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ARTICLE

Notes on Curves at a Constant Distance from the Edge of Regression on a Curve in Galilean 3-Space $\mathbb{G}_3$

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ABSTRACT

In this paper, we define the curve $r_\lambda = r + \lambda d$ at a constant distance from the edge of regression on a curve $r(s)$ with arc length parameter s in Galilean 3-space. Here, d is a non-isotropic or isotropic vector defined as a vector tightly fastened to Frenet trihedron of the curve $r(s)$ in 3-dimensional Galilean space. We build the Frenet frame $\{T_\lambda, N_\lambda, B_\lambda\}$ of the constructed curve r_λ with respect to two types of the vector d and we indicate the properties related to the curvatures of the curve r_λ. Also, for the curve r_λ, we give the conditions to be a circular helix. Furthermore, we discuss ruled surfaces of type A generated via the curve r_λ and the vector D which is defined as tangent of the curve r_λ in 3-dimensional Galilean space. The constructed ruled surfaces also appear in two ways. The first is constructed with the curve $r_\lambda(s) = r(s) + \lambda T(s)$ and the non-isotropic vector D. The second is formed by the curve $r_\lambda = r(s) + \lambda_2 N + \lambda_3 B$ and the non-isotropic vector D. We calculate the distribution parameters of the constructed ruled surfaces and we show that the ruled surfaces are developable. Finally, we provide examples and visuals to back up our research.

KEYWORDS

Edge of regression; Galilean space; curvature; helix; ruled surface

1 Introduction

Klein pronounced a different definition of geometry in his introductory speech at the University of Erlangen in 1872. He explained that geometry, given by a subgroup $\mathbb{G}$ of a set and its symmetries, is the examination of invariants under this group [1]. This concept was first presented in a lecture, and it resulted in the emergence of numerous geometries. Galilean geometry is one of these geometries whose motions are the Galilean transformations of classical kinematics. Yaglom explained the basics of Galilean geometry in 1979 [2]. Then particularly, the geometry of ruled surfaces in this space has been largely improved in Röschel's thesis [3].

One of the important research areas in differential geometry is the theory of curves examined in various spaces. In particular, it has been examined in a lot of papers and remarkable results have been obtained in the 3-dimensional Galilean space [4–10].

The notion of the curves at a constant distance from the edge of regression has been introduced by Vogler. He has studied the curves traced on a torse at a constant distance from its edge of regression. The torse of a space curve in $\mathbb{E}^3$ is dual to its pseudo-rectifying torse [11]. Later, Hacısalihoğlu obtained a more general case of Vogler's results [12].

This subject has been studied in Euclidean 3-space since the 1970s, and it is a method that generates a new curve from the curve through the Frenet frame of the curve. For the first time, we will discuss this issue in 3-dimensional Galilean space. While the curve is produced by using the unit vector which is defined by the Frenet frame apparatus of a curve in Euclidean 3-space, we will have produced the curve by considering two situations in the Galilean 3-space. This is because, in Galilean space, vectors are treated in two ways, isotropic and non-isotropic.

In this paper, we first recall the essential preliminaries on the Galilean 3-space. Then, we define curves in the Galilean 3-space and give the curvature properties of these curves. In the main part of our study, we define a curve noted by r_λ at a constant distance from its edge of regression on a unit-speed admissible curve r in the Galilean 3-space. We give relations between the Frenet apparatus and the curvatures of r and r_λ. Using these relations, we get some conclusions. Also, we investigate ruled surfaces generated via the curve r_λ. In the last section, there are examples, two of which are ruled surfaces.

2 Preliminaries

Let us consider a curve $\alpha(t)$ in 3-dimensional Euclidean space with $\{T, N, B\}$ as the Frenet frame at the point $P = \alpha(s)$ of $\alpha(t)$. d is described as a vector tightly fastened to Frenet trihedron $\{T, N, B\}$ such that $d = d_1T + d_2N + d_3B$, where d_1, d_2, d_3 are constant numbers and $d_1^2 + d_2^2 + d_3^2 = 1$. k is described as a line tightly fastened to Frenet trihedron $\{T, N, B\}$ in the direction of d and passing through point P (see Fig. 1) [12]. Let P_v denote a point on the line k at a constant distance v from P. During the movement of the Frenet trihedron along the curve $\alpha(t)$, $C_v(t)$ is geometric place of $P_v(s)$ which is defined as a curve at a constant distance from the edge of regression of the curve α (see Fig. 1) [12].

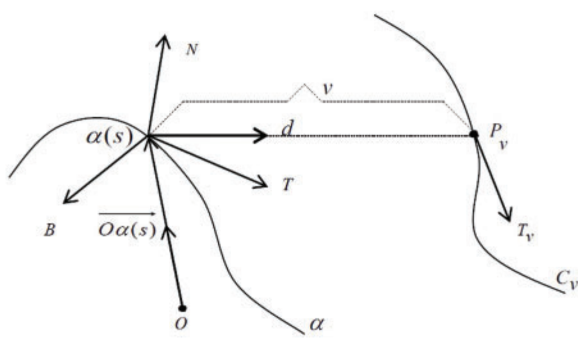

Figure 1: The curve α and the curve C_v

The Galilean space $\mathbb{G}_3$ is one of the Cayley-Klein geometries with projective signature (0, 0, +, +) as described in [6]. The absolute of the Galilean geometry is an ordered triple $\{w, f, I\}$ where w is the ideal (absolute) plane, f is the (absolute) line in w and I is the fixed elliptic involution of points of f. For more detailed information about this space, see [2,3,9,13,14].

Now, let us consider the basic definitions and notions.

Let $\vec{A} = (x, y, z)$ and $\vec{B} = (x_1, y_1, z_1)$ be two vectors in the Galilean 3-space $\mathbb{G}_3$. The Galilean scalar product of two vectors is defined by

$$\left\langle \vec{A}, \vec{B} \right\rangle_G = \vec{A}.\vec{B} = \begin{cases} xx_1, & if\ x \neq 0\ or\ x_1 \neq 0 \\ yy_1 + zz_1, & if\ x = 0\ and\ x_1 = 0 \end{cases}$$

If $\vec{A}.\vec{B} = 0$, these vectors are called perpendicular in the sense of Galilean in $\mathbb{G}_3$ [2].

Let $\vec{A} = (x, y, z)$ be a vector in the Galilean 3-space. The norm of the vector $\vec{A}$ is defined by [2]

$$\left\|\vec{A}\right\|_G = \begin{cases} |x|, & \mathrm{x} \neq 0 \\ \sqrt{y^2 + z^2}, & \mathrm{x} = 0 \end{cases}$$

The Galilean vector product of two vectors in $\mathbb{G}_3$ is

$$\vec{A} \times_G \vec{B} = \begin{vmatrix} 0 & e_2 & e_3 \\ x & y & z \\ x_1 & y_1 & z_1 \end{vmatrix},$$

where $\vec{A} = (x, y, z)$ and $\vec{B} = (x_1, y_1, z_1) \in \mathbb{G}_3$ [3,15].

A vector $\vec{A} = (x, y, z) \in \mathbb{G}_3$ is said to be isotropic if $x = 0$. On the other hand, the vector is defined as non-isotropic vector if $x \neq 0$ [15].

Definition 2.1. An angle $\boldsymbol{\theta}$ between two unit non-isotropic vectors $\vec{A} = (1, y, z)$ and $\vec{B} = (1, y_1, z_1)$ in $\mathbb{G}_3$ is described in [16] as

$$\boldsymbol{\theta} = \sqrt{(y_1 - y)^2 + (z_1 - z)^2}. \tag{1}$$

If the vectors $\vec{A} = (1, y, z)$ and $\vec{B} = (0, y_1, z_1)$ in $\mathbb{G}_3$ are taken, an angle $\boldsymbol{\theta}$ between the vectors is described as

$$\boldsymbol{\theta} = \frac{yy_1 + zz_1}{\sqrt{y_1^2 + z_1^2}}. \tag{2}$$

If the vectors $\vec{A} = (0, y, z)$ and $\vec{B} = (0, y_1, z_1)$ in $\mathbb{G}_3$ are isotropic, the cosine of the angle between two vectors is described as

$$\cos \boldsymbol{\theta} = \frac{yy_1 + zz_1}{\sqrt{y^2 + z^2}\sqrt{y_1^2 + z_1^2}}. \tag{3}$$

2.1 Curves in Galilean 3-Space

Let α be a curve given by $\alpha : I \to \mathbb{G}_3$, $\alpha(t) = (x(t), y(t), z(t))$ where $x(t), y(t), z(t) \in \mathbb{G}_3$. In this case if $x'(t) \neq 0$, $\alpha(t)$ is said to be a regular curve.

Let $\alpha : I \to \mathbb{G}_3$ be a regular curve in $\mathbb{G}_3$. Arc length of the curve α is $ds = |x'(t)dt| = |dx|$. Hence, we obtain $s = x$. Let $\alpha : I \to \mathbb{G}_3$ be a curve $\alpha(x) = (x, y(x), z(x))$ then we say that the curve is parameterized by arc length [4].

In this case, the functions $y, z : I \to \mathbb{R}$ are said to be coordinate functions of the curve. Here, differentiating $\alpha(x) = (x, y(x), z(x))$ with respect to x and using the norm definition, we obtain

$$||\alpha'(x)||_G = 1. \tag{4}$$

Then, $\alpha(x)$ is a unit speed curve. Let $\alpha : I \to \mathbb{G}_3$, $\alpha(x) = (x, y(x), z(x))$ be a regular unit speed curve in $\mathbb{G}_3$. Differentiating $\alpha(x)$, we have

$$\alpha'(x) = (1, y'(x), z'(x)). \tag{5}$$

The using Eq. (4), then the tangent vector of α is defined as

$$T(x) = (1, y'(x), z'(x)). \tag{6}$$

If we take the derivation of Eq. (5), we get

$$\alpha''(x) = (0, y''(x), z''(x)). \tag{7}$$

And from Eqs. (5) and (7), we write $\alpha'(x).\alpha''(x) = 0$. Here, the normal vector of the curve α is the vector in the direction. Then, the unit normal vector is defined as

$$N(x) = \frac{\alpha''(x)}{||\alpha''(x)||_G}. \tag{8}$$

Using Eqs. (7) and (8), we write

$$N(x) = \frac{1}{\sqrt{y''^2(x) + z''^2(x)}} (0, y''(x), z''(x)). \tag{9}$$

As a consequence, the unit binormal vector $B(x)$ of α is

$$B(x) = \frac{1}{\sqrt{y''^2(x) + z''^2(x)}} (0, -z''(x), y''(x)), \tag{10}$$

and then the frame $\{T(x), N(x), B(x)\}$ chosen in this way is called the Frenet-Serret frame for unit speed curves in the Galilean 3-space [5].

Proposition 2.1. The Frenet formulae of a unit speed curve $\alpha(x)$ in $\mathbb{G}_3$ is given by

$$\begin{pmatrix} T'(x) \\ N'(x) \\ B'(x) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(x) & 0 \\ 0 & 0 & \tau(x) \\ 0 & -\tau(x) & 0 \end{pmatrix} \begin{pmatrix} T(x) \\ N(x) \\ B(x) \end{pmatrix}, \tag{11}$$

where

$$\kappa(x) = \sqrt{y''^2(x) + z''^2(x)} \tag{12}$$

is the curvature of α and

$$\tau(x) = \frac{\det(\alpha'(x), \alpha''(x), \alpha'''(x))}{\kappa^2(x)} \tag{13}$$

is the torsion of α [17].

3 Curve at a Constant Distance from the Edge of Regression on a Curve in Galilean 3-Space

Definition 3.1. Suppose that r is a curve in Galilean 3-space and $\{T, N, B\}$ is the Frenet frame at the point $P = r(s)$ of r. Let P_λ be a point at a constant distance λ from P. During the movement of the Frenet trihedron along the curve r, $r_\lambda = r + \lambda d$ is geometric place of P_λ, where

$$d = \begin{cases} d_1 T, \text{ if d is non-isotropic} \\ d_2 N + d_3 B, \text{ if } d \text{ is isotropic} \end{cases}$$

such that $d_1^2 + d_2^2 + d_3^2 = 1, d_1, d_2, d_3 \in \mathbb{R}$ and $d^2 = 1, |d| = 1$. In this case, r_λ is called curve at a constant distance from the edge of regression of r.

Now, let us construct the Frenet frame of r_λ generated by both the non-isotropic vector d and isotropic vector d and examine its curvature properties.

Case 3.1. d is non-isotropic. In this case, $d_1^2 = 1$. Let us $d_1 = 1, d_2 = d_3 = 0$. Then, we obtain

$$r_\lambda(s) = r(s) + \lambda T(s), \tag{14}$$

where $d(s) = T(s)$.

Theorem 3.1. If $r(s)$ is a curve with arc length parameter s, then the arc length parameter of the curve r_λ is also s.

Proof. By differentiating Eq. (14)

$$\frac{d}{ds} r_\lambda(s) = r'(s) + \lambda T'(s). \tag{15}$$

If we take the norm of two sides of Eq. (15), we have

$$\left|\left|\frac{d}{ds} r_\lambda(s)\right|\right|_G = ||r'(s) + \lambda T'(s)||_G = 1$$

which completes the proof.

Theorem 3.2. Let $(r(s), r_\lambda(s))$ be given the curves pair with arc length s in $\mathbb{G}_3$. If the Frenet vectors of r and r_λ are $\{T, N, B\}$ and $\{T_\lambda, N_\lambda, B_\lambda\}$, the curvatures are κ, τ and κ_λ, τ_λ, respectively, then the following relations hold:

$$T_\lambda = T + \lambda\kappa N, \tag{16}$$

$$N_\lambda = \frac{1}{\kappa_\lambda}\left[(\kappa + \lambda\kappa') N + \lambda\kappa\tau B\right], \tag{17}$$

$$B_\lambda = \frac{1}{\kappa_\lambda}\left[-\lambda\kappa\tau N + (\kappa + \lambda\kappa') B\right], \tag{18}$$

$$\kappa_\lambda = \sqrt{\kappa^2 + 2\lambda\kappa\kappa' + \lambda^2\left(\kappa'^2 + \kappa^2\tau^2\right)} \tag{19}$$

and

$$\tau_\lambda = \frac{\kappa_\lambda^2\tau + \lambda\kappa^2\tau' + \lambda^2\left(\kappa'^2\tau + \kappa\kappa'\tau' - \kappa\kappa''\tau\right)}{\kappa_\lambda^2}. \tag{20}$$

Proof. By differentiating Eq. (14) and using Eq. (11), we have

$$r'_\lambda = T + \lambda\kappa N \tag{21}$$

which gives us Eq. (16). If we take derivation of Eq. (21) according to s, we get

$$r''_\lambda = (\kappa + \lambda\kappa')N + \lambda\kappa\tau B. \tag{22}$$

Using Eq. (22) in Eq. (12), then we get Eq. (19). From Eq. (9), we easily obtain the Eq. (17) and from Eq. (10), we get Eq. (18). Considering Eq. (13), we obtain Eq. (20).

In these calculations we used $r_\lambda''' = \left(\kappa' + \lambda\kappa'' - \lambda\kappa\tau^2\right) N + \left(\kappa\tau + 2\lambda\kappa'\tau + \lambda\kappa\tau'\right) B$

Corollary 3.1. Let $(r(s), r_\lambda(s))$ be the curves pair given with arc length s in $\mathbb{G}_3$. If the curve r is a circular helix, r_λ is also a circular helix.

Proof. We know that if the curvatures κ and τ of r are constants, $\frac{\tau}{\kappa}$ is also constant and then r is a circular helix. If we take κ and τ as constants in Eqs. (19) and (20), we get $\kappa_\lambda = \kappa\sqrt{1+\lambda^2\tau^2}$ and $\tau_\lambda = \tau$.

In this case, we have $\frac{\tau_\lambda}{\kappa_\lambda} = \frac{\tau}{\kappa\sqrt{1+\lambda^2\tau^2}}$. $\frac{\tau_\lambda}{\kappa_\lambda}$ is fixed since λ, κ and τ are constants. Then, r_λ is also a circular helix.

Case 3.2. d is isotropic. In this case, $d_2^2 + d_3^2 = 1$ and $d = d_2N + d_3B$. Hence, we have

$$r_\lambda = r(s) + \lambda_2 N + \lambda_3 B, \tag{23}$$

where $\lambda_2 = \lambda d_2$ and $\lambda_3 = \lambda d_3$.

Theorem 3.3. If $r(s)$ is a curve with arc length parameter s, then the arc length parameter of the curve r_λ is also s.

Proof. By differentiating Eq. (23), we have

$$\frac{d}{ds} r_\lambda(s) = r'(s) + \lambda_2 N'(s) + \lambda_3 B'(s). \tag{24}$$

If we take the norm of two sides of Eq. (24), we have $\left\|\frac{d}{ds} r_\lambda(s)\right\|_G = \|r'(s) + \lambda_2 N'(s) + \lambda_3 B'(s)\|_G = 1$ which completes the proof.

Theorem 3.4. Let $(r(s), r_\lambda(s))$ be the curves pair given with arc length s in $\mathbb{G}_3$. If the Frenet vectors of r and r_λ are $\{T, N, B\}$ and $\{T_\lambda, N_\lambda, B_\lambda\}$, the curvatures are κ, τ and κ_λ, τ_λ, respectively, then the following relations hold:

$$T_\lambda = T - \lambda_3\tau N + \lambda_2\tau B, \tag{25}$$

$$N_\lambda = \frac{1}{\kappa_\lambda}\left[\left(\kappa - \lambda_2\tau^2 - \lambda_3\tau'\right) N + \left(\lambda_2\tau' - \lambda_3\tau^2\right) B\right], \tag{26}$$

$$B_\lambda = \frac{1}{\kappa_\lambda}\left[\left(-\lambda_2\tau' + \lambda_3\tau^2\right) N + \left(\kappa - \lambda_2\tau^2 - \lambda_3\tau'\right) B\right], \tag{27}$$

$$\kappa_\lambda = \sqrt{\kappa^2 + \left(\lambda_2^2 + \lambda_3^2\right)\left(\tau^4 + \tau'^2\right) - 2\kappa\left(\lambda_2\tau^2 + \lambda_3\tau'\right)}, \tag{28}$$

and

$$\tau_\lambda = \frac{\kappa_\lambda^2\tau + \left(\lambda_2^2 + \lambda_3^2\right)\left(2\tau\tau'^2 - \tau^2\tau''\right) + \lambda_2\left(\kappa\tau'' - \kappa'\tau'\right) + \lambda_3\left(\kappa'\tau^2 - 2\kappa\tau\tau'\right)}{\kappa_\lambda^2}. \tag{29}$$

Proof. By differentiating Eq. (23) and using Eq. (11), we obtain $r_\lambda' = T - \lambda_3\tau N + \lambda_2\tau B$ which gives Eq. (25). If we take second derivation of r_λ according to s and use Eq. (11) again, we get $r_\lambda'' = \left(\kappa - \lambda_2\tau^2 - \lambda_3\tau'\right) N + \left(\lambda_2\tau' - \lambda_3\tau^2\right) B$. In the light of this last equation, if we take into account Eq. (12) we have Eq. (28). From Eqs. (8) and (26) is obtained and Eq. (27) is found as a consequence of Eq. (10). Considering (13), the torsion of r_λ is found as in Eq. (29).

In these calculations, we use $r_\lambda''' = \left(\kappa' - \lambda_3\tau'' - 3\lambda_2\tau\tau' + \lambda_3\tau^3\right) N + \left(\kappa\tau - \lambda_2\tau^3 - 3\lambda_3\tau\tau' + \lambda_2\tau''\right) B$.

Corollary 3.2. Let $(r(s), r_\lambda(s))$ be the curve pair given with arc length s in $\mathbb{G}_3$. If the curve r is circular helix, r_λ is also a circular helix.

Proof. If the curvatures of r are constants, $\frac{\tau}{\kappa}$ is constant and r is a circular helix. Considering κ and τ as constants in Eqs. (28) and (29), the curvatures of r_λ are $\kappa_\lambda = \sqrt{\kappa^2 + \left(\lambda_2^2 + \lambda_3^2\right)\tau^4 - 2\kappa\lambda_2\tau^2}$ and $\tau_\lambda = \tau$, respectively.

In this case, we have $\frac{\tau_\lambda}{\kappa_\lambda} = \frac{\tau}{\sqrt{\kappa^2+\left(\lambda_2^2+\lambda_3^2\right)\tau^4-2\kappa\lambda_2\tau^2}}$. Here, $\frac{\tau_\lambda}{\kappa_\lambda}$ is fixed since $\lambda_2, \lambda_3, \kappa$ and τ are constants. Then, r_λ is also a circular helix.

4 Ruled Surfaces Generated by the Curve r_λ

The ruled surfaces in $\mathbb{G}_3$ are three types. Definitions of the ruled surfaces of type A, B, C and current studies can be viewed in [6,15,17–20]. Our goal is to define the ruled surfaces using r_λ as the base curve and r'_λ as the director curve, and to see if they can be developed. Here, we take into account ruled surfaces of type A.

4.1 *Ruled Surface of Type A Generated by $r_\lambda(s) = r(s) + \lambda T(s)$*

A ruled surface of type A in $\mathbb{G}_3$ by using non-isotropic vector D can be written as

$$X_A(s, v) = r_\lambda(s) + vD(s), \tag{30}$$

where r_λ is defined as in Eq. (14) and the vector D is tangent of r_λ. Besides, the curve r_λ defined a directrix that does not lie in Euclidean plane and non-isotropic vector $D(s) = T_\lambda$ is generator. The associated orthonormal triple of $X_A(s, v)$ is given by

$$t(s) = T_\lambda(s),$$
$$n(s) = N_\lambda(s),$$
$$b(s) = B_\lambda(s), \tag{31}$$

where Frenet trihedron $\{T_\lambda, N_\lambda, B_\lambda\}$ is the Frenet frame of the unit speed curve $r_\lambda(s)$ in Galilean 3-space. From Eq. (31), we see that two orthonormal triple coincide.

Additionally, the parameter of distribution P_{X_A} of $X_A(s, v)$ is

$$P_{X_A} = -\frac{\det\left(r'_\lambda, D, D'\right)}{||D'||_G^2}. \tag{32}$$

We know that if $P_{X_A} = 0$, $X_A(s, v)$ is developable. Then, we can express the following theorem:

Theorem 4.1. Suppose that (r, r_λ) is a unit speed curves pair in Galilean 3-space with $r_\lambda = r + \lambda T$, where $\{T, N, B\}$ and $\{T_\lambda, N_\lambda, B_\lambda\}$ are the Frenet frame of r and r_λ, respectively. D is a non-isotropic vector tightly fastened to Frenet trihedron $\{T_\lambda, N_\lambda, B_\lambda\}$ of r_λ at the origin and $X_A(s, v)$ is the ruled surface of type A generated by D and r_λ. Then, $X_A(s, v)$ is a developable surface.

Proof. By taking derivative of r_λ with respect to s and by using Theorem 3.2, we obtain

$$r'_\lambda = T_\lambda = T + \lambda\kappa N. \tag{33}$$

If we take $D = T_\lambda$, we get $P_{X_A} = 0$ from Eq. (32). Thus, $X_A(s, v)$ is developable.

Now, we consider that v is a constant in the ruled surface $X_A(s, v)$. Then, $X_A(s, v) = r_\lambda(s) + vD(s)$ is the equation of parametric curve r_{λ_v} for the points K_v on the ruled surface. In this case, we have

$$T_{\lambda_v} = T_\lambda + v\kappa_\lambda N_\lambda, \tag{34}$$

where T_{λ_v} is tangent vector at a point K_v of r_{λ_v} for v −constant. If κ_λ is a non-zero constant, we deduce from Eq. (34) that r_λ is a Bertrand curve.

Finally considering Eq. (1), the angle $\boldsymbol{\theta}$ between non-isotropic vectors T_{λ_v} and D, we calculate as $\boldsymbol{\theta} = v\kappa_\lambda$.

4.2 *Ruled Surface of Type A Generated by* $r_\lambda = r(s) + \lambda_2 N + \lambda_3 B$

Similarly to Section 4.1, a ruled surface of type A in $\mathbb{G}_3$ by using $D = T_\lambda$ can be written as

$$X_A(s, v) = r_\lambda(s) + vD(s), \tag{35}$$

where the curve $r_\lambda(s) = r(s) + \lambda_2 N + \lambda_3 B$ is directrix and $D(s) = T_\lambda = T - \lambda_3 \tau N + \lambda_2 \tau B$ is generator. In this case, the associated orthonormal triple of $X_A(s, v)$ is found as Eq. (31).

Thus, the following theorem can be written:

Theorem 4.2. Suppose that (r, r_λ) is a unit speed curves pair in Galilean 3-space with $r_\lambda = r + \lambda_2 N + \lambda_3 B$, where $\{T, N, B\}$ and $\{T_\lambda, N_\lambda, B_\lambda\}$ are Frenet frame of r and r_λ, respectively. D is a non-isotropic vector tightly fastened to Frenet trihedron $\{T_\lambda, N_\lambda, B_\lambda\}$ of r_λ at the origin and $X_A(s, v)$ is the ruled surface of type A generated by D and r_λ. Then, $X_A(s, v)$ is a developable surface.

Proof. By taking derivative of r_λ with respect to s and by using Theorem 3.4, we have

$$r_\lambda' = T_\lambda = T - \lambda_3 \tau N + \lambda_2 \tau B. \tag{36}$$

We take as $D = T_\lambda$, so by a straightforward computation, the parameter of distribution for $X_A(s, v)$ is calculated as follows:

$$P_{X_A} = 0. \tag{37}$$

Hence, $X_A(s, v)$ is developable.

5 Applications

Example 5.1. Consider the curve given by the parametrization

$$\phi(\sigma) = (\sigma, -\sigma\cos(\sigma) + 2\sin(\sigma), -\sigma\sin(\sigma) - 2\cos(\sigma)). \tag{38}$$

The Frenet frame fields of the curve of $\phi(\sigma)$ are [Fig. 2]

$$T(\sigma) = (1, \sigma\sin(\sigma) + \cos(\sigma), -\sigma\cos(\sigma) + \sin(\sigma)), N(\sigma) = (0, \cos(\sigma), \sin(\sigma)), B(\sigma) = (0, -\sin(\sigma), \cos(\sigma)).$$

Considering Eq. (14), the curve $\phi_\lambda(\sigma)$ generated by non-isotropic vector d in $\mathbb{G}_3$ is

$$\phi_\lambda(\sigma) = \phi(\sigma) + \lambda T,$$

$$\phi_\lambda(\sigma) = (1 + \sigma, (1 - \sigma)\cos(\sigma) + (2 + \sigma)\sin(\sigma), (1 - \sigma)\sin(\sigma) - (2 + \sigma)\cos(\sigma))$$

for $\lambda = 1$ (see Fig. 2). If d is isotropic vector, from Eq. (23) we have

$$\phi_r(\sigma) = \left(\sigma, -\sigma\cos(\sigma) + \left(\frac{4 - \sqrt{3}}{2}\right)\sin(\sigma) + \frac{1}{2}\cos(\sigma), -\sigma\sin(\sigma) + \left(\frac{-4 + \sqrt{3}}{2}\right)\cos(\sigma) + \frac{1}{2}\sin(\sigma)\right)$$

for $r = 1, d_2 = \frac{1}{2},\ d_3 = \dfrac{\sqrt{3}}{2}$ (see Fig. 2).

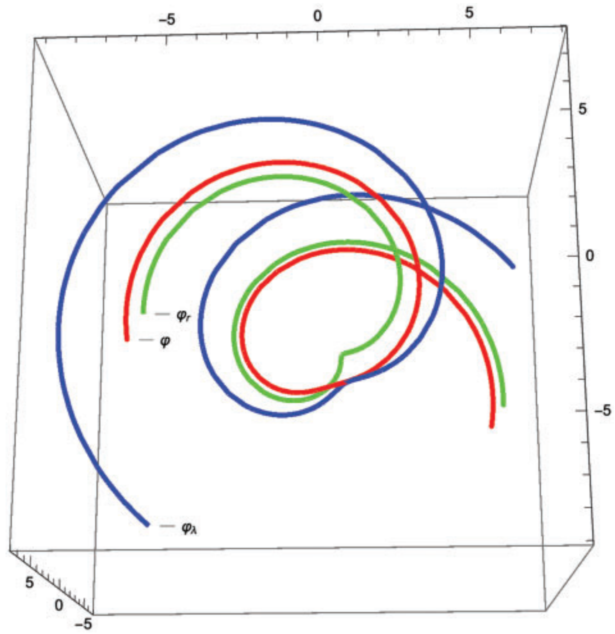

Figure 2: The red curve is $\phi(\sigma)$, the blue curve is $\phi\lambda\ (\sigma)$ and the green curve is ϕr(σ)

Example 5.2. Let us consider the curve given by Eq. (38) in Example 5.1. The ruled surface $X_A(\sigma, v)$ obtained by using Eq. (38) in light Eq. (30) is

$$X_A(\sigma, v) = \begin{pmatrix} 1 + v + \sigma, \\ (1 - \sigma + v(1 + \lambda\kappa))\cos(\sigma) + (2 + \sigma(1 + v))\sin(\sigma), \\ (1 - \sigma + v(1 + \lambda\kappa))\sin(\sigma) - (2 + \sigma(1 - v))\cos(\sigma) \end{pmatrix}. \tag{39}$$

In Eq. (39), the curve $\phi_\lambda(\sigma) = (1+\sigma, (1-\sigma)\cos(\sigma)+(2+\sigma)\sin(\sigma), (1-\sigma)\sin(\sigma)-(2+\sigma)\cos(\sigma))$ is directrix and the non-isotropic vector

$D(\sigma) = T_\lambda(s) = (1, \sigma\sin(\sigma) + (1 + \lambda\kappa)\cos(\sigma), -\sigma\cos(\sigma) + (1 + \lambda\kappa)\sin(\sigma))$

is generator. For $\kappa = \sigma$ and $\lambda = 1$, Eq. (39) is shown in the Fig. 3.

Example 5.3. Let us consider the curve given by Eq. (38) in Example 5.1. The ruled surface $X_A(\sigma, v)$ obtained by using Eq. (38) in light Eq. (35) is

$$X_A(\sigma, v) = \begin{pmatrix} \sigma + v, \\ \left(-\sigma + \frac{1}{2} + v(1 - \lambda_3\tau)\right)\cos(\sigma) + \left(\frac{4-\sqrt{3}}{2} + v(\sigma - \lambda_2\tau)\right)\sin(\sigma), \\ \left(\left(-\sigma + \frac{1}{2} + v(1 - \lambda_3\tau)\sin(\sigma) + \left(\frac{-4+\sqrt{3}}{2} + v(-\sigma + \lambda_2\tau)\right)\cos(\sigma)\right)\right. \end{pmatrix}. \tag{40}$$

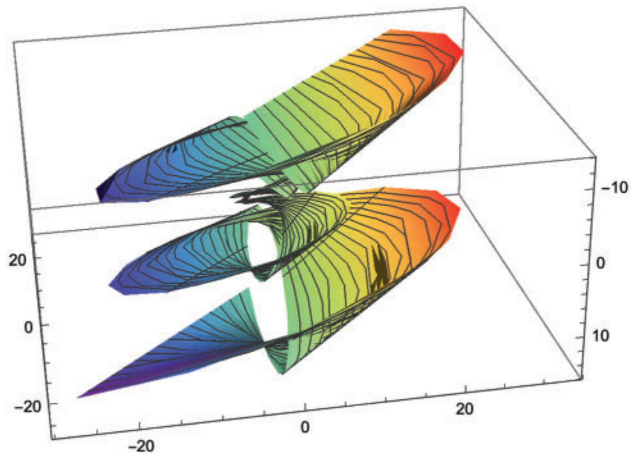

Figure 3: The surface $X_A(\sigma, v)$, for $-\pi \le \sigma \le \pi$, $-1 \le v \le 1$

In Eq. (40), the curve

$$\phi_r(\sigma) = \left(\sigma, -\sigma\cos(\sigma) + \left(\frac{4-\sqrt{3}}{2}\right)\sin(\sigma) + \frac{1}{2}\cos(\sigma), -\sigma\sin(\sigma) + \left(\frac{-4+\sqrt{3}}{2}\right)\cos(\sigma) + \frac{1}{2}\sin(\sigma)\right)$$

is directrix and the non-isotropic vector

$T_\lambda(s) = (1, (\sigma - \lambda_2\tau)\sin(\sigma) + (1-\lambda_3\tau)\cos(\sigma), (1-\lambda_3\tau)\sin(\sigma) + (-\sigma + \lambda_2\tau)\cos(\sigma))$

is generator. For $\tau = 1$ and $\lambda_2 = \lambda_3 = 1$, Eq. (40) is shown in the Fig. 4.

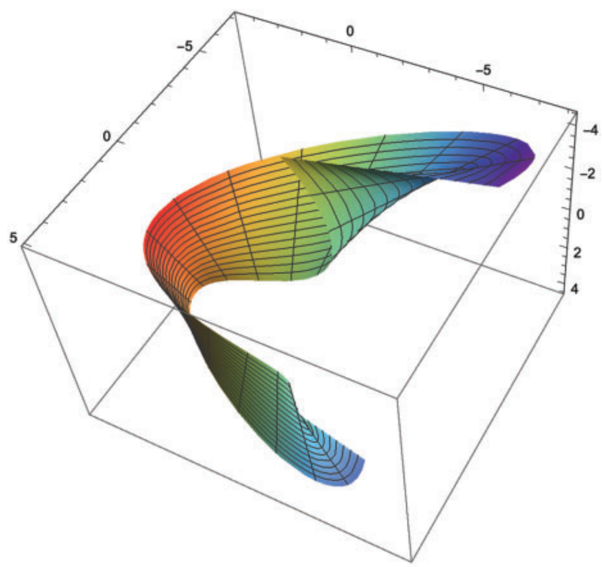

Figure 4: The surface $X_A(\sigma, v)$, for $-\pi \le \sigma \le \pi$, $-1 \le v \le 1$

6 Conclusion

In this study, we present a method that generates a new curve from the curve using the Frenet frame of a curve that is parameterized by arc length in $\mathbb{G}_3$. We show that it is possible in two ways to achieve this state in Galilean 3-space. We also calculate the Frenet frame and curvatures of the generated curve in terms of the Frenet frame and curvatures of the first curve. In this case, we reveal that if the first curve is helix, the generated curve is also the helix curve. In Section 4, for Case 3.1 and Case 3.2, ruled surfaces generated by the constructed curve and its tangents are discussed. It has been shown that the composed ruled surfaces are developable.

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