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# Quasi Controlled $\mathscr{K}$-Metric Spaces over $C^*$-Algebras with an Application to Stochastic Integral Equations

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**ABSTRACT**

Generally, the field of fixed point theory has attracted the attention of researchers in different fields of science and engineering due to its use in proving the existence and uniqueness of solutions of real-world dynamic models. $C^*$-algebra is being continually used to explain a physical system in quantum field theory and statistical mechanics and has subsequently become an important area of research. The concept of a $C^*$-algebra-valued metric space was introduced in 2014 to generalize the concept of metric space. In fact, It is a generalization by replacing the set of real numbers with a $C^*$-algebra. After that, this line of research continued, where several fixed point results have been obtained in the framework of $C^*$-algebra valued metric, as well as (more general) $C^*$-algebra-valued b-metric spaces and $C^*$-algebra-valued extended b-metric spaces. Very recently, based on the concept and properties of $C^*$-algebras, we have studied the quasi-case of such spaces to give a more general notion of relaxing the triangular inequality in the asymmetric case. In this paper, we first introduce the concept of $C^*$-algebra-valued quasi-controlled $\mathscr{K}$-metric spaces and prove some fixed point theorems that remain valid in this setting. To support our main results, we also furnish some examples which demonstrate the utility of our main result. Finally, as an application, we use our results to prove the existence and uniqueness of the solution to a nonlinear stochastic integral equation.

**KEYWORDS**

$C^*$-algebra-valued quasi controlled $\mathscr{K}$-metric spaces; left convergence; right convergence; fixed point; contraction

## 1 Introduction

One of the most relevant theories marking the passage from classical to modern analysis is the fixed point theory which was implemented by Banach [1]. Several mathematicians have created diverse generalizations of Banach fixed point theory. Wilson, on the other hand, introduced the quasi-metric space that is one of the abstractions of the metric spaces [2]. This theory, however, does not include the commutative condition. Numerous mathematicians have adopted this concept to demonstrate some fixed point outcomes, see [3].

The $b$-metric spaces concept was first set up by Bakhtin [4] and Czerwik [5]. Besides, numerous authors obtained a lot of fixed point results. For example, see [6–10]. The extended $b$-metric spaces idea was elaborated by Kamran et al. [11] and generalized by Abdeljawad et al. [12] by imposing the control or the double control of the $s$-relaxed inequality by one or two functions. Mudasir et al. [13] stated new results in the context of dislocated $b$-metric spaces and presented an application related to electrical engineering and extended the notion of Kannan maps in view of the $F$-contraction in this framework, see [14].

In [15,16], Ma et al. introduced $C^*$-algebra valued $b$-metric spaces by considering metrics that take values in the set of positive elements of a unitary $C^*$-algebra. Lately, Asim et al. [17] enlarged this class by defining $C^*$-algebra-valued extended $b$-metric spaces. Very recently, Kabbaj et al. [18] have investigated the quasi case of such a metric and they give a more general notion of relaxing the triangular inequality in the asymmetric case [19]. Recently, for some work on fixed point theory in the mentioned area, we refer to some published work as [20–34].

In this work, we introduce the notion of $C^*$-algebra-valued quasi controlled $\mathscr{K}$-metric spaces. We give basic definitions and then employ them to demonstrate fixed point results in such spaces. Examples are also provided to verify the usefulness of our main results. Finally, as an application, we verify the existence of the solution for a nonlinear stochastic integral equation in this setting.

## 2 Preliminaries

Throughout this paper, $\mathfrak{A}$ will be a unitary $C^*$-algebra with $I_{\mathfrak{A}}$ and $\sigma(\delta)$ is the spectrum of $\delta \in \mathfrak{A}$. We set

$$\mathfrak{A}_h = \{\delta \in \mathfrak{A} : \delta^* = \delta\}\ ,\ \mathfrak{A}^+ = \{\delta \in \mathfrak{A}_h : \sigma(\delta) \subset [0, +\infty[\}\,;$$

$$\mathfrak{A}_{\mathscr{I}} = \{\delta \in \mathscr{Z}_{\mathscr{I}} : \delta \succeq I_{\mathfrak{A}}\}\ ,\ \mathscr{Z}_I = \{\delta \in \mathfrak{A} : \delta\gamma = \gamma\delta\,; \forall \gamma \in \mathfrak{A}\}\,.$$

Note that $\mathfrak{A}^+$ is a cone [20], which induces a partial order $\preceq$ on $\mathfrak{A}_h$ by

$$\gamma \preceq \delta \Leftrightarrow \delta - \gamma \in \mathfrak{A}^+.$$

To prove our main results, it will be useful to introduce the following lemma.

**Lemma 2.1.** [20] Suppose that $\mathfrak{A}$ is a unital $C^*$-algebra with a unit $I_{\mathfrak{A}}$.

1. if $\gamma, \delta \in \mathfrak{A}_h$ and $\gamma \preceq \delta$, then for each $\xi \in \mathfrak{A}$, $\xi^*\gamma\xi \preceq \xi^*\delta\xi$;
2. if $\gamma, \delta \in \mathfrak{A}_h$, $\gamma, \delta \succeq 0_{\mathfrak{A}}$ and $\gamma\delta = \delta\gamma$, then $\gamma\delta \succeq 0_{\mathfrak{A}}$;
3. for all $\gamma, \delta \in \mathfrak{A}_h$, $0_{\mathfrak{A}} \preceq \gamma \preceq \delta \Leftrightarrow \|\gamma\| \leq \|\delta\|$;
4. $0 \preceq \gamma \preceq I_{\mathfrak{A}} \Leftrightarrow \|\gamma\| \leq 1$.

**Definition 2.1.** [17] Let $\Omega \neq \varnothing$ and $\Lambda : \Omega \times \Omega \to \mathfrak{A}_I$. A $C^*$-algebra-valued extended $b$-metric is a mapping $\Delta$: $\Omega \times \Omega \to \mathfrak{A}$ such that

1. $\Delta(\omega, \varpi) = 0_{\mathfrak{A}}$ if and only if $\omega = \varpi$;
2. $\Delta(\omega, \varpi) = \Delta(\varpi, \omega)$;
3. $\Delta(\omega, \varpi) \preccurlyeq \Lambda(\omega, \varpi)[\Delta(\omega, \nu) + \Delta(\nu, \varpi)]$.

The triplet $(\Omega, \mathfrak{A}, \Delta)$ is called a $C^*$-algebra valued extended $b$-metric space.

## 3 Main Results

In this section, by omitting the symmetry condition, we introduce the notion of $C^*$-algebra-valued quasi controlled $\mathscr{K}$-metric spaces, where $\mathscr{K}$ is a control function.

**Definition 3.1.** A $C^*$-algebra-valued quasi controlled $\mathscr{K}$-metric space is the triplet $(\Omega, \mathfrak{A}, \Delta)$ where $\Omega$ is a non empty set, $\mathscr{K} : \Omega \times \Omega \to \mathfrak{A}_I$ is a $C^*$-control function and $\Delta : \Omega \times \Omega \to \mathfrak{A}$ is a mapping that

1. $\Delta(\omega, \varpi) = 0_{\mathfrak{A}}$ if and only if $\omega = \varpi$;
2. $\Delta(\omega, \varpi) \preccurlyeq \mathscr{K}(\omega, \varpi)[\Delta(\omega, \nu) + \Delta(\nu, \varpi)]$ *for all* $\omega, \varpi, \vartheta \in \Omega$.

**Remark 3.1.** In particular, by taking $\mathscr{K}(\omega, \varpi) = \delta \succcurlyeq I_{\mathfrak{A}}$, $(\Omega, \mathfrak{A}, \Delta)$ is a $C^*$-algebra-valued quasi $b$-metric space [19].

**Example 3.1.** Let $\Omega = [0, 1]$ and $\mathfrak{A} = \mathbb{M}_2(\mathbb{R})$. We know that $\mathfrak{A}$ is a $C^*$-algebra where partial ordering on $\mathbb{M}_2(\mathbb{R})$ is given as

$$(\alpha_i^j)_{1\leqslant i,j\leq 2} \succeq (\beta_i^j)_{1\leqslant i,j\leq 2} \Leftrightarrow \alpha_i^j \geq \beta_i^j \text{ for } i = 1, 2.$$

Define a $C^*$-algebra-valued quasi controlled $\mathscr{K}$-metric $\Delta : \Omega \times \Omega \to \mathbb{R}^2$ by:

$$\begin{cases} \Delta(\rho, \varpi) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, & \text{iff } \rho = \varpi \\ \Delta(\rho, 0) = \Delta(0, \rho) = \begin{bmatrix} \frac{1}{\rho} & 0 \\ 0 & \frac{1}{\rho} \end{bmatrix}, & \text{if } \rho \neq 0 \\ \Delta(\rho, \varpi) = \begin{bmatrix} \frac{1+\varpi}{\rho\varpi} & 0 \\ 0 & \frac{1+\rho}{\rho\varpi} \end{bmatrix}, & \text{if } \varpi\rho \neq 0. \end{cases}$$

Given the $C^*$-control function $\mathscr{K} : \Omega \times \Omega \to \mathfrak{A}_I$ as

$$\mathscr{K}(\rho, \varpi) = \begin{bmatrix} 1 + \frac{1}{\rho + \varpi + 1} & 0 \\ 0 & 1 + \frac{1}{\rho + \varpi + 1} \end{bmatrix}.$$

Then, $(\Omega, \mathfrak{A}, \Delta)$ is a $C^*$-algebra-valued quasi controlled $\mathscr{K}$-metric space.

**Example 3.2.** Let $\Omega = [0, 1]$ and $\mathfrak{A} = M_2(\mathbb{C})$. Define a mapping $\Delta : \Omega \times \Omega \to \mathfrak{A}$ as

$$\Delta(\rho, \varpi) = \begin{bmatrix} (1 + 2|\rho| + |\varpi|)\,|\rho - \varpi|^2 & 0 \\ 0 & (1 + 2|\rho| + |\varpi|)\,|\rho - \varpi|^2 \end{bmatrix}.$$

Let the $C^*$-control function $\mathscr{K} : \Omega \times \Omega \to \mathfrak{A}$ be defined by (for all $\rho, \varpi \in \Omega$)

$$\mathscr{K}(\rho, \varpi) = 2\begin{bmatrix} 1 + 2|\rho| + |\varpi| & 0 \\ 0 & 1 + 2|\rho| + |\varpi| \end{bmatrix}.$$

**Example 3.3.** Consider $\Omega = C(\mathscr{S}, \mathbb{C})$ the space of all continuous functions where $\mathscr{S}$ is compact. Let $\mathfrak{A} = L^{\infty}(\mathscr{S})$ the usual unital $C^*$-algebra with the sup norm and given $\Delta : \Omega \times \Omega \to \mathfrak{A}^+$ for each $\varphi, \psi \in \Omega$ as

$$\Delta(\varphi, \psi)(t) = \begin{cases} 0, & \text{if } \varphi = \psi \\ 1 - \dfrac{1}{1 + |\varphi(t)|}, & \text{if } \varphi \neq 0, \psi = 0 \\ 1 - \dfrac{1}{1 + |\psi(t)|}, & \text{if } \varphi = 0, \psi \neq 0 \\ |\varphi(t)| + 2|\psi(t)|, & \text{if } \varphi\psi \neq 0 \end{cases}$$

We take

$$\mathscr{K}(\psi, \varphi)(t) = |\psi(t)| + 2|\varphi(t)| + 2.$$

Thus, $(\Omega, \Delta, L^{\infty}(\mathscr{S}))$ is a $C^*$-algebra-valued quasi controlled $\mathscr{K}$-metric space.

Next, we introduce some topological concepts on $C^*$-algebra-valued quasi controlled $\mathscr{K}$-metric spaces.

**Definition 3.2.** Let $(\Omega, \mathfrak{A}, \Delta)$ be a $C^*$-algebra-valued quasi controlled $\mathscr{K}$-metric space. The open ball $\mathscr{B}(\omega, r)$ of center $\omega \in \Omega$ and radius $r \succeq 0_{\mathfrak{A}}$ is given by

$$\mathscr{B}(\omega, r) = \{\varpi \in \Omega : \Delta(\omega, \varpi) \prec r\}.$$

**Example 3.4.** Let us define a $C^*$-algebra-valued quasi controlled $\mathscr{K}$-metric $\Delta : \mathbb{C}^* \times \mathbb{C}^* \to \mathbb{R}^2_+$ as

$$\Delta(\mathfrak{z}, \mathfrak{z}') = \begin{cases} (0, 0), & \text{if } \mathfrak{z} = \mathfrak{z} \\ \dfrac{1}{|\mathfrak{z}\mathfrak{z}'|} + \dfrac{1}{|\mathfrak{z}|}, \dfrac{1}{|\mathfrak{z}\mathfrak{z}'|} + \dfrac{1}{1|\mathfrak{z}'|}, & \text{if } \mathfrak{z} \neq \mathfrak{z}' \end{cases}$$

with the $C^*$-controlled function $\mathscr{K} : \mathbb{C}^* \times \mathbb{C}^* \to ]1, +\infty[\times]1, +\infty[$ given by

$$\mathscr{K}(\mathfrak{z}, \mathfrak{z}') = \left(\frac{1 + |\mathfrak{z}'|}{|\mathfrak{z}'|}, \frac{1 + |\mathfrak{z}|}{|\mathfrak{z}|}\right).$$

Then, it is evident that

$$\Delta(\mathfrak{z}, \mathfrak{z}') \preceq \Lambda(\mathfrak{z}, \mathfrak{z}')\left[\Delta(\mathfrak{z}, \mathfrak{z}'') + \Delta(\mathfrak{z}'', \mathfrak{z}')\right], \forall \mathfrak{z}, \mathfrak{z}', \mathfrak{z}'' \in \mathbb{C}^*.$$

The open ball $B$ is given by

$$\begin{aligned} \mathscr{B}(\mathfrak{z}_0, r.1_{\mathbb{A}}) &= B(\mathfrak{z}_0, (r, r)) \\ &= \{\mathfrak{z} \in \mathbb{C}^* : \Delta(\mathfrak{z}_0, \mathfrak{z}) \prec (r, r)\} \\ &= \{\mathfrak{z}_0\} \cup \left\{\mathfrak{z} \in \mathbb{C}^* : \mathfrak{z} \neq \mathfrak{z}_0 \text{ and } \left(\frac{1}{|\mathfrak{z}_0\mathfrak{z}|} + \frac{1}{|\mathfrak{z}_0|}, \frac{1}{|\mathfrak{z}_0\mathfrak{z}|} + \frac{1}{|\mathfrak{z}|}\right) \prec (r, r)\right\} \end{aligned}$$

$$\left(\frac{|\mathfrak{z}| + 1}{|\mathfrak{z}_0\mathfrak{z}|}, \frac{1 + |\mathfrak{z}_0|}{|\mathfrak{z}_0\mathfrak{z}|}\right) \prec (r, r) \Rightarrow \begin{cases} \dfrac{|\mathfrak{z}| + 1}{|\mathfrak{z}|} < r|\mathfrak{z}_0| \\ |\mathfrak{z}| > \dfrac{1 + |\mathfrak{z}_0|}{r|\mathfrak{z}_0|} \end{cases}$$

if $r|\mathfrak{z}_0| \leqslant 1$, then

$B(\mathfrak{z}_0, r.1_{\mathbb{A}}) = \{\mathfrak{z}_0\}$

if $r|\mathfrak{z}_0| > 1$, then

$$\mathscr{B}(\mathfrak{z}_0, r.1_{\mathbb{A}}) = \{\mathfrak{z}_0\} \cup \left\{\mathfrak{z} \in \mathbb{C}^* : |\mathfrak{z}| \in ]\max\left(\frac{1}{r|\mathfrak{z}_0| - 1} + \frac{1 + |\mathfrak{z}_0|}{|\mathfrak{z}_0|}\right), +\infty[\right\}.$$

**Remark 3.2.** We can also define the closed ball by

$\overline{\mathscr{B}}(\omega, r) = \{\varpi \in \Omega : \Delta(\omega, \varpi) \preceq r\}.$

**Definition 3.3.** Let $(\Omega, \mathfrak{A}, \Delta)$ be a $C^*$-algebra-valued quasi controlled $\mathscr{K}$-metric space and let $\{\varpi_n\}$ be a sequence in $\Omega$.

1. $\{\varpi_n\}$ is called left-converges to $\varpi \in \Omega$ with respect to $\mathfrak{A}$, if and only if $\forall \varepsilon \succ 0_{\mathfrak{A}} \exists k \in \mathbb{N}$ such that

   $n > k \Rightarrow \Delta(\varpi_n, \varpi) \prec \varepsilon.$

2. $\{\varpi_n\}$ is called right-converges to $\varpi \in \Omega$ with respect to $\mathfrak{A}$, if and only if $\forall \varepsilon \succ 0_{\mathfrak{A}} \exists k \in \mathbb{N}$ such that

   $n > k \Rightarrow \Delta(\varpi, \varpi_n) \prec \varepsilon.$

3. $\{\varpi_n\}$ is called converges to $\varpi \in \Omega$ with respect to $\mathfrak{A}$, if and only if

   $\lim_{n\to\infty} \Delta(\varpi, \varpi_n) = \lim_{n\to\infty} \Delta(\varpi_n, \varpi) = 0_{\mathfrak{A}}.$

**Definition 3.4.** Let $(X, \mathfrak{A}, \Delta)$ be a $C^*$-algebra-valued quasi controlled $\mathscr{K}$-metric space. Then

1. $\{\varpi_n\}$ is called right-Cauchy with respect to $\mathfrak{A}$, if for each $\varepsilon \succ 0_{\mathfrak{A}}$ there exists $k \in \mathbb{N}$ such that $\forall p \in \mathbb{N}$,

   $n > k \Rightarrow \Delta\left(\varpi_n, \varpi_{n+p}\right) \prec \varepsilon.$

2. $\{\varpi_n\}$ is called left-Cauchy with respect to $\mathfrak{A}$, if for each $\varepsilon \succ 0_{\mathfrak{A}}$ there exists $k \in \mathbb{N}$ such that $\forall p \in \mathbb{N}$,

   $n > k \Rightarrow \Delta\left(\varpi_{n+p}, \varpi_n\right) \prec \varepsilon.$

3. $\{\varpi_n\}$ is called Cauchy sequence with respect to $\mathfrak{A}$ if and only if $\forall p \in \mathbb{N}$,

   $\lim_{n\to\infty} \Delta\left(\varpi_n, \varpi_{n+p}\right) = \lim_{n\to\infty} \Delta\left(\varpi_{n+p}, \omega_n\right) = 0_{\mathfrak{A}}.$

4. If every Cauchy sequence $\{\varpi_n\}$ in $\Omega$ converges to some point $\varpi$ in $\Omega$, then, the triplet $(\Omega, \mathfrak{A}, \Delta)$ is said to be a complete $C^*$-algebra-valued quasi controlled $\mathscr{K}$-metric space.

**Example 3.5.** Take $\Omega = \mathbb{R}^+$ and $\mathfrak{A} = \mathbb{R}^2$

$$\Delta(\eta,\nu) = \begin{cases} (0,0), & \text{if } \eta = \nu \\ \left(\dfrac{\eta}{1+\eta}, \dfrac{\eta}{1+\eta}\right), & \text{if } \eta \neq 0, \nu = 0 \\ \left(\dfrac{\nu}{1+\nu}, \dfrac{\nu}{1+\nu}\right), & \text{if } \eta = 0, \nu \neq 0 \\ (\eta + 2\nu, \eta + 2\nu), & \text{if } \eta\nu \neq 0,\ \eta \neq \nu \end{cases}$$

Let $\mathscr{K} : \Omega \times \Omega \to \mathfrak{A}_I$ be the mapping defined by

$$\mathscr{K}(\eta,\nu) = (2\eta + 2\nu + 2, 2\eta + 2\nu + 2).$$

Then, $(\Omega, \mathfrak{A}, \Delta)$ is a complete $C^*$-algebra-valued quasi controlled $\mathscr{K}$-metric space.

**Example 3.6.** Let $\mathscr{S}$ be a compact Hausdorff space and $\mathfrak{A} = C(\mathscr{S})$ be the set of complex valued continuous functions on $\mathscr{S}$. Note that $C(\mathscr{S})$ is a unitary commutative $C^*$-algebra with the usual sup norm such that the involution is defined by $\psi^*(x) = \overline{\psi(x)} \textit{for all}\, x \in S$. Setting $\Omega = L_\infty(E)$ where $E$ is a Lebesgue mensurable set and let us define a $C^*$-algebra-valued quasi controlled $\mathscr{K}$-metric $\Delta : \Omega \times \Omega \to \mathfrak{A}$ by

$$\Delta(\phi,\psi)(t) = (1 + \|\phi\|_\infty + 2\|\psi\|_\infty)\|\phi - \psi\|_\infty e^t \quad \textit{for all}\, \phi, \psi \in \Omega\,;\ t \in [0,1].$$

Let us define the $C^*$-control operator by

$$\mathscr{K}(\phi,\psi) = (1 + \|\phi\|_\infty + 2\|\psi\|_\infty)I_{\mathfrak{A}}.$$

The condition ($i$) of Definition 3.1 is clearly satisfied by $\Delta$. Now we check the condition ($ii$). We take $\phi, \psi \in \Omega$ as arbitrary. Then

$$\begin{aligned} \Delta(\phi,\psi)(t) &= (1 + \|\phi\|_\infty + 2\|\psi\|_\infty)\|\phi - \psi\|_\infty e^t \\ &\leq (1 + \|\phi\|_\infty + 2\|\psi\|_\infty)(\|\phi - \varphi\|_\infty + \|\varphi - \psi\|_\infty)e^t \\ &\leq \mathscr{K}(\phi,\psi)\,[\Delta(\phi,\psi)(t) + \Delta(\varphi,\psi)(t)] \text{ for all } t \in [0,1]. \end{aligned}$$

Therefore,

$$\Delta(\phi,\psi) \preceq \mathscr{K}(\phi,\psi)\,(\Delta(\phi,\varphi) + \Delta(\varphi,\psi))\,\textit{for all}\, \phi, \psi\ \varphi \in \Omega.$$

This prove that $\Delta$ is a $C^*$-algebra-valued quasi controlled $\mathscr{K}$-metric. Now we want to verify that $(X, \mathfrak{A}, \Delta)$ is a complete $C^*$-algebra-valued quasi controlled $\mathscr{K}$-metric space. Let $\{\phi_n\}_{n=1}^\infty$ be a Cauchy sequence in $\Omega$ with respect to $\mathfrak{A}$. Then

$$\lim_{n\to\infty} \Delta\left(\phi_n, \phi_{n+p}\right) = \lim_{n\to\infty} \Delta\left(\phi_{n+p}, \phi_n\right) = 0_{\mathfrak{A}}.$$

We deduce $\lim\limits_{n\to\infty} \|\phi_{n+p} - \phi_n\|_\infty = 0$, so $\{\phi_n\}_{n=1}^\infty$ is a Cauchy sequence in the space $\Omega$. Since $\Omega$ is complete, $\{\phi_n\}$ has a limit $\widetilde{\phi}$ that is also in $\Omega$. Hence it follows that

$$\Delta\left(\phi_n, \widetilde{\phi}\right) \preceq e\left(1 + \|\phi_n\|_\infty + 2\|\widetilde{\phi}\|_\infty\right)\|\phi_{n+p} - \phi_n\|_\infty I_{\mathfrak{A}}$$

and

$$\Delta\left(\widetilde{\phi}, \phi_n\right) \preceq e\left(1 + \|\widetilde{\phi}\|_\infty + 2\|\phi_n\|_\infty\right)\|\phi_{n+p} - \phi_n\|_\infty I_{\mathfrak{A}}.$$

We conclude that the sequence $\{\phi_n\}_{n=1}^{\infty}$ converges to the function $\widetilde{\phi}$ in $\Omega$ respecting $\mathfrak{A}$.

We will fix the notion of a continuous metric in the context presented in this paper since in the literature during the proof of the results in fixed point certain problems arise due to the possible discontinuity of the $b$-metric with respect to the topology it generates.

**Definition 3.5.** Let $\Delta$ be a $C^*$-algebra-valued quasi controlled $\mathscr{K}$-metric. $\Delta$ is said to be continuous at $(\varpi, \omega)$ if the sequence $\{\omega_n\}_{n=0}^{\infty}$ converges to $\omega$ and $\{\varpi_n\}_{n=0}^{\infty}$ converges to $\varpi$ then

$$\Delta(\omega_n, \varpi_n) \to \Delta(\omega, \varpi) and \quad \Delta(\varpi_n, \omega_n) \to \Delta(\varpi, \omega).$$

**Lemma 3.1.** Let $(\Omega, \mathfrak{A}, \Delta)$ be a $C^*$-algebra-valued quasi controlled $\mathscr{K}$-metric space. Such $\Delta$ is continuous in each variable. If a sequence $\{\omega_n\}_{n=1}^{\infty}$ has a limit, then this limit is unique.

**Proof.** Fix $\varepsilon \succ 0_{\mathbb{A}}$. By assumption, $\varpi_n$ converges to $\omega$ so there exists $K_1 \in \mathbb{N}$ such that $d(\omega, \varpi_n) \preceq \frac{\varepsilon}{2}$ for all $n \geqslant K_1$. We also assume that $\varpi_n$ converges to $\varpi$, so there exists $K_2 \in \mathbb{N}$ such that $d(\varpi_n, \varpi) \preceq \frac{\varepsilon}{2}$ for all $n \geqslant k_2$. Then for all $n \geqslant K := \max\{K_1, K_2\}$

$$\Delta(\varpi, \omega) \preceq \mathscr{K}(\varpi, \omega)[\Delta(\varpi, \varpi_n) + \Delta(\varpi_n, \omega)] \preceq \mathscr{K}(\varpi, \omega)\varepsilon.$$

As $\varepsilon$ was arbitrary, we deduce that $\Delta(\varpi, \omega) = 0$, which implies $\varpi = \omega$.

Our main result runs as follows.

**Theorem 3.1.** Let $(\Omega, \mathfrak{A}, \Delta)$ be complete $C^*$-algebra-valued quasi controlled $\mathscr{K}$-metric space such that $\Delta$ is a continuous and $\Gamma : \Omega \to \Omega$ satisfies the following:

$$\Delta(\Gamma\varpi, \Gamma\rho) \preccurlyeq \theta^*\Delta(\varpi, \rho)\theta, \quad \forall \varpi, \rho \in \Omega \tag{1}$$

where $\theta \in \mathfrak{A}$ with $\|\theta\|_{\mathfrak{A}} < 1$ and $\lim_{n,m\to\infty} \|\mathscr{K}(\varpi_n, \varpi_m)\|_{\mathfrak{A}}\|\theta\|_{\mathfrak{A}} \prec I_{\mathfrak{A}}$ such that $\omega_n = \Gamma\varpi_{n-1} = \Gamma^n\varpi_0$ for an arbitrary $\varpi_0$. Then $\Gamma$ has a unique fixed point $\widetilde{\omega} \in \Omega$.

**Proof.** Let the sequence $\{\varpi_n\}$ be defined by $\varpi_n = \Gamma\varpi_{n-1} = \Gamma^n\varpi_0$. From Eq. (1), we obtain by induction

$$\begin{aligned}\Delta(\varpi_n, \varpi_{n+1}) &= \Delta(\Gamma\varpi_{n-1}, \Gamma\varpi_n) \preceq \theta^*\Delta(\varpi_{n-1}, \varpi_n)\theta \\ &\preccurlyeq (\theta^*)^2\Delta(\varpi_{n-2}, \varpi_{n-1})\theta^2 \\ &\quad\vdots \\ &\preccurlyeq (\theta^*)^n\Delta(\varpi_0, \varpi_1)\theta^n.\end{aligned}$$

Now we prove that $\{\varpi_n\}$ is a right-Cauchy sequence. For any $n, p \in \mathbb{N}$, we have

$$\begin{aligned}\Delta\left(\varpi_n, \varpi_{n+p}\right) &\preccurlyeq \mathscr{K}\left(\varpi_n, \varpi_{n+p}\right)\left[\Delta(\varpi_n, \varpi_{n+1}) + \Delta\left(\varpi_{n+1}, \varpi_{n+p}\right)\right] \\ &\preccurlyeq \mathscr{K}\left(\varpi_n, \varpi_{n+p}\right)\Delta(\varpi_n, \varpi_{n+1}) + \mathscr{K}\left(\varpi_n, \varpi_{n+p}\right)\mathscr{K}\left(\varpi_{n+1}, \varpi_{n+p}\right)\Delta(\varpi_{n+1}, \varpi_{n+2}) \\ &\quad\vdots \\ &\quad\mathscr{K}\left(\varpi_n, \varpi_{n+p}\right)\mathscr{K}\left(\varpi_{n+1}, \varpi_{n+p}\right)\ldots\mathscr{K}\left(\varpi_{n+p-2}, \varpi_{n+p}\right)\mathscr{K}\left(\varpi_{n+p-1}, \varpi_{n+p}\right) \\ &\quad\Delta\left(\varpi_{n+p-1}, \varpi_{n+p}\right) \\ &\preccurlyeq \mathscr{K}\left(\varpi_n, \varpi_{n+p}\right)(\theta^*)^n\mathscr{K}(\varpi_0, \varpi_1)\theta^n \\ &\quad + \mathscr{K}\left(\varpi_n, \varpi_{n+p}\right)\mathscr{K}\left(\varpi_{n+1}, \varpi_{n+p}\right)(\theta^*)^{n+1}\mathscr{K}(\varpi_0, \varpi_1)\theta^{n+1}\end{aligned}$$

$$
\begin{aligned}
&\vdots\\
&\mathscr{K}\left(\varpi_n,\varpi_{n+p}\right)\mathscr{K}\left(\varpi_{n+1},\varpi_{n+p}\right)\ldots\mathscr{K}\left(\varpi_{n+p-2},\varpi_{n+p}\right)\\
&+\mathscr{K}\left(\varpi_{n+p-1},\varpi_{n+p}\right)(\theta^*)^{n+p-1}\Delta\left(\varpi_0,\varpi_1\right)\theta^{n+p-1}\\
&=\mathscr{K}\left(\varpi_n,\varpi_{n+p}\right)(\theta^*)^{n}\left(\Delta\left(\varpi_0,\varpi_1\right)^{\frac{1}{2}}\right)^2\theta^{n}\\
&+\mathscr{K}\left(\varpi_n,\varpi_{n+p}\right)\mathscr{K}\left(\varpi_{n+1},\varpi_{n+p}\right)(\theta^*)^{n+1}\left(\Delta\left(\varpi_0,\varpi_1\right)^{\frac{1}{2}}\right)^2\theta^{n+1}\\
&\vdots\\
&+\mathscr{K}\left(\varpi_n,\varpi_{n+p}\right)\mathscr{K}\left(\varpi_{n+1},\varpi_{n+p}\right)\ldots\mathscr{K}\left(\varpi_{n+p-2},\varpi_{n+p}\right)\mathscr{K}\left(\varpi_{n+p-1},\varpi_{n+p}\right)\\
&(\theta^*)^{n+p-1}\left(\mathscr{K}\left(\varpi_0,\varpi_1\right)^{\frac{1}{2}}\right)^2\theta^{n+p-1}\\
&=\mathscr{K}\left(\varpi_n,\varpi_{n+p}\right)\left(\Delta\left(\varpi_0,\varpi_1\right)^{\frac{1}{2}}\theta^{n}\right)^*\left(\Delta\left(\varpi_0,\varpi_1\right)^{\frac{1}{2}}\theta^{n}\right)+\\
&\vdots\\
&\mathscr{K}\left(\varpi_n,\varpi_{n+p}\right)\mathscr{K}\left(\varpi_{n+1},\varpi_{n+p}\right)\ldots\mathscr{K}\left(\varpi_{n+p-1},\varpi_{n+p}\right)\left(\Delta\left(\varpi_0,\varpi_1\right)^{\frac{1}{2}}\theta^{n+p-1}\right)^*\\
&\left(\Delta\left(\varpi_0,\varpi_1\right)^{\frac{1}{2}}\theta^{n+p-1}\right)\\
&=\mathscr{K}\left(\varpi_n,\varpi_{n+p}\right)\left|\Delta\left(\varpi_0,\varpi_1\right)^{\frac{1}{2}}\theta^{n}\right|^2+\mathscr{K}\left(\varpi_n,\varpi_{n+p}\right)\mathscr{K}\left(\varpi_{n+1},\varpi_{n+p}\right)\left|\Delta\left(\varpi_0,\varpi_1\right)^{\frac{1}{2}}\theta^{n+1}\right|^2\\
&\vdots\\
&\mathscr{K}\left(\varpi_n,\varpi_{n+p}\right)\mathscr{K}\left(\varpi_{n+1},\varpi_{n+p}\right)\ldots\mathscr{K}\left(\varpi_{n+p-2},\varpi_{n+p}\right)\mathscr{K}\left(\varpi_{n+p-1},\varpi_{n+p}\right)\\
&\left|\Delta\left(\varpi_0,\varpi_1\right)^{\frac{1}{2}}\theta^{n+p-1}\right|^2\\
&=\sum_{i=0}^{n+p-1}\left|\Delta\left(\varpi_0,\varpi_1\right)^{\frac{1}{2}}\theta^{n+i}\right|^2\prod_{j=0}^{i}\mathscr{K}\left(\varpi_{n+p+j},\varpi_{n+p}\right)\\
&\preccurlyeq\left\|\sum_{i=0}^{n+p-1}\left|\Delta\left(\varpi_0,\varpi_1\right)^{\frac{1}{2}}\theta^{n+i}\right|^2\right\|_{\mathfrak{A}}\prod_{j=0}^{i}\|\mathscr{K}\left(\varpi_{n+j},\varpi_{n+p}\right)\|_{\mathfrak{A}}I_{\mathfrak{A}}\\
&\preccurlyeq\sum_{i=0}^{n+p-1}\|\Delta\left(\varpi_0,\varpi_1\right)\|_{\mathfrak{A}}\left\|\theta^{n+i}\right\|_{\mathfrak{A}}^2\prod_{j=0}^{i}\|\mathscr{K}\left(\varpi_{n+j},\varpi_{n+p}\right)\|_{\mathfrak{A}}I_{\mathfrak{A}}\\
&\preccurlyeq\|\Delta\left(\varpi_0,\varpi_1\right)\|_{\mathfrak{A}}\sum_{i=0}^{n+p-1}\left\|\theta^{n+i}\right\|_{\mathfrak{A}}^2\prod_{j=0}^{i}\|\mathscr{K}\left(\varpi_{n+j},\varpi_{n+p}\right)\|_{\mathfrak{A}}I_{\mathfrak{A}}
\end{aligned}
$$

Since $\lim_{n,m\to\infty} \|\mathscr{K}(\varpi_n, \varpi_m)\|_{\mathfrak{A}} \|\theta\|_{\mathfrak{A}} < 1$ so that the series $\sum_{n=1}^{\infty} \|\theta^n\|_{\mathfrak{A}} \prod_{i=1}^{n} \|\mathscr{K}(\varpi_i, \varpi_m)\|_{\mathfrak{A}}$ converges by ratio test for each $m \in \mathbb{N}$. Let

$$\mathscr{V}_n = \sum_{i=0}^{n} \left|\left|\theta^i\right|\right|_{\mathfrak{A}}^2 \prod_{j=0}^{i} \|\mathscr{K}\left(\varpi_j, \varpi_m\right)\|_{\mathfrak{A}} \text{ and } \mathscr{V} = \sum_{i=0}^{\infty} \left|\left|\theta^i\right|\right|_{\mathfrak{A}}^2 \prod_{j=0}^{i} \|\mathscr{K}\left(\varpi_j, \varpi_m\right)\|_{\mathfrak{A}}$$

Thus, the above inequality implies

$$\Delta\left(\varpi_n, \varpi_{n+p}\right) \preccurlyeq \|\Delta(\varpi_0, \varpi_1)\|_{\mathfrak{A}} \left|\left|\theta^{2n}\right|\right|_{\mathfrak{A}} \left[\mathscr{V}_{n+p-1} - \mathscr{V}_n\right].$$

Letting $n \to \infty$, we conclude that $\{\varpi_n\}$ is a right-Cauchy sequence. Similarly, we prove that $\{\varpi_n\}$ is a left-Cauchy sequence. The fact that $\Omega$ is complete involves $\exists\widetilde{\omega} \in \Omega$ such that

$$\lim_{n\to\infty} \Delta(\widetilde{\omega}, \omega_n) = \lim_{n\to\infty} \Delta(\omega_n, \widetilde{\omega}) = 0_{\mathfrak{A}}.$$

Remains to see that $\widetilde{\omega}$ is a fixed point of $\Gamma$. Indeed for any $n \in \mathbb{N}$, we have

$$\begin{aligned}
\Delta(\Gamma\widetilde{\omega}, \widetilde{\omega}) &\preccurlyeq \mathscr{K}(\Gamma\widetilde{\omega}, \widetilde{\omega})[\Delta(\Gamma\widetilde{\omega}, \varpi_{n+1}) + \Delta(\varpi_{n+1}, \widetilde{\omega})] \\
&= \mathscr{K}(\Gamma\widetilde{\omega}, \widetilde{\omega})[\Delta(\Gamma\widetilde{\omega}, \Gamma\varpi_n) + \Delta(\varpi_{n+1}, \widetilde{\omega})] \\
&\preccurlyeq \mathscr{K}(\mathscr{T}\widetilde{\omega}, \widetilde{\omega})[\theta^*\Delta(\widetilde{\omega}, \varpi_n)\theta + \Delta(\varpi_{n+1}, \widetilde{\omega})] \\
&\to 0_{\mathbb{A}} \text{ as } n \to \infty.
\end{aligned}$$

Therefore, $\widetilde{\omega}$ is a fixed point of $\Gamma$. To prove uniqueness, we can assume $\Gamma\widetilde{\omega} = \widetilde{\omega}$ and $\Gamma\omega^* = \omega^*$ such that $\widetilde{\omega}, \omega^* \in \Omega$. Then by employing Eq. (1), we have

$$\Delta(\widetilde{\omega}, \omega^*) = \Delta(\Gamma\widetilde{\omega}, \Gamma\omega^*) \preccurlyeq \theta^*\Delta(\widetilde{\omega}, \omega^*)\theta,$$

so that

$$\begin{aligned}
||\Delta(\widetilde{\omega}, \omega^*)||_{\mathfrak{A}} &= ||\Delta(\Gamma\widetilde{\omega}, \Gamma\omega^*)||_{\mathfrak{A}} \\
&\leq ||\theta^*\Delta(\widetilde{\omega}, \omega^*)\theta||_{\mathfrak{A}} \\
&\leq ||\theta^*|| \, ||\Delta(\widetilde{\omega}, \omega^*)|| \, \|\theta\|_{\mathfrak{A}} \\
&= \|\theta\|_{\mathfrak{A}}^2 ||\Delta(\widetilde{\omega}, \omega^*)||_{\mathfrak{A}} \\
&< ||\Delta(\widetilde{\omega}, \omega^*)||_{\mathfrak{A}}.
\end{aligned}$$

Then, we get a contradiction, as a result $\omega = \omega^*$.

Dynamic programming is a powerful technique for solving some complex problems in computer sciences. We illustrate Theorem 3.2 by studying the existence and uniqueness of the solutions of the functional equation presented in the following example.

**Example 3.7.** Let $X$ and $Y$ be Banach spaces. $S \subset X$ is the state space and $D \subset Y$ is the decision space. Let $\eta : S \times D \to S$, $\tau : S \times D \to \mathbb{R}$ and $\mathscr{T} : S \times D \times \mathbb{R} \to \mathbb{R}$. Denote by $B(S)$ the set of all real-valued bounded functions on $S$. Let $\mathfrak{A} = L^{\infty}(\mathscr{S})$ the usual unital $C^*$-algebra with the sup norm and given $\Delta : B(S) \times B(S) \to \mathfrak{A}^+$ for each $\varphi, \psi \in \Omega$ as

$$\Delta(\varphi, \psi) = \frac{1 + \|\varphi\| + \|\psi\|}{1 + \|\varphi\|} \|\varphi - \psi\|.I_{\mathfrak{A}}$$

$(B(S), \Delta, L^{\infty}(\mathscr{S}))$ is a complete $C^*$-algebra-valued quasi controlled $\mathscr{K}$-metric space. We consider the functional equation

$$\varpi(x) = \sup_{y \in D}[\tau(x, y) + \mathscr{T}(x, y, \varpi(\eta(x, y)))]\,(x \in S) \tag{2}$$

such that $\tau$ and $\mathscr{T}$ are bounded and

$$|\mathscr{T}(x, y, z_1) - \mathscr{T}(x, y, z_2)| \leqslant \frac{\alpha}{1 + 2\mathfrak{m}}|z_1 - z_2|$$

for all $(x, y, z_1), (x, y, z_2)$ in $S \times D \times \mathbb{R}$, where $0 \le \alpha < 1$ and $\mathfrak{m} = \|\mathscr{T}\|$. We define a mapping $\Gamma : B(S) \to B(S)$ by $\Gamma\varpi = \mathfrak{h}$, where

$$\mathfrak{h}(x) = \sup_{y \in D}[\tau(x, y) + \mathscr{T}(x, y, \varpi(\eta(x, y)))]\,(x \in S).$$

It is easy to get $\Delta(\Gamma\rho, \Gamma\varpi) \preccurlyeq \theta^*\Delta(\rho, \varpi)\theta$ satisfies with $\theta = \sqrt{\alpha}I_{\mathfrak{A}}$.

Therefore, the Eq. (1) possesses unique bounded solution on $S$.

**Example 3.8.** Let $\Omega = \mathbb{R}$ and $\mathfrak{A} = M_2(\mathbb{C})$. For any $A \in \mathfrak{A}$, we define its norm as $\|A\|_{\mathfrak{A}} = \max_{1 \le i \le 4}|a_i|$. Define a mapping $\Delta : \Omega \times \Omega \to \mathfrak{A}$ such that for all $\rho$ and $\varpi \in \Omega$,

$$\Delta(\rho, \varpi) = \begin{bmatrix} (1 + 2|\rho| + |\varpi|)|\rho - \varpi|^2 & 0 \\ 0 & (1 + 2|\rho| + |\varpi|)|\rho - \varpi|^2 \end{bmatrix}.$$

Let the $C^*$-control function $\mathscr{K} : \Omega \times \Omega \to \mathfrak{A}$ by:

$$\mathscr{K}(\rho, \varpi) = 2\begin{bmatrix} 1 + 2|\rho| + |\varpi| & 0 \\ 0 & 1 + 2|\rho| + |\varpi| \end{bmatrix}.$$

We define a mapping $\Gamma : \Omega \to \Omega$ by

$$\Gamma\rho = \frac{\rho}{3}, \text{ for all } \rho \in \Omega.$$

It is easy to get $\Delta(\Gamma\rho, \Gamma\varpi) \preccurlyeq \theta^*\Delta(\rho, \varpi)\theta$

where $\theta = \begin{bmatrix} \frac{\sqrt{3}}{3} & 0 \\ 0 & \frac{\sqrt{3}}{3} \end{bmatrix} \in \mathfrak{A}$ and $\|\theta\|_{\mathfrak{A}} = \dfrac{\sqrt{3}}{3} = \dfrac{1}{\sqrt{3}} < 1.$

**Definition 3.6.** Let $\Omega \neq \emptyset$ and $\mathscr{O}_\Gamma(\varpi_0) = \{\Gamma^n\varpi_0 \,|\, n \in \mathbb{N}\}$ for an arbitrary $\varpi_0 \in \Omega$. A function $\Phi : \Omega \to \mathfrak{A}$ is said to be $\Gamma$-orbitally lower semi continuous at $\varpi$ with respect to $\mathfrak{A}$ if the sequence $\{\varpi_n\}$ in $\mathscr{O}_\Gamma(\varpi_0)$ is such that $\lim_{n\to\infty}\varpi_n = \varpi$ with respect to $\mathfrak{A}$ implies

$$||\Phi(\varpi)||_{\mathfrak{A}} \leqslant \liminf ||\Phi(\varpi_n)||_{\mathfrak{A}}.$$

**Definition 3.7.** Let $(\Omega, \mathfrak{A}, \Delta)$ be a $C^*$-algebra valued quasi controlled $\mathscr{K}$-metric space. $\Gamma : \Omega \to \Omega$ is a $C^*$-left-contractive (respectively $C^*$-right-contractive mapping) if there exists $\rho \in \Omega$ and an $\delta \in \mathfrak{A}$ such that

$$\Delta\left(\Gamma\varpi, \Gamma^2\varpi\right) \preceq \delta^*\Delta(\varpi, \Gamma\varpi)\delta \quad (\text{respectively } \Delta\left(\Gamma\varpi, \Gamma^2\varpi\right) \preceq \delta^*\Delta(\Gamma, \varpi)\delta) \tag{3}$$

with $\|\delta\| < 1$ for every $\varpi \in \mathscr{O}_\Gamma(\rho)$.

**Theorem 3.2.** Let $(\Omega, \mathfrak{A}, \Delta)$ be a complete $C^*$-algebra valued quasi controlled $\mathscr{K}$-metric space such that $\Delta$ is continuous. Suppose that $\Gamma : \Omega \to \Omega$ is $C^*$-left-contractive for some $\delta \in \mathfrak{A}$, $\varpi_0 \in \Omega$ and $\lim_{n,m\to\infty}\mathscr{K}(\varpi_n, \varpi_m)$ exists for every $\{\omega_n\} \in \mathscr{O}_\Gamma(\varpi_0)$ such that $\lim_{n,m\to\infty}\|\mathscr{K}(\varpi_n, \varpi_m)\|_{\mathfrak{A}} < \frac{1}{\|\delta\|_{\mathfrak{A}}}$. Then

$\Gamma^n\varpi_0 \to \widetilde{\omega} \in \Omega$ as $n \to \infty$. Besides $\widetilde{\omega}$ is a fixed point of $\Gamma$ if and only if $\varpi \to \Delta(\varpi, \Gamma\varpi)$ is $\Gamma$-orbitally l.s.c at $\widetilde{\omega}$.

**Proof.** Similar to Theorem 3.1, we prove that $\{\varpi_n\}$ is a Cauchy sequence. Since $\Omega$ is complete then $\varpi_n \to \widetilde{\omega} \in \Omega$. Assume that $\varpi \to \Delta(\varpi, \Gamma\varpi)$ is $\Gamma$-orbitally l.s.c at $\widetilde{\omega}$, we obtain

$$\begin{aligned}\|\Delta(\widetilde{\omega}, \Gamma\widetilde{\omega})\|_{\mathfrak{A}} &\leq \liminf_{n\to\infty}\|\Delta\left(\Gamma^n\varpi_0, \Gamma^{n+1}\varpi_0\right)\|_{\mathfrak{A}} \\ &\leq \liminf_{n\to\infty}\|\delta^*\|_{\mathfrak{A}}\|\Delta\left(\Gamma^{n-1}\varpi_0, \Gamma^n\varpi_0\right)\|_{\mathfrak{A}}\|\delta\|_{\mathfrak{A}} \\ &\leq \liminf_{n\to\infty}\|\delta\|_{\mathfrak{A}}^{2n}\|\Delta(\varpi_0, \varpi_1)\|_{\mathfrak{A}} \to 0\end{aligned}$$

We find $\Delta(\widetilde{\omega}, \Gamma\widetilde{\omega}) = 0$. It follows that $\Gamma\widetilde{\omega} = \widetilde{\omega}$. Conversely, let $\widetilde{\omega} = \Gamma\widetilde{\omega}$ and $\{\varpi_n\}$ a sequence in $\mathscr{O}_\Gamma(\varpi_0)$ with $\varpi_n \to \widetilde{\omega}$. Then

$$\|\Delta(\widetilde{\omega}, \Gamma\widetilde{\omega}\|_{\mathfrak{A}}) = 0 \leq \liminf_{n\to\infty}\|\Delta(\varpi_n, \Gamma\varpi_n\|_{\mathfrak{A}}),$$

and this completes the proof.

## 4 Application

By applying the previous results and involving the $C^*$-algebra valued quasi controlled $\mathscr{K}$-metric space, we prove the existence and uniqueness of a solution of a nonlinear stochastic integral equation given by

$$\varkappa(\tau;\omega) = \Lambda(\tau;\omega) + \int_{\mathbb{R}} \Theta(\tau;\xi;\omega)\vartheta(\xi;\varkappa(\xi;\omega))d\xi \quad \tau \in \mathbb{R},\, \omega \in \Sigma, \tag{4}$$

where

1. $\Sigma$ is the support of a complete probability space;
2. $(\Sigma, \mathscr{A}, \mathscr{P})$, $\Lambda(\tau, \omega)$ is the continuous stochastic free where $\|\Lambda(\tau;.)\|_{L_2(\Sigma,\mathscr{A},\mathscr{P})} < \infty$;
3. $\Theta(\tau, \xi, \omega)$ is the stochastic kernel where $\Theta(\tau, s; .)$ belongs to $L_\infty(\Sigma, \mathscr{A}, \mathfrak{P})$ such that

   $$\sup_{\tau\in\mathbb{R}} \int_{\mathbb{R}} \|\Theta(\tau;\xi;\omega)\|_{L_\infty(\Sigma,\mathscr{A},\mathscr{P})}d\xi < \infty;$$
4. $\varkappa(\tau, \omega)$ is the unknown continuous real-valued stochastic process such that

   $$\|\varkappa(\tau;.)\|_{L_2(\Sigma,\beta,\mathfrak{P})} < \infty.$$

Let $\mathscr{E}$ be the space of all continuous functions from $\mathbb{R}$ into the space $L_2(\Sigma, \mathscr{A}, \mathfrak{P})$ such that $g(\tau, .) \in L_2(\Sigma, \mathscr{A}, \mathscr{P})$, $\|g(\tau;.)\|_{L_2(\Sigma,\mathscr{A},\mathscr{P})} < \infty$ and $\tau \to g(\tau, .)$ is continuous from $\mathbb{R}$ into $L_2(\Sigma, \mathscr{A}, \mathscr{P})$ for every $g \in \mathscr{E}$.

We consider $\mathscr{E}_{\mathscr{B}} = \left\{\varkappa \in C(\mathbb{R}, L_2(\Sigma, \beta, \mathfrak{P})) : \|\varkappa(\tau, \Sigma)\|_{\mathscr{E}_{\mathscr{B}}} = \sup_{\tau\in\mathbb{R}}\|\varkappa(\tau, \Sigma)\|_{L_2(\Sigma,\mathscr{A},\mathscr{P})} < \infty\right\}$.
Now, we define the integral operator $\Psi$ on $\mathscr{E}_{\mathscr{B}}$ by

$$(\Psi\varkappa)(\tau;\omega) = \int_{\mathbb{R}} \Theta(\tau;s;\omega)\varkappa(s;\omega)d(s)$$

We now claim $(\Psi\varkappa)(\tau;\omega))$ is bounded and continuous in mean-square. Indeed

$$\begin{aligned}\|(\Psi\varkappa)(\tau;\omega)\|_{L_2(\Sigma,\mathscr{A},\mathscr{P})} &\leq \int_{\mathbb{R}} \|\Theta(\tau;\xi;\omega)\varkappa(\xi,\omega)\|_{L_2(\Sigma,\mathscr{A},\mathscr{P})}d\xi \\ &\leq \sup_{\xi\in\mathbb{R}}\|\varkappa(\xi,\omega)\|_{L_2(\Sigma,\mathscr{A},\mathscr{P})}\int_{\mathbb{R}}\|\Theta(\tau;\xi;\omega)\|_{L_\infty(\Omega,\mathscr{A},\mathscr{P})}d\xi \\ &\leq \|\varkappa(\xi,\omega)\|_{\mathscr{E}_{\mathscr{B}}}\int_{\mathbb{R}}\|\Theta(\tau;\xi;\omega)\|_{L_\infty(\Sigma,\mathscr{A},\mathscr{P})}d\xi \\ &\leq \mathscr{M}\|\varkappa(\xi,\omega)\|_{\mathscr{E}_{\mathscr{B}}},\end{aligned}$$

where $\mathscr{M} = \sup_{\tau\in\mathbb{R}} \int_{\mathbb{R}} \|\Theta(\tau;s;\omega)\|_{L_\infty(\Sigma,\mathscr{A},\mathscr{P})}ds$. This proves $(\Psi\varkappa)(\tau;\omega)) \in \mathscr{E}_{\mathscr{B}}$, that means $\Psi$ is an operator from $\mathscr{E}_{\mathscr{B}}$ into $\mathscr{E}_{\mathscr{B}}$.

Assume now the function $\Lambda(\tau;\omega)$ is a bounded continuous function from $\mathbb{R}$ into $L_2(\Sigma,\mathscr{A},\mathscr{P})$ and the function $\vartheta(\xi,\varkappa(\xi;\omega))$ is in the $C(\mathbb{R},L_2(\Omega,\beta,\mathfrak{P}))$ satisfying the condition

$$\|\vartheta(\xi,\varkappa(\xi;\omega)) - \vartheta(\xi,\eta(\xi;\omega))\|_{L_2(\Omega,\mathscr{A},\mathscr{P})} \leq \beta\|\varkappa(\xi;\omega)-\eta(\xi;\omega)\|_{L_2(\Sigma,\mathscr{A},\mathscr{P})}, \forall\varkappa,\eta\in\mathscr{E}_\rho \tag{5}$$

where $\rho$ and $\beta$ are constants with $\beta\mathscr{M} < \frac{1}{1+3\rho}$ and $\mathscr{E}_\rho$ is defined as

$$\mathscr{E}_\rho = \left\{x\in C(\mathbb{R},L_2(\Sigma,\mathscr{A},\mathscr{P})) : \|\varkappa(\xi,\omega)\|_{\mathscr{E}_\rho} = \sup_{\xi\in\mathbb{R}}\|\varkappa(\xi,\omega)\|_{L_2(\Sigma,\mathscr{A},\mathscr{P})} < \rho\right\}.$$

Define the operator $\Gamma$ from $\mathscr{E}_\rho$ into $\mathscr{E}$ by

$$(\Gamma\varkappa)(\tau;\omega) = \Lambda(\tau;\omega) + \int_{\mathbb{R}}\Theta(\tau;\xi;\omega)\vartheta(\xi,\varkappa(\xi;\omega))d\xi.$$

Moreover, under the conditions $\|\Lambda(\tau;\omega)\|_{L_2(\Sigma,\mathscr{A},\mathscr{P})} + \mathscr{M}\|\vartheta(\tau,0)\|_{L_2(\Sigma,\mathscr{A},\mathscr{P})} \leq \rho(1-\beta\mathscr{M})$, we get

$$\begin{aligned}\|(\Gamma\varkappa)(\tau;\omega)\|_{L_2(\Sigma,\mathscr{A},\mathscr{P})} &\leq \|\Lambda(\tau;\omega)\|_{L_2(\Sigma,\mathscr{A},\mathscr{P})} + \mathscr{M}\|\vartheta(t,\varkappa(\tau;\omega))\|_{L_2(\Sigma,\mathscr{A},\mathscr{P})} \\ &\leq \|\Lambda(\tau;\omega)\|_{L_2(\Sigma,\mathscr{A},\mathscr{P})} + \mathscr{M}\|\vartheta(\tau,0)\|_{L_2(\Sigma,\mathscr{A},\mathscr{P})} + \mathscr{M}\beta\|\varkappa(\tau;\omega)\|_{L_2(\Sigma,\mathscr{A},\mathscr{P})} \quad \leq \rho.\end{aligned}$$

Hence, $(\Gamma\varkappa)(\tau;\omega) \in \mathscr{E}_\rho$ so $\Gamma$ is self mapping on $\mathscr{E}_\rho$.

We prove the existence of solutions to problem 4 utilising our deduced fixed point theorems. Now, let $\Omega = \mathscr{E}_\rho$ and $\mathscr{H} = L^2(\mathbb{R})$. We denote the set of all bounded linear operators on Hilbert space $\mathscr{H}$ by $\mathfrak{A} = \mathfrak{B}(\mathscr{H})$. Note that $\mathfrak{B}(\mathscr{H})$ is a unitary $C^*$-algebra. We define a $C^*$-algebra quasi controlled $\mathscr{K}$-metric $\Delta : \Omega\times\Omega \to \mathbb{A}$ by:

$$\Delta(\varkappa,\eta) = \pi_{\left(1+\|\varkappa\|_{L_2(\Sigma,\mathscr{A},\mathscr{P})}+2\|\eta\|_{L_2(\Sigma,\mathscr{A},\mathscr{P})}\right)\|\varkappa-\eta\|_{L_2(\Sigma,\mathscr{A},\mathscr{P})}}.$$

Similar to the Example 6, one can easily verify the completeness of $(\Omega,\mathfrak{A},\Delta)$. Then, we get by using our assumptions

$$\begin{aligned}||\Delta(\Gamma\varkappa,\Gamma\eta)||_{\mathfrak{B}(\mathscr{H})} &= \sup_{\|\phi\|=1}\left\langle\pi_{\left(1+\|\Gamma\varkappa\|_{L_2(\Sigma,\mathscr{A},\mathscr{P})}+2\|\Gamma\eta\|_{L_2(\Sigma,\mathscr{A},\mathscr{P})}\right)\|\Gamma\varkappa-\Gamma\eta\|_{L_2(\Sigma,\mathscr{A},\mathscr{P})}}\phi,\phi\right\rangle \\ &\leqslant (1+3\rho)\sup_{\|\phi\|=1}\int_{\mathbb{R}}\left\|\int_{\mathbb{R}}\Theta(\tau,\xi;\omega)\left[\vartheta(\xi,\varkappa(\xi;\omega))-\vartheta(\xi,\eta(\xi;\omega))\right]d\xi\right\|_{L_2(\Sigma,\mathscr{A},\mathscr{P})}|\phi(\tau)|^2d\tau\end{aligned}$$

$$\leq \beta\mathscr{M}(1+3\rho)\sup_{\|\phi\|=1}\int_{\mathbb{R}}|\phi(\tau)|^2\|\varkappa(\xi;\omega)-\eta(\xi;\omega)\|_{\mathscr{E}_{\mathscr{B}}}d\tau$$
$$\leq \beta\mathscr{M}(1+3\rho)\sup_{\tau\in\mathbb{R}}\|\varkappa(\xi;\omega)-\eta(\xi;\omega)\|_{L_2(\Sigma,\mathscr{A},\mathscr{P})}$$
$$\leq \beta\mathscr{M}(1+3\rho)\sup_{\|\phi\|=1}\left\langle \pi_{\left(1+\|\varkappa\|_{L_2(\Omega,\mathscr{A},\mathscr{P})}+2\|\eta\|_{L_2(\Sigma,\mathscr{A},\mathscr{P})}\right)\|\varkappa-\eta\|_{L_2(\Sigma,\mathscr{A},\mathscr{P})}}\phi,\phi\right\rangle$$
$$\leq \beta\mathscr{M}(1+3\rho)\,||\Delta(\varkappa,\eta)||_{\mathfrak{B}(\mathscr{H})}.$$

Since $\beta\mathscr{M}(1+3\rho)<1$, $\Gamma$ satisfies the inequality (1). Therefore, the integral Eq. (4) has a unique solution by Theorem 3.1.

**Example 4.1.** Let $\Sigma=]0,1[$ and $\alpha\in]0,\frac{1}{6}[$. We consider

$\Theta:\mathbb{R}\times\mathbb{R}\times\Sigma\to\mathbb{R}$

$$(\tau,\xi,\omega)\to\frac{|\tau|\alpha\omega}{(\tau^2+\tau+1)(\xi^2\tau^2+1)}$$

Note that for all $\xi\in\mathbb{R}$, the function $\tau\longmapsto\Psi(\tau,\xi;.)$ is continuous from $\mathbb{R}$ into $L_\infty(\Sigma,\beta,\mathscr{P})$.

$$\|(\Psi\varkappa)(\tau;\omega)\|_{L_2(\Sigma,\mathscr{A},\mathscr{P})}\leq\int_{\mathbb{R}}\|\Theta(\tau;\xi;\omega)\varkappa(\xi,\omega)\|_{L_2(\Sigma,\mathscr{A},\mathscr{P})}d\xi$$
$$\leq\sup_{\xi\in\mathbb{R}}\|\varkappa(\xi,\omega)\|_{L_2(\Omega,\mathscr{A},\mathscr{P})}\int_{\mathbb{R}}\left\|\frac{\tau\alpha\omega}{(\tau^2+\tau+1)(\xi^2\tau^2+1)}\right\|_{L_\infty(\Omega,\mathscr{A},\mathscr{P})}d\xi$$
$$\leq\alpha\|\varkappa(\xi,\omega)\|_{\mathscr{E}_{\mathscr{B}}}\int_{\mathbb{R}}\frac{|\tau|}{(\tau^2+\tau+1)(\xi^2\tau^2+1)}d\xi$$
$$\leq\alpha\pi\|\varkappa(\xi,\omega)\|_{\mathscr{E}_{\mathscr{B}}}$$
$$\leq\mathscr{M}\|\varkappa(\xi,\omega)\|_{\mathscr{E}_{\mathscr{B}}}.$$

Assume that $\Lambda(\tau,\omega)=0$ and we take $\vartheta(\xi,\varkappa(\xi;\omega))=\frac{e^\xi}{8(e^\xi-\xi)(1+\xi^2+|\varkappa(\xi;\omega)|)}$. Then, we can check that condition 5 is satisfied with $\beta=\frac{e}{8(e-1)}$.

Now let

$$\mathscr{E}_3=\left\{x\in C(\mathbb{R},L_2(\Sigma,\mathscr{A},\mathscr{P})):\|\varkappa(\xi,\omega)\|_{\mathscr{E}_2}=\sup_{\xi\in\mathbb{R}}\|\varkappa(\xi,\omega)\|_{L_2(\Sigma,\mathscr{A},\mathscr{P})}<2\right\}.$$

We see that $\beta\mathscr{M}(1+3\rho)=\frac{10\pi e}{48(e-1)}<1$, so all the assumptions mentioned in the application section are well insured. Hence, there exists unique solution of the nonlinear integral equation given by

$$\varkappa(\tau;\omega)=\int_{\mathbb{R}}\frac{\alpha\tau\omega e^\xi}{8(\tau^2+\tau+1)(\xi^2\tau^2+1)(e^\xi-\xi)(1+\xi^2+|\varkappa(\xi;\omega)|)}d\xi.$$

## 5 Conclusion

The results obtained are supported by non-trivial examples and complement and extend some of the most recent results from the literature. We have made a contribution by establishing some basic fixed-point problems considering a $C^*$-algebra valued quasi controlled $\mathscr{K}$-metric. We have proved some existence results for maps satisfying a new class of contractive conditions. The fixed point theorems are essential notions in the theory of integral equations. We have proved that the solution of a nonlinear stochastic integral equation of the Hammerstein type of a more general context using a $C^*$-algebra quasi controlled $\mathscr{K}$-metric spaces.

Future study is to investigate the sufficient conditions to guarantee the existence of a unique positive definite solution of the nonlinear matrix equations in the setting of $C^*$-algebra-valued quasi controlled $\mathscr{K}$-metric spaces. The conditions of Theorem 3.1 will be verified numerically by giving various values for the given matrices, and the convergence analysis of nonlinear matrix equations will be shown through graphical representations.

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