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A Novel Modified Alpha Power Transformed Weibull Distribution and Its Engineering Applications

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ABSTRACT

This paper suggests a new modified version of the traditional Weibull distribution by adding a new shape parameter utilising the modified alpha power transformed technique. We refer to the new model as modified alpha power transformed Weibull distribution. The attractiveness and significance of the new distribution lie in its power to model monotone and non-monotone failure rate functions, which are quite familiar in environmental investigations. Its hazard rate function can be decreasing, increasing, bathtub and upside-down then bathtub shaped. Diverse structural properties of the proposed model are acquired including quantile function, moments, entropies, order statistics, residual life and reversed failure rate function. The parameters of the distribution were estimated using the maximum likelihood function. The maximum likelihood method is employed to estimate the model parameters and the approximate confidence intervals are also computed. Via a simulation study, the performance of the point and interval estimates are compared using different criteria. Employing real lifetime data sets, we verify that the offered model furnishes a better fit than some other lifetime models including Weibull, gamma and alpha power Weibull models.

KEYWORDS

Weibull distribution; modified alpha power transformation method; maximum likelihood; entropy; order statistics

1 Introduction

The Weibull distribution is a wildly favoured lifetime distribution in reliability studies. It is naturally employed for studying biological, hydrological and medical data sets. The Weibull distribution is frequently employed as a suitable alternative to well-known distributions such as exponential, gamma and inverse Weibull distributions. The random variable X is expressed to have a Weibull distribution if its probability density function (PDF) is given by

$$g(x;\lambda,\theta) = \lambda\theta x^{\theta-1}e^{-\lambda x^{\theta}}, \quad x > 0, \tag{1}$$

where $\lambda > 0$ and $\theta > 0$ are the scale and shape parameters respectively. Also, its cumulative distribution function (CDF) takes the form

$$G(x;\lambda,\theta) = 1 - e^{-\lambda x^{\theta}}. \qquad x > 0. \tag{2}$$

One of the major weaknesses of the Weibull distribution is that it does not deliver an adequate fit for some applications, particularly, when the hazard rates are upside-down bathtub or bathtub shapes. To overcome this disadvantage, several investigators have developed various generalizations and modifications of the Weibull distribution to model different types of data in recent years. The generalized Weibull distribution was introduced in reference [1,2] by adding a shape parameter to the Weibull distribution. Likewise, Xie et al. [3] proposed the additive Weibull distribution, the generalized modified Weibull distribution by [4], KumaraswamyWeibull distribution by [5], beta Sarhan-Zaindin modified Weibull distribution by [6], Weibull-Weibull distribution by [7], alpha power Weibull (APW) distribution by [8], log-normal modified Weibull distribution and its reliability implications by [9], generalized extended exponential-Weibull distribution by [10], Poisson modified weibull distribution by [11], alpha logarithmic transformed Weibull distribution by [12] and logarithmic transformed Weibull by [13].

Recently, Mahdavi et al. [14] suggested a new approach to present an additional parameter to a class of distributions for more additional flexibility. The offered technique is named alpha power transformation (APT) and it is worthwhile to incorporate skewness into a family of distributions. They studied the main properties of the APT method and introduced an extension to the exponential distribution using the new approach. Many authors used the same technique to introduce new generalizations of some well-known distribution. For example, alpha power Weibull distribution by [8], alpha power generalized exponential by [15], alpha power transformed inverse Lindley distribution by [16] and alpha power Gompertz distribution by [17]. To add more flexibility to the APT method, Alotaibi et al. [18] proposed a new form of the APT method which is called the modified APT (MAPT) method. The CDF and the PDF of the MAPT method are, respectively, given by

$$F_{MAPT}(x) = \frac{\alpha^{G(x)} - 1}{(\alpha - 1)\left(1 + \alpha - \alpha^{G(x)}\right)}, \ \alpha > 0, \alpha \neq 1 \tag{3}$$

and

$$f_{MAPT}(x) = \frac{\alpha^{1+G(x)} \log(\alpha) g(x)}{(\alpha - 1)\left(1 + \alpha - \alpha^{G(x)}\right)^2}, \ \alpha > 0, \alpha \neq 1, \tag{4}$$

where α is a shape parameter, $G(x)$ is a baseline distribution and $g(x) = \acute{G}(x)$. Reference [18] investigated the major properties of the new generator. They also used the new method to develop a new version of the exponential distribution and studied some of its properties.

Modelling real data employing generalized distributions stays vital nowadays. Multiple generalized distributions have been considered and used in different domains. Nevertheless, there still remain considerable necessary issues applying real data, which are not handled by available models. The main purpose of this article is to offer a new unexplored version of the traditional Weibull distribution. To achieve this goal, we utilize the MAPT method to add a new shape parameter to the baseline CDF given by (2). We refer to the new model as modified alpha power transformed Weibull (MAPTW) distribution which contains one scale and two shape parameters. The main properties of the new model including, quantile, mixture expansion, moments, entropies and order statistics are derived. The unknown parameters are estimated using the maximum likelihood estimation method and the approximate confidence intervals (ACIs) of the unknown parameters are also constructed. We are encouraged to introduce the MAPTW distribution because

1. It contains some well-known distributions as special cases including, exponential, Weibull and MAPT exponential (MAPTE) distributions. So, at least, the MAPTW distribution contains the main properties of these important distributions which are commonly used in modelling lifetime data. Also, the main characteristics of these distributions can be obtained directly from the MAPTW distribution.
2. It is appropriate to model positively skewed, negatively skewed and approximately symmetric data which may not be adequately modelled by other competitive models.
3. It can model decreasing, increasing, bathtub and upside-down then bathtub hazard rates which are often faced in real-life problems.
4. It is indicated in Section 2 that the MAPTW distribution can be shown as a combination of the Weibull distribution. This property is especially helpful to derive the properties of the MAPTW distribution directly from the Weibull distribution.
5. The analysis of two real data sets demonstrate that the MAPTW distribution compares satisfactorily with different competing lifetime distributions in modelling engineering data sets. As a results, the MAPTW distribution can be able to model engineering applications rather that some well known and recently proposed generalized models. This is due to the flexibility of its PDF and hazard rate function (HRF).

One of the main advantages of this distribution over many other generalized distributions is that it gives the main forms of the HRF with two shape parameters only. The same advantage can be found in its PDF which can be used to model positively and negatively skewed as well as approximately symmetric data which may not be adequately fitted by other distributions. This makes the new distribution more flexible, especially for modelling data sets when studying reliability experiments, electronics, materials, automotive industries and many engineering applications. Another motivation for this study is examining the behaviour of the maximum likelihood estimators as well as the ACIs by considering different sample sizes and different true parameter values. The rest of the paper is classified as follows: We present the MAPTW distribution in Section 2. The main properties of the MAPTW distribution are derived in Section 3. In Section 4, we discuss the maximum likelihood estimates (MLEs) as well as the ACIs of the model parameter. In Section 5, a simulation study is conducted to compare the performance of point and interval estimates. The effectiveness of the MAPTW distribution is depicted by studying two real data sets from the engineering field in Section 6. Finally, in Section 7 we conclude the paper.

2 The Modified Alpha Power Transformed Weibull Distribution

The MAPTW distribution is discussed in this section. If X has a Weibull distribution with PDF and CDF, respectively, provided by (1) and (2), then the CDF of the MAPTW distribution is given by (3).

$$F(x;\alpha,\lambda,\theta) = \frac{\alpha^{1-e^{-\lambda x^{\theta}}} - 1}{(\alpha-1)\left(1+\alpha-\alpha^{1-e^{-\lambda x^{\theta}}}\right)}, \quad x>0, \alpha,\theta,\lambda>0, \alpha\neq 1, \tag{5}$$

and the corresponding PDF is given by

$$f(x;\alpha,\lambda,\theta) = \frac{\lambda\theta\log(\alpha)x^{\theta-1}e^{-\lambda x^{\theta}}\alpha^{2-e^{-\lambda x^{\theta}}}}{(\alpha-1)\left(1+\alpha-\alpha^{1-e^{-\lambda x^{\theta}}}\right)^{2}}, \quad x>0, \alpha,\theta,\lambda>0, \alpha\neq 1, \tag{6}$$

where α and θ are the shape parameters while λ is the scale parameter. It is observed that $F(x; \alpha, \lambda, 1)$ reduces to the MAPTE by [18], $F(x; \to 1, \lambda, 1)$ tends to the exponential distribution, $F(x; \to 1, \lambda, 2)$ tends to the Rayleigh distribution, $F(x; \alpha, \lambda, 2)$ tends to the MAPT Rayleigh distribution and $F(x; \to 1, \lambda, \theta)$ tends to the Weibull distribution. It is to mentioned hare that the MAPTW distribution can be considered as a weighted version of the APW distribution with weight function $\left(1 + \alpha - \alpha^{1-e^{-\lambda x^{\theta}}}\right)^{-2}$. The reliability function (RF) and HRF of the MAPTW distribution are given, respectively, by

$$R(x; \alpha, \lambda, \theta) = 1 - \frac{\alpha^{1-e^{-\lambda x^{\theta}}} - 1}{(\alpha - 1)\left(1 + \alpha - \alpha^{1-e^{-\lambda x^{\theta}}}\right)} \tag{7}$$

and

$$h(x; \alpha, \lambda, \theta) = \frac{\lambda\theta \log(\alpha) x^{\theta-1} e^{-\lambda x^{\theta}} \alpha^{1-e^{-\lambda x^{\theta}}}}{\left(1 + \alpha - \alpha^{1-e^{-\lambda x^{\theta}}}\right)\left(\alpha - \alpha^{1-e^{-\lambda x^{\theta}}}\right)}. \tag{8}$$

Henceforward, X is used to denote the random variable that has the PDF in (6). Fig. 1 shows the various forms of the PDF of the MAPTW distribution given by (6) by assuming different values of the shape parameters and by considering the scale parameter λ to be one in all the issues. Fig. 1 indicates that the MAPTW distribution can model the left-skewed, right-skewed and approximated symmetric data. Also, Fig. 2 presents the different shapes of the HRF given by (8) by considering various values of the shape parameters and by assuming the scale parameter to be one in all the cases. Fig. 2 reveals that the HRF of the MAPTW distribution can be decreasing, increasing, bathtub and upside-down then bathtub shaped.

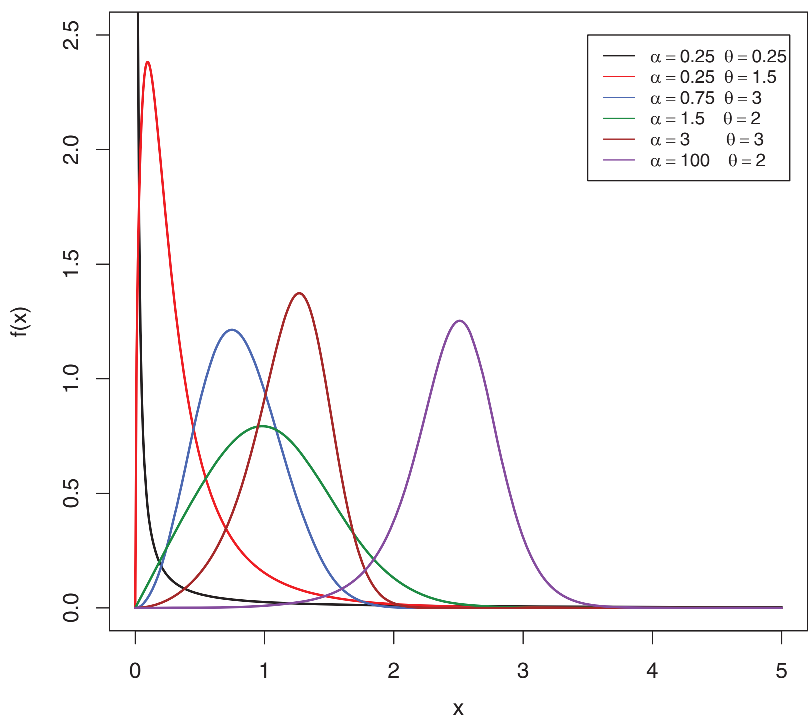


Figure 1: Plots for the PDF of the MAPTW distribution

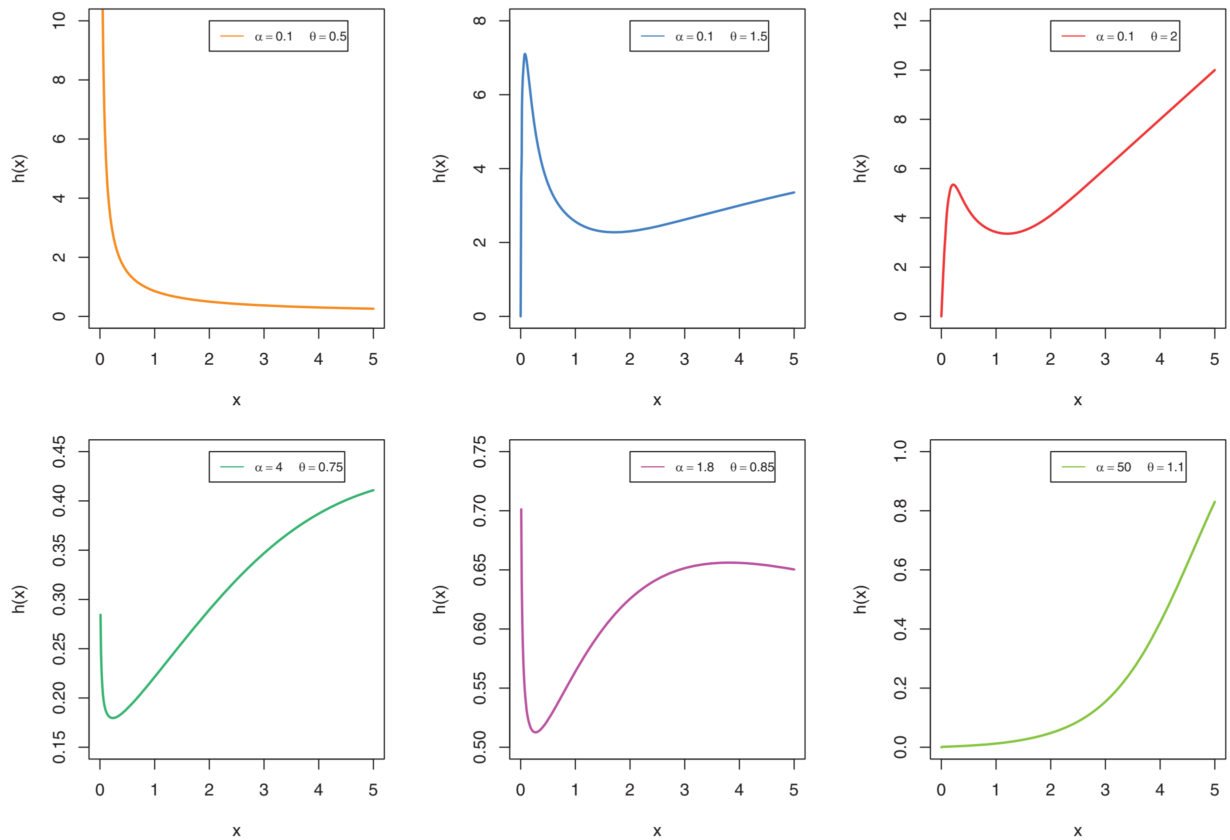


Figure 2: Plots for the HRF of the MAPTW distribution

To derive the main properties of the MAPTW distribution, we provide a valuable expansion for the PDF in (6). Consider the following power series:

$$\alpha^{y} = \sum_{m=0}^{\infty} \frac{(\log \alpha)^{m}}{m!} y^{m} \tag{9}$$

and the generalized binomial expansion in the form

$$(1-y)^{-2} = \sum_{k=0}^{\infty} (k+1)\, y^{k},\ |y| < 1. \tag{10}$$

Using the series in (9) and (10), one can write the PDF in (6) as a mixture representation of the Weibull densities as follows:

$$f(x) = \sum_{m=0}^{\infty} \sum_{j=0}^{m} \varpi_{m,j}\, g\left(x; (j+1)\lambda, \theta\right). \tag{11}$$

where $g(x; (j+1)\lambda, \theta)$ is the PDF of the Weibull distribution with scale parameter $(j+1)\lambda$ and shape parameter θ, where

$$\varpi_{m,j} = (-1)^{j} \binom{m+1}{j+1} \sum_{k=0}^{\infty} \frac{\alpha (k+1)^{m+1} (\log \alpha)^{m+1}}{(m+1)!\, (\alpha-1)(\alpha+1)^{k+2}}.$$

Moreover, we can write the linear representation of the CDF of the MAPTW distribution after integrating (11), as follows:

$$F_{MAPTW}(x) = \sum_{m=0}^{\infty}\sum_{j=0}^{m} \varpi_{m,j} G(x;(j+1)\lambda). \tag{12}$$

where $G(x;(j+1)\lambda)$ is the CDF of the Weibull distribution with scale and shape parameters $(j+1)\lambda$ and θ, respectively.

3 Structure Properties

We provide some essential properties for the MAPTW distribution in this section, such as quantile, moments, entropies, order statistics, residual life, and reversed failure rate function.

3.1 Quantile Function

The quantile function (QF) of the MAPTW distribution, say $x = Q(p)$, can be obtained from the CDF in (5) as

$$x_p = Q(p) = \left\{ \frac{-1}{\lambda} \log\left[1 - \left[\frac{\log\left(\frac{1+p(\alpha^2-1)}{1+p(\alpha-1)}\right)}{\log\alpha}\right]\right]\right\}^{\frac{1}{\theta}}, 0 < p < 1. \tag{13}$$

The QF in (13) can be used to generate a random sample from the MAPTW distribution by taking p to be $U(0,1)$.

3.2 Moments

Moments play an essential feature in statistical theory and numerous necessary characteristics of any distribution can be examined via moments. For the MAPTW distribution, the n^{th} moment can be derived as

$$\begin{aligned}
\mu_n^{'} = E(X^n) &= \int_{-\infty}^{\infty} x^n f(x)dx \\
&= \sum_{m=0}^{\infty}\sum_{j=0}^{m} \varpi_{m,j} \int_0^{\infty} x^n g(x;(j+1)\lambda)\, dx \\
&= \sum_{m=0}^{\infty}\sum_{j=0}^{m} \varpi_{m,j} \frac{\Gamma\left(\frac{n}{\theta}+1\right)}{[(j+1)\lambda]^{\frac{n}{\theta}}}, \qquad n = 1,2,3,\ldots,
\end{aligned} \tag{14}$$

where $\Gamma(.)$ is the gamma function. Similarly, for the MAPTW distribution we can obtain the n^{th} inverse moment by using (6) as follows:

$$E\left(\frac{1}{X^n}\right) = \sum_{m=0}^{\infty}\sum_{j=0}^{m} \varpi_{m,j}\Gamma\left(1-\frac{n}{\theta}\right)[(j+1)\lambda]^{\frac{n}{\theta}}, \quad \frac{n}{\theta} \geq 1.$$

Practically, the mean of the MAPTW distribution can be obtained from (14) by setting $n = 1$ as follows:

$$\mu_1^{'} = E(X) = \sum_{m=0}^{\infty} \sum_{j=0}^{m} \varpi_{m,j} \frac{\Gamma\left(\frac{1}{\theta}+1\right)}{[(j+1)\lambda]^{\frac{1}{\theta}}}.$$

Similarly, the second moment can be obtained by putting $n = 2$ in (14) as

$$\mu_2^{'} = E\left(X^2\right) = \sum_{m=0}^{\infty} \sum_{j=0}^{m} \varpi_{m,j} \frac{\Gamma\left(\frac{2}{\theta}+1\right)}{[(j+1)\lambda]^{\frac{2}{\theta}}}$$

Then, once can obtain the variance of the MAPTW distribution using the following relation:

$$Var(X) = \mu_2^{'} - \mu_1^{'2}.$$

The r^{th} central moment μ_r of X is derived as Moreover, The skewness (Sk) and kurtosis (Ku) measures can be calculated employing the next formulas

$$Sk = \frac{\mu_3}{[Var(X)]^{\frac{3}{2}}}$$

and

$$Ku = \frac{\mu_4}{[Var(X)]^2},$$

where $\mu_3 = \mu_3^{'} - 3\mu_2^{'}\mu_1^{'} + 2\mu_1^{'3}$ and $\mu_4 = \mu_4^{'} - 4\mu_3^{'}\mu_1^{'} + 6\mu_2^{'}\mu_1^{'2} - 3\mu_1^{'4}$ are the third and fourth central moments, receptively. Based on the above measures of mean, variance, skewness and kurtosis and for various values for α and θ with $\lambda = 1$, Fig. 3 displays the plots for the mean, variance, skewness and kurtosis of the MAPTW distribution. It is seen from Fig. 3 the skewness of the MAPTW distribution can be positive or negative and decreases as α and θ increase. It is also observed that the kurtosis of the MAPTW distribution decreases as α and θ increase.

The following Lemma explain various kinds of moments of the MAPTW distribution such as incomplete moments (IMs), moment generating function (MGF), characteristic generating function (CGF) and conditional moments (CMs).

Lemma 3.1. If $X \sim$ MAPTIW, then

1. The IMs of X is

$$\psi_r(t) = \sum_{m=0}^{\infty} \sum_{j=0}^{m} \varpi_{m,j} \frac{\gamma\left(r+1, (j+1)\lambda t^{\theta}\right)}{[(j+1)\lambda]^{\frac{r}{\theta}}},$$

where $\gamma(.,.)$ is the incomplete gamma function.

2. The MGF of X is

$$M_X(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} E\left(X^k\right)$$
$$= \sum_{k,m=0}^{\infty} \sum_{j=0}^{m} \varpi_{m,j} \frac{t^k}{k!} \frac{\Gamma\left(\frac{k}{\theta}+1\right)}{[(j+1)\lambda]^{\frac{k}{\theta}}}.$$

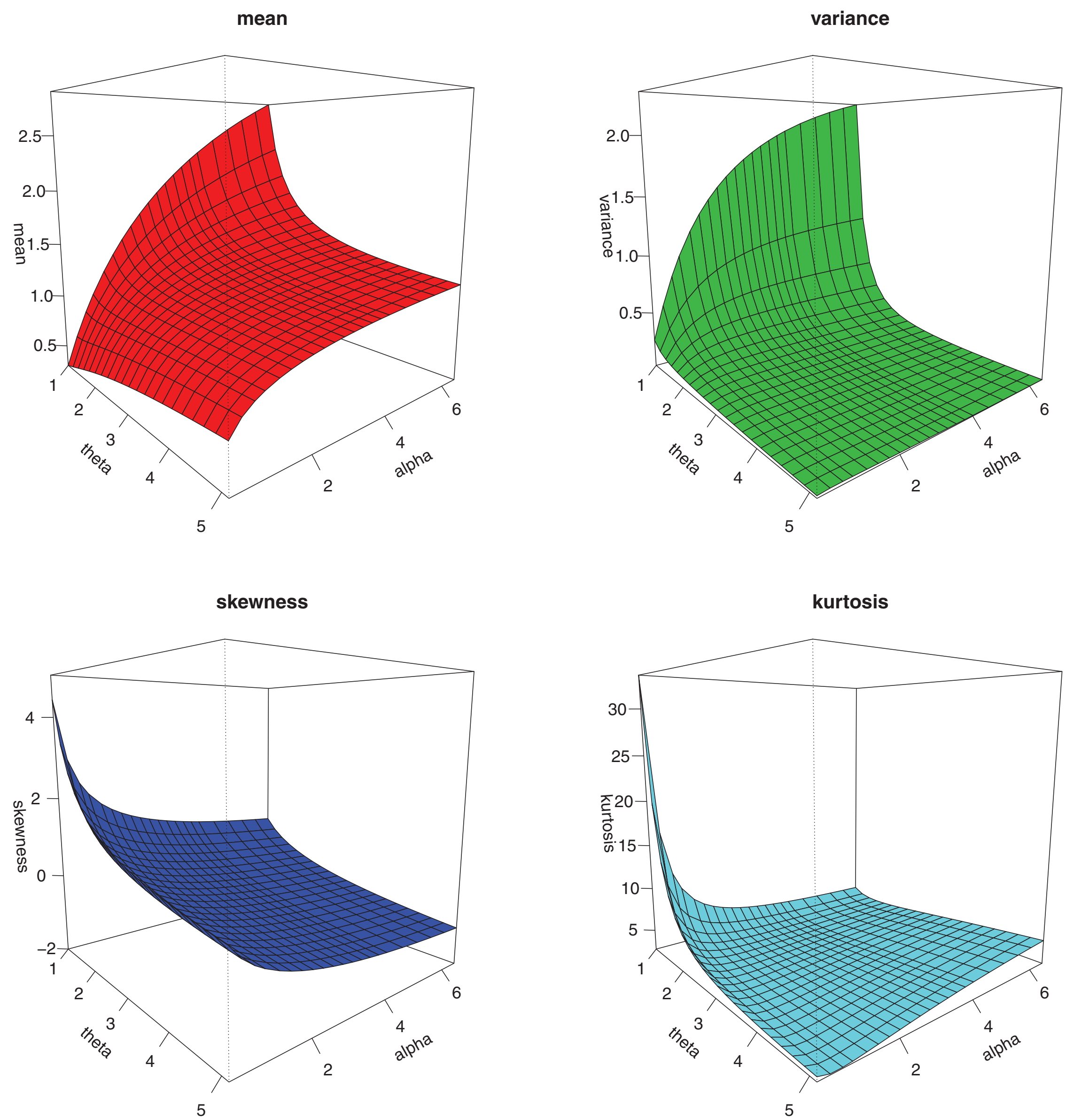


Figure 3: Mean, variance, skewness and kurtosis of the MAPTW distribution

3. The CGF of X is

$$M_X(it) = \sum_{k=0}^{\infty} \frac{(it)^k}{k!} E\left(X^k\right)$$
$$= \sum_{k,m=0}^{\infty} \sum_{j=0}^{m} \varpi_{m,j} \frac{(it)^k}{k!} \frac{\Gamma\left(\frac{k}{\theta}+1\right)}{\left[(j+1)\lambda\right]^{\frac{k}{\theta}}}.$$

4. The CMs of X is

$$C_X(t) = E\left(X^n | X > x\right) = \frac{\psi_n(x)}{1 - F(x)},$$

where $\psi_n(t)$ is the IMs and $F(x)$ is the CDF given by (5).

3.3 Entropies

Entropy is one of the numerous popular techniques used to calculate the uncertainty corresponding to a random variable and it was initially employed in the field of physics. Estimating entropy is a significant matter in many fields including statistics and biological phenomenon. High entropy is guided to minor information found in data. For the MAPTW distribution and from (6), the Renyi entropy (RE) can be acquired as follows:

$$\begin{aligned} I_\rho(x) &= \frac{1}{1-\rho} \log \left\{ \left(\frac{\lambda\theta \log\alpha}{\alpha - 1} \right)^\rho \int_0^\infty \frac{x^\theta \alpha^{(2+e^{-\lambda x^\theta})\rho} e^{-\lambda\rho x^\theta}}{\left(\alpha - (\alpha+1)\alpha^{e^{-\lambda x^\theta}}\right)^{2\rho}} dx \right\} \\ &= \frac{1}{1-\rho} \log \left\{ \left(\frac{\lambda\theta \log\alpha}{\alpha - 1} \right)^\rho \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} \binom{2\rho + j - 1}{j} \left(\frac{\alpha+1}{\alpha} \right)^j \frac{(j+\rho)^m (\log\alpha)^m}{m!} \frac{\Gamma\left(\frac{1}{\theta} + 1\right)}{\left[\lambda\left(m+\rho\right)\right]^{\frac{1}{\theta}}} \right\}. \end{aligned} \tag{15}$$

Also, Shannon's entropy can be derived by Limiting $\rho \uparrow 1$ in (15) as

$$\begin{aligned} E\left[-\log f\left(X\right)\right] = &-\log\left[\frac{\lambda\theta\log\alpha}{\alpha-1}\right] - (\theta - 1)\, E\left(\log(x)\right) + \lambda E\left(x^\theta\right) + \log(\alpha)\, E\left(2 - e^{-\lambda x^\theta}\right) \\ &- 2E\left(\log\left[1 + \alpha - \alpha^{1-e^{-\lambda x^\theta}}\right]\right) \\ = &-\log\left[\frac{\lambda\theta\log\alpha}{\alpha-1}\right] + \sum_{m=0}^{\infty}\sum_{j=0}^{\infty} \varpi_{m,j} \frac{(\theta-1)\left(\log((j+1)\lambda) + \gamma\right)}{\theta} + \sum_{m=0}^{\infty}\sum_{j=0}^{\infty} \frac{\varpi_{m,j}}{(1+j)\lambda} \\ &+ \log(\alpha) \sum_{m=0}^{\infty}\sum_{j=0}^{\infty} \varpi_{m,j} \left(\frac{j+3}{j+2}\right) - 2E\left(\log\left[1 + \alpha - \alpha^{1-e^{-\lambda x^\theta}}\right]\right) \end{aligned}$$

3.4 Order Statistics

Order statistics contain a broad range of applications in statistics, including nonparametric statistics, life testing, quality control monitoring and reliability analysis. Let $X_1, \ldots, X_n$ be a random sample from the MAPTW distribution with CDF and PDF given by (5) and (6), respectively. Suppose also that the order statistics of $X_1, \ldots, X_n$ are denoted by $X_{(1)}, \ldots, X_{(n)}$. Then, the PDF of the i^{th} order statistic, $X_{(i)}$, is expressed as

$$\begin{aligned} f_{X_{(i)}}(x) &= \frac{f(x)}{B(i, n-i+1)} F^{i-1}(x)\left[1 - F(x)\right]^{n-i} \\ &= \frac{f(x)}{B(i, n-i+1)} \sum_{k=0}^{n-i} (-1)^k \binom{n-i}{k} F^{k+i-1}(x), \end{aligned} \tag{16}$$

where $B(.,.)$ is the beta function. Using (5), (6) and a power series expansion, we obtain

$$f(x)\,F^{k+i-1}(x) = \frac{\alpha\log\alpha}{(\alpha^2-1)^{i+k}} \sum_{a,b,c,d=0}^{\infty} \frac{(-1)^{i+k+b+d-1}}{(\alpha+1)^{a+1}} \frac{((a+b+1)\log\alpha)^c}{c!} \binom{a+i+k}{a} \times \binom{i+k-1}{b}\binom{c}{d} \lambda\theta x^{\theta-1} e^{-\lambda(d+1)x^{\theta}}, \tag{17}$$

Substituting (17) in Eq. (16), the PDF of $X_{(i)}$ reduces to

$$f_{X_{(i)}}(x) = \sum_{d=0}^{\infty} \omega_d\, g(x;(d+1)\lambda,\theta) \tag{18}$$

where $g(x;(d+1)\lambda,\theta)$ is the PDF of the Weibull distribution with scale parameter $(d+1)\lambda$ and shape parameter θ, where

$$\omega_d = \sum_{k=0}^{n-i}\sum_{a,b,c=0}^{\infty} \frac{(-1)^{i+2k+b+d-1}}{(\alpha+1)^{a+1}} \frac{\alpha\log\alpha}{(\alpha^2-1)^{i+k} B(i,n-s+1)} \frac{((a+b+1)\log\alpha)^c}{c!} \times \binom{a+i+k}{a}\binom{i+k-1}{b}\binom{n-i}{k}\binom{c}{d}.$$

3.5 Probability Weighted Moments

The $(u,v)^{th}$ probability weighted moments can be defined as

$$\vartheta_{u,v} = E\{X^u F(X)^v\} = \int_{-\infty}^{\infty} x^u f(x) F^v(X)\,dx. \tag{19}$$

For the of the MAPTW distribution, it follows from (5) and (6) that

$$f(x)\,F^{v}(x) = \frac{\alpha\log\alpha}{(\alpha^2-1)^{v+1}} \sum_{a,b,c,d=0}^{\infty} \frac{(-1)^{v+b+d}}{(\alpha+1)^{a+1}} \frac{((a+b+1)\log\alpha)^c}{c!} \binom{a+v+1}{a} \times \binom{v}{b}\binom{c}{d} \lambda\theta x^{\theta-1} e^{-\lambda(d+1)x^{\theta}}, \tag{20}$$

Substituting (20) in Eq. (19), the probability weighted moments of the MAPTW distribution is given by

$$\vartheta_{u,v} = \sum_{a,b,c,d=0}^{\infty} \frac{(-1)^{v+b+d}}{(\alpha+1)^{a+1}} \frac{\alpha\log\alpha}{(\alpha^2-1)^{v+1}} \frac{[(a+b+1)\log\alpha]^c}{c!} \binom{a+v+1}{a} \times \binom{v}{b}\binom{c}{d} \frac{\Gamma\left(1+\frac{u}{\theta}\right)}{\theta\,[(1+d)\lambda]^{1+\frac{u}{\theta}}}$$

3.6 Residual Life and Reversed Failure Rate Function

The r^{th} moment of the residual life (RL) of the random variable X is defined as follows:

$$R_r(t) = E\left((X-t)^r \,|X > t\right) = \frac{1}{1-F(t)}\int_t^{\infty}(x-t)^r f(x)\,dx, \qquad r \geq 1$$

Using (6) and applying the binomial expansion of $(x-t)^r$, we have

$$R_r(t) = \frac{1}{1-F(t)}\sum_{k=0}^{r}(-t)^{r-k}\binom{r}{k}\int_t^{\infty} x^r f(x)\,dx$$
$$= \frac{1}{1-F(t)}\sum_{k=0}^{r}\sum_{m,j=0}^{\infty}\varpi_{m,j}(-t)^{r-k}\binom{r}{k}\int_t^{\infty} x^r g(x;(j+1)\lambda,\theta)\,dx$$

where $g(x;(j+1)\lambda,\theta)$ is the PDF of the Weibull distribution, then

$$R_r(t) = \frac{(\alpha-1)\left(1+\alpha-\alpha^{1-e^{-\lambda x^{\theta}}}\right)}{\alpha^2+(\alpha-2)\alpha^{1-e^{-\lambda x^{\theta}}}-1}\sum_{k=0}^{r}\sum_{m,j=0}^{\infty}\varpi_{m,j}(-t)^{r-k}\binom{r}{k}\frac{\Gamma\left(\frac{r}{\theta}+1,(j+1)\lambda t^{\theta}\right)}{[(j+1)\lambda]^{\frac{r}{\theta}}},$$

where $\Gamma(a,y) = \int_y^{\infty} z^{a-1}e^{-z}dz$ denotes the incomplete gamma function. The r^{th} moment of the reversed RL can be derived using the general formula

$$m_r(t) = E\left((t-X)^r \,|X \leq t\right) = \frac{1}{F(t)}\int_0^{t}(t-x)^r f(x)\,dx, \qquad r \geq 1.$$

Using (5), (6) and applying the binomial expansion of $(t-x)^r$, we can write

$$m_r(t) = \frac{(\alpha-1)\left(1+\alpha-\alpha^{1-e^{-\lambda x^{\theta}}}\right)}{\alpha^{1-e^{-\lambda x^{\theta}}}-1}\sum_{k=0}^{r}\sum_{m,j,a=0}^{\infty}\varpi_{m,j}(-1)^{a+r-k}\binom{r}{k}\frac{\theta\left((j+1)\lambda\right)^{a+1}t^{2r-k+(a+1)\theta}}{r+(a+1)\theta}$$

3.7 Stress-Strength Model

Let $Y_1 \sim \text{MAPTW}(\alpha_1,\theta,\lambda)$ and $Y_2 \sim \text{MAPTW}(\alpha_2,\theta,\lambda)$. If Y_1 represents stress and Y_2 represents strength, then the stress-strength parameter, denoted by R, for the MAPTW distribution. Here, we consider two cases as follows:

Case one: When $\alpha_1 \neq \alpha_2$

$$R = P(Y_2 > Y_1) = \int_0^{\infty} F_1(y)dF_2(y)dy$$
$$= \frac{\lambda\theta\log(\alpha_2)}{(\alpha_1-1)(\alpha_1+1)(\alpha_2-1)(\alpha_2+1)^2}\int_0^{\infty} y^{\theta-1}e^{-\lambda y^{\theta}}\alpha_2^{2-e^{-\lambda y^{\theta}}}\left(\alpha_1^{1-e^{-\lambda y^{\theta}}}-1\right)$$
$$\times\left(1-\frac{\alpha_1^{1-e^{-\lambda y^{\theta}}}}{\alpha_1+1}\right)^{-1}\left(1-\frac{\alpha_2^{1-e^{-\lambda y^{\theta}}}}{\alpha_2+1}\right)^{-2}dy$$

Using the series expansion in (10), the last equation takes the form

$$R=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}T_1\left(j,k\right)\int_0^{\infty}\lambda\theta y^{\theta-1}e^{-\lambda y^{\theta}}\left(\alpha_1\alpha_1^{e^{-(j+1)\lambda y^{\theta}}}-\alpha_1^{e^{-j\lambda y^{\theta}}}\right)\alpha_2^{-(k+1)e^{-\lambda y^{\theta}}}dy.$$

Employing the series expansion in (9), we obtain

$$R=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}T_1\left(j,k\right)\int_0^{\infty}\lambda\theta y^{\theta-1}e^{-\lambda y^{\theta}}\left(\alpha_1^{e^{-\lambda y^{\theta}}}-1\right)\alpha_1^{je^{-\lambda y^{\theta}}}\alpha_2^{-(k+1)e^{-\lambda y^{\theta}}}dy$$

$$=\sum_{j,k=0}^{\infty}\sum_{n,m=0}^{\infty}T_1\left(j,k\right)T_2\left(n,m\right)\int_0^{\infty}\lambda\theta y^{\theta-1}e^{-(n+m+1))\lambda y^{\theta}}dy$$

$$=\sum_{j,k=0}^{\infty}\sum_{n,m=0}^{\infty}\frac{T_1(j,k)T_2(n,m)}{n+m+1}$$

where

$$T_1\left(j,k\right)=\frac{(k+1)\alpha_2^{k+2}\alpha_1^{j}\log(\alpha_2)}{(\alpha_1-1)(\alpha_2-1)(\alpha_1+1)^{j+1}(\alpha_2+1)^{k+2}}$$

and

$$T_2\left(n,m\right)=\frac{(-1)^{n+m}(k+1)^m\left(\log(\alpha_1)\right)^n\left(\log(\alpha_2)\right)^m}{n!\,m!}\left(\alpha_1(j+1)^n-j^n\right).$$

Case two: When $\alpha_1\neq\alpha_2$, $\lambda_1\neq\lambda_2$ and $\theta_1\neq\theta_2$

Let $Y_1\sim\text{MAPTW}(\alpha_1,\theta_1,\lambda_1)$ and $Y_2\sim\text{MAPTW}(\alpha_2,\theta_1,\lambda_2)$. If Y_1 represents stress and Y_2 represents strength, then the stress-strength parameter, denoted by R, for the MAPTW distribution is given by

$$R=P(Y_2>Y_1)=\int_0^{\infty}F_1(y)dF_2(y)dy$$

$$=\frac{\lambda_2\theta_2\log(\alpha_2)}{(\alpha_1-1)(\alpha_1+1)(\alpha_2-1)(\alpha_2+1)^2}\int_0^{\infty}y^{\theta_2-1}e^{-\lambda_2y^{\theta_2}}\alpha_2^{2-e^{-\lambda_2y^{\theta_2}}}\left(\alpha_1^{1-e^{-\lambda_1y^{\theta_1}}}-1\right)$$

$$\times\left(1-\frac{\alpha_1^{1-e^{-\lambda_1y^{\theta_1}}}}{\alpha_1+1}\right)^{-1}\left(1-\frac{\alpha_2^{1-e^{-\lambda_2y^{\theta_2}}}}{\alpha_2+1}\right)^{-2}dy$$

Using the series expansion in (10), we have

$$R=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}T_1\left(j,k\right)\int_0^{\infty}\lambda_2\theta_2y^{\theta_2-1}e^{-\lambda_2y^{\theta_2}}\left(\alpha_1^{1-e^{-\lambda_1y^{\theta_1}}}-1\right)\alpha_1^{-je^{-\lambda_1y^{\theta_1}}}\alpha_2^{-(k+1)e^{-\lambda_2y^{\theta_2}}}dy.$$

Utilizing the series expansion in (9), it comes

$$R=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}T_1\left(j,k\right)\int_0^{\infty}\lambda_2\theta_2y^{\theta_2-1}e^{-\lambda_2y^{\theta_2}}\left(\alpha_1^{1-e^{-\lambda_1y^{\theta_1}}}-1\right)\alpha_1^{-je^{-\lambda_1y}}\alpha_2^{-(k+1)e^{-\lambda_2y^{\theta_2}}}dy$$

$$=\sum_{j,k=0}^{\infty}\sum_{n,m=0}^{\infty}T_1\left(j,k\right)T_2\left(n,m\right)\int_0^{\infty}\lambda_2\theta_2y^{\theta_2-1}e^{-\left(n\lambda_1y^{\theta_1}+(m+1)\lambda_2y^{\theta_2}\right)}dy,$$

where

$$T_1(j,k) = \frac{(k+1)\alpha_1^j \alpha_2^{2+k} \log(\alpha_2)}{(\alpha_1 - 1)(\alpha_2 - 1)(\alpha_1 + 1)^{j+1}(\alpha_2 + 1)^{k+2}}$$

and

$$T_2(n,m) = (-1)^{n+m} \frac{(k+1)^m (\log(\alpha_1))^n (\log(\alpha_2))^m}{n!\, m!} \left(\alpha_1 (j+1)^n - j^n\right).$$

4 Estimation of the Parameters

Here, the method of maximum likelihood is employed to estimate the unknown parameters of the MAPTW distribution. Moreover, the approximate confidence intervals (ACIs) of the unknown parameters are obtained. Assume that we have a random sample of size n taken from the MAPTW distribution with PDF given by (6), then we can express the log-likelihood function in this case as follows:

$$\ell(\alpha,\lambda,\theta) = n \log\left(\frac{\lambda\theta \log\alpha}{\alpha - 1}\right) + (\theta - 1)\sum_{i=1}^{n} \log x_i - \lambda \sum_{i=1}^{n} x_i^{\theta} + (\log\alpha)\sum_{i=1}^{n}\left(2 - e^{-\lambda x_i^{\theta}}\right)$$
$$- 2\sum_{i=1}^{n} \log\left(1 + \alpha - \alpha^{1-e^{-\lambda x_i^{\theta}}}\right). \tag{21}$$

To get the maximum likelihood estimates (MLEs) of α, λ and θ denoted by $\hat{\alpha}$, $\hat{\lambda}$ and $\hat{\theta}$, one can maximize the objective function in (21) with respect to α, λ and θ. An alternative approach to compute the required estimates is to solve the following normal equations simultaneously:

$$\frac{\partial \ell(\alpha,\lambda,\theta)}{\partial\alpha} = \frac{n}{\alpha\log(\alpha)} - \frac{n}{\alpha - 1} + \frac{1}{\alpha}\sum_{i=1}^{n}\left(2 - e^{-\lambda x_i^{\theta}}\right) - 2\sum_{i=1}^{n}\varphi_{\alpha} = 0, \tag{22}$$

$$\frac{\partial \ell(\alpha,\lambda,\theta)}{\partial\lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i^{\theta} + \log(\alpha)\sum_{i=1}^{n} x_i^{\theta} e^{-\lambda x_i^{\theta}} - 2\sum_{i=1}^{n}\varphi_{\lambda} = 0 \tag{23}$$

and

$$\frac{\partial \ell(\alpha,\lambda,\theta)}{\partial\theta} = \frac{n}{\theta} - 2\sum_{i=1}^{n}\varphi_{\theta} + \sum_{i=1}^{n}\log(x_i)\left[1 + \lambda x_i^{\theta}\left(\log(\alpha) e^{-\lambda x_i^{\theta}} - 1\right)\right] = 0, \tag{24}$$

where

$$\varphi_{\alpha} = \frac{1 - \left(1 - e^{-\lambda x_i^{\theta}}\right)\alpha^{-e^{-\lambda x_i^{\theta}}}}{\alpha - \alpha^{1-e^{-\lambda x_i^{\theta}}} + 1},$$

$$\varphi_{\lambda} = -\frac{\log(\alpha) x_i^{\theta} e^{-\lambda x_i^{\theta}} \alpha^{1-e^{-\lambda x_i^{\theta}}}}{\alpha - \alpha^{1-e^{-\lambda x_i^{\theta}}} + 1}$$

and

$$\varphi_\theta = -\frac{\alpha\lambda\log(\alpha)x_i^\theta\log(x_i)\,e^{-\lambda x_i^\theta}}{(\alpha+1)\alpha^{e^{-\lambda x_i^\theta}}-\alpha}.$$

It is to note that there are no closed-form solutions for Eqs. (22)–(24). To get the MLEs in this case, one can direct to any numerical technique for this purpose. Using the asymptotic properties of the MLEs, we can obtain the ACIs of α, λ and θ. It is known that $(\alpha,\lambda,\theta)\sim N\left(\left(\hat{\alpha},\hat{\lambda},\hat{\theta}\right), I_0^{-1}(\alpha,\lambda,\theta)\right)$, where $I_0^{-1}(\alpha,\lambda,\theta)$ is the asymptotic variance-covariance (AVC) of the MLEs. Practically, it is not easy to compute $I_0^{-1}(\alpha,\lambda,\theta)$, thus, the approximate AVC of the MLEs denoted by $I_0^{-1}\left(\hat{\alpha},\hat{\lambda},\hat{\theta}\right)$ can be used as follows:

$$I_0\left(\hat{\alpha},\hat{\lambda},\hat{\theta}\right)=\begin{bmatrix}-I_{\alpha\alpha} & -I_{\alpha\lambda} & -I_{\alpha\theta}\\ -I_{\alpha\lambda} & -I_{\lambda\lambda} & -I_{\lambda\theta}\\ -I_{\alpha\theta} & -I_{\lambda\theta} & -I_{\theta\theta}\end{bmatrix}^{-1}_{(\alpha,\lambda,\theta)=\left(\hat{\alpha},\hat{\lambda},\hat{\theta}\right)}=\begin{bmatrix}\widehat{var}\left(\hat{\alpha}\right) & \widehat{cov}\left(\hat{\alpha},\hat{\lambda}\right) & \widehat{cov}\left(\hat{\alpha},\hat{\theta}\right)\\ \widehat{cov}\left(\hat{\lambda},\hat{\alpha}\right) & \widehat{var}\left(\hat{\lambda}\right) & \widehat{cov}\left(\hat{\lambda},\hat{\theta}\right)\\ \widehat{cov}\left(\hat{\theta},\hat{\alpha}\right) & \widehat{cov}\left(\hat{\theta},\hat{\lambda}\right) & \widehat{var}\left(\hat{\theta}\right)\end{bmatrix}. \tag{25}$$

where $I_{\alpha\alpha}, I_{\alpha\lambda}=I_{\lambda\alpha}, I_{\alpha\theta}=I_{\theta\alpha}, I_{\lambda\theta}=I_{\theta\lambda}, I_{\lambda\lambda}$ and $I_{\theta\theta}$ are the second derivatives of (21) and given by

$$I_{\alpha\alpha}=n\left[\frac{1}{(\alpha-1)^2}-\frac{\log(\alpha)+1}{\alpha^2\log^2(\alpha)}\right]-\frac{1}{\alpha^2}\sum_{i=1}^{n}\left(2-e^{-\lambda x_i{}^\theta}\right)-2\sum_{i=1}^{n}\varphi_{\alpha\alpha},$$

$$I_{\lambda\lambda}=-\frac{n}{\lambda^2}-\log(\alpha)\sum_{i=1}^{n}x_i^{2\theta}e^{-\lambda x_i^\theta}-2\sum_{i=1}^{n}\varphi_{\lambda\lambda},$$

$$I_{\theta\theta}=\frac{n}{\theta^2}+\lambda\log(\alpha)\sum_{i=1}^{n}\left[x_i^\theta\left(1-x_i^\theta\right)e^{-\lambda x_i^\theta}\log^2(x_i)\right]-\lambda\sum_{i=1}^{n}x_i^\theta\log^2(x_i)-2\sum_{i=1}^{n}\varphi_{\theta\theta},$$

$$I_{\alpha\lambda}=\frac{\sum_{i=1}^{n}x_i^\theta e^{-\lambda x_i^\theta}}{\alpha}-2\sum_{i=1}^{n}\varphi_{\alpha\lambda},$$

$$I_{\alpha\theta}=\frac{1}{\alpha}\sum_{i=1}^{n}e^{-\lambda x_i^\theta}\lambda x_i^\theta\log(x_i)-2\sum_{i=1}^{n}\varphi_{\alpha\theta}$$

and

$$I_{\lambda\theta}=-\sum_{i=1}^{n}x_i^\theta\log(x_i)-\log(\alpha)\sum_{i=1}^{n}x_i^\theta\log(x_i)\,e^{-\lambda x_i^\theta}\left(\lambda x_i^\theta-1\right)-2\sum_{i=1}^{n}\varphi_{\lambda\theta},$$

where

$$\varphi_{\alpha\alpha}=\frac{e^{-\lambda x_i^\theta}\left(1-e^{-\lambda x_i^\theta}\right)\alpha^{-e^{-\lambda x_i^\theta}-1}}{\alpha-\alpha^{1-e^{-\lambda x_i^\theta}}+1}-\frac{\left(1-\left(1-e^{-\lambda x_i^\theta}\right)\alpha^{-e^{-\lambda x_i^\theta}}\right)^2}{\left(\alpha-\alpha^{1-e^{-\lambda x_i^\theta}}+1\right)^2},$$

$$\varphi_{\lambda\lambda}=\frac{\alpha\log(\alpha)x_i^{2\theta}e^{-2\lambda x_i^\theta}\left(e^{\lambda x_i^\theta}-\log(\alpha)\right)}{(\alpha+1)\alpha^{e^{-\lambda x_i^\theta}}-\alpha}-\frac{\log^2(\alpha)x_i^{2\theta}e^{-2\lambda x_i^\theta}\alpha^{2-2e^{-\lambda x_i^\theta}}}{\left(\alpha-\alpha^{1-e^{-\lambda x_i^\theta}}+1\right)^2},$$

$$\varphi_{\theta\theta} = \frac{\alpha\lambda \log(\alpha) x_i^{\theta} \log^2(x_i)\, e^{-2\lambda x_i^{\theta}} \left((\alpha+1)\alpha^{e^{-\lambda x_i^{\theta}}} \left(\lambda x_i^{\theta} \left(e^{\lambda x_i^{\theta}} - \log(\alpha) \right) - e^{\lambda x_i^{\theta}} \right) - \alpha e^{\lambda x_i^{\theta}} \left(\lambda x_i^{\theta} - 1 \right) \right)}{\left(\alpha - (\alpha+1)\alpha^{e^{-\lambda x_i^{\theta}}} \right)^2},$$

$$\varphi_{\alpha\theta} = \frac{\alpha \log(\alpha) x_i^{2\theta} e^{-2\lambda x_i^{\theta}} \left(e^{\lambda x_i^{\theta}} - \log(\alpha) \right)}{(\alpha+1)\alpha^{e^{-\lambda x_i^{\theta}}} - \alpha} - \frac{\log^2(\alpha) x_i^{2\theta} e^{-2\lambda x_i^{\theta}} \alpha^{2-2e^{-\lambda x_i^{\theta}}}}{\left(\alpha - \alpha^{1-e^{-\lambda x_i^{\theta}}} + 1 \right)^2},$$

$$\varphi_{\alpha\lambda} = \frac{\log(\alpha) x_i^{\theta} e^{-\lambda x_i^{\theta}} \left(1 - \left(1 - e^{-\lambda x_i^{\theta}} \right) \alpha^{-e^{-\lambda x_i^{\theta}}} \right) \alpha^{1-e^{-\lambda x_i^{\theta}}}}{\left(\alpha - \alpha^{1-e^{-\lambda x_i^{\theta}}} + 1 \right)^2}$$

$$+ \frac{x_i^{\theta} \left(-e^{-\lambda x_i^{\theta}} \right) \alpha^{-e^{-\lambda x_i^{\theta}}} - \log(\alpha) x_i^{\theta} e^{-\lambda x_i^{\theta}} \left(1 - e^{-\lambda x_i^{\theta}} \right) \alpha^{-e^{-\lambda x_i^{\theta}}}}{\alpha - \alpha^{1-e^{-\lambda x_i^{\theta}}} + 1},$$

$$\varphi_{\lambda\theta} = \frac{\alpha \log(\alpha) x_i^{\theta} \log(x_i)\, e^{-\lambda x_i^{\theta}} \left(\lambda x_i^{\theta} - 1 \right)}{(\alpha+1)\alpha^{e^{-\lambda x_i^{\theta}}} - \alpha} - \frac{(\alpha+1)\lambda \log^2(\alpha) x_i^{2\theta} \log(x_i)\, e^{-2\lambda x_i^{\theta}} \alpha^{e^{-\lambda x_i^{\theta}}+1}}{\left(\alpha - (\alpha+1)\alpha^{e^{-\lambda x_i^{\theta}}} \right)^2}.$$

Thus, the $(1-\gamma)\%$ ACIs of α, λ and θ can be obtained as follows:

$$\hat{\alpha} \pm z_{\gamma/2}\sqrt{\widehat{var}\left(\hat{\alpha}\right)},\quad \hat{\lambda} \pm z_{\gamma/2}\sqrt{\widehat{var}\left(\hat{\lambda}\right)} \quad \text{and} \quad \hat{\theta} \pm z_{\gamma/2}\sqrt{\widehat{var}\left(\hat{\theta}\right)}$$

where $z_{\gamma/2}$ is the upper $(\gamma/2)^{th}$ percentile point of a standard normal distribution.

5 Simulation Study

To evaluate the performance of the MLEs of the MAPTW distribution, a Monte Carlo simulation is conducted based on a sufficiently large 5,000 independent from the MAPTW distribution using different sample sizes $n(= 50, 100, 150, 200)$ and different choices for true values of the model parameters α, λ and θ as $\alpha(= 0.1, 0.3, 0.5, 0.7)$, $\lambda(= 0.5, 0.8, 1.2, 1.6)$ and $\theta(= 0.5, 0.8, 1.5, 2.5)$. The simulation procedure is carried out based on following the steps:

Step 1: Determine the sample size and the starting values for the parameters.

Step 2: Generate a random sample of size n from the MAPTW distribution from (9).

Step 3: Compute the average estimates with their root mean squared-errors (RMSEs) and relative absolute biases (RABs) of α, λ and θ.

Step 4: Obtain the $(1-\gamma)\%$CLs of the parameters α, λ and θ.

Step 5: Repeat steps 2–5 5000 times.

Step 6: Calculate the average values (AV) of MLEs, RMSEs, RABs and CLs of any function of α, λ and θ (say φ) are given, respectively, by

$$AV - MLE\left(\varphi\right) = \frac{1}{5000}\sum_{i=1}^{5000}\hat{\varphi}_i,$$

$$AV - RMSE(\varphi) = \sqrt{\frac{1}{5000}\sum_{i=1}^{5000}\left(\hat{\varphi}_i - \varphi\right)^2},$$

$$AV - RAB(\varphi) = \frac{1}{5000}\sum_{i=1}^{5000}\frac{1}{\varphi}\left|\hat{\varphi}^{(i)} - \varphi\right|,$$

and

$$AV - CL(\varphi) = \frac{1}{5000}\sum_{i=1}^{5000}(\varphi i^U - \varphi i^L),$$

where ϕ_i^L and ϕ_i^U are the lower and upper ACI bounds, respectively.

All required numerical results are obtained via R software version 4.0.4 utilizing 'maxLik' package proposed by [19] and later recommended by [20,21]. The simulation results are shown in Tables 1 and 2. From Table 1, we observe that the maximum likelihood estimates of the parameters α, λ and θ are quite satisfactory in terms of minimum RMSEs and RABs. It can also be shown that, as n increases, the AV-MLEs of the parameters α, λ and θ close to the true parameter values. This result implies the fact that the MLEs behaved as asymptotically unbiased estimators. For each set of parameter values, as n increases, the AV-RMSEs and AV-RABs decrease which implying that the MLEs are also consistent. To be more specific, it is noted that the AV-RMSE and AV-RAB for the parameter θ decrease rapidly after the sample size exceeds 50. On the other hand, the AV-RMSEs and AV-RABs for the parameters α and λ need a larger sample size to get a small value for these measures. After revising the computation results, we have observed that some samples from the 1000 generated samples give high estimated values for α and λ and this is the reason for the high values of AV-RMSEs and AV-RABs in this case, but the majority of the generated samples get satisfactory results. Further, Table 2 shows that the AV-CLs decrease are narrowed down with the increase in the sample size n as expected. Graphically, the AV-MSEs, AV-RABs and AV-CLs of the parameters α, λ and θ are displayed in Fig. 4 (when $(\alpha, \lambda, \theta) = (0.1, 0.5, 0.5)$ as an example). This conclusion refers to the fact that when n increases more additional information is gathered.

Table 1: The AV-MLEs (first-line), AV-RMSEs (second-line) and AV-RABs (third-line) of α, λ and θ

n	α	λ	θ	α	λ	θ	α	λ	θ
$(\alpha, \lambda, \theta) \to$	(0.1, 0.5, 0.5)			(0.1, 0.5, 1.5)			(0.1, 1.2, 0.5)		
50	0.3276	1.2799	0.4944	0.4080	1.3106	1.4792	0.3531	2.8568	0.4929
	1.4922	1.6136	0.0836	2.3084	1.6559	0.2496	1.6978	3.3836	0.0839
	2.7765	2.0645	0.1303	3.5617	2.1072	0.1267	3.0082	1.8548	0.1297
100	0.2009	0.9377	0.4953	0.2008	0.9295	1.4898	0.2007	2.0963	0.4959
	0.8258	0.9819	0.0570	0.8590	0.9726	0.1737	0.9016	2.0010	0.0571
	1.4394	1.3183	0.0893	1.4596	1.3221	0.0905	1.4403	1.1803	0.0890
150	0.1535	0.7856	0.4975	0.1600	0.7883	1.4926	0.1505	1.8220	0.4962
	0.3216	0.7001	0.0464	0.4828	0.7482	0.1428	0.1289	1.5486	0.0453
	0.9391	0.9892	0.0733	1.0271	1.0205	0.0749	0.9018	0.9222	0.0723
200	0.1377	0.7191	0.4976	0.1364	0.7157	1.4969	0.1382	1.6788	0.4979

(Continued)

Table 1 (continued)

n	α	λ	θ	α	λ	θ	α	λ	θ
$(\alpha, \lambda, \theta) \rightarrow$	(0.1, 0.5, 0.5)			(0.1, 0.5, 1.5)			(0.1, 1.2, 0.5)		
	0.1061	0.5815	0.0389	0.1052	0.5781	0.1184	0.1075	1.3015	0.0399
	0.7503	0.8300	0.0620	0.7578	0.8404	0.0625	0.7620	0.7865	0.0640
$(\alpha, \lambda, \theta) \rightarrow$	(0.3, 0.8, 0.8)			(0.3, 1.2, 1.5)			(0.3, 1.6, 2.5)		
50	0.8343	1.1866	0.7943	0.8676	1.7296	1.4894	0.8664	2.2263	2.4707
	3.4966	1.1367	0.1625	4.1765	1.4595	0.3045	3.4567	1.7301	0.5169
	2.1779	0.8791	0.1535	2.2763	0.8074	0.1518	2.2832	0.7594	0.1549
100	0.5576	1.0018	0.7976	0.5904	1.4675	1.4905	0.5373	1.9295	2.4833
	2.4068	0.7699	0.1165	2.5529	0.9918	0.2202	2.1014	1.1634	0.3536
	1.2053	0.5942	0.1079	1.3094	0.5490	0.1079	1.1229	0.5186	0.1058
150	0.4300	0.9444	0.7947	0.4632	1.3864	1.4941	0.4557	1.8158	2.4902
	1.3920	0.5907	0.0920	1.8020	0.7941	0.1750	1.8408	0.9196	0.2840
	0.7281	0.4753	0.0862	0.8502	0.4489	0.0868	0.8161	0.4101	0.0847
200	0.3810	0.9026	0.7977	0.3706	1.3209	1.4972	0.3593	1.7507	2.5000
	1.0746	0.4749	0.0774	1.0369	0.6135	0.1429	0.8092	0.7428	0.2406
	0.5402	0.3976	0.0745	0.5134	0.3684	0.0732	0.4771	0.3508	0.0747
$(\alpha, \lambda, \theta) \rightarrow$	(0.5, 0.5, 0.8)			(0.5, 0.8, 1.5)			(0.5, 1.6, 2.5)		
50	1.2764	0.7463	0.7995	1.2572	1.0986	1.5003	1.2717	2.0291	2.4811
	4.9076	0.8618	0.1859	4.2794	1.0341	0.3509	4.4629	1.4009	0.5789
	1.9483	0.9083	0.1769	1.9087	0.7590	0.1766	1.9226	0.6017	0.1755
100	0.9119	0.6511	0.7939	0.9276	0.9833	1.4885	0.9421	1.8211	2.4773
	2.9873	0.6381	0.1384	2.9742	0.7861	0.2609	2.8717	1.0544	0.4385
	1.1460	0.6484	0.1285	1.1812	0.5536	0.1286	1.2125	0.4313	0.1287
150	0.7530	0.6026	0.7964	0.8178	0.9306	1.4911	0.7789	1.7474	2.4880
	2.2514	0.5091	0.1145	2.6348	0.6551	0.2180	2.4189	0.8289	0.3535
	0.7939	0.5127	0.1056	0.9239	0.4464	0.1060	0.8409	0.3405	0.1037
200	0.6947	0.5761	0.7962	0.6465	0.8806	1.4976	0.6705	1.7198	2.4906
	1.9819	0.4328	0.0988	1.5406	0.4796	0.1783	1.8017	0.6925	0.3041
	0.6494	0.4317	0.0909	0.5497	0.3548	0.0879	0.5956	0.2967	0.0908
$(\alpha, \lambda, \theta) \rightarrow$	(0.7, 0.5, 0.5)			(0.7, 1.6, 1.5)			(0.7, 1.6, 2.5)		
50	1.5141	0.7029	0.5075	1.4719	1.8578	1.5230	1.5725	1.8803	2.5346
	4.4358	0.7941	0.1263	4.552	1.2110	0.3800	5.2958	1.2553	0.6355
	1.5795	0.8625	0.1957	1.5251	0.5269	0.1967	1.6656	0.5406	0.1975
100	1.3351	0.6632	0.4974	1.2923	1.7848	1.4982	1.4450	1.8164	2.4838
	3.6671	0.6911	0.1012	3.8406	1.0201	0.3005	4.2214	1.0754	0.5137
	1.2460	0.7013	0.1528	1.1877	0.4165	0.1501	1.3992	0.4269	0.1523
150	1.1256	0.6121	0.4987	1.0566	1.7266	1.4971	1.2184	1.1778	2.4779
	2.9534	0.5669	0.0841	2.5628	0.8228	0.2474	3.1269	0.9301	0.4434
	0.9055	0.5509	0.1241	0.8068	0.3350	0.1226	1.0327	0.3608	0.1293

(Continued)

Table 1 (continued)

n	α	λ	θ	α	λ	θ	α	λ	θ
$(\alpha, \lambda, \theta) \rightarrow$	(0.1, 0.5, 0.5)			(0.1, 0.5, 1.5)			(0.1, 1.2, 0.5)		
200	1.0145	0.5887	0.4984	1.0047	1.7126	1.4948	1.0113	1.7155	2.4858
	2.3841	0.4989	0.0746	2.4321	0.7464	0.2218	2.2830	0.7586	0.3716
	0.7220	0.4772	0.1090	0.7055	0.2979	0.1095	0.7122	0.2995	0.1091

Table 2: The AV-CLs for 95% ACIs of α, λ and θ

n	α	λ	θ	α	λ	θ	α	λ	θ
$(\alpha, \lambda, \theta) \rightarrow$	(0.1, 0.5, 0.5)			(0.1, 0.5, 1.5)			(0.1, 1.2, 0.5)		
50	0.6398	1.6286	0.2888	0.4501	1.5871	0.8745	0.6675	4.3819	0.2438
100	0.4366	0.9167	0.1933	0.3436	0.8953	0.5931	0.4720	2.2415	0.1705
150	0.3712	0.6272	0.1626	0.2990	0.6644	0.4963	0.4158	1.6898	0.1504
200	0.3520	0.5294	0.1542	0.2699	0.4933	0.4217	0.3767	1.3960	0.1370
$(\alpha, \lambda, \theta) \rightarrow$	(0.3, 0.8, 0.8)			(0.3, 1.2, 1.5)			(0.3, 1.6, 2.5)		
50	0.5462	0.7186	0.5815	0.4617	1.2692	1.1405	0.3509	1.2701	1.8358
100	0.3766	0.4567	0.3843	0.3346	0.7653	0.7495	0.2430	0.8869	1.2782
150	0.3061	0.3577	0.3065	0.2734	0.6031	0.5935	0.2036	0.7296	1.0769
200	0.2700	0.2970	0.2635	0.2397	0.4736	0.4829	0.1724	0.5984	0.8662
$(\alpha, \lambda, \theta) \rightarrow$	(0.5, 0.5, 0.8)			(0.5, 0.8, 1.5)			(0.5, 1.6, 2.5)		
50	0.4802	0.3169	0.6563	0.5882	0.4514	1.3570	0.3967	0.8468	2.1047
100	0.3484	0.2384	0.4952	0.4186	0.3152	0.9789	0.2739	0.6045	1.5690
150	0.2783	0.1952	0.3957	0.3287	0.2537	0.7459	0.2232	0.5058	1.3428
200	0.2446	0.1764	0.3568	0.2908	0.2236	0.6775	0.1903	0.4388	1.1480
$(\alpha, \lambda, \theta) \rightarrow$	(0.7, 0.5, 0.5)			(0.7, 1.6, 1.5)			(0.7, 1.6, 2.5)		
50	0.3924	0.3150	0.5406	0.4856	0.7322	1.4961	0.4695	0.6705	2.5082
100	0.2705	0.2503	0.4005	0.3476	0.5040	1.0909	0.3268	0.4647	1.8220
150	0.2333	0.2277	0.3589	0.2832	0.4254	0.9454	0.2502	0.3844	1.5078
200	0.2000	0.2112	0.3252	0.2473	0.3814	0.8705	0.2123	0.3497	1.4222

6 Engineering Applications

To show how our proposed model works in practice, in this section we present the analysis of two real data sets from engineering science for illustrative purposes. The first data set (denoted by Data-I) consists of 40 observations of the active repair times for airborne communication transceiver, see [22].

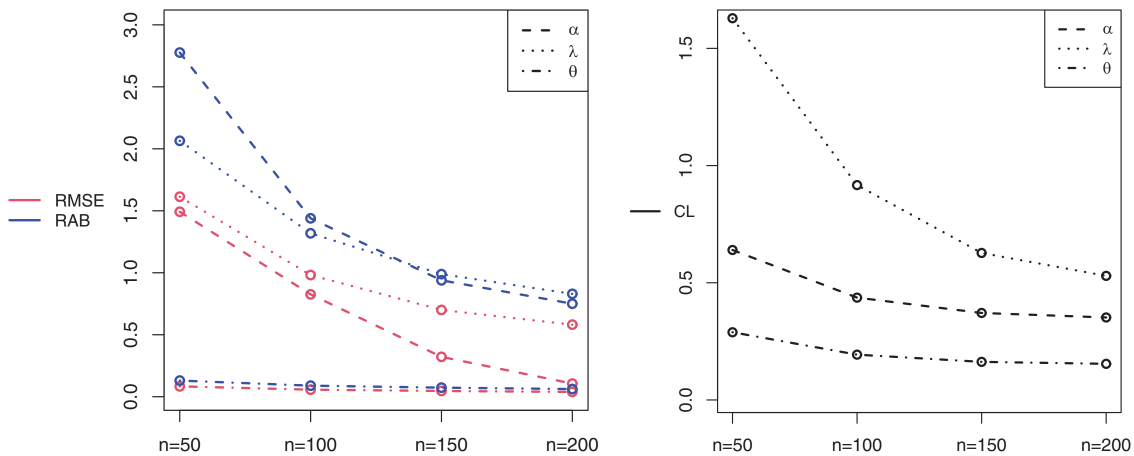


Figure 4: Plot of the RMSEs, RABs (left-panel) and CLs (right-panel) of α, λ and θ

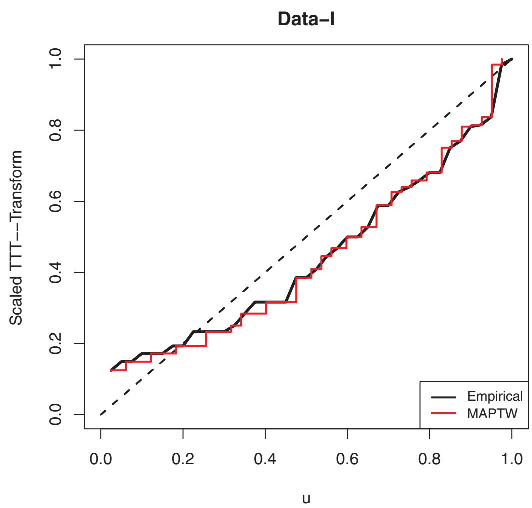
The second data set (denoted by Data-II) represents the failure times of 84 Aircraft Windshields and been obtained from [23]. The values of both data sets I and II are arranged with ascending order and reported in Table 3.

Table 3: The failure times of communication transceiver and aircraft windshields

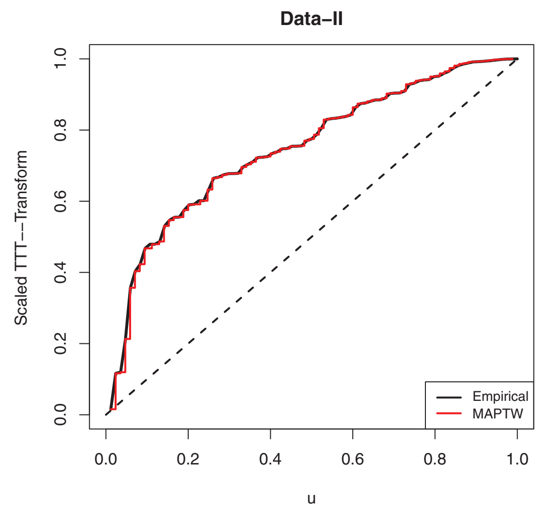
Data	Failure times
I	0.50, 0.60, 0.60, 0.70, 0.70, 0.70, 0.80, 0.80, 1.00, 1.00, 1.00, 1.00, 1.10, 1.30, 1.50, 1.50, 1.50, 1.50, 2.00, 2.00, 2.20, 2.50, 2.70, 3.00, 3.00, 3.30, 4.00, 4.00, 4.50, 4.70, 5.00, 5.40, 5.40, 7.00, 7.50, 8.80, 9.00, 10.2, 22.0, 24.5
II	0.040, 0.301, 0.309, 0.557, 0.943, 1.070, 1.124, 1.248, 1.281, 1.281, 1.303, 1.432, 1.480, 1.505, 1.506, 1.568, 1.615, 1.619, 1.652, 1.652, 1.757, 1.866, 1.876, 1.899, 1.911, 1.912, 1.914, 1.981, 2.010, 2.038, 2.085, 2.089, 2.097, 2.135, 2.154, 2.190, 2.194, 2.223, 2.224, 2.229, 2.300, 2.324, 2.385, 2.481, 2.610, 2.625, 2.632, 2.646, 2.661, 2.688, 2.823, 2.890, 2.902, 2.934, 2.962, 2.964, 3.000, 3.103, 3.114, 3.117, 3.166, 3.344, 3.376, 3.443, 3.467, 3.478, 3.578, 3.595, 3.699, 3.779, 3.924, 4.035, 4.121, 4.167, 4.240, 4.255, 4.278, 4.305, 4.376, 4.449, 4.485, 4.570, 4.602, 4.663

First, to identify the shape of the HRF of MAPTW distribution, we shall consider a graphical method, given by [24], based on the total time on test (TTT) plot. However, the scaled TTT transform based on both observed data sets I and II is given by

$G_n(m/n) = \left(\sum_{i=1}^{m} z_{(i)} + (n-m)\, z_{(m)}\right) / \sum_{i=1}^{n} z_{(i)}$, $m = 1, 2, \ldots, n$, where $z_{(i)}$ is the ith order statistic of the observed sample. Graphically, the scaled TTT transform is displayed by plotting $(m/n, G_n(m/n))$. Using both data sets I and II in Table 3, plots of the empirical and estimated scaled TTT-Transform of the MAPTW distribution are provided in Fig. 5. It shows that the TTT plot for the data sets I and II is convex and concave, respectively. So, the failure rate shape for the Data-I set decreases while it for the Data-II set increases.



(a) Data-I



(b) Data-II

Figure 5: Empirical and estimated scaled TTT-transform plot of the MAPTW distribution

To monitored the applicability and flexibility of the MAPTW distribution, we compare fit the MAPTW distribution to both data sets I and II with several other competitive models exhibiting various hazard rates, namely: alpha power transformed generalized-exponential (APTGE), alpha power exponential (APE), APW, generalized-exponential (GE), Nadarajah-Haghighi (NH), gamma (G) and Weibull (W) distributions. The corresponding PDFs of the competing models (for $x > 0$) are listed in Table 4.

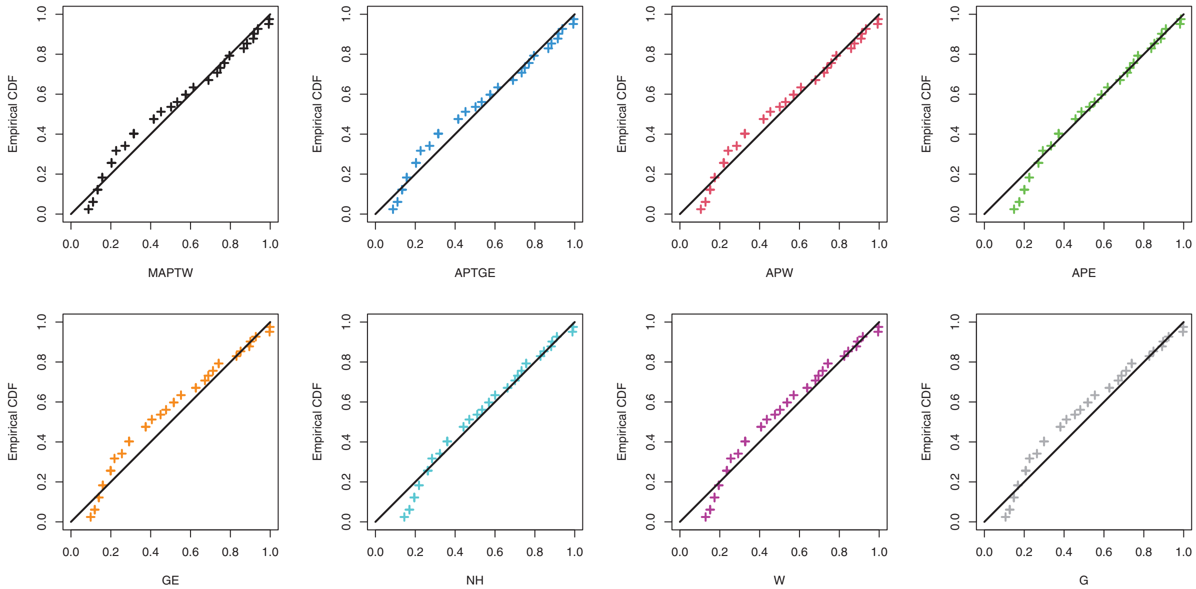
Table 4: Some competing lifetime models of the MAPTW distribution

Model	PDF	Author(s)
W	$\theta\lambda x^{\theta-1}\exp(-\lambda x^{\theta})$	[25]
G	$(\lambda^{\theta}/\Gamma(\theta))x^{\theta-1}\exp(-\lambda x)$	[26]
GE	$\theta\lambda(1-e^{-\lambda x})^{\theta-1}\exp(-\lambda x)$	[27]
NH	$\theta\lambda(1+\lambda * x)^{\theta-1}\exp(1-(1+\lambda x)^{\theta})$	[28]
APE	$\lambda\log(\theta)\exp(-\lambda x)\theta^{(1-\exp(}-\lambda x))/(\theta-1)$	[14]
APW	$\lambda\theta\log(\alpha)x^{\theta-1}\exp(-\lambda x^{\theta})\alpha^{1-\exp(-\lambda x^{\theta})}/(\alpha-1)$	[8]
APTGE	$\lambda\theta\log(\alpha)\alpha^{(1-\exp(-\lambda x))^{\theta}}\exp(-\lambda x)(1-\exp(-\lambda x))^{\theta-1}/(\alpha-1)$	[15]

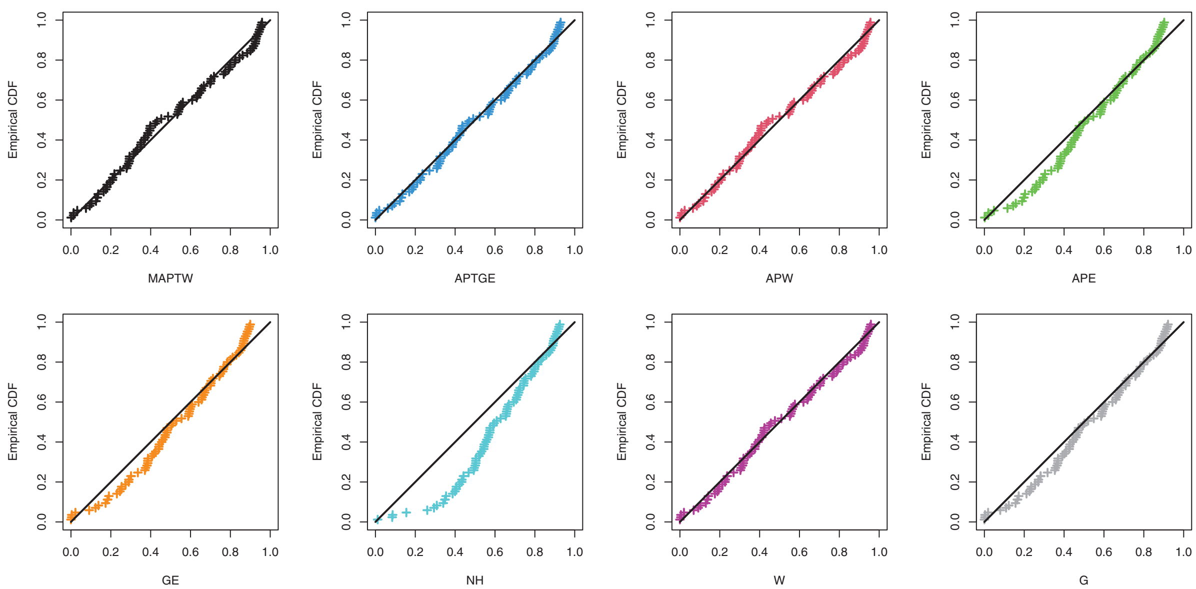
To show the feasibility and validity of the MAPTW model along with other seven competing models, several goodness-of-fit measures namely; estimated negative log-likelihood (NL), Akaike information criterion (AIC), consistent Akaike information criterion (CAIC), Bayesian information criterion (BIC) and Hannan-Quinn information criterion (HQIC) are considered. Based on both data sets I and II, the selection criteria as well as the MLEs along with their standard errors (SEs) of the model parameters are calculated and provided in Table 5. It shows that the MAPTW distribution has the smallest values of NCL, AIC, CAIC, BIC, HQIC, K-S and the highest *P*-value. Moreover, Fig. 6 shows graphically the quantile-quantile (QQ) plots of all competitive distributions for given data sets. Furthermore, the relative histograms of both data sets and the fitted densities, as well as the fitted/empirical survival functions, are plotted in Fig. 7. It can be seen from Figs. 6 and 7, that the graphical presentations support the same numerical findings presented in Table 5. Thus, the proposed MAPTW distribution provides the best fit, for the given data sets, than the APTGE, APTW and other popular lifetime distributions. Since the empirical HRFs of the two data sets are decreasing and increasing, respectively, and most of the competitive models such as Weibull and APW distributions can model such HRFs, the results show that the MAPTW distribution provides a better fit for the two engineering applications in terms of goodness of fit statistics. This is because the MAPTW distribution has a very flexible HRF with different shape behaviour as well as a flexible PDF, especially for skewed data. To sum up, we recommend the use of MAPTW for real practical purposes.

Table 5: The MLEs(SEs) and selection criteria of the MAPTW distribution and other competing models

Model	Estimate (SE)			NL	AIC	CAIC	BIC	HQIC	KS (*P*-value)
	α	θ	λ						
				Data-I					
MAPTW	0.0599(0.0387)	1.6043(0.1847)	0.0052(0.0040)	91.761	189.52	190.19	194.59	191.35	0.1136(0.681)
APTGE	0.0434(0.0741)	1.3722(0.2458)	0.1539(0.0724)	92.937	191.87	192.54	196.94	193.71	0.1345(0.464)
APW	0.0282(0.0527)	1.1866(0.1375)	0.0698(0.0396)	93.472	192.94	193.61	198.01	194.77	0.1239(0.570)
APE	-	0.0175(0.0975)	0.0798(0.1136)	94.448	192.90	193.22	196.27	194.12	0.1504(0.326)
GE	-	1.1138(0.2446)	0.2677(0.0561)	95.458	194.92	195.24	198.29	196.14	0.1584(0.268)
NH	-	0.7095(0.1764)	0.4556(0.2186)	94.745	193.49	193.81	196.87	194.71	0.1456(0.365)
W	-	0.9604(0.1089)	3.9271(0.6872)	95.511	195.02	195.35	198.40	196.24	0.1290(0.518)
G	-	1.0615(0.2101)	0.2646(0.0663)	95.532	195.06	195.39	198.41	196.28	0.1507(0.324)
				Data-II					
MAPTW	6.0494(6.6481)	1.2419(0.4767)	0.8456(0.8135)	128.29	262.58	262.88	269.87	265.51	0.0762(0.713)
APTGE	78.609(83.751)	1.8663(0.5869)	1.0131(0.0972)	131.18	268.35	268.65	275.64	271.28	0.0704(0.799)
APW	23.887(36.364)	1.5881(0.3198)	0.3736(0.2026)	128.53	263.07	263.37	270.36	265.99	0.0673(0.841)
APE	-	85.771(43.101)	0.8121(0.0612)	274.53	272.58	272.72	277.44	274.53	0.1186(0.188)
GE	-	3.5605(0.6110)	0.7579(0.0769)	139.84	283.68	283.83	288.54	285.64	0.1209(0.171)
NH	-	33.663(26.334)	0.0083(0.0066)	143.79	291.58	291.72	296.44	293.54	0.2570(0.001)
W	-	2.3747(0.2096)	2.8630(0.1375)	130.05	264.11	264.25	269.97	266.06	0.0536(0.969)
G	-	3.4923(0.5125)	1.3655(0.2166)	136.94	277.87	278.02	282.74	279.83	0.1035(0.329)



(a) Data-I



(b) Data-II

Figure 6: The QQ plots of MAPTW distribution and its competing models for Data-I and Data-II

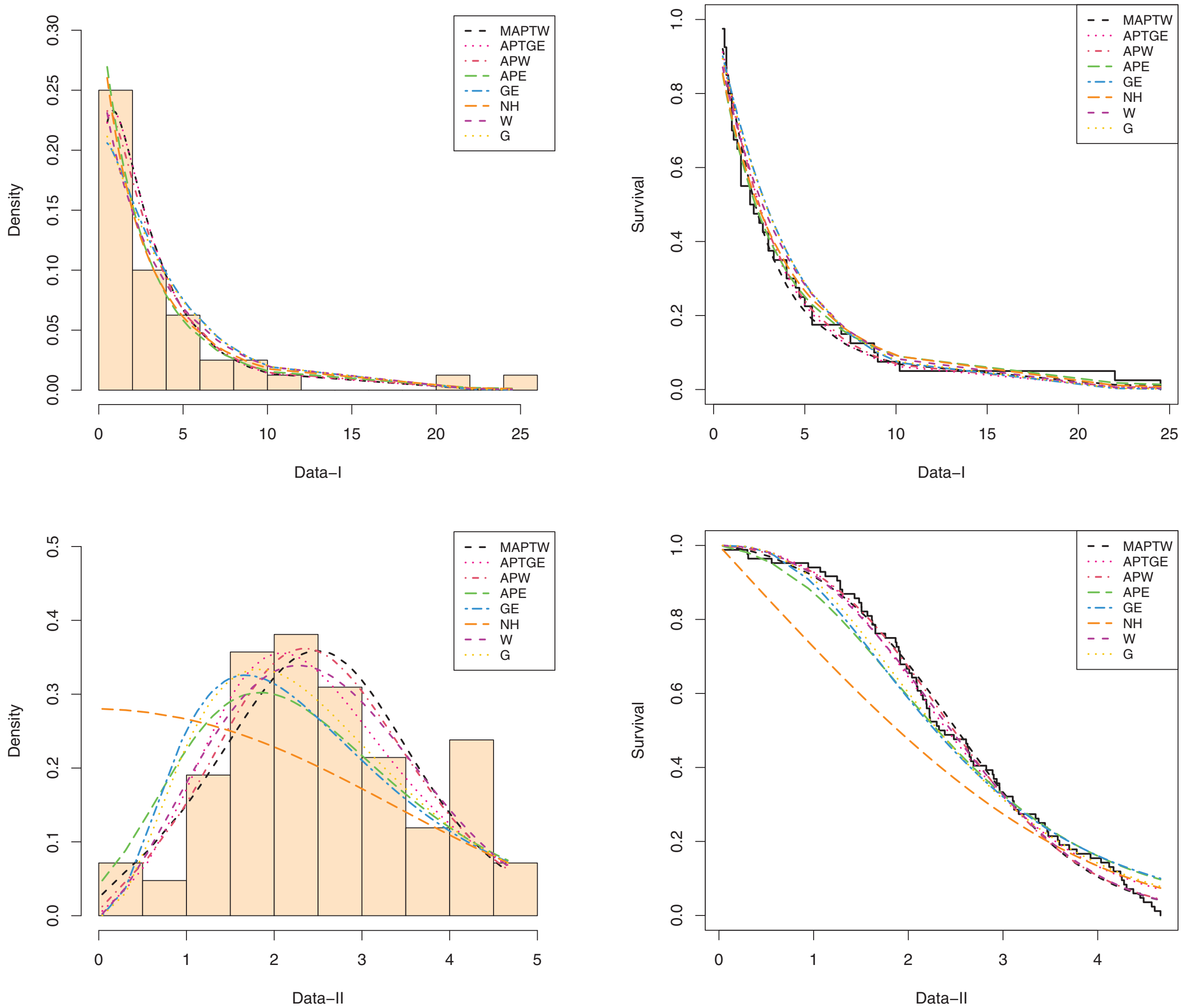


Figure 7: Histogram and fitted densities (left panel), empirical and fitted survival functions (right panel) of MAPTW distribution and some competing models

7 Conclusion

In this paper, we have offered a fresh version of Weibull distribution named the modified alpha power transformed Weibull distribution. The proposed distribution is acquired by taking the Weibull distribution as the baseline distribution in the modified alpha power transformed method. Some properties of the new distribution are derived. The hazard rate function of the new distribution can take different shapes possessing decreasing, increasing, bathtub and upside-down then bathtub shaped. Accordingly, it can be viewed perfectly effectively in modelling lifetime data. Via the maximum likelihood method, the point and interval estimates of the model parameters are evaluated. The performance of the point and interval estimates is assessed through a simulation study. The simulation yields verified that the estimates are asymptotically unbiased and consistent. In addition, two real data sets to active repair times for airborne communication transceiver and the failure times of aircraft windshields are investigated. From the empirical results, we can conclude that the new distribution provides a more adequate fit than some known distributions including gamma, Weibull and alpha power Weibull distributions. For future work, it is of interest to study the proposed distribution under different censoring schemes including Type-II and progressive Type-II censoring. Another future work is to investigate the Bayesian estimation of the unknown parameters of the suggested model.

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Conflicts of Interest: The authors declare that they have no conflicts of interest to report regarding the present study.

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