mu_n) + (\alpha+\beta+\gamma)d(\mu_n, \mu_m) + (1+\gamma)d(\mu_m, \mathscr{S}\mu_m). \end{aligned}$$

This implies

$$(\alpha+\gamma)d(\mu_n, \mu_m) \le (1+\beta)d(\mu_n, \mathscr{S}\mu_n) + (1+\gamma)d(\mu_m, \mathscr{S}\mu_m).$$

As $n, m \to \infty$, we have $\lim_{n,m\to\infty} d(\mu_n, \mu_m) = 0$. Hence, $\{\mu_n\}$ is a Cauchy sequence in Λ. By completeness of Λ, this sequence $\{\mu_n\}$ converges to $\nu \in \Lambda$. Since $\lim_{n\to\infty} d(\mu_n, \mathscr{S}\mu_n) = 0$, continuity of $\mathscr{S}$ implies that ν is a fixed point of $\mathscr{S}$. The uniqueness of the fixed point follows from the inequality (21).

Theorem 2.11. Let (Λ, d) be a complete metric space and $\mathscr{S}, \mathscr{T}:\Lambda \to \Lambda$. Suppose that $\mathscr{S}$ is asymptotically regular with respect to $\mathscr{T}$. Assume that there exist non-negative numbers α, β, γ such that $2\alpha + \beta + 2\gamma = 1$, satisfying

$$d(\mathscr{S}\mu, \mathscr{S}\omega) \le \alpha d(\mathscr{T}\mu, \mathscr{T}\omega) + \beta d(\mathscr{T}\omega, \mathscr{S}\mu) + \gamma d(\mathscr{T}\mu, \mathscr{S}\omega), \tag{22}$$

for each $\mu, \omega \in \Lambda$. Furthermore, suppose that $\mathscr{S}$ and $\mathscr{T}$ are $(\mathscr{S}, \mathscr{T})$-orbitally continuous and compatible. Then $C(\mathscr{S}, \mathscr{T}) \neq \phi$ and $\mathscr{S}$ and $\mathscr{T}$ have a unique common fixed point.

Proof. Since $\mathscr{S}$ is asymptotically regularity with respect to $\mathscr{T}$ at $\mu_0 \in \Lambda$. So, there exists a sequence $\{\omega_n\} \in \Lambda$ such that $\omega_n = \mathscr{S}\mu_n = \mathscr{T}\mu_{n+1}$ for each $n \ge 0$, and $\lim_{n\to\infty} d(\mathscr{T}\mu_{n+1}, \mathscr{T}\mu_{n+2}) = \lim_{n\to\infty} d(\omega_n, \omega_{n+1}) = 0$. We show that $\{\omega_n\}$ is a Cauchy sequence. By triangle inequality and (22), for each n and $\kappa > 0$, we obtain

$$\begin{aligned} d(\mathscr{S}\mu_{n+\kappa}, \mathscr{S}\mu_n) = d(\omega_{n+\kappa}, \omega_n) &\le d(\omega_{n+\kappa}, \omega_{n+\kappa+1}) + d(\omega_{n+\kappa+1}, \omega_{n+1}) + d(\omega_{n+1}, \omega_n) \\ &\le d(\omega_{n+\kappa}, \omega_{n+\kappa+1}) + \alpha d(\omega_{n+\kappa}, \omega_n) + \beta d(\omega_n, \omega_{n+\kappa+1}) + \gamma d(\omega_{n+\kappa}, \omega_{n+1}) + d(\omega_{n+1}, \omega_n) \\ &\le d(\omega_{n+\kappa}, \omega_{n+\kappa+1}) + \alpha d(\omega_{n+\kappa}, \omega_n) + \beta\{d(\omega_n, \omega_{n+\kappa}) + d(\omega_{n+\kappa}, \omega_{n+\kappa+1})\} \\ &\quad + \gamma\{d(\omega_{n+\kappa}, \omega_n) + d(\omega_n, \omega_{n+1})\} + d(\omega_{n+1}, \omega_n) \\ &\le (1+\beta)d(\omega_{n+\kappa}, \omega_{n+\kappa+1}) + (\alpha+\beta+\gamma)d(\omega_{n+\kappa}, \omega_n) + (1+\gamma)d(\omega_n, \omega_{n+1}). \end{aligned}$$

Thus,

$$d(\omega_{n+\kappa}, \omega_n) \le \frac{1+\beta}{\alpha+\beta} d(\omega_{n+\kappa}, \omega_{n+\kappa+1}) + \frac{1+\gamma}{\alpha+\beta} d(\omega_{n+1}, \omega_n).$$

Since $\mathscr{S}$ is asymptotically regularity with respect to $\mathscr{T}$, it implies that $d(\omega_{n+\kappa}, \omega_n) \to 0$ as $n \to \infty$. Therefore, $\{\omega_n\}$ is a Cauchy sequence. Since Λ is complete, there exists a point $\nu \in \Lambda$ such that $\omega_n \to \nu$ as $n \to \infty$, and $\omega_n = \mathscr{S}\mu_n = \mathscr{T}\mu_{n+1} \to \nu$.

Assuming that $\mathscr{S}$ and $\mathscr{T}$ are compatible mappings, orbital continuity of $\mathscr{S}$ and $\mathscr{T}$ implies that

$\lim_{n\to\infty} \mathscr{S}\mathscr{S}\mu_n = \lim_{n\to\infty} \mathscr{S}\mathscr{T}\mu_n = \mathscr{S}\nu,$

further,

$\lim_{n\to\infty} \mathscr{S}\mathscr{T}\mu_n = \lim_{n\to\infty} \mathscr{S}\mathscr{S}\mu_n = \mathscr{S}\nu,$

Then, compatibility of $\mathscr{S}$ and $\mathscr{T}$ yields $\lim_{n\to\infty} d(\mathscr{S}\mathscr{T}\mu_n, \mathscr{T}\mathscr{S}\mu_n) = 0$. Taking limit as $n \to \infty$ we have $\mathscr{S}\nu = \mathscr{T}\nu$, which implies, $C(\mathscr{S}, \mathscr{T}) \neq \phi$. Again the compatibility of $\mathscr{S}$ and $\mathscr{T}$ implies commutativity at a coincidence point ν. Hence $\mathscr{T}\mathscr{S}\nu = \mathscr{S}\mathscr{S}\nu = \mathscr{T}\mathscr{T}\nu$. Using (22), we have

$$d(\mathscr{S}\nu, \mathscr{S}\mathscr{S}\nu) \leq \alpha d(\mathscr{T}\nu, \mathscr{T}\mathscr{S}\nu) + \beta d(\mathscr{T}\mathscr{S}\nu, \mathscr{S}\nu) + \gamma d(\mathscr{T}\nu, \mathscr{S}\mathscr{S}\nu) < d(\mathscr{S}\nu, \mathscr{S}\mathscr{S}\nu),$$

that is, $\mathscr{S}\nu = \mathscr{S}\mathscr{S}\nu$. Then $\mathscr{S}\nu = \mathscr{S}\mathscr{S}\nu = \mathscr{T}\mathscr{S}\nu$ and $\mathscr{S}\nu$ are common fixed points of $\mathscr{S}$ and $\mathscr{T}$. The uniqueness of the common fixed point follows from (22).

Example 2.12. Let $\Lambda = [2, 20]$ and d be the usual metric. Define $\mathscr{S}, \mathscr{T} : \Lambda \to \Lambda$ by

$$\mathscr{S}\mu = \begin{cases} 2 & \text{if } \mu = 2, \\ 3 & \text{if } 2 < \mu \leq 5, \\ 2 & \text{if } \mu > 5, \end{cases} \quad \textit{and} \quad \mathscr{T}\mu = \begin{cases} 2 & \text{if } \mu = 2, \\ 11 & \text{if} 2 < \mu \leq 5, \\ \dfrac{\mu+1}{3} & \text{if } \mu > 5. \end{cases}$$

Then $\mathscr{S}$ and $\mathscr{T}$ satisfy all the conditions of Theorem 2.11 for $\alpha = \frac{1}{15}$, $\beta = \frac{1}{5}$ and $\gamma = \frac{1}{3}$. In fact, if $\mu = 2$, $\omega > 5$, we have

$$\begin{aligned} \alpha d(\mathscr{T}\mu, \mathscr{T}\omega) + \beta d(\mathscr{T}\omega, \mathscr{S}\mu) + \gamma d(\mathscr{T}\mu, \mathscr{S}\omega) &= \frac{1}{15}\left|2 - \frac{\omega+1}{3}\right| + \frac{1}{5}\left|\frac{\omega+1}{3} - 2\right| + \frac{1}{3}|2-2| \\ &= \frac{1}{75}|\omega - 5| + \frac{1}{15}|\omega - 5| \geq 0 = d(\mathscr{S}\mu, \mathscr{S}\omega). \end{aligned}$$

Similarly, if take $2 < \mu \leq 5$ and $\omega > 5$ we obtain

$$\begin{aligned} \alpha d(\mathscr{T}\mu, \mathscr{T}\omega) + \beta d(\mathscr{T}\omega, \mathscr{S}\mu) + \gamma d(\mathscr{T}\mu, \mathscr{S}\omega) &= \frac{1}{15}\left|11 - \frac{\omega+1}{3}\right| + \frac{1}{5}\left|\frac{\omega+1}{3} - 3\right| + \frac{1}{3}|11-2| \\ &= \frac{1}{75}|32 - \omega| + \frac{1}{15}|\omega - 8| + \frac{8}{3} \\ &> \frac{8}{3} > 1 = d(\mathscr{S}\mu, \mathscr{S}\omega). \end{aligned}$$

If $\mu > 5$ and $\omega > 5$, we get

$$\begin{aligned} \alpha d(\mathscr{T}\mu, \mathscr{T}\omega) + \beta d(\mathscr{T}\omega, \mathscr{S}\mu) + \gamma d(\mathscr{T}\mu, \mathscr{S}\omega) &= \frac{1}{15}\left|\frac{\mu+1}{3} - \frac{\omega+1}{3}\right| + \frac{1}{5}\left|\frac{\omega+1}{3} - 2\right| \\ &\quad + \frac{1}{3}\left|\frac{\mu+1}{3} - 2\right| \\ &= \frac{1}{75}|\mu - \omega| + \frac{1}{15}|\omega - 5| + \frac{1}{9}|\mu - 5| \\ &> 0 = d(\mathscr{S}\mu, \mathscr{S}\omega). \end{aligned}$$

Otherwise, if $\mu = 1$ and $\omega = 2$ or $2 < \omega \leq 5$ (or $2 < \mu, \omega \leq 5$), it is easy to verify that the mappings $\mathscr{S}$ and $\mathscr{T}$ satisfy the conditions of Theorem 2.11. The mappings $\mathscr{S}$ and $\mathscr{T}$ have a unique common fixed point $\mu = 2$.

3 Fixed Point and Common Fixed Point Results in Banach Spaces

In this section, we present some fixed point and common fixed point theorems for Reich and Chatterjea nonexpansive mappings in a Banach space.

Consider a fixed point iteration, which is given by

$$\mu_{n+1} = \mathscr{S}\mu_n = \mathscr{S}^n\mu_0, \quad n \in \mathbb{N}. \tag{23}$$

with an arbitrary $\mu_0 \in \Lambda$. The iterative method (23) is also known as Picard iteration. For the Banach contraction mapping theorem [1], the Picard iteration converges to the unique fixed point of $\mathscr{S}$.

Define $\mathscr{S}^0 = I$ (the identity map on Λ) and $\mathscr{S}^n = \mathscr{S}^{n-1} \circ \mathscr{S}$, called the n^{th} iterate of $\mathscr{S}$ for $n \in \mathbb{N}$. The Krasnoselskii-Ishikawa iteration method associated with $\mathscr{S}$ is the sequence $\{\mu_n\}_{n=0}^{\infty}$ defined by

$$\mu_{n+1} = (1 - \lambda)\mu_n + \lambda\mathscr{S}\mu_n, \tag{24}$$

for each $n \geq 0$, and $\lambda \in [0, 1]$. The Krasnoselskii-Ishikawa sequence $\{\mu_n\}_{n=0}^{\infty}$ is exactly the Picard iteration corresponding to an averaged operator:

$$\mathscr{S}_\lambda = (1 - \lambda)I + \lambda\mathscr{S}, \tag{25}$$

However, if $\lambda = 1$, the Krasnoselskii-Ishikawa iteration given by (24) is reduced to the Picard iteration. Moreover, $Fix(\mathscr{S}) = Fix(\mathscr{S}_\lambda)$, for each $\lambda \in (0, 1]$.

In the following, we prove basic lemmas for the Reich nonexpansive mapping which in turn are useful to proving the results of this section.

Lemma 3.1. Let $(\Lambda, \|.\|)$ be a normed space. Assume that there exist non-negative numbers α, β and γ with $2\alpha+2\beta+\gamma = 1$, where $2\alpha < 1$ and the Reich nonexpansive mapping $\mathscr{S} : \Lambda \to \Lambda$ satisfying the following inequality

$$d(\mathscr{S}\mu, \mathscr{S}\omega) \leq \alpha d(\mu, \omega) + \beta d(\mu, \mathscr{S}\mu) + \gamma d(\omega, \mathscr{S}\omega), \tag{26}$$

for all $\mu, \omega \in \Lambda$. Then for any positive integer n, there exists $\lambda \in (0, 1)$ such that for all $\mu \in \Lambda$ and for each p, q in $\mathbb{N}$, we have

$$\|\mathscr{S}_\lambda^p\mu - \mathscr{S}_\lambda^q\mu\| \leq \eta\delta[O(\mathscr{S}_\lambda, \mu, n)], \tag{27}$$

where $\eta = max\{2\alpha, \beta, \gamma\}$ and $\delta[A] = \sup\{\|\mu - \omega\| : \mu, \omega \in A\}$.

Proof. Let us choose $\lambda = \dfrac{1 - 2\alpha}{1 - \alpha}$, where $2\alpha < 1$, clearly, $0 < \lambda < 1$. Considering the operator given by (25), we have

$$\lambda\|\mu - \mathscr{S}\mu\| = \|\mu - \mathscr{S}_\lambda\mu\|, \tag{28}$$

and

$$\lambda\|\omega - \mathscr{S}\omega\| = \|\omega - \mathscr{S}_\lambda\omega\|, \tag{29}$$

for all $\mu, \omega \in \Lambda$. Moreover, we obtain

$$\begin{aligned}\|\mathscr{S}_\lambda\mu - \mathscr{S}_\lambda\omega\| &= \|(1-\lambda)(\mu-\omega) + \lambda(\mathscr{S}\mu - \mathscr{S}\omega)\\ &\leq (1-\lambda)\|\mu-\omega\| + \lambda\|\mathscr{S}\mu - \mathscr{S}\omega\|, \end{aligned} \tag{30}$$

Since $\mathscr{S}$ is a Reich nonexpansive mapping, from (26), we have

$$\lambda\|\mathscr{S}\mu - \mathscr{S}\omega\| \leq \alpha\lambda\|\mu-\omega\| + \beta\lambda\|\mu - \mathscr{S}\mu\| + \lambda\gamma\|\omega - \mathscr{S}\omega\|. \tag{31}$$

Using (28), (29) and (31), we obtain

$$\lambda\|\mathscr{S}\mu - \mathscr{S}\omega\| \leq \alpha\lambda\|\mu-\omega\| + \beta\|\mu - \mathscr{S}_\lambda\mu\| + \gamma\|\omega - \mathscr{S}_\lambda\omega\|. \tag{32}$$

Now, (30) and (32) imply that

$$\begin{aligned}\|\mathscr{S}_\lambda\mu - \mathscr{S}_\lambda\omega\| &\leq (1-\lambda)\|\mu-\omega\| + \alpha\lambda\|\mu-\omega\| + \beta\|\mu - \mathscr{S}_\lambda\mu\| + \gamma\|\omega - \mathscr{S}_\lambda\omega\|\\ &\leq (1-\lambda+\alpha\lambda)\|\mu-\omega\| + \beta\|\mu - \mathscr{S}_\lambda\mu\| + \gamma\|\omega - \mathscr{S}_\lambda\omega\|. \end{aligned} \tag{33}$$

Since $\lambda = \dfrac{1-2\alpha}{1-\alpha}$ and $2\alpha < 1$, we assume that $1-\lambda+\alpha\lambda = 2\alpha < 1$. Then (33) becomes

$$\|\mathscr{S}_\lambda\mu - \mathscr{S}_\lambda\omega\| \leq 2\alpha\|\mu-\omega\| + \beta\|\mu - \mathscr{S}_\lambda\mu\| + \gamma\|\omega - \mathscr{S}_\lambda\omega\|. \tag{34}$$

Let $\mu \in \Lambda$ be arbitrary and fixed positive integer n. Therefore, using (34), we have

$$\begin{aligned}\|\mathscr{S}_\lambda^p\mu - \mathscr{S}_\lambda^q\mu\| &= \|\mathscr{S}_\lambda\mathscr{S}_\lambda^{p-1}\mu - \mathscr{S}_\lambda\mathscr{S}_\lambda^{q-1}\mu\|\\ &\leq 2\alpha\|\mathscr{S}_\lambda^{p-1}\mu - \mathscr{S}_\lambda^{q-1}\mu\| + \beta\|\mathscr{S}_\lambda^{p-1}\mu - \mathscr{S}_\lambda^{p}\mu\| + \gamma\|\mathscr{S}_\lambda^{q-1}\mu - \mathscr{S}_\lambda^{q}\mu\|,\end{aligned}$$

which implies that

$$\|\mathscr{S}_\lambda^p\mu - \mathscr{S}_\lambda^q\mu\| \leq \eta\delta[O(\mathscr{S}_\lambda, \mu, n)],$$

such that $\eta = max\{2\alpha, \beta, \gamma\}$.

Remark 3.1. It follows from Lemma 3.1 that if $\mathscr{S}$ is a Reich nonexpansive mapping given by (26) and $\mu \in \Lambda$, then for any $n \in \mathbb{N}$, there exists a positive integer $\kappa \leq n$, such that

$$\|\mu - \mathscr{S}_\lambda^\kappa\mu\| = \delta[O(\mathscr{S}_\lambda, \mu, n)].$$

Lemma 3.2. Let $(\Lambda, \|.\|)$ be a normed space. Suppose that $\mathscr{S}$ is a Reich nonexpansive mapping give by (26), such that $2\alpha + 2\beta + \gamma = 1$, and $2\alpha < 1$. Then, there exists $\lambda \in (0, 1)$ such that

$$\delta\left[O\left(\mathscr{S}_\lambda, \mu, \infty\right)\right] \leq \frac{1}{1-\eta}\|\mu - \mathscr{S}_\lambda^\kappa\mu\|, \tag{35}$$

holds for each $\mu \in \mathscr{S}$, and $\eta = max\{2\alpha, \beta, \gamma\}$.

Proof. Let $\mu \in \Lambda$ be arbitrary and $\lambda \in (0, 1)$. Since

$$\delta[O(\mathscr{S}_\lambda, \mu, 1)] \leq \delta[O(\mathscr{S}_\lambda, \mu, 2)] \leq \ldots,$$

we see that

$$\delta[O(\mathscr{S}_\lambda, \mu, \infty)] = \sup\{\delta[O(\mathscr{S}_\lambda, \mu, n)] : n \in \mathbb{N}\}.$$

Then, (35) implies that

$$\delta\left[O\left(\mathscr{S}_{\lambda}, \mu, n\right)\right] \leq \frac{1}{1-\eta}\|\mu - \mathscr{S}_{\lambda}^{\kappa}\mu\|.$$

Let n be any positive integer. From Remark 3.1, there exists $\mathscr{S}_{\lambda}^{\kappa} \in O(\mathscr{S}_{\lambda}, \mu, n)$ where $1 \leq \kappa \leq n$, such that

$$\|\mu - \mathscr{S}_{\lambda}^{\kappa}\mu\| = \delta[O(\mathscr{S}_{\lambda}, \mu, n)].$$

Applying a triangle inequality and Lemma 3.1, we obtain

$$\begin{aligned}\|\mu - \mathscr{S}_{\lambda}^{\kappa}\mu\| &\leq \|\mu - \mathscr{S}_{\lambda}\mu\| + \|\mathscr{S}_{\lambda}\mu - \mathscr{S}_{\lambda}^{\kappa}\mu\| \\ &\leq \|\mu - \mathscr{S}_{\lambda}\mu\| + \eta\delta[O(\mathscr{S}_{\lambda}, \mu, n)] \\ &= \|\mu - \mathscr{S}_{\lambda}\mu\| + \eta\|\mu - \mathscr{S}_{\lambda}^{\kappa}\mu\|.\end{aligned}$$

Therefore,

$$\delta\left[O\left(\mathscr{S}_{\lambda}, \mu, n\right)\right] = \|\mu - \mathscr{S}_{\lambda}^{\kappa}\mu\| \leq \frac{1}{1-\eta}\|\mu - \mathscr{S}_{\lambda}\mu\|.$$

Since n was arbitrary, the proof is completed.

Now, we state and prove our main results of this section:

Theorem 3.2. Let $(\Lambda, \|.\|)$ be a Banach space and a self-mapping $\mathscr{S}$ be a Reich nonexpansive given by (26), with $2\alpha + 2\beta + \gamma = 1$ and $2\alpha < 1$. Then, the Krasnoselskii-Ishikawa iteration $\{\mu_n\}$ defined by

$$\mu_{n+1} = (1-\lambda)\mu_{n-1} + \lambda\mathscr{S}\mu_{n-1}, \ n \geq 1,$$

converges to a unique fixed point ν for any $\mu_0 \in \Lambda$, provided that Λ is $\mathscr{S}_{\lambda}$-orbitally complete.

Proof. Following a similar lines of the proof of Lemma 3.1, we have

$$\|\mathscr{S}_{\lambda}\mu - \mathscr{S}_{\lambda}\omega\| \leq 2\alpha\|\mu - \omega\| + \beta\|\mu - \mathscr{S}_{\lambda}\mu\| + \gamma\|\omega - \mathscr{S}_{\lambda}\omega\|, \tag{36}$$

where $\lambda = \dfrac{1-2\alpha}{1-\alpha}$ and $2\alpha < 1$.

Let μ_0 be an arbitrary point of Λ. Given the iteration (23), the Krasnoselskii-Ishikawa sequence $\{\mu_n\}$ is exactly the Picard iteration associated with $\mathscr{S}_{\lambda}$, that is

$$\mu_n = \mathscr{S}_{\lambda}\mu_{n-1} = \mathscr{S}_{\lambda}^{n}\mu_0, \ n \geq 0. \tag{37}$$

We shall show that the sequence of iterates $\{\mu_n\}$ given by (37) is a Cauchy sequence. Let n and m $(n < m)$ be any positive integers. From Lemma 3.1, we obtain

$$\begin{aligned}\|\mu_n - \mu_m\| &= \|\mathscr{S}_{\lambda}^{n}\mu_0 - \mathscr{S}_{\lambda}^{m}\mu_0\| \\ &= \|\mathscr{S}_{\lambda}\mathscr{S}_{\lambda}^{n-1}\mu_0 - \mathscr{S}_{\lambda}^{m-n+1}\mathscr{S}_{\lambda}^{n-1}\mu_0\| \\ &= \|\mathscr{S}_{\lambda}\mu_{n-1} - \mathscr{S}_{\lambda}^{m-n+1}\mu_{n-1}\| \\ &\leq \eta\delta[O(\mathscr{S}_{\lambda}, \mu_{n-1}, m-n+1)],\end{aligned} \tag{38}$$

where $\eta = max\{2\alpha, \beta, \gamma\}$. According to Remark 3.1, there exists an integer κ, $1 \leq \kappa \leq m-n+1$, such that

$\delta[O(\mathscr{S}_\lambda, \mu_{n-1}, m-n+1)] = \|\mu_{n-1} - \mu_{n+\kappa-1}\|$.

Again, by Lemma 3.1, we have

$$\begin{aligned}\|\mu_{n-1} - \mu_{n+\kappa-1}\| &= \|\mathscr{S}_\lambda \mu_{n-2} - \mathscr{S}_\lambda^{\kappa+1}\mu_{n-2}\| \\ &\leq \eta\delta[O(\mathscr{S}_\lambda, \mu_{n-2}, \kappa+1)],\end{aligned}$$

which implies that

$$\|\mu_{n-1} - \mu_{n+\kappa-1}\| \leq \eta\delta[O(\mathscr{S}_\lambda, \mu_{n-2}, m-n+2)].$$

Moreover, by (38) we get

$$\|\mu_n - \mu_m\| \leq \eta\delta[O(\mathscr{S}_\lambda, \mu_{n-1}, m-n+1)] \leq \eta^2\delta[O(\mathscr{S}_\lambda, \mu_{n-2}, m-n+2)].$$

Continuing this process, we obtain

$$\|\mu_n - \mu_m\| \leq \eta\delta[O(\mathscr{S}_\lambda, \mu_{n-1}, m-n+1)] \leq \ldots \leq \eta^n\delta[O(\mathscr{S}_\lambda, \mu_0, m)],$$

and it follows from Lemma 3.2 that

$$\|\mu_n - \mu_m\| \leq \frac{\eta^n}{1-\eta}\|\mu_0 - \mathscr{S}_\lambda\mu_0\|.$$

Taking limit as $n \to \infty$, we find that $\{\mu_n\}$ is a Cauchy sequence. Since Λ is $\mathscr{S}_\lambda$-orbital complete, there exists $\nu \in \Lambda$ such that $\lim_{n\to\infty}\mu_n = \nu$. Next, we prove that ν is a fixed point of $\mathscr{S}_\lambda$. In (36), we consider the following inequalities:

$$\begin{aligned}\|\nu - \mathscr{S}_\lambda\nu\| &\leq \|\nu - \mu_{n+1}\| + \|\mu_{n+1} - \mathscr{S}_\lambda\nu\| \\ &= \|\nu - \mu_{n+1}\| + \|\mathscr{S}_\lambda\mu_n - \mathscr{S}_\lambda\nu\| \\ &\leq \|\nu - \mu_{n+1}\| + 2\alpha\|\mu_n - \nu\| + \beta\|\mu_n - \mu_{n+1}\| + \gamma\|\nu - \mathscr{S}_\lambda\nu\|.\end{aligned}$$

Hence,

$$\|\nu - \mathscr{S}_\lambda\nu\| \leq \frac{1}{1-\gamma}\left(\|\nu - \mu_{n+1}\| + 2\alpha\|\mu_n - \nu\| + \beta\|\mu_n - \mu_{n+1}\|\right).$$

Since $\lim_{n\to\infty}\mu_n = \nu$, we have $\|\nu - \mathscr{S}_\lambda\nu\| = 0$, that is $\mathscr{S}_\lambda$ has a fixed point. We claim that there is a unique common fixed point of $\mathscr{S}_\lambda$. Assume on the contrary that, $\mathscr{S}_\lambda\nu = \nu$ and $\mathscr{S}_\lambda\nu^* = \nu^*$ but $\nu \neq \nu^*$. By supposition, we obtain

$$\begin{aligned}\|\nu - \nu^*\| = \|\mathscr{S}_\lambda\nu - \mathscr{S}_\lambda\nu^*\| &\leq 2\alpha\|\nu - \nu^*\| + \beta\|\nu^* - \mathscr{S}_\lambda\nu\| + \gamma\|\nu^* - \mathscr{S}_\lambda\nu^*\| \\ &\leq 2\alpha\|\nu - \nu^*\| < \|\nu - \nu^*\|,\end{aligned}$$

which is a contradiction, hence, $\nu = \nu^*$. Since $Fix(\mathscr{S}_\lambda) = Fix(\mathscr{S})$, we get that $\mathscr{S}$ has a unique fixed point.

In the next theorem, we present a common fixed point result for Reich mappings.

Theorem 3.3. Let $(\Lambda, \|.\|)$ be a Banach space and $\mathscr{S}$ and $\mathscr{T}$ be self-mappings on Λ. Assume that there exist non-negative numbers α, β, γ such that $2\alpha + 2\beta + \gamma = 1$ and $2\alpha < 1$, satisfying

$$\|\mathscr{S}\mu - \mathscr{T}\omega\| \leq \alpha\|\mu - \omega\| + \beta\|\mu - \mathscr{S}\mu\| + \gamma\|\omega - \mathscr{T}\omega\|, \tag{39}$$

for all $\mu, \omega \in \Lambda$. Then, the iteration sequence $\{\mu_n\}_{n=0}^{\infty}$ defined by (24) converges to a unique common fixed point ν for any $\mu_0 \in \Lambda$, provided that Λ is $(\mathscr{S}_\lambda, \mathscr{T}_\lambda)$-orbitally complete.

Proof. Suppose that μ_0 is an arbitrary point in Λ. Consider the iterative process $\{\mu_n\}_{n=0}^{\infty}$ defined by (24), which is, in fact, the Picard iteration associated with $\mathscr{S}_\lambda$, that is

$$\mu_{n+1} = \mathscr{S}_\lambda \mu_n. \tag{40}$$

Now, using the operator defined by (25), we obtain

$$\begin{aligned} &\mathscr{S}_\lambda \mu_{2n+1} = (1-\lambda)\mu_{2n+1} + \lambda\mathscr{S}\mu_{2n+1} = \mu_{2n+2}, \\ &\mathscr{T}_\lambda \mu_{2x} = (1-\lambda)\mu_{2n} + \lambda\mathscr{T}\mu_{2n} = \mu_{2n+1}. \end{aligned} \tag{41}$$

From (39) and (41), we get the following:

$$\begin{aligned} \|\mathscr{S}_\lambda\mu_{2n+1} - \mathscr{T}_\lambda\mu_{2n}\| &= \|\mu_{2n+2} - \mu_{2n+1}\| \\ &\leq (1-\lambda)\|\mu_{2n+1} - \mu_{2n}\| + \alpha\lambda\|\mu_{2n+1} - \mu_{2n}\| + \beta\|\mu_{2n+1} - \mathscr{S}_\lambda\mu_{2n+1}\| \\ &\quad + \gamma\|\mu_{2n} - \mathscr{T}_\lambda\mu_{2n}\| \\ &= (1-\lambda+\alpha\lambda)\|\mu_{2n+1} - \mu_{2n}\| + \beta\|\mu_{2n+1} - \mu_{2n+2}\| + \gamma\|\mu_{2n} - \mu_{2n+1}\|, \end{aligned} \tag{42}$$

such that $1 - \lambda + \alpha\lambda = 2\alpha < 1$, $\lambda \in (0, 1)$. This implies

$$\begin{aligned} &(1-\beta)\|\mu_{2n+2} - \mu_{2n+1}\| \leq (2\alpha+\gamma)|\mu_{2n+1} - \mu_{2n}\| \\ &\quad\Rightarrow \|\mu_{2n+2} - \mu_{2n+1}\| \leq \frac{2\alpha+\gamma}{1-\beta}\|\mu_{2n+1} - \mu_{2n}\| \\ &\qquad\qquad\qquad\qquad\quad \leq \frac{1-2\beta}{1-\beta}\delta\left[O\left(\mathscr{S}_\lambda, \mathscr{T}_\lambda, \mu_{2n}, 2n+1\right)\right] \\ &\qquad\qquad\qquad\qquad\quad < \eta\delta[O(\mathscr{S}_\lambda\mathscr{T}_\lambda, \mu_{2n}, 2n+1)], \end{aligned}$$

where $\eta = \dfrac{1-2\beta}{1-\beta} < 1$ and $\beta < 1$.

Similarly,

$$\|\mu_{2n+3} - \mu_{2n+2}\| \leq \eta\|\mu_{2n+2} - \mu_{2n+1}\| = \eta\delta[O(\mathscr{S}_\lambda, \mathscr{T}_\lambda, \mu_{2n+1}, 2n+2)] \leq \eta^2\delta[O(\mathscr{S}_\lambda, \mathscr{T}_\lambda, \mu_{2n}, 2n+1)].$$

Continuing the process, we get the following:

$$\begin{aligned} \|\mu_{m+1} - \mu_m\| &< \eta\delta[O(\mathscr{S}_\lambda, \mathscr{T}_\lambda, \mu_{m-1}, m)] < \eta^2\delta[O(\mathscr{S}_\lambda, \mathscr{T}_\lambda, \mu_{m-2}, m-1)] < \ldots \\ &< \eta^m\delta[O(\mathscr{S}_\lambda, \mathscr{T}_\lambda, \mu_0, m+1)], \quad m \in \mathbb{N}. \end{aligned} \tag{43}$$

Now, we show that $\{\mu_n\}$ is a Cauchy sequence converging to $\nu \in \Lambda$. Then there exists $m \in \mathbb{N}$ such that $n < m$, from (43), and Lemma 3.2, we obtain

$$\|\mu_m - \mu_n\| < \eta\delta[O(\mathscr{S}_\lambda, \mathscr{T}_\lambda, \mu_n, n+1)] < \dots . < \eta^n\delta[O(\mathscr{S}_\lambda, \mathscr{T}_\lambda, \mu_0, m)]$$
$$\leq \frac{\eta^n}{1-\eta}\|\mu_0 - \mathscr{T}_\lambda\mu_0\|, \ n < m.$$

Since $\lim_{n\to\infty} \dfrac{\eta^n}{1-\eta} = 0$, so, $\{\mu_n\}$ is a Cauchy sequence on Λ. Since Λ is $(\mathscr{S}_\lambda, \mathscr{T}_\lambda)$-orbital complete, there exists $\nu \in \Lambda$ such that $\lim_{n\to\infty} \mu_n = \nu$. Now, we prove that ν is a fixed point of $\mathscr{S}_\lambda$. From (41) and (42), we consider the following inequalities:

$$\begin{aligned}\|\mathscr{S}_\lambda\nu - \nu\| &\leq \|\mathscr{S}_\lambda\nu - \mu_{2n+1}\| + \|\mu_{2n+1} - \nu\| \\ &= \|\mathscr{S}_\lambda\nu - \mathscr{T}_\lambda\mu_{2n}\| + \|\mu_{2n+1} - \nu\| \\ &\leq 2\alpha\|\mu_{2n} - \nu\| + \beta\|\nu - \mathscr{S}_\lambda\nu\| + \gamma\|\mu_{2n} - \mathscr{T}_\lambda\mu_{2n}\| \\ &\leq \frac{2\alpha}{1-\beta}\|\mu_{2n} - \nu\| + \frac{\gamma}{1-\beta}\|\mu_{2n} - \mu_{2n+1}\|,\end{aligned}$$

where we have $\|\nu - \mathscr{S}_\lambda\nu\| = 0$, as $n \to \infty$, that is $\mathscr{S}_\lambda$ has a fixed point. Similarly, the mapping $\mathscr{T}_\lambda$ has a fixed point. We claim that there is a unique common fixed point of $\mathscr{S}_\lambda$ and $\mathscr{T}_\lambda$. We assume on the contrary, such that $\mathscr{T}_\lambda\nu = \mathscr{S}_\lambda\nu = \nu$ and $\mathscr{T}_\lambda\nu^* = \mathscr{S}_\lambda\nu^* = \nu^*$ but $\nu \neq \nu^*$. By supposition, we can replace μ_{2n+2} by ν and μ_{2n+1} by ν^* in (42) to obtain

$$\begin{aligned}\|\nu - \nu^*\| = \|\mathscr{S}_\lambda\nu - \mathscr{T}_\lambda\nu^*\| &\leq 2\alpha\|\nu - \nu^*\| + \beta\|\nu - \mathscr{S}_\lambda\nu\| + \gamma\|\nu^* - \mathscr{T}_\lambda\nu^*\| \\ &\leq 2\alpha\|\nu - \nu^*\| < \|\nu - \nu^*\|,\end{aligned}$$

this is a contradiction, hence $\nu = \nu^*$. Since $Fix(\mathscr{S}_\lambda) = Fix(\mathscr{S})$, we get that $\mathscr{S}$ and $\mathscr{T}$ have a unique common fixed point.

Theorem 3.4. Let $(\Lambda, \|.\|)$ be a Banach space and $\mathscr{S}, \mathscr{T} : \Lambda \to \Lambda$. Assume that there exist non-negative numbers α, β, γ such that $2\alpha + 3\beta + 2\gamma = 1$, satisfying

$$\|\mathscr{S}\mu - \mathscr{T}\omega\| \leq \alpha\|\mu - \omega\| + \beta\|\omega - \mathscr{S}\mu\| + \gamma\|\mu - \mathscr{T}\omega\|, \tag{44}$$

for all $\mu, \omega \in \Lambda$. Then, the iteration sequence $\{\mu_n\}_{n=1}^{\infty}$ defined by (24) converges to a unique common fixed point ν for any $\mu_0 \in \Lambda$, provided that Λ is $(\mathscr{S}_\lambda, \mathscr{T}_\lambda)$-orbitally complete.

Proof. Let μ_0 be arbitrary. Since $(\mathscr{S}_\lambda, \mathscr{T}_\lambda)$-orbitally complete in Λ, we define the operators $\mathscr{S}_\lambda$ and $\mathscr{T}_\lambda$ such that

$$\begin{aligned}\mathscr{S}_\lambda\mu_{2n} &= \mu_{2n+1}, \\ \mathscr{T}_\lambda\mu_{2n+1} &= \mu_{2n+2}.\end{aligned} \tag{45}$$

Following similar lines of the proof of Theorem 3.2, we obtain

$$\begin{aligned}\|\mathscr{S}_\lambda\mu_{2n+1} - \mathscr{T}_\lambda\mu_{2n}\| = \|\mu_{2n+2} - \mu_{2n+1}\| &\leq (1-\lambda)\|\mu_{2n+1} - \nu_{2n}\| + \lambda(\alpha+\beta+\gamma)\|\mu_{2n+1} - \mu_{2n}\| + \beta\|\mu_{2n+1} - \mathscr{S}_\lambda\mu_{2n+1}\| \\ &\quad + \gamma\|\mu_{2n} - \mathscr{T}_\lambda\mu_{2n}\| \\ &\leq (2\alpha+\beta+\gamma)\|\mu_{2n+1} - \mu_{2n}\| + \beta\|\mu_{2n+1} - \mu_{2n+2}\|) + \gamma\|\mu_{2n} - \mu_{2n+1}\|.\end{aligned} \tag{46}$$

Let us choose $1-\lambda+\lambda(\alpha+\beta+\gamma) \leq 2\alpha+\beta+\gamma$. Since $2\alpha+3\beta+2\gamma = 1$, we have $2(\alpha+\gamma) = 1-3\beta$. This implies that

$$(1-\beta)\|\mu_{2n+2}-\mu_{2n+1}\| \leq (2\alpha+2\gamma+\beta)|\mu_{2n+1}-\mu_{2n}\|$$
$$\Rightarrow \|\mu_{2n+2}-\mu_{2n+1}\| \leq \frac{1-2\beta}{1-\beta}\|\mu_{2n+1}-\mu_{2n}\|$$
$$\leq \eta\delta[O(\mathscr{S}_\lambda, \mathscr{T}_\lambda, \mu_{2n}, 2n+1)],$$

where $\eta = \dfrac{1-2\beta}{1-\beta}$ and $\beta < 1$.

Similarly, we obtain

$$\|\mu_{2n+3}-\mu_{2n+2}\| \leq \eta\|\mu_{2n+2}-\mu_{2n+1}\| = \eta\delta[O(\mathscr{S}_\lambda, \mathscr{T}_\lambda, \mu_{2n+1}, 2n+2)] \leq \eta^2\delta[O(\mathscr{S}_\lambda, \mathscr{T}_\lambda, \mu_{2n}, 2n+1)].$$

Continuing the process, we get the following:

$$\|\mu_{m+1}-\mu_m\| < \eta\delta[O(T_\lambda, S_\lambda, \mu_{m-1}, m)] < \eta^2\delta[O(T_\lambda, S_\lambda, \mu_{m-2}, m-1)] < \ldots$$
$$< \eta^m\delta[O(T_\lambda, S_\lambda, \mu_0, m+1)], \quad m \in \mathbb{N}. \tag{47}$$

Next, we show that $\{\mu_n\}$ is a Cauchy sequence converging to $\nu \in \Lambda$. There exists $m \in \mathbb{N}$ such that $n < m$, by Lemma 3.2, and (47), we have

$$\|\mu_m-\mu_n\| < \eta\delta[O(\mathscr{S}_\lambda, \mathscr{T}_\lambda, \mu_n, n+1)] < \ldots . < \eta^n\delta[O(\mathscr{S}_\lambda, \mathscr{T}_\lambda, \mu_0, m)]$$
$$\leq \frac{\eta^n}{1-\eta}\delta\|\mu_0-\mathscr{T}_\lambda\mu_0\|, \quad n < m.$$

Since $\lim_{n\to\infty}\dfrac{\eta^n}{1-\eta} = 0$, so, $\{\mu_n\}$ is a Cauchy sequence in Λ. Since Λ is $(\mathscr{S}_\lambda, \mathscr{T}_\lambda)$-orbital complete, there exists $\nu \in \Lambda$ such that $\lim_{n\to\infty}\mu_n = \nu$. We prove now that ν is a fixed point of $\mathscr{S}_\lambda$. From (45) and (46), we consider the following inequalities:

$$\|\mathscr{S}_\lambda\nu-\nu\| \leq \|\mathscr{S}_\lambda\nu-\mu_{2n+1}\| + \|\mu_{2n+1}-\nu\|$$
$$= \|\mathscr{S}_\lambda\nu-\mathscr{T}_\lambda\mu_{2n}\| + \|\mu_{2n+1}-\nu\|$$
$$\leq (\alpha+\beta+\gamma)\|\mu_{2n}-\nu\| + \beta\|\mu_{2n}-\mathscr{S}_\lambda\nu\| + \gamma\|\nu-\mathscr{T}_\lambda\mu_{2n}\| + \|\mu_{2n+1}-\nu\|$$
$$\leq (\alpha+\beta+\gamma)\|\mu_{2n}-\nu\| + \beta(\|\mu_{2n}-\nu\| + \|\nu-\mathscr{S}_\lambda\nu\|) + (\gamma+1)\|\nu-\mu_{2n+1}\|$$
$$\leq \frac{\alpha+2\beta+\gamma}{1-\beta}\|\mu_{2n}-\nu\| + \frac{\gamma+1}{1-\beta}\|\nu-\mu_{2n+1}\|.$$

Since $\mu_{2n} \to \nu$ as $n \to \infty$ we have $\|\nu-\mathscr{S}_\lambda\nu\| = 0$, as $n \to \infty$, that is $\mathscr{S}_\lambda$ has a fixed point. Similarly, the mapping $\mathscr{T}_\lambda$ has a fixed point. We claim that there is a unique common fixed point of $\mathscr{S}_\lambda$ and $\mathscr{T}_\lambda$. We assume the contrary that $\mathscr{T}_\lambda\nu = \mathscr{S}_\lambda\nu = \nu$ and $\mathscr{T}_\lambda\nu^* = \mathscr{S}_\lambda\nu^* = \nu^*$ but $\nu \neq \nu^*$ with supposition, we can replace μ_{2n+2} by ν and μ_{2n+1} by ν^* in (41) to obtain

$$\|\nu-\nu^*\| = \|\mathscr{S}_\lambda\nu-\mathscr{T}_\lambda\nu^*\| \leq (2\alpha+\beta+\gamma)\|\nu-\nu^*\| + \beta\|\nu-\mathscr{S}_\lambda\nu\| + \gamma\|\nu^*-\mathscr{T}_\lambda\nu^*\|$$
$$\leq (2\alpha+\beta+\gamma)\|\nu-\nu^*\| < \|\nu-\nu^*\|,$$

which is a contradiction, hence $\nu = \nu^*$. Since $Fix(\mathscr{S}_\lambda) = Fix(\mathscr{S})$, we obtain that $\mathscr{S}$ and $\mathscr{T}$ have a unique common fixed point.

Example 3.5. Let $\Lambda = [0,1]$ be equipped with the usual metric d defined by $d(\mu,\omega) = |\mu - \omega|$. Consider the following self-mappings defined by

$$\mathscr{S}\mu = \begin{cases} \frac{1}{2} & \text{if } 0 \le \mu \le \frac{1}{2}, \\ \frac{1}{2}\mu + \frac{1}{4} & \text{if } \frac{1}{2} \le \mu \le 1. \end{cases} \qquad \mathscr{T}\mu = \begin{cases} \frac{1}{2} & \text{if } 0 \le \mu \le \frac{1}{2}, \\ \frac{1+\mu}{3} & \text{if } \frac{1}{2} \le \mu \le 1. \end{cases}$$

Take $\mu_0 = \frac{2}{3}$, then $O(\mathscr{S}_\lambda, \mathscr{T}_\lambda, \mu_n, n) \subset \left\{2/(2^p 3^{q-1}), p, q \in \mathbb{N}\right\}$.

Let $\alpha = \frac{1}{9}$, $\beta = \frac{1}{27}$, and $\gamma = \frac{1}{3}$. Now, we show that $\mathscr{S}$ and $\mathscr{T}$ satisfy the condition (39). We consider the following cases:

Case 1: Let $\mu \in \left[0, \frac{1}{2}\right]$, and $\omega \in \left[0, \frac{1}{2}\right]$, such that $\mu \le \omega$. We have

$$\begin{aligned} \alpha d(\mu,\omega) + \beta d(\mu, \mathscr{S}\mu) + \gamma d(\omega, \mathscr{T}\omega) &= \frac{1}{9}|\mu - \omega| + \frac{1}{27}|\mu - \frac{1}{2}| + \frac{1}{3}|\omega - \frac{1}{2}| \\ &\ge \frac{1}{9}(\omega - \mu) + \frac{1}{27}\left(\frac{1}{2} - \mu\right) + \frac{1}{3}\left(\frac{1}{2} - \omega\right) \\ &= \frac{5}{27} - \frac{2}{9}\omega - \frac{4}{27}\mu \\ &\ge d(\mathscr{S}\mu, \mathscr{T}\omega). \end{aligned}$$

Case 2: Let $\mu \in \left[0, \frac{1}{2}\right]$, $\omega \in \left[\frac{1}{2}, 1\right]$, such that $\mu \le \omega$. We get

$$\begin{aligned} \alpha d(\mu,\omega) + \beta d(\mu, \mathscr{S}\mu) + \gamma d(\omega, \mathscr{T}\omega) &= \frac{1}{9}|\mu - \omega| + \frac{1}{27}|\mu - \frac{1}{2}| + \frac{1}{3}|\omega - \frac{1}{3} - \frac{1}{3}\omega| \\ &\ge \frac{1}{9}(\omega - \mu) + \frac{1}{27}\left(\frac{1}{2} - \mu\right) + \frac{1}{9}(2\omega - 1) \\ &= \frac{1}{54}[18\omega - 8\mu - 5] \ge |\frac{1}{6} - \frac{1}{3}\omega| = d(\mathscr{S}\mu, \mathscr{T}\omega). \end{aligned}$$

Case 3: Let $\mu \in \left[\frac{1}{2}, 1\right]$, $\omega \in \left[\frac{1}{2}, 1\right]$ and $\mu \le \omega$. We obtain

$$\begin{aligned} \alpha d(\mu,\omega) + \beta d(\mu, \mathscr{S}\mu) + \gamma d(\omega, \mathscr{T}\omega) &= \frac{1}{9}|\mu - \omega| + \frac{1}{27}|\mu - \frac{\mu}{2} - \frac{1}{4}| + \frac{1}{3}|\omega - \frac{1}{3} - \frac{1}{3}\omega| \\ &\ge \frac{1}{9}(\omega - \mu) + \frac{1}{108}(2\mu - 1) + \frac{1}{9}(2\omega - 1) \\ &= \frac{1}{3}\omega - \frac{5}{54}\mu - \frac{13}{108} \\ &= \frac{1}{108}[36\omega - 10\mu - 13] \ge \frac{1}{12}|6\mu - 4\omega - 1| = d(\mathscr{S}\mu, \mathscr{T}\omega). \end{aligned}$$

Therefore, in all the cases, $\mathscr{S}$ and $\mathscr{T}$ satisfy the condition of (39) for all $\mu, \omega \in \Lambda$. Moreover, as all the assumptions of Theorem 3.3 hold, so $\mathscr{S}$ and $\mathscr{T}$ have $\mu = \frac{1}{2}$ as their unique common fixed point.

4 Application to Nonlinear Integral Equations

Let $\Lambda = C([0,1],\mathbb{R})$ be the set of real continuous functions defined on [0, 1]. Define the metric $d : \Lambda \times \Lambda \to [0,\infty)$ by $d(\mu,\omega) = \sup_{t\in[0,1]} |\mu(t) - \omega(t)|$, for each $\mu, \omega \in \Lambda$. Then, (Λ, d) is a complete metric space.

We consider the following integral equations formulated as a common fixed point problem of the following nonlinear mappings:

$$\begin{cases} \mathscr{S}\mu(t) = \alpha\mu(t) - \int_0^1 K(t,r,\mu(r),\mathscr{S}\mu(r))dr, \\ \mathscr{T}\omega(t) = \alpha\omega(t) - \int_0^1 K(t,r,\omega(r),\mathscr{T}\omega(r))dr, \end{cases} \tag{48}$$

such that $0 \leq \alpha < 1$, where the investigation is essentially based on the properties of the kernel $K(.,.,.,.)$. The following assumptions apply:

(i) $K_1(t,r,\mu(r),\mathscr{S}\mu(r)) \geq 0$ and $K_2(t,r,\omega(r),\mathscr{T}\omega(r)) \geq 0$ for $t, r \in [0,1]$ such that $K_i(.,.,0,.) \neq 0$, for $i = 1,2$. In addition, $\mathscr{S}(E), \mathscr{T}(E) \subseteq E$ where $E = \{\nu \in \Lambda : 0 \leq \nu(t) \leq 1\}$.

(ii) The two mappings G and G^* are defined by

$$G\mu(t) = \int_0^t K_1(t,r,\mu(r),\mathscr{S}\mu(r))dr,$$

and

$$G^*\omega(t) = \int_0^t K_2(t,r,\omega(r),\mathscr{T}\omega(r))dr,$$

satisfying $G\mu,\ G^*\omega$ in E, for all $\mu(t), \omega(t) \in E$ and

$\|G\mu(t) - G^*\omega(t)\| < \alpha(1-\alpha)\|\mu(t) - \omega(t)\|$,

for all $\mu(t), \omega(t) \in E$, such that $\mu(t) \neq \omega(t)$ and $t \in [0,1]$.

Theorem 4.1. Under the assumption *(i)* and *(ii)*, then the system of (48) has a common fixed point in E.

Proof. We have

$$\mathscr{S}\mu(t) = \alpha\mu(t) - \int_0^t K_1(t,r,\mu(r),\mathscr{S}\mu(r))dr.$$

We also have,

$$\mathscr{T}\omega(t) = \alpha\omega(t) - \int_0^t K_2(t,r,\omega(r),\mathscr{T}\omega(r))dr.$$

Suppose that $\mu, \omega \in E$. Then, by using our assumptions, we obtain

$$\begin{aligned}\left|\mathscr{S}\mu(t) - \mathscr{T}\omega(t)\right| &= \left|\alpha(\mu(t) - \omega(t)) - \int_0^1 K_1(t, r, \mu(r), \mathscr{S}\mu(r))dr + \int_0^1 K_2(t, r, \omega(r), \mathscr{T}\omega(r))dr\right| \\ &\le \alpha\left|\mu(t) - \omega(t)\right| + \left|\int_0^1 [K_1(t, r, \mu(r), \mathscr{S}\mu(r)) - K_2(t, r, \omega(r), \mathscr{T}\omega(r))]dr\right| \\ &\le \alpha\left|\mu(t) - \omega(t) - \mathscr{S}\mu(t) + \mathscr{S}\mu(t) - \mathscr{T}\omega(t) + \mathscr{T}\omega(t)\right| + \left|G\mu(t) - G^*\omega(t)\right| \\ &\le \alpha\left\|\mu(t) - \mathscr{S}\mu(t)\right\| + \left\|\omega(t) - \mathscr{T}\omega(t)\right\| \\ &\quad + \left\|\mathscr{S}\mu(t) - \mathscr{T}\omega(t)\right\| + \alpha(1-\alpha)\left\|\mu(t) - \omega(t)\right\|,\end{aligned}$$

which implies that

$$(1-\alpha)\|\mathscr{S}\mu(t) - \mathscr{T}\omega(t)\| \le \alpha\|\mu(t) - \mathscr{S}\mu(t)\| + \alpha\|\omega(t) - \mathscr{T}\omega(t)\| + \alpha(1-\alpha)\|\mu(t) - \omega(t)\|.$$

Therefore,

$$\|\mathscr{S}\mu(t) - \mathscr{T}\omega(t)\| \le \frac{\alpha}{1-\alpha}\|\mu(t) - \mathscr{S}\mu(t)\| + \|\omega(t) - \mathscr{T}\omega(t)\| + \frac{\alpha(1-\alpha)}{1-\alpha}\|\mu(t) - \omega(t)\|.$$

Now, since $0 \le \alpha < 1$ and $\beta = \dfrac{\alpha}{1-\alpha} \le \gamma$, then

$$\|\mathscr{S}\mu(t) - \mathscr{T}\omega(t)\| \le \alpha\|\mu(t) - \omega(t)\| + \beta\|\mu(t) - \mathscr{T}\mu(t)\| + \gamma\|\omega(t) - \mathscr{S}\omega(t)\|.$$

By Theorem 2.1 there exists a common fixed point of $\mathscr{S}$ and $\mathscr{T}$ which is a common solution for the system (48).

5 Application to Nonlinear Fractional Differential Equation

Fractional differential equations have applications in various fields of engineering and science including diffusive transport, electrical networks, fluid flow and electricity. Many researchers have studied this topic because it has many applications. Related to this matter, we suggest the recent literature [29–35] and the references therein.

The classical Caputo fractional derivative is defined by

$$\mathscr{D}^\xi\mu(t) := \frac{1}{\Gamma(m-\xi)}\int_0^t (t-r)^{m-\xi-1}\frac{d^m}{dr^m}\mu(r)dr$$

where $m-1 < Re(\xi) < m$, $m \in \mathbb{N}$.

Now, we consider the following fractional differential equation:

$$\begin{cases}\mathscr{D}^\xi\mu(t) + h(t, \mu(t)) = 0, \quad (0 \le \mu \le 1, 1 < \xi < 2), \\ \mu(0) = \mu(1) = 0,\end{cases} \tag{49}$$

where $\mathscr{D}^\xi$ is the Caputo fractional derivative of order ξ and $h : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function. Suppose that $\Lambda = C[0, 1]$ be a Banach space of a continuous function endowed with the maximum norm and Green's function associated with (49) is defined as

$$G(\mu,\omega)=\begin{cases}\dfrac{1}{\Gamma(\xi)}\left(\mu(1-\omega)^{\xi-1}-(\mu-\omega)^{\xi-1}\right), & 0\le\omega\le\mu\le 1,\\ \dfrac{1}{\Gamma(\xi)}\mu(1-\omega)^{\xi-1}, & 0\le\mu\le\omega\le 1.\end{cases} \tag{50}$$

Assume that

$$|h(t,\mu)-h(t,\omega)|\le\frac{q}{2(1-q)}|\mu-\omega|,$$

for all $t\in[0,1]$, $q\in\left(0,\dfrac{2}{3}\right)$ and $\mu,\omega\in\mathbb{R}$.

Theorem 5.1. Let $\Lambda=C[0,1]$ and the operators $\mathscr{S},\mathscr{T}:\Lambda\to\Lambda$ be defined as

$$\mathscr{S}(\mu(t))=\int_0^1 G(\mu,\omega)h(\mu(t),\omega(t))dt,$$

and

$$\mathscr{T}(\mu(t))=\int_0^1 G(\mu,\omega)h(\mu(t),\omega(t))dt,$$

for all $t\in[0,1]$. If condition (50) is satisfied then the system (49) has a common solution in Λ.

Proof. It is easy to see that $\mu\in\Lambda$ is a solution of (49) if and only if μ is a solution of the following integral equation:

$$\mu(t)=\int_0^1 G(t,r)h(s,\mu(r))dr.$$

Let $\mu,\omega\in\Lambda$ and $t\in[0,1]$. Then,

$$\begin{aligned}|\mathscr{S}\mu(t)-\mathscr{T}\omega(t)|&\le\int_0^1 G(t,r)(h(r,\mu(r))-h(r,\omega(r)))dr\\&\le\frac{q}{2(1-q)}\int_0^1 G(t,r)\,|\mu(r)-\omega(r)|dr\end{aligned}$$

$$\begin{aligned}\Rightarrow\|\mathscr{S}\mu(t)-\mathscr{T}\omega(t)\|&\le\frac{q}{2(1-q)}\|\mu-\omega\|\\&\le\frac{q}{2(1-q)}\|\mu-\omega\|+\frac{1-3q}{(1-q)}\|\mu-\mathscr{S}\omega\|+\frac{q}{2(1-q)}\|\omega-\mathscr{T}\mu\|,\end{aligned}$$

where $q<\dfrac{1}{3}$. Take $2\alpha=\dfrac{2q}{2(1-q)}, 2\gamma=\dfrac{2q}{2(1-q)}$ and $\beta=\dfrac{1-3q}{1-q}$ such that $2\alpha+\beta+2\gamma=1$. By Theorem 2.7 there exists a common fixed point of $\mathscr{S}$ and $\mathscr{T}$ which is a common solution for the system (49).

Example 5.2. Consider the following fractional differential equation:

$$\begin{cases}\mathscr{D}^{\xi}\mu(t)+t^2=0, & (0\le t\le 1, 1<\xi<2),\\ \mu(0)=\mu(1)=0.\end{cases} \tag{51}$$

The exact solution of the above problem (51) is given by

$$\mu(t) = \frac{1}{\Gamma(\xi)} \int_0^t (t(1-\mu^*)^{\xi-1} - (t-\mu^*)^{\xi-1})\mu^{*^2} d\mu^* + \frac{t}{\Gamma(\xi)} \int_t^1 \mu^{*^2} (1-\mu^*)^{\xi-1} d\mu^*.$$

The operator $\mathscr{S}, \mathscr{T} : C[0,1] \to C[0,1]$ are defined by

$$\mathscr{S}\mu(t) = \frac{1}{6\Gamma(\xi)} \int_0^t (t(1-\mu^*)^{\xi-1} - (t-\mu^*)^{\xi-1})\mu^{*^2} d\mu^* + \frac{t}{6\Gamma(\xi)} \int_t^1 \mu^{*^2} (1-\mu^*)^{\xi-1} d\mu^*,$$

and

$$\mathscr{T}\omega(t) = \frac{1}{3\Gamma(\xi)} \int_0^t (t(1-\omega^*)^{\xi-1} - (t-\omega^*)^{\xi-1})\omega^{*^2} d\omega^* + \frac{t}{3\Gamma(\xi)} \int_t^1 \omega^{*^2} (1-\omega^*)^{\xi-1} d\omega^*.$$

By taking $\xi = \frac{1}{2}$, $\alpha = \frac{1}{33}$, $\beta = \frac{45}{50}$ and $\gamma = \frac{13}{660}$ with $2\alpha + \beta + 2\gamma = 1$ and the initial value $\mu_0(t) = t(1-t)$, $t \in [0,1]$. We have

$$\|\mathscr{S}\mu(t) - \mathscr{T}\omega(t)\| \le \|\frac{1}{6}\mu(t) - \frac{1}{3}\omega(t)\| \le \frac{1}{3}\|\mu(t) - \omega(t)\| + \frac{1}{6}\|\mu(t)\|.$$

On the other hand, we obtain that

$$\begin{aligned}
&\alpha\|\mu(t) - \omega(t)\| + \beta\|\omega(t) - \mathscr{S}\mu(t)\| + \gamma\|\mu(t) - \mathscr{T}\omega(t)\| \\
&= \frac{1}{33}\|\mu(t) - \omega(t)\| + \frac{45}{50}\|\omega(t) - \mathscr{S}\mu(t)\| + \frac{13}{660}\|\mu(t) - \mathscr{T}\omega(t)\| \\
&\le \frac{1}{33}\|\mu(t) - \omega(t)\| + \frac{45}{50}\{\|\omega(t) - \mu(t)\| + \|\mu(t) - \mathscr{S}\mu(t)\|\} \\
&\quad + \frac{13}{660}\{\|\mu(t) - \omega(t)\| + \|\omega(t) - \mathscr{T}\omega(t)\|\} \\
&= \left(\frac{1}{33} + \frac{45}{50} + \frac{13}{660}\right)\|\mu(t) - \omega(t)\| + \frac{45}{50}\left\|\mu(t) - \frac{1}{6}\mu(t)\right\| + \frac{13}{660}\left\|\omega(t) - \frac{1}{3}\omega(t)\right\| \\
&= \frac{19}{20}\|\mu(t) - \omega(t)\| + \frac{3}{4}\|\mu(t)\| + \frac{13}{990}\|\omega(t)\|
\end{aligned}$$

which implies that

$$\|\mathscr{S}\mu(t) - \mathscr{T}\omega(t)\| \le \alpha\|\mu(t) - \omega(t)\| + \beta\|\omega(t) - \mathscr{S}\mu(t)\| + \gamma\|\mu(t) - \mathscr{T}\omega(t)\|.$$

By Theorem 2.7 there exists a common fixed point of $\mathscr{S}$ and $\mathscr{T}$ which is a common solution for the system (51).

6 Conclusion

This paper contains the study of Reich and Chatterjea nonexpansive mappings on complete metric and Banach spaces. The existence of fixed points of these mappings which are asymptotically regular or continuous mappings in complete metric space is discussed. Furthermore, we provide some fixed point and common fixed point theorems for Reich and Chatterjea nonexpansive mappings by employing the Krasnoselskii-Ishikawa iteration method associated with $\mathscr{S}_\lambda$. We have established application of our results to nonlinear integral equations and nonlinear fractional differential equations.

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