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Neutrosophic $\mathcal{N}$-Structures Applied to Sheffer Stroke BL-Algebras

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ABSTRACT

In this paper, we introduce a neutrosophic $\mathcal{N}$-subalgebra, a (ultra) neutrosophic $\mathcal{N}$-filter, level sets of these neutrosophic $\mathcal{N}$-structures and their properties on a Sheffer stroke BL-algebra. By defining a quasi-subalgebra of a Sheffer stroke BL-algebra, it is proved that the level set of neutrosophic $\mathcal{N}$-subalgebras on the algebraic structure is its quasi-subalgebra and vice versa. Then we show that the family of all neutrosophic $\mathcal{N}$-subalgebras of a Sheffer stroke BL-algebra forms a complete distributive lattice. After that a (ultra) neutrosophic $\mathcal{N}$-filter of a Sheffer stroke BL-algebra is described, we demonstrate that every neutrosophic $\mathcal{N}$-filter of a Sheffer stroke BL-algebra is its neutrosophic $\mathcal{N}$-subalgebra but the inverse is generally not true. Finally, it is presented that a level set of a (ultra) neutrosophic $\mathcal{N}$-filter of a Sheffer stroke BL-algebra is also its (ultra) filter and the inverse is always true. Moreover, some features of neutrosophic $\mathcal{N}$-structures on a Sheffer stroke BL-algebra are investigated.

KEYWORDS

Sheffer stroke BL-algebra; (ultra) filter; neutrosophic $\mathcal{N}$-subalgebra; (ultra) neutrosophic $\mathcal{N}$-filter

1 Introduction

Fuzzy set theory, which has the truth (t) (membership) function and state positive meaning of information, is introduced by Zadeh [1] as a generalization the classical set theory. This led scientists to find negative meaning of information. Hence, intuitionistic fuzzy sets [2] which are fuzzy sets with the falsehood (f) (nonmembership) function were introduced by Atanassov. However, there exist uncertainty and vagueness in the language, as well as positive ana negative meaning of information. Thus, Smarandache defined neutrosophic sets which are intuitionistic fuzzy sets with the indeterminacy/neutrality (i) function [3,4]. Thereby, neutrosophic sets are determined on three components: $(t, i, f) : (truth,\ indeterminacy,\ falsehood)$ [5]. Since neutrosophy enables that information in language can be comprehensively examined at all points, many researchers applied neutrosophy to different theoretical areas such as BCK/BCI-algebras, BE-algebras, semigroups, metric spaces, Sheffer stroke Hilbert algebras and strong Sheffer stroke non-associative MV-algebras [6–15] so as to improve devices imitating human behaviours and thoughts, artificial intelligence and technological tools.

Sheffer stroke (or Sheffer operation) was originally introduced by Sheffer [16]. Since Sheffer stroke can be used by itself without any other logical operators to build a logical system which is easy to control, Sheffer stroke can be applied to many logical algebras such as Boolean algebras [17], ortholattices [18], Sheffer stroke Hilbert algebras [19]. On the other side, BL-algebras were introduced by Hájek as an axiom system of his Basic Logic (BL) for fuzzy propositional logic, and he widely studied many types of filters [20]. Moreover, Oner et al. [21] introduced BL-algebras with Sheffer operation and investigated some types of (fuzzy) filters.

We give fundamental definitions and notions about Sheffer stroke BL-algebras, $\mathcal{N}$-functions and neutrosophic $\mathcal{N}$-structures defined by these functions on a crispy set X. Then a neutrosophic $\mathcal{N}$-subalgebra and a (τ, γ, ρ)-level set of a neutrosophic $\mathcal{N}$-structure are presented on Sheffer stroke BL-algebras. By defining a quasi-subalgebra of a Sheffer stroke BL-algebra, it is proved that every (τ, γ, ρ)-level set of a neutrosophic $\mathcal{N}$-subalgebra of the algebra is the quasi-subalgebra and the inverse is true. Also, we show that the family of all neutrosophic $\mathcal{N}$-subalgebras of this algebraic structure forms a complete distributive lattice. Some properties of neutrosophic $\mathcal{N}$-subalgebras of Sheffer stroke BL-algebras are examined. Indeed, we investigate the case which $\mathcal{N}$-functions defining a neutrosophic $\mathcal{N}$-subalgebra of a Sheffer stroke BL-algebra are constant. Moreover, we define a (ultra) neutrosophic $\mathcal{N}$-filter of a Sheffer stroke BL-algebra by $\mathcal{N}$-functions and analyze many features. It is demonstrated that (τ, γ, ρ)-level set of a neutrosophic $\mathcal{N}$-filter of a Sheffer stroke BL-algebra is its filter but the inverse does not hold in general. In fact, we propound that (τ, γ, ρ)-level set of a (ultra) neutrosophic $\mathcal{N}$-filter of a Sheffer stroke BL-algebra is its (ultra) filter and the inverse is true. Finally, new subsets of a Sheffer stroke BL-algebra are defined by the $\mathcal{N}$-functions and special elements of the algebra. It is illustrated that these subsets are (ultra) filters of a Sheffer stroke BL-algebra for the (ultra) neutrosophic $\mathcal{N}$-filter but the special conditions are necessary to prove the inverse.

2 Preliminaries

In this section, basic definitions and notions on Sheffer stroke BL-algebras and neutrosophic $\mathcal{N}$-structures.

Definition 2.1. [18] Let $\mathcal{H} = \langle H, | \rangle$ be a groupoid. The operation $|$ is said to be a *Sheffer stroke (or Sheffer operation)* if it satisfies the following conditions:

$(S1)$ $x \mid y = y \mid x$,

$(S2)$ $(x \mid x) \mid (x \mid y) = x$,

$(S3)$ $x \mid ((y \mid z) \mid (y \mid z)) = ((x \mid y) \mid (x \mid y)) \mid z$,

$(S4)$ $(x \mid ((x \mid x) \mid (y \mid y))) \mid (x \mid ((x \mid x) \mid (y \mid y))) = x$.

Definition 2.2. [21] A Sheffer stroke BL-algebra is an algebra $(C, \vee, \wedge, |, 0, 1)$ of type $(2, 2, 2, 0, 0)$ satisfying the following conditions:

$(sBL-1)$ $(C, \vee, \wedge, 0, 1)$ is a bounded lattice,

$(sBL-2)$ $(C, |)$ is a groupoid with the Sheffer stroke,

$(sBL-3)$ $c_1 \wedge c_2 = (c_1 \mid (c_1 \mid (c_2 \mid c_2))) \mid (c_1 \mid (c_1 \mid (c_2 \mid c_2)))$,

$(sBL-4)$ $(c_1 \mid (c_2 \mid c_2)) \vee (c_2 \mid (c_1 \mid c_1)) = 1$,

for all $c_1, c_2 \in C$.

$1 = 0 \mid 0$ is the greatest element and $0 = 1 \mid 1$ is the least element of C.

Proposition 2.1. [21] In any Sheffer stroke BL-algebra C, the following features hold, for all $c_1, c_2, c_3 \in C$:

(1) $c_1 \mid ((c_2 \mid (c_3 \mid c_3)) \mid (c_2 \mid (c_3 \mid c_3))) = c_2 \mid ((c_1 \mid (c_3 \mid c_3)) \mid (c_1 \mid (c_3 \mid c_3)))$,
(2) $c_1 \mid (c_1 \mid c_1) = 1$,
(3) $1 \mid (c_1 \mid c_1) = c_1$,
(4) $c_1 \mid (1 \mid 1) = 1$,
(5) $(c_1 \mid 1) \mid (c_1 \mid 1) = c_1$,
(6) $(c_1 \mid c_2) \mid (c_1 \mid c_2) \leq c_3 \Leftrightarrow c_1 \leq c_2 \mid (c_3 \mid c_3)$
(7) $c_1 \leq c_2$ iff $c_1 \mid (c_2 \mid c_2) = 1$,
(8) $c_1 \leq c_2 \mid (c_1 \mid c_1)$,
(9) $c_1 \leq (c_1 \mid c_2) \mid c_2$,
(10) (a) $(c_1 \mid (c_1 \mid (c_2 \mid c_2))) \mid (c_1 \mid (c_1 \mid (c_2 \mid c_2))) \leq c_1$,
(b) $(c_1 \mid (c_1 \mid (c_2 \mid c_2))) \mid (c_1 \mid (c_1 \mid (c_2 \mid c_2))) \leq c_2$.
(11) If $c_1 \leq c_2$, then
(i) $c_3 \mid (c_1 \mid c_1) \leq c_3 \mid (c_2 \mid c_2)$,
(ii) $(c_1 \mid c_3) \mid (c_1 \mid c_3) \leq (c_2 \mid c_3) \mid (c_2 \mid c_3)$,
(iii) $c_2 \mid (c_3 \mid c_3) \leq c_1 \mid (c_3 \mid c_3)$.
(12) $c_1 \mid (c_2 \mid c_2) \leq (c_3 \mid (c_1 \mid c_1)) \mid ((c_3 \mid (c_2 \mid c_2)) \mid (c_3 \mid (c_2 \mid c_2)))$,
(13) $c_1 \mid (c_2 \mid c_2) \leq (c_2 \mid (c_3 \mid c_3)) \mid ((c_1 \mid (c_3 \mid c_3)) \mid (c_1 \mid (c_3 \mid c_3)))$,
(14) $((c_1 \vee c_2) \mid c_3) \mid ((c_1 \vee c_2) \mid c_3) = ((c_1 \mid c_3) \mid (c_1 \mid c_3)) \vee ((c_2 \mid c_3) \mid (c_2 \mid c_3))$,
(15) $c_1 \vee c_2 = ((c_1 \mid (c_2 \mid c_2)) \mid (c_2 \mid c_2)) \wedge ((c_2 \mid (c_1 \mid c_1)) \mid (c_1 \mid c_1))$.

Lemma 2.1. [21] Let C be a Sheffer stroke BL-algebra. Then

$$(c_1 \mid (c_2 \mid c_2)) \mid (c_2 \mid c_2) = (c_2 \mid (c_1 \mid c_1)) \mid (c_1 \mid c_1),$$

for all $c_1, c_2 \in C$.

Corollary 2.1. [21] Let C be a Sheffer stroke BL-algebra. Then

$$c_1 \vee c_2 = (c_1 \mid (c_2 \mid c_2)) \mid (c_2 \mid c_2),$$

for all $c_1, c_2 \in C$.

Lemma 2.2. [21] Let C be a Sheffer stroke BL-algebra. Then

$$c_1 \mid ((c_2 \mid (c_3 \mid c_3)) \mid (c_2 \mid (c_3 \mid c_3))) = (c_1 \mid (c_2 \mid c_2)) \mid ((c_1 \mid (c_3 \mid c_3)) \mid (c_1 \mid (c_3 \mid c_3))),$$

for all $c_1, c_2, c_3 \in C$.

Definition 2.3. [21] A filter of C is a nonempty subset $P \subseteq C$ satisfying

$(SF-1)$ if $c_1, c_2 \in P$, then $(c_1 \mid c_2) \mid (c_1 \mid c_2) \in P$,

$(SF-2)$ if $c_1 \in P$ and $c_1 \leq c_2$, then $c_2 \in P$.

Proposition 2.2. [21] Let P be a nonempty subset of C. Then P is a filter of C if and only if the following hold:

$(SF-3)$ $1 \in P$,

$(SF-4)$ $c_1 \in P$ and $c_1 \mid (c_2 \mid c_2) \in P$ imply $c_2 \in P$.

Definition 2.4. [21] Let P be a filter of C. Then P is called an ultra filter of C if it satisfies $c \in P$ or $c \mid c \in P$, for all $c \in C$.

Lemma 2.3. [21] A filter P of C is an ultra filter of C if and only if $c_1 \vee c_2 \in P$ implies $c_1 \in P$ or $c_2 \in P$, for all $c_1, c_2 \in C$.

Definition 2.5. [8] $\mathcal{F}(X, [-1, 0])$ denotes the collection of functions from a set X to $[-1, 0]$ and an element of $\mathcal{F}(X, [-1, 0])$ is called a negative-valued function from X to $[-1, 0]$ (briefly, $\mathcal{N}$-function on X). An $\mathcal{N}$-structure refers to an ordered pair (X, f) of X and $\mathcal{N}$-function f on X.

Definition 2.6. [12] A neutrosophic $\mathcal{N}$-structure over a nonempty universe X is defined by

$$X_N := \frac{X}{(T_N, I_N, F_N)} = \left\{ \frac{x}{(T_N(x), I_N(x), F_N(x))} : x \in X \right\}$$

where T_N, I_N and F_N are $\mathcal{N}$-functions on X, called the negative truth membership function, the negative indeterminacy membership function and the negative falsity membership function, respectively.

Every neutrosophic $\mathcal{N}$-structure X_N over X satisfies the condition $(\forall x \in X)(-3 \leq T_N(x) + I_N(x) + F_N(x) \leq 0)$.

Definition 2.7. [13] Let X_N be a neutrosophic $\mathcal{N}$-structure on a set X and τ, γ, ρ be any elements of $[-1, 0]$ such that $-3 \leq \tau + \gamma + \rho \leq 0$. Consider the following sets:

$$T_N^\tau := \{x \in X : T_N(x) \leq \tau\},$$

$$I_N^\gamma := \{x \in X : I_N(x) \geq \gamma\}$$

and

$$F_N^\rho := \{x \in X : F_N(x) \leq \rho\}.$$

The set

$$X_N(\tau, \gamma, \rho) := \{x \in X : T_N(x) \leq \tau, I_N(x) \geq \gamma \text{ and } T_N(x) \leq \rho\}$$

is called the (τ, γ, ρ)-level set of X_N. Moreover, $X_N(\tau, \gamma, \rho) = T_N^\tau \cap I_N^\gamma \cap F_N^\rho$.

Consider sets

$$X_N^{c_t} := \{x \in X : T_N(x) \leq T_N(c_t)\},$$

$$X_N^{c_i} := \{x \in X : I_N(x) \geq I_N(c_i)\}$$

and

$$X_N^{c_f} := \{x \in X : F_N(x) \leq F_N(c_f)\},$$

for any $c_t, c_i, c_f \in X$. Obviously, $c_t \in X_N^{c_t}, c_i \in X_N^{c_i}$ and $c_f \in X_N^{c_f}$ [13].

3 Neutrosophic $\mathcal{N}$-Structures

In this section, neutrosophic $\mathcal{N}$-subalgebras and neutrosophic $\mathcal{N}$-filters on Sheffer stroke BL-algebras. Unless otherwise specified, C denotes a Sheffer stroke BL-algebra.

Definition 3.1. A neutrosophic $\mathcal{N}$-structure C_N on a Sheffer stroke BL-algebra C is called a neutrosophic $\mathcal{N}$-subalgebra of C if the following condition is valid:

$$\begin{aligned} &\min\{T_N(c_1), T_N(c_2)\} \leq T_N(c_1 \mid (c_2 \mid c_2)),\\ &\max\{I_N(c_1), I_N(c_2)\} \geq I_N(c_1 \mid (c_2 \mid c_2)) \ \textit{and} \\ &\max\{F_N(c_1), F_N(c_2)\} \geq F_N(c_1 \mid (c_2 \mid c_2)), \end{aligned} \tag{1}$$

for all $c_1, c_2 \in C$.

Example 3.1. Consider a Sheffer stroke BL-algebra C where the set $C = \{0, a, b, c, d, e, f, 1\}$ and the Sheffer operation $|$, the join operation $\vee$ and the meet operation $\wedge$ on C has the Cayley tables in Tab. 1 [21]. Then a neutrosophic $\mathcal{N}$-structure

$$C_N = \left\{\frac{x}{(-0.08, -0.999, -0.26)} : x = d, 1\right\} \cup \left\{\frac{x}{(-0.92, -0.52, -0.0012)} : x \in C - \{d, 1\}\right\}$$

on C is a neutrosophic $\mathcal{N}$-subalgebra of C.

Table 1: Tables of the Sheffer operation $|$, the join operation $\vee$ and the meet operation $\wedge$ on C

$\mid$	0	a	b	c	d	e	f	1
0	1	1	1	1	1	1	1	1
a	1	f	1	1	f	f	1	f
b	1	1	e	1	e	1	e	e
c	1	1	1	d	1	d	d	d
d	1	f	e	1	c	f	e	c
e	1	f	1	d	f	b	d	b
f	1	1	e	d	e	d	a	a
1	1	f	e	d	c	b	a	1

$\vee$	0	a	b	c	d	e	f	1
0	0	a	b	c	d	e	f	1
a	a	a	d	e	d	e	1	1
b	b	d	b	f	d	1	f	1
c	c	e	f	c	1	e	f	1
d	d	d	d	1	d	1	1	1
e	e	e	1	e	1	e	1	1
f	f	1	f	f	1	1	f	1
1	1	1	1	1	1	1	1	1

$\wedge$	0	a	b	c	d	e	f	1
0	0	0	0	0	0	0	0	0
a	0	a	0	0	a	a	0	a
b	0	0	b	0	b	0	b	b
c	0	0	0	c	0	c	c	c
d	0	a	b	0	d	a	b	d
e	0	a	0	c	a	e	c	e
f	0	0	b	c	b	c	f	f
1	0	a	b	c	d	e	f	1

Definition 3.2. Let C_N be a neutrosophic $\mathcal{N}$-structure on a Sheffer stroke BL-algebra C and τ, γ, ρ be any elements of $[-1,0]$ such that $-3 \leq \tau+\gamma+\rho \leq 0$. For the sets

$T_N^{\tau} := \{c \in C : T_N(c) \geq \tau\}$,

$I_N^{\gamma} := \{c \in C : I_N(c) \leq \gamma\}$

and

$F_N^{\rho} := \{c \in C : F_N(c) \leq \rho\}$,

the set

$C_N(\tau,\gamma,\rho) := \{c \in C : T_N(c) \geq \tau, I_N(c) \leq \gamma \ and \ F_N(c) \leq \rho\}$

is called the (τ,γ,ρ)-level set of C_N. Moreover, $C_N(\tau,\gamma,\rho) = T_N^{\tau} \cap I_N^{\gamma} \cap F_N^{\rho}$.

Definition 3.3. A subset D of a Sheffer stroke BL-algebra C is called a quasi-subalgebra of C if $c_1 \mid (c_2 \mid c_2) \in D$, for all $c_1, c_2 \in D$. Obviously, C itself and $\{1\}$ are quasi-subalgebras of C.

Example 3.2. Consider the Sheffer stroke BL-algebra C in Example 3.1. Then $\{0, a, f, 1\}$ is a quasi-subalgebra of C.

Theorem 3.1. Let C_N be a neutrosophic $\mathcal{N}$-structure on a Sheffer stroke BL-algebra C and τ, γ, ρ be any elements of $[-1,0]$ such that $-3 \leq \tau+\gamma+\rho \leq 0$. If C_N is a neutrosophic $\mathcal{N}$-subalgebra of C, then the nonempty level set $C_N(\tau,\gamma,\rho)$ of C_N is a quasi-subalgebra of C.

Proof. Let C_N be a neutrosophic $\mathcal{N}$-subalgebra of C and c_1, c_2 be any elements of $C_N(\tau,\gamma,\rho)$, for $\tau,\gamma,\rho \in [-1,0]$ with $-3 \leq \tau+\gamma+\rho \leq 0$. Then $T_N(c_1), T_N(c_2) \geq \tau, I_N(c_1), I_N(c_2) \leq \gamma$ and $F_N(c_1), F_N(c_2) \leq \rho$. Since

$\tau \leq \min\{T_N(c_1), T_N(c_2)\} \leq T_N(c_1 \mid (c_2 \mid c_2))$,

$I_N(c_1 \mid (c_2 \mid c_2)) \leq \max\{I_N(c_1), I_N(c_2)\} \leq \gamma$

and

$F_N(c_1 \mid (c_2 \mid c_2)) \leq \max\{F_N(c_1), F_N(c_2)\} \leq \rho$,

for all $c_1, c_2 \in C$, we obtain that $c_1 \mid (c_2 \mid c_2) \in T_N^{\tau}$, $c_1 \mid (c_2 \mid c_2) \in I_N^{\gamma}$ and $c_1 \mid (c_2 \mid c_2) \in F_N^{\rho}$, and so, $c_1 \mid (c_2 \mid c_2) \in T_N^{\tau} \cap I_N^{\gamma} \cap F_N^{\rho} = C_N(\tau,\gamma,\rho)$. Hence, $C_N(\tau,\gamma,\rho)$ is a quasi-subalgebra of C.

Theorem 3.2. Let C_N be a neutrosophic $\mathcal{N}$-structure on a Sheffer stroke BL-algebra C and T_N^{τ}, I_N^{γ} and F_N^{ρ} be quasi-subalgebras of C, for all $\tau,\gamma,\rho \in [-1,0]$ with $-3 \leq \tau+\gamma+\rho \leq 0$. Then C_N is a neutrosophic $\mathcal{N}$-subalgebra of C.

Proof. Let C_N be a neutrosophic $\mathcal{N}$-structure on a Sheffer stroke BL-algebra C, and T_N^{τ}, I_N^{γ} and F_N^{ρ} be quasi-subalgebras of C, for all $\tau,\gamma,\rho \in [-1,0]$ with $-3 \leq \tau+\gamma+\rho \leq 0$. Suppose that c_1 and c_2 be any elements of C such that $w_1 = T_N(c_1 \mid (c_2 \mid c_2)) < \min\{T_N(c_1), T_N(c_2)\} = w_2$, $t_1 = \max\{I_N(c_1), I_N(c_2)\} < I_N(c_1 \mid (c_2 \mid c_2)) = t_2$ and $r_1 = \max\{F_N(c_1), F_N(c_2)\} < F_N(c_1 \mid (c_2 \mid c_2)) = r_2$. If $\tau_1 = \dfrac{1}{2}(w_1+w_2) \in [-1,0)$, $\gamma_1 = \dfrac{1}{2}(t_1+t_2) \in [-1,0)$ and $\rho_1 = \dfrac{1}{2}(r_1+r_2) \in [-1,0)$, then $w_1 < \tau_1 < w_2$, $t_1 < \gamma_1 < t_2$ and $r_1 < \rho_1 < r_2$. Thus, $c_1, c_2 \in T_N^{\tau_1}$, $c_1, c_2 \in I_N^{\gamma_1}$ and $c_1, c_2 \in F_N^{\rho_1}$ but $c_1 \mid (c_2 \mid c_2) \notin T_N^{\tau_1}$, $c_1 \mid (c_2 \mid c_2) \notin I_N^{\gamma_1}$ and $c_1 \mid (c_2 \mid c_2) \notin F_N^{\rho_1}$, which are contradictions. Hence, $\min\{T_N(c_1), T_N(c_2)\} \leq T_N(c_1 \mid (c_2 \mid c_2))$, $I_N(c_1 \mid (c_2 \mid c_2)) \leq \max\{I_N(c_1), I_N(c_2)\}$ and $F_N(c_1 \mid (c_2 \mid c_2)) \leq \max\{F_N(c_1), F_N(c_2)\}$, for all $c_1, c_2 \in C$. Thereby, C_N is a neutrosophic $\mathcal{N}$-subalgebra of C.

Theorem 3.3. Let $\{C_{N_i} : i \in \mathbb{N}\}$ be a family of all neutrosophic $\mathcal{N}$-subalgebras of a Sheffer stroke BL-algebra C. Then $\{C_{N_i} : i \in \mathbb{N}\}$ forms a complete distributive lattice.

Proof. Let D be a nonempty subset of $\{C_{N_i} : i \in \mathbb{N}\}$. Since C_{N_i} is a neutrosophic $\mathcal{N}$-subalgebra of C, for all $i \in \mathbb{N}$, it satisfies the condition (1). Then $\bigcap D$ satisfies the condition (1). Thus, $\bigcap D$ is a neutrosophic $\mathcal{N}$-subalgebra of C. Let E be a family of all neutrosophic $\mathcal{N}$-subalgebras of C containing $\bigcup\{C_{N_i} : i \in \mathbb{N}\}$. Thus, $\bigcap E$ is also a neutrosophic $\mathcal{N}$-subalgebra of C. If $\bigwedge_{i\in\mathbb{N}} C_{N_i} = \bigcap_{i\in\mathbb{N}} C_{N_i}$ and $\bigvee_{i\in\mathbb{N}} C_{N_i} = \bigcap E$, then $(\{C_{N_i} : i \in \mathbb{N}\}, \bigvee, \bigwedge)$ forms a complete lattice. Also, it is distibutive by the definitions of $\bigvee$ and $\bigwedge$.

Lemma 3.1. Let C_N be a neutrosophic $\mathcal{N}$-subalgebra of a Sheffer stroke BL-algebra C. Then $T_N(c) \leq T_N(1)$, $I_N(c) \geq I_N(1)$ and $F_N(c) \geq F_N(1)$, for all $c \in C$.

Proof. Let C_N be a neutrosophic $\mathcal{N}$-subalgebra of C. Then it follows from Poposition 2.1 (2) that

$$T_N(c) = \min\{T_N(c), T_N(c)\} \leq T_N(c \mid (c \mid c)) = T_N(1),$$

$$I_N(1) = I_N(c \mid (c \mid c)) \leq \max\{I_N(c), I_N(c)\} = I_N(c)$$

and

$$F_N(1) = F_N(c \mid (c \mid c)) \leq \max\{F_N(c), F_N(c)\} = F_N(c),$$

for all $c \in C$.

The inverse of Lemma 3.1 is not true in general.

Example 3.3. Consider the Sheffer stroke BL-algebra C in Example 3.1. Then a neutrosophic $\mathcal{N}$-structure

$$C_N = \left\{\frac{x}{(-0.01, -0.1, -0.11)} : x = a, b, 1\right\} \cup \left\{\frac{x}{(-0.1, -0.01, -0.01)} : x \in C - \{a, b, 1\}\right\}$$

on C is not a neutrosophic $\mathcal{N}$-subalgebra of C since $\max\{F_N(a), F_N(b)\} = -0.11 < -0.01 = F_N(f) = F_N(a \mid (b \mid b))$.

Lemma 3.2. A neutrosophic $\mathcal{N}$-subalgebra C_N of a Sheffer stroke BL-algebra C satisfies $T_N(c_1) \leq T_N(c_1 \mid (c_2 \mid c_2))$, $I_N(c_1) \geq I_N(c_1 \mid (c_2 \mid c_2))$ and $F_N(c_1) \geq F_N(c_1 \mid (c_2 \mid c_2))$, for all $c_1, c_2 \in C$ if and only if T_N, I_N and F_N are constant.

Proof. Let C_N be a a neutrosophic $\mathcal{N}$-subalgebra of C such that $T_N(c_1) \leq T_N(c_1 \mid (c_2 \mid c_2))$, $I_N(c_1) \geq I_N(c_1 \mid (c_2 \mid c_2))$ and $F_N(c_1) \geq F_N(c_1 \mid (c_2 \mid c_2))$, for all $c_1, c_2 \in C$. Since $T_N(1) \leq T_N(1 \mid (c \mid c)) = T_N(c)$, $I_N(1) \geq I_N(1 \mid (c \mid c)) = I_N(c)$ and $F_N(1) \geq F_N(1 \mid (c \mid c)) = F_N(c)$ from Proposition 2.1 (3), it is obtained from Lemma 3.1 that $T_N(c) = T_N(1)$, $I_N(c) = I_N(1)$ and $F_N(c) = F_N(1)$, for all $c \in C$. Hence, T_N, I_N and F_N are constant.

Conversely, it is obvious since T_N, I_N and F_N are constant.

Definition 3.4. A neutrosophic $\mathcal{N}$-structure C_N on a Sheffer stroke BL-algebra C is called a neutrosophic $\mathcal{N}$-filter of C if

1. $c_1 \leq c_2$ implies $T_N(c_1) \leq T_N(c_2)$, $I_N(c_2) \leq I_N(c_1)$ and $F_N(c_2) \leq F_N(c_1)$,
2. $\min\{T_N(c_1), T_N(c_2)\} \leq T_N((c_1 \mid c_2) \mid (c_1 \mid c_2))$, $I_N((c_1 \mid c_2) \mid (c_1 \mid c_2)) \leq \max\{I_N(c_1), I_N(c_2)\}$ and $F_N((c_1 \mid c_2) \mid (c_1 \mid c_2)) \leq \max\{F_N(c_1), F_N(c_2)\}$,

for all $c_1, c_2 \in C$.

Example 3.4. Consider the Sheffer stroke BL-algebra C in Example 3.1. Then a neutrosophic $\mathcal{N}$-structure

$$C_N = \left\{ \frac{x}{(-0.3, -1, -0.15)} : x = c, e, f, 1 \right\} \cup \left\{ \frac{x}{(-1, -0.7, 0)} : x = 0, a, b, d \right\}$$

on C is a neutrosophic $\mathcal{N}$-filter of C.

Theorem 3.4. Let C_N be a a neutrosophic $\mathcal{N}$-structure on a Sheffer stroke BL-algebra C. Then C_N is a neutrosophic $\mathcal{N}$-filter of C if and only if

$$\begin{aligned}
&\min\{T_N(c_1), T_N(c_1 \mid (c_2 \mid c_2))\} \leq T_N(c_2) \leq T_N(1),\\
&I_N(1) \leq I_N(c_2) \leq \max\{I_N(c_1), I_N(c_1 \mid (c_2 \mid c_2))\} \textit{ and} \\
&F_N(1) \leq F_N(c_2) \leq \max\{F_N(c_1), F_N(c_1 \mid (c_2 \mid c_2))\},
\end{aligned} \tag{2}$$

for all $c_1, c_2 \in C$.

Proof. Let C_N be a neutrosophic $\mathcal{N}$-filter of C. Then it follows from (sBL-3) and Definition 3.4 that

$$\min\{T_N(c_1), T_N(c_1 \mid (c_2 \mid c_2))\} \leq T_N((c_1 \mid (c_1 \mid (c_2 \mid c_2))) \mid (c_1 \mid (c_1 \mid (c_2 \mid c_2)))) = T_N(c_1 \wedge c_2) \leq T_N(c_2) \leq T_N(1),$$

$$I_N(1) \leq I_N(c_2) \leq I_N(c_1 \wedge c_2) = I_N((c_1 \mid (c_1 \mid (c_2 \mid c_2))) \mid (c_1 \mid (c_1 \mid (c_2 \mid c_2)))) \leq \max\{I_N(c_1), I_N(c_1 \mid (c_2 \mid c_2))\}$$

and

$$F_N(1) \leq F_N(c_2) \leq F_N(c_1 \wedge c_2) = F_N((c_1 \mid (c_1 \mid (c_2 \mid c_2))) \mid (c_1 \mid (c_1 \mid (c_2 \mid c_2)))) \leq \max\{F_N(c_1), F_N(c_1 \mid (c_2 \mid c_2))\},$$

for all $c_1, c_2 \in C$.

Conversely, let C_N be a a neutrosophic $\mathcal{N}$-structure on C satisfying the condition (2). Assume that $c_1 \leq c_2$. Then $c_1 \mid (c_2 \mid c_2) = 1$ from Proposition 2.1 (7). Thus,

$$T_N(c_1) = \min\{T_N(c_1), T_N(1)\} = \min\{T_N(c_1), T_N(c_1 \mid (c_2 \mid c_2))\} \leq T_N(c_2),$$

$$I_N(c_2) \leq \max\{I_N(c_1), I_N(c_1 \mid (c_2 \mid c_2))\} = \max\{I_N(c_1), I_N(1)\} = I_N(c_1)$$

and

$$F_N(c_2) \leq \max\{F_N(c_1), F_N(c_1 \mid (c_2 \mid c_2))\} = \max\{F_N(c_1), F_N(1)\} = F_N(c_1),$$

for all $c_1, c_2 \in C$. Also, it follows from Proposition 2.1 (9), (S1) and (S2) that

$$\begin{aligned}
\min\{T_N(c_1), T_N(c_2)\} &\leq \min\{T_N(c_1), T_N(c_1 \mid (c_1 \mid c_2))\} \\
&= \min\{T_N(c_1), T_N(c_1 \mid (((c_1 \mid c_2) \mid (c_1 \mid c_2)) \mid ((c_1 \mid c_2) \mid (c_1 \mid c_2))))\} \\
&\leq T_N((c_1 \mid c_2) \mid (c_1 \mid c_2)),
\end{aligned}$$

$$\begin{aligned}
I_N((c_1 \mid c_2) \mid (c_1 \mid c_2)) &\leq \max\{I_N(c_1), I_N(c_1 \mid (((c_1 \mid c_2) \mid (c_1 \mid c_2)) \mid ((c_1 \mid c_2) \mid (c_1 \mid c_2))))\} \\
&= \max\{I_N(c_1), I_N(c_1 \mid (c_1 \mid c_2))\} \\
&\leq \max\{I_N(c_1), I_N(c_2)\}
\end{aligned}$$

and

$$\begin{aligned} F_N((c_1 \mid c_2) \mid (c_1 \mid c_2)) &\leq \max\{F_N(c_1), F_N(c_1 \mid (((c_1 \mid c_2) \mid (c_1 \mid c_2)) \mid ((c_1 \mid c_2) \mid (c_1 \mid c_2))))\} \\ &= \max\{F_N(c_1), F_N(c_1 \mid (c_1 \mid c_2))\} \\ &\leq \max\{F_N(c_1), F_N(c_2)\}, \end{aligned}$$

for all $c_1, c_2 \in C$. Thus, C_N is a neutrosophic $\mathcal{N}$-filter of C.

Corollary 3.1. Let C_N be a neutrosophic $\mathcal{N}$-filter of a Sheffer stroke BL-algebra C. Then

1. $\min\{T_N(c_3), T_N(c_3 \mid (((c_2 \mid (c_1 \mid c_1)) \mid (c_1 \mid c_1)) \mid ((c_2 \mid (c_1 \mid c_1)) \mid (c_1 \mid c_1))))\} \leq T_N((c_1 \mid (c_2 \mid c_2)) \mid (c_2 \mid c_2))$,
   $I_N((c_1 \mid (c_2 \mid c_2)) \mid (c_2 \mid c_2)) \leq \max\{I_N(c_3), I_N(c_3 \mid (((c_2 \mid (c_1 \mid c_1)) \mid (c_1 \mid c_1)) \mid ((c_2 \mid (c_1 \mid c_1)) \mid (c_1 \mid c_1))))\}$
   and $F_N((c_1 \mid (c_2 \mid c_2)) \mid (c_2 \mid c_2)) \leq \max\{F_N(c_3), F_N(c_3 \mid (((c_2 \mid (c_1 \mid c_1)) \mid (c_1 \mid c_1)) \mid ((c_2 \mid (c_1 \mid c_1)) \mid (c_1 \mid c_1))))\}$,
2. $\min\{T_N(c_3), T_N(c_3 \mid ((c_1 \mid (c_2 \mid c_2)) \mid\mid (c_1 \mid (c_2 \mid c_2))))\} \leq T_N(c_1 \mid (c_2 \mid c_2))$,
   $I_N(c_1 \mid (c_2 \mid c_2)) \leq \max\{I_N(c_3), I_N(c_3 \mid ((c_1 \mid (c_2 \mid c_2)) \mid (c_1 \mid (c_2 \mid c_2))))\}$ *and*
   $F_N(c_1 \mid (c_2 \mid c_2)) \leq \max\{F_N(c_3), F_N(c_3 \mid ((c_1 \mid (c_2 \mid c_2)) \mid (c_1 \mid (c_2 \mid c_2))))\}$,
3. $\min\{T_N(c_1 \mid ((c_2 \mid (c_3 \mid c_3)) \mid\mid (c_2 \mid (c_3 \mid c_3)))), T_N(c_1 \mid (c_2 \mid c_2))\} \leq T_N(c_1 \mid (c_3 \mid c_3))$,
   $I_N(c_1 \mid (c_3 \mid c_3)) \leq \max\{I_N(c_1 \mid ((c_2 \mid (c_3 \mid c_3)) \mid (c_2 \mid (c_3 \mid c_3)))), I_N(c_1 \mid (c_2 \mid c_2))\}$ *and*
   $F_N(c_1 \mid (c_3 \mid c_3)) \leq \max\{F_N(c_1 \mid ((c_2 \mid (c_3 \mid c_3)) \mid (c_2 \mid (c_3 \mid c_3)))), F_N(c_1 \mid (c_2 \mid c_2))\}$,
4. $T_N(c_1 \mid (c_2 \mid c_2)) = T_N(1)$, $I_N(c_1 \mid (c_2 \mid c_2)) = I_N(1)$ and $F_N(c_1 \mid (c_2 \mid c_2)) = F_N(1)$ imply $T_N(c_1) \leq T_N(c_2)$, $I_N(c_2) \leq I_N(c_1)$ and $F_N(c_2) \leq F_N(c_1)$,

for all $c_1, c_2, c_3 \in C$.

Proof. It is proved from Theorem 3.4, Lemma 2.1 and Lemma 2.2.

Lemma 3.3. Let C_N be a neutrosophic $\mathcal{N}$-structure on a Sheffer stroke BL-algebra C. Then C_N is a neutrosophic $\mathcal{N}$-filter of C if and only if

$$c_1 \leq c_2 \mid (c_3 \mid c_3) \ \textit{implies} \ \begin{pmatrix} \min\{T_N(c_1), T_N(c_2)\} \leq T_N(c_3), \\ I_N(c_3) \leq \max\{I_N(c_1), I_N(c_2)\} \quad \textit{and} \\ F_N(c_3) \leq \max\{F_N(c_1), F_N(c_2)\}, \end{pmatrix} \tag{3}$$

for all $c_1, c_2, c_3 \in C$.

Proof. Let C_N be a neutrosophic $\mathcal{N}$-filter of C and $c_1 \leq c_2 \mid (c_3 \mid c_3)$. Then it is obtained from Definition 3.4 (1) and Theorem 3.4 that

$$\min\{T_N(c_1), T_N(c_2)\} \leq \min\{T_N(c_2), T_N(c_2 \mid (c_3 \mid c_3))\} \leq T_N(c_3),$$

$$I_N(c_3) \leq \max\{I_N(c_2), I_N(c_2 \mid (c_3 \mid c_3))\} \leq \max\{I_N(c_1), I_N(c_2)\}$$

and

$$F_N(c_3) \leq \max\{F_N(c_2), F_N(c_2 \mid (c_3 \mid c_3))\} \leq \max\{F_N(c_1), F_N(c_2)\},$$

for all $c_1, c_2, c_3 \in C$.

Conversely, let C_N be a neutrosophic $\mathcal{N}$-structure on C satisfying the condition (3). Since it is known from Proposition 2.1 (4) that $c \leq 1 = c \mid (1 \mid 1)$, for all $c \in C$, we get that $T_N(c) = \min\{T_N(c), T_N(c)\} \leq T_N(1)$, $I_N(1) \leq \max\{I_N(c), I_N(c)\} = I_N(c)\}$ and $F_N(1) \leq \max\{F_N(c), F_N(c)\} = F_N(c)\}$, for all $c \in C$. Suppose that $c_1 \leq c_2$. Since we have $c_1 \leq c_2 = 1 \mid (c_2 \mid c_2)$ from Proposition 2.1

(3), it is obtained that $T_N(c_1) = \min\{T_N(c_1), T_N(1)\} \le T_N(c_2)$, $I_N(c_2) \le \max\{I_N(c_1), I_N(1)\} = I_N(c_1)$ and $F_N(c_2) \le \max\{F_N(c_1), F_N(1)\} = F_N(c_1)$. Since $c_1 \le (c_1 \mid c_2) \mid c_2 = c_2 \mid (((c_1 \mid c_2) \mid (c_1 \mid c_2)) \mid ((c_1 \mid c_2) \mid (c_1 \mid c_2)))$ from Proposition 2.1 (9), (S1) and (S2), it follows that

$$\min\{T_N(c_1), T_N(c_2)\} \le T_N((c_1 \mid c_2) \mid (c_1 \mid c_2)),$$

$$I_N((c_1 \mid c_2) \mid (c_1 \mid c_2)) \le \max\{I_N(c_1), I_N(c_2)\}$$

and

$$F_N((c_1 \mid c_2) \mid (c_1 \mid c_2)) \le \max\{F_N(c_1), F_N(c_2)\},$$

for all $c_1, c_2 \in C$. Thus, C_N is a neutrosophic $\mathcal{N}$-filter of C.

Lemma 3.4. Every neutrosophic $\mathcal{N}$-filter of a Sheffer stroke BL-algebra C is a neutrosophic $\mathcal{N}$-subalgebra of C.

Proof. Let C_N be a neutrosophic $\mathcal{N}$-filter of C. Since

$$\begin{aligned}
&((c_1 \mid c_2) \mid (c_1 \mid c_2)) \mid ((c_1 \mid (c_2 \mid c_2)) \mid (c_1 \mid (c_2 \mid c_2))) \\
&\quad = c_1 \mid ((((c_1 \mid c_2) \mid (c_1 \mid c_2)) \mid (c_2 \mid c_2)) \mid (((c_1 \mid c_2) \mid (c_1 \mid c_2)) \mid (c_2 \mid c_2))) \\
&\quad = c_1 \mid ((c_1 \mid ((c_2 \mid (c_2 \mid c_2)) \mid (c_2 \mid (c_2 \mid c_2)))) \mid (c_1 \mid ((c_2 \mid (c_2 \mid c_2)) \mid (c_2 \mid (c_2 \mid c_2))))) \\
&\quad = c_1 \mid ((c_1 \mid (1 \mid 1)) \mid (c_1 \mid (1 \mid 1))) \\
&\quad = c_1 \mid (1 \mid 1) \\
&\quad = 1
\end{aligned}$$

from Proposition 2.1 (1), (2), (4) and (S3), it follows from Proposition 2.1 (7) that $(c_1 \mid c_2) \mid (c_1 \mid c_2) \le c_1 \mid (c_2 \mid c_2)$, for all $c_1, c_2 \in C$. Then

$$\min\{T_N(c_1), T_N(c_2)\} \le T_N((c_1 \mid c_2) \mid (c_1 \mid c_2)) \le T_N(c_1 \mid (c_2 \mid c_2)),$$

$$I_N(c_1 \mid (c_2 \mid c_2)) \le I_N((c_1 \mid c_2) \mid (c_1 \mid c_2)) \le \max\{I_N(c_1), I_N(c_2)\}$$

and

$$F_N(c_1 \mid (c_2 \mid c_2)) \le F_N((c_1 \mid c_2) \mid (c_1 \mid c_2)) \le \max\{F_N(c_1), F_N(c_2)\},$$

for all $c_1, c_2 \in C$. Thereby, C_N is a neutrosophic $\mathcal{N}$-subalgebra of C.

The inverse of Lemma 3.4 is usually not true.

Example 3.5. Consider the Sheffer stroke BL-algebra C in Example 3.1. Then a neutrosophic $\mathcal{N}$-structure

$$C_N = \left\{\frac{0}{(-1,0,0)}, \frac{1}{(0,-1,-1)}\right\} \cup \left\{\frac{x}{(-0.5,-0.5,-0.5)} : x \in C - \{0,1\}\right\}$$

on C is a neutrosophic $\mathcal{N}$-subalgebra of C whereas it is not a neutrosophic $\mathcal{N}$-filter of C since $\min\{T_N(a), T_N(b)\} = -0.5 > -1 = T_N((a \mid b) \mid (a \mid b))$.

Definition 3.5. Let C_N be a neutrosophic $\mathcal{N}$-structure on a Sheffer stroke BL-algebra C. Then an ultra neutrosophic $\mathcal{N}$-filter C_N of C is a neutrosophic $\mathcal{N}$-filter of C satisfying $T_N(c) = T_N(1)$, $I_N(c) = I_N(1)$, $F_N(c) = F_N(1)$ or $T_N(c \mid c) = T_N(1)$, $I_N(c \mid c) = I_N(1)$, $F_N(c \mid c) = F_N(1)$, for all $c \in C$.

Example 3.6. Consider the Sheffer stroke BL-algebra C in Example 3.1. Then a neutrosophic $\mathcal{N}$-structure

$$C_N = \left\{ \frac{x}{(-0.02, -0.77, -0.6)} : x = b, d, f, 1 \right\} \cup \left\{ \frac{x}{(-0.79, -0.05, -0.41)} : x = 0, a, c, e \right\}$$

on C is an ultra neutrosophic $\mathcal{N}$-filter of C.

Remark 3.1. By Definition 3.5, every ultra neutrosophic $\mathcal{N}$-filter of a Sheffer stroke BL-algebra C is a neutrosophic $\mathcal{N}$-filter of C but the inverse does not generally hold.

Example 3.7. Consider the Sheffer stroke BL-algebra C in Example 3.1. Then a neutrosophic $\mathcal{N}$-filter

$$C_N = \left\{ \frac{x}{(-0.18, -0.82, -0.57)} : \ x = e, 1 \right\} \cup \left\{ \frac{x}{(-1, -0.64, -0.43)} : x \in C - \{e, 1\} \right\}$$

of C is not ultra since $T_N(a) \neq T_N(1) \neq T_N(a \mid a) = T_N(f)$, $I_N(a) \neq I_N(1) \neq I_N(a \mid a) = I_N(f)$ and $F_N(a) \neq F_N(1) \neq TF_N(a \mid a) = F_N(f)$.

Lemma 3.5. Let C_N be a neutrosophic $\mathcal{N}$-filter of a Sheffer stroke BL-algebra C. Then C_N is an ultra neutrosophic $\mathcal{N}$-filter of C if and only if $T_N(c_1) \neq T_N(1), T_N(c_2) \neq T_N(1)$, $I_N(c_1) \neq I_N(1), I_N(c_2) \neq I_N(1)$ and $F_N(c_1) \neq F_N(1), F_N(c_2) \neq F_N(1)$ imply $T_N(c_1 \mid (c_2 \mid c_2)) = T_N(1) = T_N(c_2 \mid (c_1 \mid c_1))$, $I_N(c_1 \mid (c_2 \mid c_2)) = I_N(1) = I_N(c_2 \mid (c_1 \mid c_1))$ and $F_N(c_1 \mid (c_2 \mid c_2)) = F_N(1) = F_N(c_2 \mid (c_1 \mid c_1))$, for all $c_1, c_2 \in C$.

Proof. Let C_N be an ultra neutrosophic $\mathcal{N}$-filter of C, and $T_N(c_1) \neq T_N(1), T_N(c_2) \neq T_N(1)$, $I_N(c_1) \neq I_N(1), I_N(c_2) \neq I_N(1)$ and $F_N(c_1) \neq F_N(1), F_N(c_2) \neq F_N(1)$, for any $c_1, c_2 \in C$. Then $T_N(c_1 \mid c_1) = T_N(1) = T_N(c_2 \mid c_2)$, $I_N(c_1 \mid c_1) = I_N(1) = I_N(c_2 \mid c_2)$ and $F_N(c_1 \mid c_1) = F_N(1) = F_N(c_2 \mid c_2)$. Since

$$(c_1 \mid c_1) \mid ((c_1 \mid (c_2 \mid c_2)) \mid (c_1 \mid (c_2 \mid c_2))) = (c_2 \mid c_2) \mid ((c_1 \mid (c_1 \mid c_1)) \mid (c_1 \mid (c_1 \mid c_1))) = (c_2 \mid c_2) \mid (1 \mid 1) = 1$$

and

$$(c_2 \mid c_2) \mid ((c_2 \mid (c_1 \mid c_1)) \mid (c_2 \mid (c_1 \mid c_1))) = (c_1 \mid c_1) \mid ((c_2 \mid (c_2 \mid c_2)) \mid (c_2 \mid (c_2 \mid c_2))) = (c_1 \mid c_1) \mid (1 \mid 1) = 1$$

from (S1), (S3), Proposition 2.1 (2) and (4), it follows from Theorem 3.4 that

$$T_N(1) = \min\{T_N(1), T_N(1)\} = \min\{T_N(c_1 \mid c_1), T_N((c_1 \mid c_1) \mid ((c_1 \mid (c_2 \mid c_2)) \mid (c_1 \mid (c_2 \mid c_2))))\} \leq T_N(c_1 \mid (c_2 \mid c_2)),$$

$$I_N(c_1 \mid (c_2 \mid c_2)) \leq \max\{I_N(c_1 \mid c_1), I_N((c_1 \mid c_1) \mid ((c_1 \mid (c_2 \mid c_2)) \mid (c_1 \mid (c_2 \mid c_2))))\} = \max\{I_N(1), I_N(1)\} = I_N(1),$$

$$F_N(c_1 \mid (c_2 \mid c_2)) \leq \max\{F_N(c_1 \mid c_1), F_N((c_1 \mid c_1) \mid ((c_1 \mid (c_2 \mid c_2)) \mid (c_1 \mid (c_2 \mid c_2))))\} = \max\{F_N(1), F_N(1)\} = F_N(1),$$

and similarly, $T_N(1) \leq T_N(c_2 \mid (c_1 \mid c_1))$, $I_N(c_2 \mid (c_1 \mid c_1)) \leq I_N(1)$, $F_N(c_2 \mid (c_1 \mid c_1)) \leq F_N(1)$. Hence, we obtain from Theorem 3.4 that $T_N(c_1 \mid (c_2 \mid c_2)) = T_N(1) = T_N(c_2 \mid (c_1 \mid c_1))$, $I_N(c_1 \mid (c_2 \mid c_2)) = I_N(1) = I_N(c_2 \mid (c_1 \mid c_1))$ and $F_N(c_1 \mid (c_2 \mid c_2)) = F_N(1) = F_N(c_2 \mid (c_1 \mid c_1))$, for all $c_1, c_2 \in C$.

Conversely, let C_N be a neutrosophic $\mathcal{N}$-filter of C such that $T_N(c_1) \neq T_N(1), T_N(c_2) \neq T_N(1)$, $I_N(c_1) \neq I_N(1), I_N(c_2) \neq I_N(1)$ and $F_N(c_1) \neq F_N(1), F_N(c_2) \neq F_N(1)$ imply $T_N(c_1 \mid (c_2 \mid c_2)) = T_N(1) = T_N(c_2 \mid (c_1 \mid c_1))$, $I_N(c_1 \mid (c_2 \mid c_2)) = I_N(1) = I_N(c_2 \mid (c_1 \mid c_1))$ and $F_N(c_1 \mid (c_2 \mid c_2)) = F_N(1) = F_N(c_2 \mid (c_1 \mid c_1))$, for all $c_1, c_2 \in C$. Assume that $T_N(c) \neq T_N(1) \neq T_N(0) = T_N(1 \mid 1)$, $I_N(c) \neq I_N(1) \neq I_N(0) = I_N(1 \mid 1)$ and $F_N(c) \neq F_N(1) \neq F_N(0) = F_N(1 \mid 1)$. Hence, $T_N(c \mid c) = T_N(1 \mid ((c \mid c) \mid (c \mid c))) = T_N(c \mid 1) = T_N(c \mid ((1 \mid 1) \mid (1 \mid 1))) = T_N(1)$, $T_N((1 \mid 1) \mid (c \mid c)) = T_N(1)$, $I_N(c \mid c) = I_N(1 \mid ((c \mid c) \mid (c \mid c))) = I_N(c \mid 1) = I_N(c \mid ((1 \mid 1) \mid (1 \mid 1))) = I_N(1)$, $I_N((1 \mid 1) \mid (c \mid c)) = I_N(1)$ and $F_N(c \mid c) = F_N(1 \mid ((c \mid c) \mid (c \mid c))) = F_N(c \mid 1) = F_N(c \mid ((1 \mid 1) \mid (1 \mid 1))) = F_N(1)$, $F_N((1 \mid 1) \mid$

$(c \mid c)) = F_N(1)$ from Proposition 2.1 (3), (4), (S1) and (S2). Suppose that $T_N(c \mid c) \neq T_N(1) \neq T_N(0) = T_N(1 \mid 1)$, $I_N(c) \neq I_N(1) \neq I_N(0) = I_N(1 \mid 1)$ and $F_N(c) \neq F_N(1) \neq F_N(0) = F_N(1 \mid 1)$. Thus, $T_N(c) = T_N(1 \mid (c \mid c)) = T_N((c \mid c) \mid ((1 \mid 1) \mid (1 \mid 1))) = T_N(1)$, $T_N((1 \mid 1) \mid ((c \mid c) \mid (c \mid c))) = T_N(1)$, $I_N(c) = I_N(1 \mid (c \mid c)) = I_N((c \mid c) \mid ((1 \mid 1) \mid (1 \mid 1))) = I_N(1)$, $I_N((1 \mid 1) \mid ((c \mid c) \mid (c \mid c))) = I_N(1)$ and $F_N(c) = F_N(1 \mid (c \mid c)) = F_N((c \mid c) \mid ((1 \mid 1) \mid (1 \mid 1))) = F_N(1)$, $F_N((1 \mid 1) \mid ((c \mid c) \mid (c \mid c))) = F_N(1)$ from Proposition 2.1 (3), (4), (S1) and (S2). Therefore, C_N is an ultra neutrosophic $\mathcal{N}$-filter of C.

Lemma 3.6. Let C_N be a neutrosophic $\mathcal{N}$-filter of a Sheffer stroke BL-algebra C. Then C_N is an ultra neutrosophic $\mathcal{N}$-filter of C if and only if $T_N(c_1 \vee c_2) \leq T_N(c_1) \vee T_N(c_2)$, $I_N(c_1) \vee I_N(c_2) \leq I_N(c_1 \vee c_2)$ and $F_N(c_1) \vee F_N(c_2) \leq F_N(c_1 \vee c_2)$, for all $c_1, c_2 \in C$.

Proof. Let C_N be an ultra neutrosophic $\mathcal{N}$-filter of C. If $T_N(c_1) = T_N(1), I_N(c_1) = I_N(1)$, $F_N(c_1) = F_N(1)$ or $T_N(c_2) = T_N(1), I_N(c_2) = I_N(1)$, $F_N(c_2) = F_N(1)$, then the proof is completed from Theorem 3.4. Assume that $T_N(c_1) \neq T_N(1) \neq T_N(c_2), I_N(c_1) \neq I_N(1) \neq I_N(c_2)$ and $F_N(c_1) \neq F_N(1) \neq F_N(c_2)$. Thus, we have from Lemma 3.5 that $T_N(c_1 \mid (c_2 \mid c_2)) = T_N(1) = T_N(c_2 \mid (c_1 \mid c_1))$, $I_N(c_1 \mid (c_2 \mid c_2)) = I_N(1) = I_N(c_2 \mid (c_1 \mid c_1))$ and $F_N(c_1 \mid (c_2 \mid c_2)) = F_N(1) = F_N(c_2 \mid (c_1 \mid c_1))$, for all $c_1, c_2 \in C$. Since

$$T_N(c_1 \vee c_2) = \min\{T_N(1), T_N(c_1 \vee c_2)\} = \min\{T_N(c_1 \mid (c_2 \mid c_2)), T_N((c_1 \mid (c_2 \mid c_2)) \mid (c_2 \mid c_2))\} \leq T_N(c_2),$$

$$I_N(c_2) \leq \max\{I_N(c_1 \mid (c_2 \mid c_2)), I_N((c_1 \mid (c_2 \mid c_2)) \mid (c_2 \mid c_2))\} = \max\{I_N(1), I_N(c_1 \vee c_2)\} = I_N(c_1 \vee c_2),$$

$$F_N(c_2) \leq \max\{F_N(c_1 \mid (c_2 \mid c_2)), F_N((c_1 \mid (c_2 \mid c_2)) \mid (c_2 \mid c_2))\} = \max\{F_N(1), I_N(c_1 \vee c_2)\} = F_N(c_1 \vee c_2),$$

and similarly, $T_N(c_1 \vee c_2) = T_N(c_2 \vee c_1) \leq T_N(c_1)$, $I_N(c_1) \leq I_N(c_2 \vee c_1) = I_N(c_1 \vee c_2)$, $F_N(c_1) \leq F_N(c_2 \vee c_1) = F_N(c_1 \vee c_2)$ from Corollary 2.1 and Theorem 3.4, it follows that $T_N(c_1 \vee c_2) \leq T_N(c_1) \vee T_N(c_2)$, $I_N(c_1) \vee I_N(c_2) \leq I_N(c_1 \vee c_2)$ and $F_N(c_1) \vee F_N(c_2) \leq F_N(c_1 \vee c_2)$, for all $c_1, c_2 \in C$.

Conversely, let C_N be a neutrosophic $\mathcal{N}$-filter of C satisfying that $T_N(c_1 \vee c_2) \leq T_N(c_1) \vee T_N(c_2)$, $I_N(c_1) \vee I_N(c_2) \leq I_N(c_1 \vee c_2)$ and $F_N(c_1) \vee F_N(c_2) \leq F_N(c_1 \vee c_2)$, for any $c_1, c_2 \in C$. Since

$$T_N(1) = T_N(c \mid (c \mid c)) = T_N((c \mid ((c \mid c) \mid (c \mid c))) \mid ((c \mid c) \mid (c \mid c))) = T_N(c \vee (c \mid c)) \leq T_N(c) \vee T_N(c \mid c),$$

$$I_N(c) \vee I_N(c \mid c) \leq I_N(c \vee (c \mid c)) = I_N((c \mid ((c \mid c) \mid (c \mid c))) \mid ((c \mid c) \mid (c \mid c))) = I_N(c \mid (c \mid c)) = I_N(1)$$

and

$$F_N(c) \vee F_N(c \mid c) \leq F_N(c \vee (c \mid c)) = F_N((c \mid ((c \mid c) \mid (c \mid c))) \mid ((c \mid c) \mid (c \mid c))) = F_N(c \mid (c \mid c)) = F_N(1)$$

from Proposition 2.1 (2), (S1), (S2) and Corollary 2.1, it is obtained from Theorem 3.4 that $T_N(c) \vee T_N(c \mid c) = T_N(1)$, $I_N(c) \vee I_N(c \mid c) = I_N(1)$ and $F_N(c) \vee F_N(c \mid c) = F_N(1)$, and so, $T_N(c) = T_N(1)$, $I_N(c) = I_N(1)$, $F_N(c) = F_N(1)$ or $T_N(c \mid c) = T_N(1)$, $I_N(c \mid c) = I_N(1)$, $F_N(c \mid c) = F_N(1)$, for all $c \in C$. Thus, C_N is an ultra neutrosophic $\mathcal{N}$-filter of C.

Theorem 3.5. Let C_N be a neutrosophic $\mathcal{N}$-structure on a Sheffer stroke BL-algebra C and τ, γ, ρ be any elements of $[-1, 0]$ with $-3 \leq \tau + \gamma + \rho \leq 0$. If C_N is a (ultra) neutrosophic $\mathcal{N}$-filter of C, then the nonempty subset $C_N(\tau, \gamma, \rho)$ is a (ultra) filter of C.

Proof. Let C_N be a neutrosophic $\mathcal{N}$-filter of C and $C_N(\tau, \gamma, \rho) \neq \emptyset$, for $\tau, \gamma, \rho \in [-1, 0]$ with $-3 \leq \tau + \gamma + \rho \leq 0$. Asumme that $c_1, c_2 \in C_N(\tau, \gamma, \rho)$. Since $\tau \leq T_N(c_1), \tau \leq T_N(c_2)$, $I_N(c_1) \leq \gamma, I_N(c_2) \leq \gamma$, $F_N(c_1) \leq \rho$ and $F_N(c_2) \leq \rho$, it follows that

$$\tau \leq \min\{T_N(c_1), T_N(c_2)\} \leq T_N((c_1 \mid c_2) \mid (c_1 \mid c_2)),$$

$$I_N((c_1 \mid c_2) \mid (c_1 \mid c_2)) \leq \max\{I_N(c_1), I_N(c_2)\} \leq \gamma$$

and

$F_N((c_1 \mid c_2) \mid (c_1 \mid c_2)) \leq \max\{F_N(c_1), f_N(c_2)\} \leq \rho.$

Then $(c_1 \mid c_2) \mid (c_1 \mid c_2) \in T_N^\tau, I_N^\gamma, F_N^\rho$, and so, $(c_1 \mid c_2) \mid (c_1 \mid c_2) \in C_N(\tau, \gamma, \rho)$. Suppose that $c_1 \in C_N(\tau, \gamma, \rho)$ and $c_1 \leq c_2$. Since $\tau \leq T_N(c_1) \leq T_N(c_2)$, $I_N(c_2) \leq I_N(c_1) \leq \gamma$ and $F_N(c_2) \leq F_N(c_1) \leq \rho$, we have that $c_2 \in T_N^\tau, I_N^\gamma, F_N^\rho$, and so, $c_2 \in C_N(\tau, \gamma, \rho)$. Hence, $C_N(\tau, \gamma, \rho)$ is a filter of C. Moreover, let C_N be an ultra neutrosophic $\mathcal{N}$-filter of C. Assume that $c_1 \vee c_2 \in C_N(\tau, \gamma, \rho)$. Since $\tau \leq T_N(c_1 \vee c_2)$, $I_N(c_1 \vee c_2) \leq \gamma$ and $F_N(c_1 \vee c_2) \leq \rho$, it is obtained from Lemma 3.6 that $\tau \leq T_N(c_1 \vee c_2) \leq T_N(c_1) \vee T_N(c_2)$, $I_N(c_1) \vee I_N(c_2) \leq I_N(c_1 \vee c_2) \leq \gamma$ and $F_N(c_1) \vee F_N(c_2) \leq F_N(c_1 \vee c_2) \leq \rho$, for all $c_1, c_2 \in C$. Thus, $\tau \leq T_N(c_1)$, $I_N(c_1) \leq \gamma$, $F_N(c_2) \leq \rho$ or $\tau \leq T_N(c_2)$, $I_N(c_2) \leq \gamma$, $F_N(c_2) \leq \rho$, and so, $c_1 \in C_N(\tau, \gamma, \rho)$ or $c_2 \in C_N(\tau, \gamma, \rho)$. By Lemma 2.3, $C_N(\tau, \gamma, \rho)$ is an ultra filter of C.

Theorem 3.6. Let C_N be a neutrosophic $\mathcal{N}$-structure on a Sheffer stroke BL-algebra C, and T_N^τ, I_N^γ and F_N^ρ be (ultra) filters of C, for all $\tau, \gamma, \rho \in [-1, 0]$ with $-3 \leq \tau + \gamma + \rho \leq 0$. Then C_N is a (ultra) neutrosophic $\mathcal{N}$-filter of C.

Proof. Let C_N be a neutrosophic $\mathcal{N}$-structure on C, and T_N^τ, I_N^γ and F_N^ρ be filters of C, for all $\tau, \gamma, \rho \in [-1, 0]$ with $-3 \leq \tau + \gamma + \rho \leq 0$. Assume that

$\tau_1 = T_N((c_1 \mid c_2) \mid (c_1 \mid c_2)) < \min\{T_N(c_1), T_N(c_2)\} = \tau_2,$

$\gamma_1 = \max\{I_N(c_1), I_N(c_2)\} < I_N((c_1 \mid c_2) \mid (c_1 \mid c_2)) = \gamma_2$

and

$\rho_1 = \max\{F_N(c_1), f_N(c_2)\} < F_N((c_1 \mid c_2) \mid (c_1 \mid c_2)) = \rho_2,$

for some $c_1, c_2 \in C$. If $\tau_0 = \frac{1}{2}(\tau_1 + \tau_2)$, $\gamma_0 = \dfrac{1}{2}(\gamma_1 + \gamma_2)$, $\rho_0 = \dfrac{1}{2}(\rho_1 + \rho_2) \in [-1, 0)$, then $\tau_1 < \tau_0 < \tau_2$, $\gamma_1 < \gamma_0 < \gamma_2$ and $\rho_1 < \rho_0 < \rho_2$. So, $(c_1 \mid c_2) \mid (c_1 \mid c_2) \notin T_N^{\tau_0}, I_N^{\gamma_0}, F_N^{\rho_0}$ when $c_1, c_2 \in T_N^{\tau_0}, I_N^{\gamma_0}, F_N^{\rho_0}$, which contradict with (SF-1). Thus

$\min\{T_N(c_1), T_N(c_2)\} \leq T_N((c_1 \mid c_2) \mid (c_1 \mid c_2)),$

$I_N((c_1 \mid c_2) \mid (c_1 \mid c_2)) \leq \max\{I_N(c_1), I_N(c_2)\}$

and

$F_N((c_1 \mid c_2) \mid (c_1 \mid c_2)) \leq \max\{F_N(c_1), f_N(c_2)\},$

for all $c_1, c_2 \in C$. Let $c_1 \leq c_2$. Suppose that $T_N(c_2) < T_N(c_1)$, $I_N(c_1) < I_N(c_2)$ and $F_N(c_1) < F_N(c_2)$, for some $c_1, c_2 \in C$. If $\tau^* = \dfrac{1}{2}(T_N(c_1) + T_N(c_2))$, $\gamma^* = \dfrac{1}{2}(I_N(c_1) + I_N(c_2))$, $\rho^* = \dfrac{1}{2}(F_N(c_1) + F_N(c_2)) \in [-1, 0)$, then $T_N(c_2) < \tau^* < T_N(c_1)$, $I_N(c_1) < \gamma^* < I_N(c_2)$ and $F_N(c_1) < \rho^* < F_N(c_2)$. Hence, $c_1 \in T_N^{\tau^*}, I_N^{\gamma^*}, F_N^{\rho^*}$ but $c_2 \notin T_N^{\tau^*}, I_N^{\gamma^*}, F_N^{\rho^*}$ which is a contradiction with (SF-2). Therefore, $T_N(c_1) \leq T_N(c_2)$, $I_N(c_2) \leq I_N(c_1)$ and $F_N(c_2) \leq F_N(c_1)$, for all $c_1, c_2 \in C$. Thereby, C_N is a neutrosophic $\mathcal{N}$-filter of C.

Also, let T_N^τ, I_N^γ and F_N^ρ be ultra filters of C, for all $\tau, \gamma, \rho \in [-1, 0]$ with $-3 \leq \tau + \gamma + \rho \leq 0$, and $T_N(c_1 \vee c_2) = \tau$, $I_N(c_1 \vee c_2) = \gamma$ and $F_N(c_1 \vee c_2) = \rho$. Since $c_1 \vee c_2 \in T_N^\tau, I_N^\gamma, F_N^\rho$, it follows from Lemma 2.3 that $c_1 \in T_N^\tau, I_N^\gamma, F_N^\rho$ or $c_2 \in T_N^\tau, I_N^\gamma, F_N^\rho$. Thus, $T_N(c_1 \vee c_2) = \tau \leq T_N(c_1), T_N(c_2)$,

$I_N(c_1), I_N(c_2) \leq \gamma = I_N(c_1 \vee c_2)$ and $F_N(c_1), F_N(c_2) \leq \rho = F_N(c_1 \vee c_2)$, and so, $T_N(c_1 \vee c_2) \leq T_N(c_1) \vee T_N(c_2)$, $I_N(c_1) \vee I_N(c_2) \leq I_N(c_1 \vee c_2)$ and $F_N(c_1) \vee F_N(c_2) \leq F_N(c_1 \vee c_2)$, for all $c_1, c_2 \in C$. By Lemma 3.6, C_N is an ultra neutrosophic $\mathcal{N}$-filter of C.

Definition 3.6. Let C be a Sheffer stroke BL-algebra. Define

$$C_N^{c_t} := \{c \in C : T_N(c_t) \leq T_N(c)\},$$

$$C_N^{c_i} := \{c \in C : I_N(c) \leq I_N(c_i)\}$$

and

$$C_N^{c_f} := \{c \in C : F_N(c) \leq F_N(c_f)\},$$

for all $c_t, c_i, c_f \in C$. It is obvious that $c_t \in C_N^{c_t}, c_i \in C_N^{c_i}$ and $c_f \in C_N^{c_f}$.

Example 3.8. Consider the Sheffer stroke BL-algebra C in Example 3.1. Let $c_t = a, c_i = b$, $c_f = c \in C$,

$$T_N(x) = \begin{cases} -0.18 & \text{if } x = 0, a, f, 1 \\ -0.29 & \text{otherwise,} \end{cases} \quad I_N(x) = \begin{cases} 0 & \text{if } x = d, e, f \\ -1 & \text{otherwise} \end{cases} \quad \textit{and } F_N(x) = \begin{cases} -0.55 & \text{if } x = 0, 1 \\ -0.56 & \text{if } x = a, b, c \\ -0.57 & \text{if } x = d, e, f. \end{cases}$$

Then

$$C_N^a = \{x \in C : T_N(a) \leq T_N(x)\} = \{x \in C : -0.18 \leq T_N(x)\} = \{0, a, f, 1\},$$

$$C_N^{xb} = \{x \in C : I_N(x) \leq I_N(b)\} = \{x \in C : I_N(x) \leq -1\} = \{0, a, b, c, 1\}$$

and

$$C_N^c = \{x \in C : F_N(x) \leq F_N(c)\} = \{x \in C : F_N(x) \leq -0.56\} = \{a, b, c, d, e, f\}.$$

Theorem 3.7. Let c_t, c_i and c_f be any elements of a Sheffer stroke BL-algebra C. If C_N is a (ultra) neutrosophic $\mathcal{N}$-filter of C, then $C_N^{c_t}, C_N^{c_i}$ and $C_N^{c_f}$ are (ultra) filters of C.

Proof. Let c_t, c_i and c_f be any elements of C and C_N be a neutrosophic $\mathcal{N}$-filter of C. Assume that $c_1, c_2 \in C_N^{c_t}, C_N^{c_i}, C_N^{c_f}$. Since $T_N(c_t) \leq T_N(c_1), T_N(c_t) \leq T_N(c_2)$, $I_N(c_1) \leq I_N(c_i), I_N(c_2) \leq I_N(c_i)$ and $F_N(c_1) \leq F_N(c_f), F_N(c_2) \leq F_N(c_f)$, we get that

$$T_N(c_t) \leq \min\{T_N(c_1), T_N(c_2)\} \leq T_N((c_1 \mid c_2) \mid (c_1 \mid c_2)),$$

$$I_N((c_1 \mid c_2) \mid (c_1 \mid c_2)) \leq \max\{I_N(c_1), I_N(c_2)\} \leq I_N(c_i)$$

and

$$F_N((c_1 \mid c_2) \mid (c_1 \mid c_2)) \leq \max\{F_N(c_1), F_N(c_2)\} \leq F_N(c_f).$$

Then $(c_1 \mid c_2) \mid (c_1 \mid c_2) \in C_N^{c_t}, C_N^{c_i}, C_N^{c_f}$. Suppose that $c_1 \in C_N^{c_t}, C_N^{c_i}, C_N^{c_f}$ and $c_1 \leq c_2$. Since $T_N(c_t) \leq T_N(c_1) \leq T_N(c_2)$, $I_N(c_2) \leq I_N(c_1) \leq I_N(c_i)$ and $F_N(c_2) \leq F_N(c_1) \leq F_N(c_f)$, it is obtained that $c_2 \in C_N^{c_t}, C_N^{c_i}, C_N^{c_f}$. Thus, $C_N^{c_t}, C_N^{c_i}$, $C_N^{c_f}$ are filters of C.

Let C_N be an ultra neutrosophic $\mathcal{N}$-filter of C and $c_1 \vee c_2 \in C_N^{c_t}, C_N^{c_i}, C_N^{c_f}$. Since

$T_N(c_t) \leq T_N(c_1 \vee c_2) \leq T_N(c_1) \vee T_N(c_2)$,

$I_N(c_1) \vee I_N(c_2) \leq I_N(c_1 \vee c_2) \leq I_N(c_i)$

and

$F_N(c_1) \vee F_N(c_2) \leq F_N(c_1 \vee c_2) \leq F_N(c_f)$

from Lemma 3.6, it follows that $T_N(c_t) \leq T_N(c_1)$, $I_N(c_1) \leq I_N(c_i)$, $F_N(c_1) \leq F_N(c_f)$ or $T_N(c_t) \leq T_N(c_2)$, $I_N(c_2) \leq I_N(c_i)$, $F_N(c_2) \leq F_N(c_f)$. Hence, $c_1 \in C_N^{c_t}, C_N^{c_i}, C_N^{c_f}$ or $c_2 \in C_N^{c_t}, C_N^{c_i}, C_N^{c_f}$. Therefore, $C_N^{c_t}, C_N^{c_i}$ and $C_N^{c_f}$ are ultra filters of C from Lemma 2.3.

Example 3.9. Consider the Sheffer stroke BL-algebra C in Example 3.1. For a neutrosophic $\mathcal{N}$-filter

$$C_N = \left\{\frac{x}{(-0.21, -0.41, -0.61)} : x = 0, a, b, d\right\} \cup \left\{\frac{x}{(-0.13, -0.53, -0.93)} : x = c, e, f, 1\right\}$$

of C, $c_t = b$, $c_i = c$ and $c_f = f \in C$, the subsets

$C_N^b = \{x \in C : T_N(b) \leq T_N(x)\} = \{x \in C : -0.21 \leq T_N(x)\} = C$,

$C_N^c = \{x \in C : I_N(x) \leq I_N(c)\} = \{x \in C : I_N(x) \leq -0.53\} = \{c, e, f, 1\}$

and

$C_N^f = \{x \in C : F_N(x) \leq F_N(f)\} = \{x \in C : F_N(x) \leq -0.93\} = \{c, e, f, 1\}$

of C are filters of C. Also, C_N^b, C_N^c and C_N^f are ultra since C_N is ultra.

The inverse of Theorem 3.7 does not hold in general.

Example 3.10. Consider the Sheffer stroke BL-algebra C in Example 3.1. Then

$C_N^c = \{x \in C : T_N(c) \leq T_N(x)\} = \{x \in C : -0.11 \leq T_N(x)\} = C$,

$C_N^d = \{x \in C : I_N(x) \leq I_N(d)\} = \{x \in C : I_N(x) \leq 0\} = C$

and

$C_N^e = \{x \in C : F_N(x) \leq F_N(e)\} = \{x \in C : F_N(x) \leq -0.12\} = C$

of C are filters of C but a neutrosophic $\mathcal{N}$-structure

$$C_N = \left\{\frac{x}{(-0.11, 0, -0.12)} : x = 0, c, d, e\right\} \cup \left\{\frac{x}{(0, -1, -0.87)} : x = a, b, f, 1\right\}$$

is not a neutrosophic $\mathcal{N}$-filter of C since $T_N(d) = -0.11 < 0 = T_N(a)$ when $a \leq d$.

Theorem 3.8. Let c_t, c_i and c_f be any elements of a Sheffer stroke BL-algebra C and C_N be a neutrosophic $\mathcal{N}$-structure on C.

1. If $C_N^{c_t}, C_N^{c_i}$ and $C_N^{c_f}$ are filters of C, then

$$T_N(c_1) \leq \min\{T_N(c_2 \mid (c_3 \mid c_3)), T_N(c_2)\} \Rightarrow T_N(c_1) \leq T_N(c_3),$$
$$\max\{I_N(c_2 \mid (c_3 \mid c_3)), I_N(c_2)\} \leq I_N(c_1) \Rightarrow I_N(c_3) \leq I_N(c_1) \text{ and} \quad (4)$$

$\max\{F_N(c_2 \mid (c_3 \mid c_3)), F_N(c_2)\} \leq F_N(c_1) \Rightarrow F_N(c_3) \leq F_N(c_1)$,

for all $c_1, c_2, c_3 \in C$.

2. If C_N satisfies the condition (4) and

$$c_1 \leq c_2 \textit{ implies } T_N(c_1) \leq T_N(c_2),\ I_N(c_2) \leq I_N(c_1) \textit{ and } F_N(c_2) \leq F_N(c_1), \tag{5}$$

for all $c_1, c_2, c_3 \in C$, then $C_N^{c_t}, C_N^{c_i}$ and $C_N^{c_f}$ are filters of C, for all $c_t \in T_N^{-1}$, $c_i \in I_N^{-1}$ and $c_f \in F_N^{-1}$.

Proof. Let C_N be a neutrosophic $\mathcal{N}$-structure on C.

1. Assume that $C_N^{c_t}, C_N^{c_i}$ and $C_N^{c_f}$ are filters of C, for all $c_t, c_i, c_f \in C$, and c_1, c_2 and c_3 are any elements of C such that $T_N(c_1) \leq \min\{T_N(c_2 \mid (c_3 \mid c_3)), T_N(c_2)\}$, $\max\{I_N(c_2 \mid (c_3 \mid c_3)), I_N(c_2)\} \leq I_N(c_1)$ and $\max\{F_N(c_2 \mid (c_3 \mid c_3)), F_N(c_2)\} \leq F_N(c_1)$. Since $c_2 \mid (c_3 \mid c_3), c_2 \in C_N^{c_t}, C_N^{c_i}, C_N^{c_f}$ where $c_t = c_i = c_f = c_1$, we have from (SF-4) that $c_3 \in C_N^{c_t}, C_N^{c_i}, C_N^{c_f}$ where $c_t = c_i = c_f = c_1$. So, $T_N(c_1) \leq T_N(c_3)$, $I_N(c_3) \leq I_N(c_1)$ and $F_N(c_3) \leq F_N(c_1)$, for all $c_1, c_2, c_3 \in C$.
2. Suppose that C_N be a neutrosophic $\mathcal{N}$-structure on C satisfying the conditions (4) and (5), for any $c_t \in T_N^{-1}, c_i \in I_N^{-1}$ and $c_f \in F_N^{-1}$. Let $c_1, c_2 \in C_N^{c_t}, C_N^{c_i}, C_N^{c_f}$. Since $c_2 \leq (c_2 \mid c_1) \mid c_1 = c_1 \mid (((c_1 \mid c_2) \mid (c_1 \mid c_2)) \mid ((c_1 \mid c_2) \mid (c_1 \mid c_2)))$ from Proposition 2.1 (9), (S1)–(S2), and $T_N(c_t) \leq T_N(c_1), T_N(c_t) \leq T_N(c_2)$, $I_N(c_1) \leq I_N(c_i), I_N(c_2) \leq I_N(c_i)$, $F_N(c_1) \leq F_N(c_f)$ and $F_N(c_2) \leq F_N(c_f)$, it follows from the condition (5) that

$$T_N(c_t) \leq \min\{T_N(c_1), T_N(c_2)\} \leq \min\{T_N(c_1), T_N(c_1 \mid (((c_1 \mid c_2) \mid (c_1 \mid c_2)) \mid ((c_1 \mid c_2) \mid (c_1 \mid c_2))))\},$$

$$\max\{I_N(c_1), I_N(c_1 \mid (((c_1 \mid c_2) \mid (c_1 \mid c_2)) \mid ((c_1 \mid c_2) \mid (c_1 \mid c_2))))\} \leq \max\{I_N(c_1), I_N(c_2)\} \leq I_N(c_i)$$

and $\max\{F_N(c_1), F_N(c_1 \mid (((c_1 \mid c_2) \mid (c_1 \mid c_2)) \mid ((c_1 \mid c_2) \mid (c_1 \mid c_2))))\} \leq \max\{F_N(c_1), F_N(c_2)\} \leq F_N(c_f)$.

Thus, $T_N(c_t) \leq T_N((c_1 \mid c_2) \mid (c_1 \mid c_2))$, $I_N((c_1 \mid c_2) \mid (c_1 \mid c_2)) \leq I_N(c_i)$ and $F_N((c_1 \mid c_2) \mid (c_1 \mid c_2)) \leq F_N(c_f)$ from the condition (4), and so, $(c_1 \mid c_2) \mid (c_1 \mid c_2) \in C_N^{c_t}, C_N^{c_i}, C_N^{c_f}$. Let $c_1 \leq c_2$ and $c_1 \in C_N^{c_t}, C_N^{c_i}, C_N^{c_f}$. Since $T_N(c_t) \leq T_N(c_1) \leq T_N(c_2)$, $I_N(c_2) \leq I_N(c_1) \leq I_N(c_i)$ and $F_N(c_2) \leq F_N(c_1) \leq F_N(c_f)$ from condition (5), it is obtained that $c_2 \in C_N^{c_t}, C_N^{c_i}, C_N^{c_f}$. Thereby, $C_N^{c_t}, C_N^{c_i}$ and $C_N^{c_f}$ are filters of C.

Example 3.11. Consider the Sheffer stroke BL-algebra C in Example 3.1. Let

$$T_N(x) = \begin{cases} -0.07 & \text{if } x = 1 \\ -0.77 & \text{otherwise,} \end{cases} \quad I_N(x) = \begin{cases} -0.63 & \text{if } x = e, 1 \\ 0 & \text{otherwise,} \end{cases} \quad \textit{and } F_N(x) = \begin{cases} -0.84 & \text{if } x = a, d, e, 1 \\ -0.42 & \text{otherwise.} \end{cases}$$

Then the filters $C_N^{c_t} = C, C_N^{c_i} = \{e.1\}$ and $C_N^{c_f} = \{a, d, e, 1\}$ of C satisfy the condition (4), for the elements $c_t = a, c_i = e$ and $c_f = d$ of C.

Also, let

$$C_N = \left\{ \frac{x}{(-0.91, -0.23, -0.001)} : x \in C - \{1\} \right\} \cup \left\{ \frac{1}{(-0.17, -0.86, -0.79)} \right\}$$

be a neutrosophic $\mathcal{N}$-structure on C satisfying the conditions (4) and (5). Then the subsets

$C_N^{c_t} = \{x \in C : T_N(f) \le T_N(x)\} = \{x \in C : -0.91 \le T_N(x)\} = C$,

$C_N^{c_i} = \{x \in C : I_N(x) \le I_N(b)\} = \{x \in A : I_N(x) \le -0.23\} = C$

and

$C_N^{c_f} = \{x \in C : F_N(x) \le F_N(1)\} = \{x \in C : F_N(x) \le -0.79\} = \{1\}$

of C are filters of C, where $c_t = f, c_i = b$ and $c_f = 1$ of C.

4 Conclusion

In the study, neutrosophic $\mathcal{N}$-structures defined by $\mathcal{N}$-functions on Sheffer stroke BL-algebras have been examined. By giving basic definitions and notions of Sheffer stroke BL-algebras and neutrosophic $\mathcal{N}$-structures on a crispy set X, a neutrosophic $\mathcal{N}$-subalgebra and a (τ, γ, ρ)-level set of a neutrosophic $\mathcal{N}$-structure are defined on Sheffer stroke BL-algebras. We determine a quasi-subalgebra of a Sheffer stroke BL-algebra and prove that the (τ, γ, ρ)-level set of a neutrosophic $\mathcal{N}$-subalgebra of a Sheffer stroke BL-algebra is its quasi-subalgebra and vice versa. Besides, it is stated that the family of all neutrosophic $\mathcal{N}$-subalgebras of the algebra forms a complete distributive lattice. It is illustrated that every neutrosophic $\mathcal{N}$-subalgebra of a Sheffer stroke BL-algebra satisfies $T_N(x) \le T_N(1)$, $I_N(1) \le I_N(x)$ and $F_N(1) \le F_N(x)$, for all elements x of the algebra but the inverse does not generally hold. We interpret the case which $\mathcal{N}$-functions defining a neutrosophic $\mathcal{N}$-subalgebra of a Sheffer stroke BL-algebra are constant. Also, a (ultra) neutrosophic $\mathcal{N}$-filter of a Sheffer stroke BL-algebra is described and some properties are analysed. Indeed, it is proved that every neutrosophic $\mathcal{N}$-filter of a Sheffer stroke BL-algebra is the neutrosophic $\mathcal{N}$-subalgebra but the inverse is not true in general, and that the (τ, γ, ρ)-level set of a (ultra) neutrosophic $\mathcal{N}$-filter of a Sheffer stroke BL-algebra is its (ultra) filter and the inverse is always true. After that the subsets $C_N^{c_t}, C_N^{c_i}$ and $C_N^{c_f}$ of a Sheffer stroke BL-algebra are described by means of $\mathcal{N}$-functions and any elements c_t, c_i and c_f of this algebraic structure, it is demonstrated that these subsets are (ultra) filters of a Sheffer stroke BL-algebra if C_N is the (ultra) neutrosophic $\mathcal{N}$-filter.

In future works, we wish to study on plithogenic structures and relationships between neutrosophic $\mathcal{N}$-structures on some algebraic structures.

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