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ARTICLE

Parameter Estimation Based on Censored Data under Partially Accelerated Life Testing for Hybrid Systems due to Unknown Failure Causes

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ABSTRACT

In general, simple subsystems like series or parallel are integrated to produce a complex hybrid system. The reliability of a system is determined by the reliability of its constituent components. It is often extremely difficult or impossible to get specific information about the component that caused the system to fail. Unknown failure causes are instances in which the actual cause of system failure is unknown. On the other side, thanks to current advanced technology based on computers, automation, and simulation, products have become incredibly dependable and trustworthy, and as a result, obtaining failure data for testing such exceptionally reliable items have become a very costly and time-consuming procedure. Therefore, because of its capacity to produce rapid and adequate failure data in a short period of time, accelerated life testing (ALT) is the most utilized approach in the field of product reliability and life testing. Based on progressively hybrid censored (PrHC) data from a three-component parallel series hybrid system that failed to owe to unknown causes, this paper investigates a challenging problem of parameter estimation and reliability assessment under a step stress partially accelerated life-test (SSPALT). Failures of components are considered to follow a power linear hazard rate (PLHR), which can be used when the failure rate displays linear, decreasing, increasing or bathtub failure patterns. The Tempered random variable (TRV) model is considered to reflect the effect of the high stress level used to induce early failure data. The maximum likelihood estimation (MLE) approach is used to estimate the parameters of the PLHR distribution and the acceleration factor. A variance covariance matrix (VCM) is then obtained to construct the approximate confidence intervals (ACIs). In addition, studentized bootstrap confidence intervals (ST-B CIs) are also constructed and compared with ACIs in terms of their respective interval lengths (ILs). Moreover, a simulation study is conducted to demonstrate the performance of the estimation procedures and the methodology discussed in this paper. Finally, real failure data from the air conditioning systems of an airplane is used to illustrate further the performance of the suggested estimation technique.

KEYWORDS

Step stress partially accelerated life test; progressive hybrid censoring; data masking; power linear hazard rate distribution; hybrid system

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1 Introduction

Computers, cellphones, and many other systems of competitive innovation and modernization are getting increasingly complex, with numerous subsystems and sub-assemblies in each. Furthermore, these subsystems and subassemblies are made up of several components, making the life testing procedure for such systems more complicated. The study of reliability and life testing of systems, subsystems, or components is entirely depend on lifetime data, which is a combination of two important pieces of information. The first is to determine the product's failure time, and the second is to determine the reasons for its failure. The failure times of the system can easily be recorded, but the cause of the failure is not always recognized due to a variety of factors, including a lack of adequate diagnosis, time and expense constraints for comprehensive failure analysis, and numerous component failures with destructive consequences. As a result, in the literature of reliability and life testing analysis, such data in which the true cause of system failure is unknown and only a minimum random subset of the reasons that are responsible for system failure can be recognized is referred to as masked data. See [1,2] for more details.

The overall quality of today's products due to the existing advanced technology based on computers, automation and simulation has been improved drastically, which makes them extremely reliable and trustworthy. Consequently, gathering failure data for testing of such extremely reliable products using ordinary reliability tests (ORTs) has become a very costly and time-consuming process, making the use of ORTs impractical. Hence, ALTs are a more advanced approach for obtaining fast failure data by testing items under higher stress than normal, and then, a life stress relationship is used to get the product's life characteristics under normal usage settings. According to [3], stress loading in ALT may be performed in a variety of ways, although the most commonly used stress loadings are constant, step, and progressive stress loadings. Many scholars so far have looked at the ALT models, including [4–10]. Assuming a lognormal lifetime distribution, Li et al. [5] proposed two types of Bayesian accelerated acceptance sampling plans for illustrating product reliability based on the product's operating characteristic curve under Type-I censoring. The first plan addresses both producer and customer risks at the same time, whereas the second exclusively considers consumer risk. Rahman et al. [9] used MLE methods for estimating the parameters of Burr-X life distribution parameters assuming that failure under arithmetically increasing stress levels of CSALT forms a geometric process. Ma et al. [10] proposed an optimum hybrid accelerated test plan under many experimental design restrictions by combining ALT with accelerated degradation testing and modeling the degradation process with an inverse Gaussian process.

However, there are situations when these sorts of relationships are not possible. Because of the prevalence of this issue in ALTs, PALTs are considered to be preferable option and are usually implemented in two ways. The first is constant stress PALT, while the second is SSPALT. In SSPALT, a sample of components or systems is first tested under normal usage circumstances for a certain amount of time, and then systems or components that have not failed are allocated to testing at accelerated conditions until all items fail or a predetermined censorship scheme is met. SSPALT analysis has been considered by many authors since it was first suggested by [11,12] as a TRV model. Bhattacharyya et al. [13] developed a tampered failure rate model using the TRV model. Bai et al. [14–16] are others who considered SSPALT using the same concept of the TRV model for different distributions and censoring schemes. Assuming the TRV model, Zhang et al. [17] discussed the MLEs of the unknown parameters for the extended Weibull distribution. Ismail et al. [18,19] investigated the MLEs of the parameters of the Weibull and Burr Type-XII distributions, respectively, and compared the results based on two different PrHC schemes. Mahmoud et al. [20–22] deals with SSPALT based on an adaptive type PrHC scheme to obtain estimates of parameters using MLEs and Bayesian estimates (BEs) of generalized Pareto, two-parameter exponentiated Weibull and Lindley distributions, respectively.

A considerable number of studies have been carried out on parameter estimation using mask data based on ORTs since it was first introduced by [1,2]. Considering different failure distributions, Guess et al. [23–33] utilizes the MLE technique for estimating model parameters for a single component or a series or parallel systems of two or three components, whereas Reiser et al. [34–43] considered BE technique based on different priors. As it was discussed earlier, many real-life systems or machines these days are made of hybrid structures which are a combination of series and parallel subsystems. More complex systems can have many hybrid subsystems which are generally connected in parallel-series, series-parallel, series-parallel-series, and parallel-series-parallel configurations and so on. Peng et al. [44] developed a Bayesian approach for system reliability evaluation and prediction in which pass-fail data, lifespan data, and deterioration data are integrated coherently at multiple system levels. Yang et al. [45] developed an Adaptive Bayesian Melding reliability evaluation technique for analyzing and assessing the reliability of a hierarchical system with imperfect prior knowledge. They also expanded the concept to a broader multi-level hierarchical structure by employing a more effective method of pooling inconsistent priors. Yang et al. [46] proposed a Bayesian reliability approach for complex systems with dependent life metrics and developed a likelihood decomposition method to convert the overall likelihood into a product of explicit and implicit evidence-based likelihood functions. As of now, only a limited number of research papers considering hybrid systems based on ORTs have been published. Considering three component parallel-series and series-parallel systems, Wang et al. [47] obtained the MLEs of the parameters based on masked data assuming constant and linear hazard rate of independent components. Sha et al. [48] considered a hybrid system of three dependent components and obtained the MLEs based on mask data assuming a bivariate exponential distribution. Cai et al. [49] considered the same system as in [48] and derived the MLEs based copula function under mask cause of failure. Recently, Rodrigues et al. [50,51] used more complex structures based on four or five components and obtained MLEs and BEs under incomplete data.

Only a few studies on ALTs so far that focused on hybrid systems and masked data have been published in available research. Reference [52] under SSALT, describes the procedure of obtaining MLEs of the parameters of the Weibull distribution for a series system based on an expectation minimization algorithm assuming symmetric masking. Considering masked interval data in SSALT, Fan et al. [53] obtained estimates of the parameters of the exponential distribution. Xu et al. [54,55] described the general Bayesian analysis of the series system masked failure data for the log-location-scale and Weibull distributions respectively under SSALT. Assuming the same exponential hazard rate for a four components hybrid system, Shi et al. [56,57] obtained the MLEs of unknown parameters of the model based on the masked data under SSPALT and CSPALT respectively. Shi et al. [58] considered two different hybrid systems of three components and then obtained MLEs of the modified Weibull distribution.

To the best of our knowledge so far, no article has been published that considers SSALT for PLHR distribution for a hybrid system under masked data. Much of the sources listed above considered hazard rates that are monotonic in nature, but there are cases where the hazard rate is not monotonic. The primary goal of this work is to describe the SSPALT using a more flexible PLHR that can be used when failure rates indicate non-monotonic characteristics for hybrid systems. To illustrate the considered estimation procedure under SSPALT using PrHC masked data, a three-component hybrid system is considered. The rest of the paper is organized as follows, Section 2 addresses the formulation of the SSPALT model for PrHC masked data from a hybrid system and some useful assumptions. The MLEs and corresponding ACIs and ST-B CIs of the parameters and acceleration factor are discussed in Section 3. In Section 4, we conducted a simulation study to demonstrate the performance of the estimation procedures and the methodology discussed in this paper. Section 5 addresses the suggested approach's real-life data applicability. Finally, we conclude our study in Section 6.

2 Design and Assumptions of the Model

2.1 PLHR Distribution

In reliability and life testing experiments/studies, hazard rate is the most important function since it plays a very important role in characterizing the aging process of the systems and hence in classifying failure time distributions, more details can be seen in [59]. Commonly used hazard rates are constant, linear and power hazard rates. However, distributions derived from these hazard rates are very useful and popular in reliability and life testing theory when the failure rate depicts monotonic properties, but they cannot be used to fit non-monotonic failure rates. Therefore, in this paper, a more flexible PLHR distribution introduced by [60] that is derived from the combination of linear and power hazard rates is considered. Different shapes of hazard rate and density function of PLHR distribution with different values of parameters are given in Figs. 1 and 2, respectively. The Probability density function (PDF), cumulative distribution function (CDF), reliability function (RF) and the hazard rate (HR) of PLHR distribution are given respectively by the following equations:

$$f(t,\gamma,\kappa) = \left(t+\gamma t^{\kappa}\right) exp\left(-\frac{1}{2}t^2 - \frac{\gamma}{\kappa+1}t^{\kappa+1}\right);\ t>0,\ \gamma>0,\ \kappa>-1\ \&\ \kappa\neq 1 \tag{1}$$

$$F(t) = 1 - exp\left(-\frac{1}{2}t^2 - \frac{\gamma}{\kappa+1}t^{\kappa+1}\right) \tag{2}$$

$$R(t) = exp\left(-\frac{1}{2}t^2 - \frac{\gamma}{\kappa+1}t^{\kappa+1}\right) \tag{3}$$

$$h(t) = \left(t+\gamma t^{\kappa}\right) \tag{4}$$

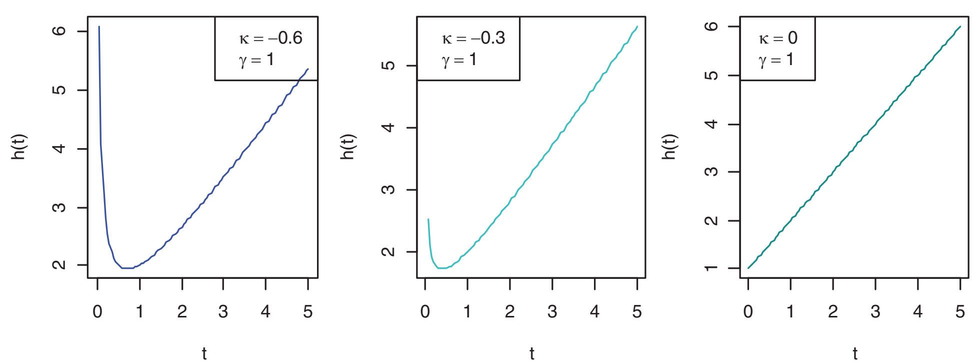

Figure 1: (Continued)

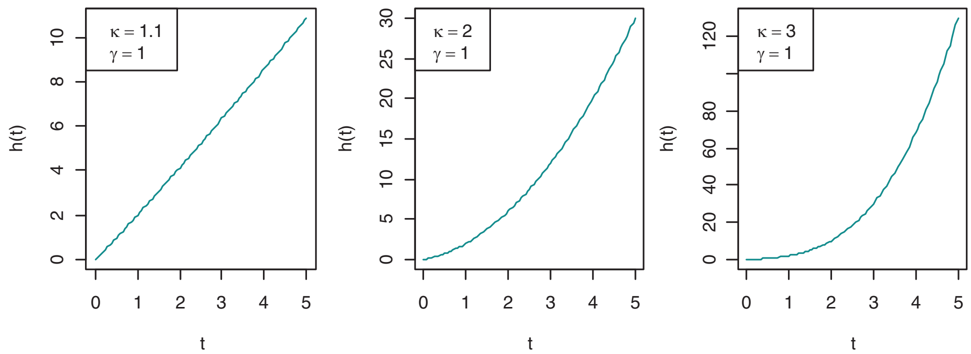

Figure 1: Shapes of the HR function of PLHR distribution with different values of the parameters

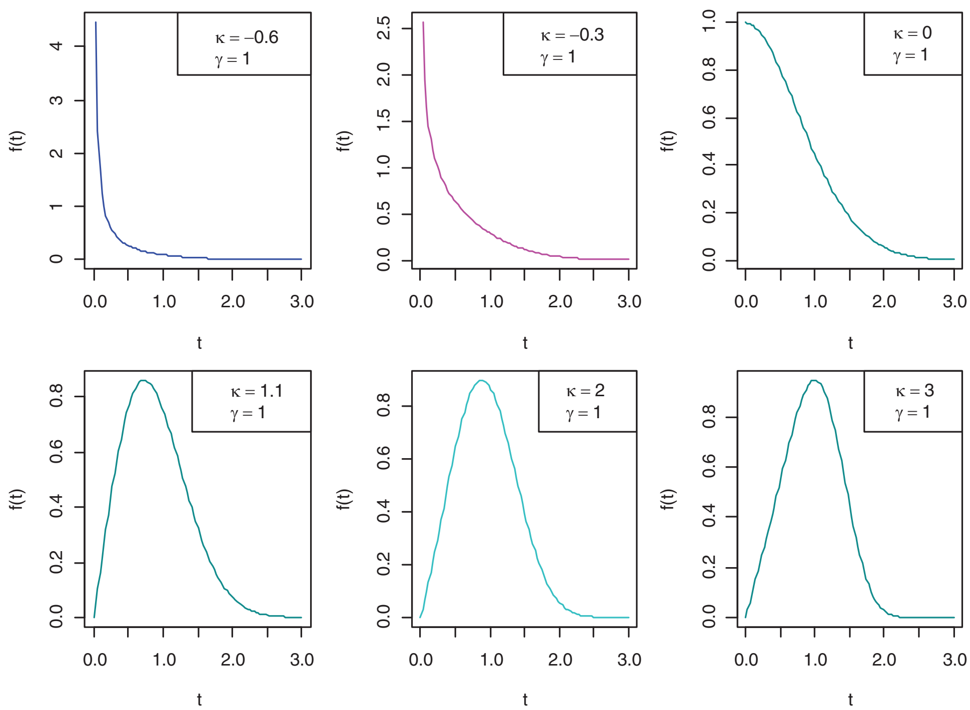

Figure 2: Shapes of the PDF of PLHR distribution using different values of the parameters

2.2 Basic Assumptions

In this paper, the following assumptions are made in order to describe the SSPALT model for a hybrid structure under PrHC masked data:

1. $\mathfrak{X}_\xi, \xi = 1, 2$ represents the two stress levels, i.e., $\mathfrak{X}_1$ and $\mathfrak{X}_2$ are normal and accelerated stress level, respectively, used in SSPALT.
2. The system under consideration is a series parallel system containing three independent components $j = 1, 2, 3$ which is described by Fig. 3 as follows:

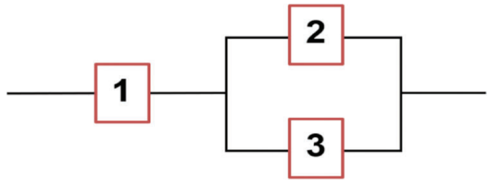

Figure 3: A series parallel hybrid system of three components

3. The lifetime $T_{\xi i}$, $\xi = 1, 2$; $i = 1, 2, \ldots, n$ of system i tested under SSPALT are i.i.d. at both normal and accelerated test conditions.
4. The data masking mechanism is statistically independent of the various stress conditions used in the experiment and the actual cause of failure of the system.
5. The lifetime of jth component at normal stress $\mathfrak{X}_1$, in the considered hybrid system follows PLHR distribution and the corresponding PDF, CDF, SF and HR are given respectively by Eqs. (1)–(4).
6. Let the total lifetime T of a component in hybrid system under SSPALT is explained by TRV model and can be written as follows:

$$T = \begin{cases} X, & if \;\; X < \tau \\ \tau + \vartheta^{-1}(X - \tau), & if \;\; X > \tau \end{cases} \tag{5}$$

where X represents lifetime of the component at $\mathfrak{X}_1$, τ is the time when items are switched from $\mathfrak{X}_1$ to $\mathfrak{X}_2$ and is known as stress change time and $\vartheta > 1$ is used to reflect the effect of stress change and is known as acceleration factor. Now lifetime of the component at $\mathfrak{X}_2$ by using Eqs. (1)–(5) can be obtained as

$$f_2(t, \gamma, \kappa, \vartheta) = \vartheta\left\{(\tau + \vartheta(t - \tau)) + \gamma(\tau + \vartheta(t - \tau))^{\kappa}\right\} \times exp\left\{-\frac{1}{2}(\tau + \vartheta(t - \tau))^2 - \frac{\gamma}{\kappa + 1}(\tau + \vartheta(t - \tau))^{\kappa+1}\right\} \tag{6}$$

$$F_2(t, \gamma, \kappa, \vartheta) = 1 - exp\left\{-\frac{1}{2}(\tau + \vartheta(t - \tau))^2 - \frac{\gamma}{\kappa + 1}(\tau + \vartheta(t - \tau))^{\kappa+1}\right\} \tag{7}$$

$$R_2(t, \gamma, \kappa, \vartheta) = exp\left\{-\frac{1}{2}(\tau + \vartheta(t - \tau))^2 - \frac{\gamma}{\kappa + 1}(\tau + \vartheta(t - \tau))^{\kappa+1}\right\} \tag{8}$$

$$h_2(t, \gamma, \kappa) = \vartheta\left\{(\tau + \vartheta(t - \tau)) + \gamma(\tau + \vartheta(t - \tau))^{\kappa}\right\} \tag{9}$$

Definition 2.1: Masking Probability

Suppose there are n system that are tested in the experiment and $t_{\xi i}$ represents observed value of $T_{\xi i}$, where $\xi = 1, 2$ represents the stress levels. Suppose also that $t_{\xi ij}$, $i = 1, 2, 3, \ldots, n$; $j = 1, 2, 3$ are realizations of the life time $T_{\xi ij}$ of the j^{th} component of the system i. Let $\omega_{\xi i} \epsilon \Omega_{\xi i}$ be a masked event corresponding to the j^{th} component of the system i that is one of the possible causes for system failure. If ω_i include only single component, e.g., $\omega_i = \{1\}$, then it can be said that the failure cause of the system is exact, otherwise cause is called masked [1,47,56]. Let $C_{\xi ij}$ be the

j^{th} component that is the exact cause of failure of system i with masked event $\omega_{\xi i} \epsilon \Omega_{\xi i}$, then the masking probability can be defined as follows:

$$P\left(\Omega_{\xi i} = \omega_{\xi i} | t_{\xi i} < T_{\xi i} < t_{\xi i} + dt_{\xi i}, C_{\xi ij} = j\right) \tag{10}$$

Now, using the Assumption 4, above expression for masking probability can be written as follows:

$$P\left(\Omega_{\xi i} = \omega_{\xi i} | t_{\xi i} < T_{\xi i} < t_{\xi i} + dt_{\xi i}, C_{\xi ij} = j\right) = P\left(\Omega_{\xi i} = \omega_{\xi i} | C_{\xi ij} = j\right) = \zeta_{\xi i} \tag{11}$$

2.3 Formulation of the SSPALT with PrHC Mask Data

PrHC scheme was first proposed and applied by [61] in traditional life tests. In SSPALT, suppose experiment is first started by randomly choosing n similar systems of three components described in the Fig. 3 to be assigned to test at the normal stress level $\mathfrak{X}_1$ with some predefined progressive values of random removals patterns $e_1, e_2 \ldots, e_m$, stress change time τ and a test termination time t_0. Now, test will progress by removing e_1 systems randomly at the time t_1 of first failure. Similarly, e_2 systems will be removed randomly at the time t_2 of second failure and so on. If, n_1 is the number of total observed failures occurred at normal use condition $\mathfrak{X}_1$ and e_{n_1} is total systems removed at $\mathfrak{X}_1$ before stress change time τ, then total remaining surviving systems will be $(n - n_u - e_{n_1})$. At this point τ of the experiment, all the remaining $(n - n_1 - e_{n_1})$ survival systems are removed from the test and are assigned to test at accelerated test condition $\mathfrak{X}_2 (> \mathfrak{X}_1)$ with hybrid test termination time $t_0{}^* = min(t_{m,m,n}, t_0)$ following the same procedure as it was on stress $\mathfrak{X}_1$. If m^{th} failure $t_{m,m,n}$ is observed before the pre-set time constraint t_0, then we will stop the test by removing all the remaining $e_m = n - m - (e_1 + e_2 + \cdots + e_{m-1})$ test systems from the experiment. If m^{th} failure $t_{m,m,n}$ is not observed before t_0, then test will be terminated by removing all the remaining $e_d = n - d - (e_1 + e_2 + \cdots + e_{d-1})$ test systems, where d is the number system failures before t_0. Let n_2 be the total number of observed failures at accelerated test condition $\mathfrak{X}_1$ during time interval $[\tau,\ t_0]$, for more details see [62,63]. In many complicated systems these days, the cause that exactly responsible for system failure is not known. Let $\Omega_{\xi i} = \{\{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$ be the set of all the possible events that can be a reason for the system failure and $\omega_{\xi i}$ be a subset (observed value) of the $\Omega_{\xi i}$. If observed subset $\omega_{\xi i}$ contains exactly one element, then cause of the system failure is known. In contrast, if observed subset $\omega_{\xi i}$ contains more than one element, then the exact reason of system failure is not known (or Masked). In this scenario, following two cases of failure data are observed in general:

$$\begin{aligned} I &: \left(t_{\xi,1,m,n}, \omega_{\xi,1,m,n}\right), \left(t_{\xi,2,m,n}, \omega_{\xi,2,m,n}\right), \ldots, \left(t_{\xi,n_1,m,n}, \omega_{\xi,n_1,m,n}\right) < \tau < \left(t_{\xi,n_1+1,m,n}, \omega_{\xi,n_1+1,m,n}\right) \ldots \\ &\qquad (t_{\xi,m,m,n}, \omega_{\xi,m,m,n}),\ if \ \ t_{\xi,m,m,n} \le t_0 \\ II &: \left(t_{\xi,1,m,n}, \omega_{\xi,1,m,n}\right), \left(t_{\xi,2,m,n}, \omega_{\xi,2,m,n}\right), \ldots, \left(t_{\xi,n_1,m,n}, \omega_{\xi,n_1,m,n}\right) < \tau < \left(t_{\xi,n_1+1,m,n}, \omega_{\xi,n_1+1,m,n}\right) \ldots \\ &\qquad (t_{\xi,(n_1+n_2),m,n}, \omega_{\xi,m,m,n},\ if \ \ t_{\xi,m,m,n} > t_0 \end{aligned} \tag{12}$$

Now, using Eqs. (10), (12) and the PrHC masked data given in Eq. (16), we can write the likelihood function for the hybrid system under SSPALT as follows:

$$L(\vartheta, \gamma, \kappa) \propto \prod_{i=1}^{n_1} \left[\left(\sum_{j \in \omega_{\xi i}} \zeta_{\xi i} f_{\xi ij} \right) \{R_1(t_i)\}^{r_i} \right] \prod_{i=n_1+1}^{r} \left[\left(\sum_{j \in \omega_{\xi i}} f_{\xi ij} \right) \{R_2(t_i)\}^{r_i} \right] [R_2(t_0)]^{r^*} \tag{13}$$

where, $t_i = t_{\xi,i,m,n}$, $r = m, r^* = 0$ for first data set and $r = d = n_1 + n_2$, $r^* = n - d - (e_1 + e_2 + \cdots + e_{d-1})$ for second data set given in Eq. (12).

Theorem 2.1: For a hybrid system consist of j independent components described in Fig. 3 having lifetime $T_{\xi i}$ with masking probability $P\left(\Omega_{\xi i} = \omega_{\xi i} | C_{\xi ij} = j\right)$, the density function of the hybrid system due to masked event $\omega_{\xi i} \epsilon \Omega_{\xi i}$ with exact failure component j at time $t_{\xi i}$ is given by following expression:

$$P\left(t_{\xi i} < T_{\xi i} < t_{\xi i} + dt_{\xi i}, \Omega_{\xi i} = \omega_{\xi i}\right) = \sum_{j \in \omega_{\xi i}} \zeta_{\xi i} f_{\xi ij} \tag{14}$$

Proof:

Probability that the system i is failed due to component j under mask occurrence $\omega_{\xi i}$ at time $t_{\xi i}$ can be obtained as follows [47,56]:

$$P\left(t_{\xi i} < T_{\xi i} < t_{\xi i} + dt_{\xi i}, \Omega_{\xi i} = \omega_{\xi i}\right)$$

$$= \sum_{j=1}^{3} P\left(t_{\xi i} < T_{\xi i} < t_{\xi i} + dt_{\xi i}, \Omega_{\xi i} = \omega_{\xi i}, C_{\xi ij} = j\right)$$

$$= \sum_{j=1}^{3} P\left(t_{\xi i} < T_{\xi i} < t_{\xi i} + dt_{\xi i}, C_{\xi ij} = j\right) P\left(\Omega_{\xi i} = \omega_{\xi i} | t_{\xi i} < T_{\xi i} < t_{\xi i} + dt_{\xi i}, C_{\xi ij} = j\right)$$

$$= \sum_{j \in \omega_{\xi i}} P\left(t_{\xi i} < T_{\xi i} < t_{\xi i} + dt_{\xi i}, C_{\xi ij} = j\right) P\left(\Omega_{\xi i} = \omega_{\xi i} | t_{\xi i} < T_{\xi i} < t_{\xi i} + dt_{\xi i}, C_{\xi ij} = j\right) \tag{15}$$

Now, the lifetime $T_{\xi i}$ of the hybrid system i given in Fig. 3 is

$$T_{\xi i} = min\left[T_{\xi i1}, \max\left(T_{\xi i2}, T_{\xi i2}\right)\right]$$

And, therefore the RF of the system i can be obtained as follows:

$$P\left(T_{\xi i} > t_{\xi i}\right) = P\left(T_{\xi i1} > t_{\xi i}\right) P\left[max\left(T_{\xi i2}, T_{\xi i3}\right)\right]$$
$$= P\left(T_{\xi i1} > t_{\xi i}\right)\left[1 - P\left(T_{\xi i2} \le t_{\xi i}\right) P\left(T_{\xi i2} \le t_{\xi i}\right)\right] = R_{\xi i1}\left(t_{\xi i}\right)\left[1 - F_{\xi i2}(t_{\xi i}) F_{\xi i3}(t_{\xi i})\right] \tag{16}$$

Similarly, the probability densities of hybrid system i which is failed due to component j at time $t_{\xi i}$ can be calculated as

$$P_{\xi i1} = P\left(t_{\xi i} < T_{\xi i} < t_{\xi i} + dt_{\xi i}, C_{\xi j} = 1\right) = P\left(t_{\xi i} < T_{\xi i1} < t_{\xi i} + dt_{\xi i}\right)\left[1 - P\left(T_{\xi i2} \le t_{\xi i}\right) P\left(T_{\xi i3} \le t_{\xi i}\right)\right]$$
$$= f_{\xi 1}\left(t_{\xi i}\right)\left[1 - F_{\xi 2}\left(t_{\xi i}\right) F_{\xi 3}\left(t_{\xi i}\right)\right] dt_{\xi i} = h_{\xi 1}\left(t_{\xi i}\right) R_{\xi 1}\left(t_{\xi i}\right)\left[1 - \left(1 - R_{\xi 2}\left(t_{\xi i}\right)\right)\left(1 - R_{\xi 3}\left(t_{\xi i}\right)\right)\right] dt_{\xi i}$$

$$P_{\xi i2} = P\left(t_{\xi i} < T_{\xi i} < t_{\xi i} + dt_{\xi i}, C_{\xi j} = 2\right) = P\left(T_{\xi i1} > t_{\xi i}\right) P\left(t_{\xi i} < T_{\xi i2} < t_{\xi i} + dt_{\xi i}\right) P\left(T_{\xi i3} \le t_{\xi i}\right)$$
$$= R_{\xi 1}\left(t_{\xi i}\right) f_{\xi 2}\left(t_{\xi i}\right) F_{\xi 3}\left(t_{\xi i}\right) dt_{\xi i} = R_{\xi 1}\left(t_{\xi i}\right) h_{\xi 2}\left(t_{\xi i}\right) R_{\xi 2}\left(t_{\xi i}\right)\left[1 - R_{\xi 3}\left(t_{\xi i}\right)\right] dt_{\xi i}$$

$$P_{\xi i3} = P\left(t_{\xi i} < T_{\xi i} < t_{\xi i} + dt_{\xi i}, C_{\xi j} = 3\right) = P\left(T_{\xi i1} > t_{\xi i}\right) P\left(T_{\xi i2} \le t_{\xi i}\right) P\left(t_{\xi i} < T_{\xi i3} < t_{\xi i} + dt_{\xi i}\right)$$
$$= R_{\xi 1}\left(t_{\xi i}\right) F_{\xi 2}\left(t_{\xi i}\right) f_{\xi 3}\left(t_{\xi i}\right) dt_{\xi i} = R_{\xi 1}\left(t_{\xi i}\right)\left[1 - R_{\xi 2}\left(t_{\xi i}\right)\right] h_{\xi 3}\left(t_{\xi i}\right) R_{\xi 3}\left(t_{\xi i}\right) dt_{\xi i}$$

For simplifying notations, let denote

$$\begin{cases} f_{\xi i1} = h_{\xi 1}\left(t_{\xi i}\right) R_{\xi 1}\left(t_{\xi i}\right)\left[1-(1-R_{\xi 2}\left(t_{\xi i}\right))(1-R_{\xi 3}\left(t_{\xi i}\right))\right] = f_{\xi 1}(t_{\xi i})\left[1-F_{\xi 2}(t_{\xi i})F_{\xi 3}(t_{\xi i})\right] \\ f_{\xi i2} = R_{\xi 1}\left(t_{\xi i}\right) h_{\xi 2}\left(t_{\xi i}\right) R_{\xi 2}\left(t_{\xi i}\right)\left[1-R_{\xi 3}\left(t_{\xi i}\right)\right] = R_{\xi 1}\left(t_{\xi i}\right) f_{\xi 2}(t_{\xi i})F_{\xi 3}(t_{\xi i}) \\ f_{\xi i3} = R_{\xi 1}\left(t_{\xi i}\right)\left[1-R_{\xi 2}\left(t_{\xi i}\right)\right] h_{\xi 3}\left(t_{\xi i}\right) R_{\xi 3}\left(t_{\xi i}\right) = R_{\xi 1}\left(t_{\xi i}\right) F_{\xi 2}(t_{\xi i})f_{\xi 3}(t_{\xi i}) \end{cases} \tag{17}$$

Now using Eqs. (11), (15), (17) and Assumption 4, we obtained following:

$$P\left(t_{\xi i} < T_{\xi i} < t_{\xi i} + dt_{\xi i},\ \Omega_{\xi i} = \omega_{\xi i}\right) = \sum_{j \in \omega_{\xi i}} \zeta_{\xi i} f_{\xi ij}$$

which completes the proof.

Applying Eqs. (11), (14) and (16) in Eq. (13), we obtained the following form of the likelihood function [56]:

$$\begin{aligned} L(\vartheta,\gamma,\kappa) \propto & \prod_{i=1}^{n_1}\left(t_i+\gamma t_i^{\kappa}\right) exp\left(-\frac{1}{2}t_i^2-\frac{\gamma}{\kappa+1}t_i^{\kappa+1}\right)\left[1-\left(1-exp\left(-\frac{1}{2}t_i^2-\frac{\gamma}{\kappa+1}t_i^{\kappa+1}\right)\right)^2\right] \\ & \times \prod_{i=1}^{n_1}\left[exp\left(-\frac{1}{2}t_i^2-\frac{\gamma}{\kappa+1}t_i^{\kappa+1}\right)\left\{1-\left(1-exp\left(-\frac{1}{2}t_i^2-\frac{\gamma}{\kappa+1}t_i^{\kappa+1}\right)\right)^2\right\}\right]^{r_i} \\ & \times \prod_{i=n_1+1}^{r}\left[\vartheta\left\{(\tau+\vartheta\,(t_i-\tau))+\gamma\,(\tau+\vartheta\,(t_i-\tau))^{\kappa}\right\} exp\left(-\frac{1}{2}(\tau+\vartheta\,(t_i-\tau))^2-\frac{\gamma}{\kappa+1}(\tau+\vartheta\,(t_i-\tau))^{\kappa+1}\right)\right. \\ & \left.\times\left\{1-\left(1-exp\left(-\frac{1}{2}(\tau+\vartheta\,(t_i-\tau))^2-\frac{\gamma}{\kappa+1}(\tau+\vartheta\,(t_i-\tau))^{\kappa+1}\right)\right)^2\right\}\right] \\ & \times \prod_{i=n_1+1}^{r}\left[exp\left(-\frac{1}{2}(\tau+\vartheta\,(t_i-\tau))^2-\frac{\gamma}{\kappa+1}(\tau+\vartheta\,(t_i-\tau))^{\kappa+1}\right)\right. \\ & \left.\times\left\{1-\left(1-exp\left(-\frac{1}{2}(\tau+\vartheta\,(t_i-\tau))^2-\frac{\gamma}{\kappa+1}(\tau+\vartheta\,(t_i-\tau))^{\kappa+1}\right)\right)^2\right\}\right]^{r_i} \\ & \times\left[exp\left(-\frac{1}{2}(\tau+\vartheta(t_0-\tau))^2-\frac{\gamma}{\kappa+1}(\tau+\vartheta(t_0-\tau))^{\kappa+1}\right)\right. \\ & \left.\times\left\{1-\left(1-exp\left(-\frac{1}{2}(\tau+\vartheta(t_0-\tau))^2-\frac{\gamma}{\kappa+1}(\tau+\vartheta(t_0-\tau))^{\kappa+1}\right)\right)^2\right\}\right]^{r^*} \end{aligned} \tag{18}$$

where, at use stress level $\mathfrak{X}_1$, $f_1(t_i)$ and $R_1(t_i)$ are failure density function and reliability of the system i respectively caused by each component and are obtained by using Eqs. (1), (3), (16), (17) and Assumption 4 as follows, respectively:

$$f_1(t_i) = \left(t_i+\gamma t_i^{\kappa}\right) exp\left(-\frac{1}{2}t_i^2-\frac{\gamma}{\kappa+1}t_i^{\kappa+1}\right)\left[1-\left(1-exp\left(-\frac{1}{2}t_i^2-\frac{\gamma}{\kappa+1}t_i^{\kappa+1}\right)\right)^2\right]$$

$$R_1(t_i) = exp\left(-\frac{1}{2}t_i^2-\frac{\gamma}{\kappa+1}t_i^{\kappa+1}\right)\left\{1-\left(1-exp\left(-\frac{1}{2}t_i^2-\frac{\gamma}{\kappa+1}t_i^{\kappa+1}\right)\right)^2\right\}$$

similarly, at accelerated stress level $\mathfrak{X}_2$, $f_2(t_i)$ and $R_2(t_i)$ are failure density function and reliability of the system i respectively caused by each component and are obtained by using Eqs. (6), (8), (16), (17) and Assumption 6 as follows, respectively:

$$f_2(t_i) = \vartheta\left\{(\tau+\vartheta(t_i-\tau)) + \gamma(\tau+\vartheta(t_i-\tau))^{\kappa}\right\} exp\left(-\frac{1}{2}(\tau+\vartheta(t_i-\tau))^2 - \frac{\gamma}{\kappa+1}(\tau+\vartheta(t_i-\tau))^{\kappa+1}\right)$$
$$\times\left\{1-\left(1-exp\left(-\frac{1}{2}(\tau+\vartheta(t_i-\tau))^2-\frac{\gamma}{\kappa+1}(\tau+\vartheta(t_i-\tau))^{\kappa+1}\right)\right)^2\right\}$$

$$R_2(t_i) = exp\left(-\frac{1}{2}(\tau+\vartheta(t_i-\tau))^2 - \frac{\gamma}{\kappa+1}(\tau+\vartheta(t_i-\tau))^{\kappa+1}\right)$$
$$\times\left\{1-\left(1-exp\left(-\frac{1}{2}(\tau+\vartheta(t_i-\tau))^2-\frac{\gamma}{\kappa+1}(\tau+\vartheta(t_i-\tau))^{\kappa+1}\right)\right)^2\right\}$$

3 Estimation of Model Parameters

3.1 Point Estimates

To obtain the estimates of parameters, the MLE technique is used. Now, taking log on both sides of Eq. (18) and making some adjustments, the log likelihood function (or Score function) $\ell = L(\vartheta, \gamma, \kappa)$ is derived in the form of following equation:

$$\ell \propto r^*(r-n_u)\, log\phi_0 + r^*(r-n_u)\, log\left(1-(1-\phi_0)^2\right) + \sum_{i=1}^{n_u} log\left(t_i+\gamma t_i^{\kappa}\right) + \sum_{i=1}^{n_u}(1+r_i)\, log\Psi_1$$
$$+\sum_{i=1}^{n_u}(1+r_i) log\left(1-(1-\Psi_1)^2\right) + \sum_{i=n_u+1}^{r} log\vartheta + \sum_{i=n_u+1}^{r} log\left(\Psi_2+\gamma\Psi_2^{\kappa}\right) + \sum_{i=n_u+1}^{r}(1+r_i)\, log\phi_1$$
$$+\sum_{i=n_u+1}^{r}(1+r_i) log\left(1-(1-\phi_1)^2\right) \tag{19}$$

where, we assume $exp\left(-\frac{1}{2}t_i^2 - \frac{\gamma}{\kappa+1}t_i^{\kappa+1}\right) = \Psi_1$; $(\tau+\vartheta(t_i-\tau)) = \Psi_2; \tau + \vartheta(t_0-\tau) = \Psi_0$ $exp\left(-\frac{1}{2}\Psi_2^2 - \frac{\gamma}{\kappa+1}\Psi_2^{\kappa+1}\right) = \phi_1$ and $exp\left(-\frac{1}{2}\Psi_0^2 - \frac{\gamma}{\kappa+1}\Psi_0^{\kappa+1}\right) = \phi_0$ to make equations simple.

Now, differentiating Eq. (19) partially with respect to model parameter ϑ, γ and κ, we obtained following likelihood equations:

$$\frac{\partial\ell}{\partial\vartheta} = -(t_0-\tau)\, r^*(r-n_u)\left(\Psi_0+\gamma\Psi_0^{\kappa}\right)\left\{1+\frac{2(1-\phi_0)\phi_0}{\left(1-(1-\phi_0)^2\right)}\right\} + \frac{(r-n_u)}{\vartheta} + \sum_{i=n_u+1}^{r}\frac{(t_i-\tau)\left(1+\gamma\kappa\Psi_2^{\kappa-1}\right)}{\left(\Psi_2+\gamma\Psi_2^{\kappa}\right)}$$
$$-\sum_{i=n_u+1}^{r}(t_i-\tau)(1+r_i)\left(\Psi_2+\gamma\Psi_2^{\kappa}\right)\left\{1+\frac{2(1-\phi_1)\phi_1}{\left(1-(1-\phi_1)^2\right)}\right\} = 0 \tag{20}$$

$$\frac{\partial \ell}{\partial \gamma} = -\frac{r^* (r-n_u) \Psi_0^{\kappa+1}}{\kappa+1} \left\{ 1 - \frac{2(1-\phi_0)\phi_0}{\left(1-(1-\phi_0)^2\right)} \right\} - \sum_{i=1}^{n_u} \frac{(1+r_i) t_i^{\kappa+1}}{\kappa+1} \left\{ 1 - \frac{2(1-\Psi_1)\Psi_1}{\left(1-(1-\Psi_1)^2\right)} \right\}$$
$$+ \sum_{i=1}^{n_u} \frac{t_i^{\kappa}}{\left(t_i+\gamma t_i^{\kappa}\right)} + \sum_{i=n_u+1}^{r} \frac{\Psi_2^{\kappa}}{\left(\Psi_2+\gamma \Psi_2^{\kappa}\right)} - \sum_{i=n_u+1}^{r} \frac{(1+r_i)\Psi_2^{\kappa+1}}{\kappa+1} \left\{ 1 - \frac{2(1-\phi_1)\phi_1}{\left(1-(1-\phi_1)^2\right)} \right\} = 0 \quad (21)$$

$$\frac{\partial \ell}{\partial \kappa} = \frac{r^* (r-n_u) \gamma \Psi_0^{\kappa+1} (1-(\kappa+1) \log\Psi_0)}{(\kappa+1)^2} \left\{ 1 - \frac{2(1-\phi_0)}{\left(1-(1-\phi_0)^2\right)} \right\} + \sum_{i=1}^{n_u} \frac{\gamma t_i^{\kappa} \log t_i}{\left(t_i+\gamma t_i^{\kappa}\right)}$$
$$+ \sum_{i=1}^{n_u} \frac{(1+r_i)\gamma t_i^{\kappa+1} (1-(\kappa+1)\log t_i)}{(\kappa+1)^2} \left\{ 1 - \frac{2(1-\Psi_1)}{\left(1-(1-\Psi_1)^2\right)} \right\} + \sum_{i=n_u+1}^{r} \frac{\gamma \Psi_2^{\kappa} \log\Psi_2}{\left(\Psi_2+\gamma\Psi_2^{\kappa}\right)}$$
$$+ \sum_{i=n_u+1}^{r} \frac{(1+r_i)\gamma \Psi_2^{\kappa+1} (1-(\kappa+1)\log\Psi_2)}{(\kappa+1)^2} \left\{ 1 - \frac{2(1-\phi_1)}{\left(1-(1-\phi_1)^2\right)} \right\} = 0 \quad (22)$$

The MLEs $\hat{\vartheta}, \hat{\gamma}$ and $\hat{\kappa}$ of the parameters ϑ, γ and κ can be obtained by solving Eqs. (20)–(22) simultaneously but these equations are very complex non-linear equations to be solved analytically. In fact, above equations cannot be solved analytically, therefore Newton–Raphson technique is used to solve these equations numerically.

3.2 Interval Estimates

3.2.1 ACIs

One of the best redeeming features of MLE is due to its large sample properties. For large sets of data, the distribution of MLEs of the parameters is approximately normal. As it is discussed in the last subsection, the likelihood equations for finding MLEs are virtually impossible to solve in close form and, hence the exact distribution of the MLEs is nearly impossible to find for a complex situation such as the one we are dealing with. Therefore, we utilized the asymptotic properties of MLEs to construct the ACIs. The distribution of MLEs $\hat{\vartheta}, \hat{\gamma}$ and $\hat{\kappa}$ of the unknown parameters ϑ, γ and κ is asymptotically normal and can be described as follows:

$$\begin{pmatrix} \hat{\vartheta} \\ \hat{\gamma} \\ \hat{\kappa} \end{pmatrix} = \left(\begin{pmatrix} \vartheta \\ \gamma \\ \kappa \end{pmatrix}, \hat{V} \right) \quad (23)$$

where, $\hat{V}$ is asymptotic variance covariance which can be calculated by inverting observed information matrix F and replacing parameters ϑ, γ and κ with their corresponding MLEs $\hat{\vartheta}, \hat{\gamma}$ and $\hat{\kappa}$ as follows:

$$\hat{V} = \begin{pmatrix} \hat{V}_{11} & \hat{V}_{12} & \hat{V}_{13} \\ \hat{V}_{21} & \hat{V}_{22} & \hat{V}_{23} \\ \hat{V}_{31} & \hat{V}_{32} & \hat{V}_{33} \end{pmatrix} = F^{-1} = \begin{pmatrix} -\frac{\partial^2 \ell}{\partial \vartheta^2} & -\frac{\partial^2 \ell}{\partial \vartheta \partial \gamma} & -\frac{\partial^2 \ell}{\partial \vartheta \partial \kappa} \\ -\frac{\partial^2 \ell}{\partial \gamma \partial \vartheta} & -\frac{\partial^2 \ell}{\partial \gamma^2} & -\frac{\partial^2 \ell}{\partial \gamma \partial \kappa} \\ -\frac{\partial^2 \ell}{\partial \kappa \partial \vartheta} & -\frac{\partial^2 \ell}{\partial \kappa \partial \gamma} & -\frac{\partial^2 \ell}{\partial \kappa^2} \end{pmatrix}^{-1} \quad (24)$$

where the elements of F are given in Appendix A. Now $100(1-\alpha)\%$ ACIs for model parameters ϑ, γ and κ can be obtained as follows:

$$\hat{\vartheta} \pm z_{\alpha/2}\sqrt{\hat{V}_{11}}; \quad \hat{\gamma} \pm z_{\alpha/2}\sqrt{\hat{V}_{22}}; \quad \hat{\kappa} \pm z_{\alpha/2}\sqrt{\hat{V}_{22}} \tag{25}$$

where, $\hat{V}_{11} = var(\hat{\vartheta})$, $\hat{V}_{11} = var(\hat{\gamma})$, $\hat{V}_{11} = var(\hat{\kappa})$ and $z_{\alpha/2}$ represent $(1-\alpha/2)$ quantile of standard normal distribution.

3.2.2 Bootstrap CIs

This subsection deals with the bootstrap re-sampling approach for constructing parameters CIs in which the original data is assumed to be a population and then several samples are generated using the original data to create the CIs, for more details see [64–66]. We use the following algorithm to construct ST-B CIs [56]:

1. Find the MLEs $\hat{\mho} = \left(\hat{\vartheta}, \hat{\gamma}, \hat{\kappa}\right)$ of the parameters $\mho = (\vartheta, \gamma, \kappa)$ based on PrHC masked data generated from PLHR distribution under SSPALT which is obtained by following the Steps 1–6 in Section 4.
2. Now using MLEs $\hat{\mho} = \left(\hat{\vartheta}, \hat{\gamma}, \hat{\kappa}\right)$ and PrHC masked data in Step 1, generate following bootstrap sample:

$$\left(I: t^*_{\xi,1,m,n}, \omega^*_{\xi,1,m,n}\right), \left(t^*_{\xi,1,m,n}, \omega^*_{\xi,1,m,n}\right), \ldots, \left(t^*_{\xi,n_1,m,n}, \omega^*_{\xi,n_1,m,n}\right) < \tau < \left(t^*_{\xi,n_1,m,n}, \omega^*_{\xi,n_1,m,n}\right) \ldots$$
$$\left(t^*_{\xi,m,m,n}, \omega^*_{\xi,m,m,n}\right), \;\; if \;\; t^*_{\xi,m,m,n} \leq t_0$$
$$\left(II: \left(t^*_{\xi,1,m,n}, \omega^*_{\xi,1,m,n}\right), \left(t^*_{\xi,1,m,n}, \omega^*_{\xi,1,m,n}\right), \ldots, \left(t^*_{\xi,n_1,m,n}, \omega^*_{\xi,n_1,m,n}\right) < \tau < \left(t^*_{\xi,n_1,m,n}, \omega^*_{\xi,n_1,m,n}\right) \ldots\right.$$
$$\left.\left(t^*_{\xi,(n_1+n_2),,m,n}, \omega^*_{\xi,(n_1+n_2),m,n}\right), \;\; if \;\; t^*_{\xi,m,m,n} > t_0\right)$$

3. Using bootstrap sample obtained in Step 2, find estimate $\hat{\mho}^* = \left(\hat{\vartheta}^*, \hat{\gamma}^*, \hat{\kappa}^*\right)$.
4. Repeat above 3 steps $\mathcal{N}$ times, say for example $\mathcal{N} = 5000$ and obtain a set $\left\{\hat{\mho}^*_i, \, i = 1, 2, \ldots \mathcal{N}\right\}$ of bootstrap estimates and the corresponding statistics

   $$\mathfrak{T}^*_i = \left(\hat{\mho}^*_i - \hat{\mho}_i\right) \Big/ \sqrt{var(\hat{\mho}^*_i)}, i = 1, 2, \ldots \mathcal{N}.$$

5. Compute $100(1-\alpha)\%$ ST-B CIs now, therefore can be constructed for $\hat{\mho}$ as

   $$\left[\hat{\mho} + \mathfrak{T}^*_{i(1-\alpha/2)}\sqrt{var\left(\hat{\mho}^*\right)}, \hat{\mho} + \mathfrak{T}^*_{i(\alpha/2)}\sqrt{var\left(\hat{\mho}^*\right)}\right]$$

4 Simulation Study

Here in this section of the paper, we are going to perform a simulation study to investigate and compare the performance of estimates for the hybrid system under SSPALT for PrHC masked data by utilizing the Monte Carlo simulation technique. The performance of MLEs assessed with their respective mean square errors (MSEs) and relative absolute biases (RABs). 95% ACIs are also constructed and their performance is investigated in terms of their respective ILs. First, we generated the hybrid progressive censored data from the considered distribution under SSPALT following some of the steps given in [58,64,67] and the complete steps of the algorithm are given as:

1. Specify the values of $[\tau, n, m, t_0, (e_1, e_2, \ldots, e_m)]$ and the values of the parameters $\vartheta, \gamma, \kappa$.

2. To obtain a random censored sample of size r, first generate a random sample of size r from uniform distribution $U(0,1)$, suppose these generated samples are $(\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_r)$.

3. Specify, $\mathcal{X}_i = \mathcal{U}_i^{1/(i+\sum_{m=n-i+1}^{r} e_m}$ for given values of censoring scheme $(e_1, e_2, \ldots, e_m)$, where, $i = 1, 2, \ldots r$, fore case I, $r = m$, and $r = d$, for Case II.

4. Define $\mathcal{U}_{i,m,n} = 1 - \prod_{m=n-i+1}^{r} e_m$ in order to generate progressive censored sample of size r from uniform distribution as $\mathcal{U}_{1,m,n}, \mathcal{U}_{2,m,n} \ldots \mathcal{U}_{r,m,n}$.

5. Set $\mathcal{U}_{n_1,m,n} < F_1(\tau) \leq \mathcal{U}_{n_1+1,m,n}$, in order to find sample of size n_1 from PLHR distribution at normal stress level for fixed values of parameters γ, κ and ϑ using the expression $log(1 - \mathcal{U}_{i,m,n}) + \frac{1}{2} t_i^2 + \frac{\gamma}{\kappa+1} t_i^{\kappa+1} = 0$.

6. Similarly, find a sample at accelerated condition for fixed values of parameters $\vartheta, \gamma, \kappa, \tau$ using the following equation $log(1 - \mathcal{U}_{i,m,n}) + \frac{1}{2} (\tau + \vartheta (t_i - \tau))^2 + \frac{\gamma}{\kappa+1} (\tau + \vartheta (t_i - \tau))^{\kappa+1} = 0$.

7. Following above steps, we generated the desired PrHC masked data from PLHR distribution in the form of Eq. (12).

8. Now, using the data obtained in Step 7, compute MLEs $\hat{\mho} = \left(\hat{\vartheta}, \hat{\gamma}, \hat{\kappa}\right)$ of the parameters $\mho = (\vartheta, \gamma, \kappa)$ by solving Eqs. (20)–(22) and their respective MSEs and RABs.

9. Replicate Steps 2–9, 10,000 times, and obtain average MLEs, MSEs and RABs of the parameters.

10. Compute the 95% ACIs for the parameters of PLHR distribution and acceleration factor.

11. For different values of $[\tau, n, m, t_0, (e_1, e_2, \ldots, e_m)]$, $\vartheta, \gamma, \kappa$ and test schemes, the above procedure from Steps 1 to 10 should be repeated.

In our study, we used four different censoring schemes generated from six different combinations of n and m with different fixed values of τ and t_0. The censoring schemes, initially fixed values of parameters and the values of τ, n, m, t_0 that we used in this paper are given in Table 1. Averages of MLEs based on 10,000 replications described in simulation study and their respective MSEs and RABs are reported in Tables 2 and 3. 95% ACIs and ST-B CIs along with their expected interval length are reported in Tables 4 and 5.

Fig. 4 presents the plots of simulated samples and the histograms of the parameters to verify the convergence of the parameters. For this demonstration, plots are made using the Scheme 1 simulation results. The graphs indicate that the parameters are converging, and the histograms reveal that they are asymptotically normal over large number of simulation runs. As a result, the simulation study is consistent with the statistical assumptions for parameter estimation. The step-by-step process of the proposed estimation method is demonstrated through a flow chart given in Fig. 5.

Table 1: PrHC schemes, combinations of (n,m) and (τ, t_0) with fixed values of parameters $\vartheta, \gamma, \kappa$

Censoring scheme	Values of parameters	Values of τ, t_0, and p	(n, m)
1: $r_1 = r_2 = \ldots = r_{m-1} = 0,\ r_m = n - m$	$\vartheta = 1.6,\ \gamma = 1.1,\ \kappa = 1.2$	$(\tau = 0.75,\ t_0 = 1.0,\ p = 0.20)$ $(\tau = 1.0,\ t_0 = 1.25,\ p = 0.20)$ $(\tau = 1.25,\ t_0 = 1.5,\ p = 0.20)$	(60, 35), (60, 45), (70, 35), (70, 45), (80, 35), (80, 45)
2: $r_1 = n - m,\ r_2 = \ldots = r_{m-1} = r_m = 0$	$\vartheta = 1.6,\ \gamma = 1.1,\ \kappa = 1.2$	$(\tau = 0.75,\ t_0 = 1.0,\ p = 0.20)$ $(\tau = 1.0,\ t_0 = 1.25,\ p = 0.20)$ $(\tau = 1.25,\ t_0 = 1.5,\ p = 0.20)$	(60, 35), (60, 45), (70, 35), (70, 45), (80, 35), (80, 45)
3: $r_1 = r_2 = \ldots = r_{m-6} = 0,\ r_{m-5} = r_{m-4} = \ldots = r_{m-1} = 1,\ r_m = n - m - 5$	$\vartheta = 1.6,\ \gamma = 1.1,\ \kappa = 1.2$	$(\tau = 0.75,\ t_0 = 1.0,\ p = 0.20)$ $(\tau = 1.0,\ t_0 = 1.2,\ p = 0.20)$ $(\tau = 1.25,\ t_0 = 1.5,\ p = 0.20)$	(60, 35), (60, 45), (70, 35), (70, 45), (80, 35), (80, 45)
4: $r_1 = r_2 = \ldots = r_m = \frac{(n-m)}{m}$	$\vartheta = 1.6,\ \gamma = 1.1,\ \kappa = 1.2$	$(\tau = 0.75,\ t_0 = 1.0,\ p = 0.20)$ $(\tau = 1.0,\ t_0 = 1.25,\ p = 0.20)$ $(\tau = 1.25,\ t_0 = 1.5,\ p = 0.20)$	(60, 35), (60, 45), (70, 35), (70, 45), (80, 35), (80, 45)

Table 2: The MLEs of parameters of with their respective MSEs and RABs with Schemes 1 and 2

(n, m)	ϑ			γ			κ		
	MLE	MSE	RAB	MLE	MSE	RAB	MLE	MSE	RAB
	CS 1 with $(\vartheta = 1.6, \gamma = 1.1, \kappa = 1.2)$ and $(\tau = 0.75, t_0 = 1.0)$								
(60, 35)	1.483	0.73	0.359	0.933	0.047	0.002	0.983	0.023	0.001
(60, 45)	1.532	0.662	0.286	0.982	0.116	0.014	1.032	0.046	0.002
(70, 35)	1.625	0.53	0.173	1.075	0.248	0.057	1.125	0.177	0.028
(70, 45)	1.647	0.499	0.151	1.097	0.279	0.071	1.147	0.208	0.038
(80, 35)	1.623	0.53	0.173	1.071	0.244	0.057	1.126	0.178	0.028
(80, 45)	1.544	0.645	0.27	0.994	0.132	0.018	1.044	0.062	0.004

(Continued)

Table 2 (continued).

(n, m)	ϑ			γ			κ		
	MLE	MSE	RAB	MLE	MSE	RAB	MLE	MSE	RAB
CS 1 with $(\vartheta = 1.6, \gamma = 1.1, \kappa = 1.2)$ and $(\tau = 1.0, t_0 = 1.25)$									
(60, 35)	1.626	0.529	0.172	1.076	0.249	0.058	1.126	0.178	0.028
(60, 45)	1.995	0.008	0.099	1.045	0.77	0.411	1.495	0.699	0.327
(70, 35)	1.731	0.381	0.084	1.181	0.397	0.133	1.231	0.326	0.086
(70, 45)	1.627	0.528	0.171	1.077	0.25	0.058	1.127	0.179	0.029
(80, 35)	1.626	0.529	0.172	1.076	0.249	0.058	1.126	0.178	0.028
(80, 45)	1.65	0.495	0.149	1.1	0.282	0.073	1.15	0.212	0.039
CS 1 with $(\vartheta = 1.6, \gamma = 1.1, \kappa = 1.2)$ and $(\tau = 1.25, t_0 = 1.5)$									
(60, 35)	1.627	0.528	0.171	1.077	0.25	0.058	1.127	0.179	0.029
(60, 45)	1.73	0.382	0.084	1.18	0.396	0.133	1.23	0.326	0.086
(70, 35)	1.626	0.529	0.172	1.076	0.249	0.058	1.126	0.179	0.025
(70, 45)	1.627	0.527	0.171	1.077	0.251	0.058	1.127	0.18	0.029
(80, 35)	1.624	0.532	0.174	1.074	0.246	0.056	1.124	0.176	0.027
(80, 45)	1.647	0.499	0.151	1.097	0.279	0.071	1.147	0.208	0.038
CS 2 with $(\vartheta = 1.6, \gamma = 1.1, \kappa = 1.2)$ and $(\tau = 0.75, t_0 = 1.0)$									
(60, 35)	1.627	0.527	0.171	1.077	0.251	0.058	1.127	0.18	0.029
(60, 45)	1.627	0.528	0.172	1.077	0.25	0.058	1.127	0.179	0.028
(70, 35)	1.543	0.646	0.27	0.993	0.132	0.018	1.043	0.061	0.004
(70, 45)	1.624	0.532	0.175	1.074	0.245	0.056	1.124	0.175	0.027
(80, 35)	1.626	0.53	0.173	1.076	0.248	0.057	1.126	0.177	0.028
(80, 45)	1.625	0.53	0.173	1.075	0.248	0.057	1.125	0.177	0.028
CS 2 with $(\vartheta = 1.6, \gamma = 1.1, \kappa = 1.2)$ and $(\tau = 1.0, t_0 = 1.25)$									
(60, 35)	1.628	0.527	0.171	1.078	0.251	0.058	1.128	0.18	0.029
(60, 45)	1.627	0.527	0.171	1.077	0.251	0.058	1.127	0.18	0.029
(70, 35)	1.625	0.53	0.173	1.075	0.248	0.057	1.125	0.177	0.028
(70, 45)	1.648	0.498	0.151	1.098	0.279	0.071	1.148	0.209	0.038
(80, 35)	1.613	0.547	0.186	1.063	0.231	0.05	1.113	0.16	0.023
(80, 45)	1.625	0.531	0.173	1.075	0.247	0.057	1.125	0.176	0.028
CS 2 with $(\vartheta = 1.6, \gamma = 1.1, \kappa = 1.2)$ and $(\tau = 1.25, t_0 = 1.5)$									
(60, 35)	1.628	0.526	0.17	1.078	0.251	0.059	1.128	0.181	0.029
(60, 45)	1.648	0.497	0.15	1.098	0.281	0.072	1.148	0.21	0.038
(70, 35)	1.544	0.645	0.27	0.994	0.132	0.018	1.044	0.062	0.004
(70, 45)	1.626	0.529	0.172	1.076	0.248	0.057	1.126	0.178	0.028
(80, 35)	1.652	0.492	0.147	1.102	0.285	0.074	1.152	0.215	0.04
(80, 45)	1.544	0.645	0.27	0.994	0.132	0.018	1.044	0.062	0.004

Table 3: The MLEs of parameters of with their respective MSEs and RABs with Schemes 3 and 4

(n, m)	ϑ			γ			κ		
	MLE	MSE	RAB	MLE	MSE	RAB	MLE	MSE	RAB
CS 3 with ($\vartheta = 1.6, \gamma = 1.1, \kappa = 1.2$) and ($\tau = 0.75, t_0 = 1.0$)									
(60, 35)	1.628	0.526	0.17	1.078	0.252	0.059	1.128	0.181	0.029
(60, 45)	1.629	0.525	0.169	1.079	0.253	0.059	1.129	0.182	0.029
(70, 35)	1.627	0.527	0.171	1.077	0.25	0.058	1.127	0.18	0.029
(70, 45)	1.626	0.529	0.172	1.076	0.249	0.058	1.126	0.178	0.028
(80, 35)	1.626	0.529	0.172	1.076	0.249	0.058	1.126	0.178	0.028
(80, 45)	1.625	0.53	0.173	1.075	0.248	0.057	1.125	0.177	0.028
CS 3 with ($\vartheta = 1.6, \gamma = 1.1, \kappa = 1.2$) and ($\tau = 1.0, t_0 = 1.25$)									
(60, 35)	1.625	0.531	0.173	1.075	0.247	0.057	1.125	0.176	0.028
(60, 45)	1.626	0.529	0.172	1.076	0.249	0.053	1.126	0.178	0.028
(70, 35)	1.645	0.502	0.153	1.095	0.276	0.07	1.145	0.205	0.037
(70, 45)	1.977	0.033	0.001	1.427	0.745	0.389	1.477	0.674	0.308
(80, 35)	1.648	0.498	0.151	1.098	0.28	0.071	1.148	0.209	0.038
(80, 45)	1.645	0.502	0.153	1.095	0.276	0.07	1.145	0.205	0.037
CS 3 with ($\vartheta = 1.6, \gamma = 1.1, \kappa = 1.2$) and ($\tau = 1.25, t_0 = 1.5$)									
(60, 35)	1.649	0.497	0.15	1.099	0.281	0.072	1.149	0.21	0.038
(60, 45)	1.625	0.531	0.173	1.075	0.247	0.057	1.125	0.176	0.028
(70, 35)	1.636	0.515	0.162	1.086	0.262	0.063	1.136	0.192	0.032
(70, 45)	1.625	0.531	0.173	1.075	0.247	0.057	1.125	0.176	0.028
(80, 35)	1.646	0.501	0.153	1.096	0.276	0.07	1.146	0.206	0.037
(80, 45)	1.625	0.531	0.173	1.075	0.247	0.057	1.125	0.177	0.028
CS 4 with ($\vartheta = 1.6, \gamma = 1.1, \kappa = 1.2$) and ($\tau = 0.75, t_0 = 1.0$)									
(60, 35)	1.647	0.499	0.151	1.097	0.279	0.071	1.147	0.208	0.038
(60, 45)	1.627	0.528	0.172	1.077	0.25	0.058	1.127	0.179	0.028
(70, 35)	1.543	0.646	0.27	0.993	0.132	0.018	1.043	0.061	0.004
(70, 45)	1.624	0.532	0.175	1.074	0.245	0.056	1.124	0.175	0.027
(80, 35)	1.542	0.662	0.286	0.982	0.116	0.014	1.032	0.046	0.002
(80, 45)	1.635	0.53	0.173	1.075	0.248	0.057	1.125	0.177	0.028
CS 4 with ($\vartheta = 1.6, \gamma = 1.1, \kappa = 1.2$) and ($\tau = 1.0, t_0 = 1.25$)									
(60, 35)	1.627	0.527	0.171	1.077	0.251	0.058	1.127	0.18	0.029
(60, 45)	1.731	0.381	0.084	1.181	0.397	0.133	1.231	0.326	0.086
(70, 35)	1.625	0.530	0.173	1.075	0.248	0.057	1.125	0.177	0.028
(70, 45)	1.578	0.498	0.151	1.098	0.279	0.071	1.148	0.209	0.038
(80, 35)	1.613	0.547	0.186	1.063	0.231	0.05	1.113	0.16	0.023
(80, 45)	1.595	0.228	0.099	1.075	0.247	0.057	1.125	0.176	0.028

(Continued)

Table 3 (continued).

(n, m)	ϑ			γ			κ		
	MLE	MSE	RAB	MLE	MSE	RAB	MLE	MSE	RAB
				CS 4 with $(\vartheta = 1.6, \gamma = 1.1, \kappa = 1.2)$ and $(\tau = 1.25, t_0 = 1.5)$					
(60, 35)	1.652	0.492	0.147	1.102	0.285	0.074	1.152	0.215	0.04
(60, 45)	1.627	0.527	0.171	1.077	0.25	0.058	1.127	0.18	0.029
(70, 35)	1.625	0.531	0.173	1.075	0.247	0.057	1.125	0.176	0.028
(70, 45)	1.626	0.529	0.172	1.076	0.249	0.053	1.126	0.178	0.028
(80, 35)	1.613	0.447	0.186	1.063	0.231	0.05	1.113	0.16	0.023
(80, 45)	1.574	0.345	0.127	0.994	0.132	0.018	1.044	0.062	0.004

Table 4: Lower (L) and Upper (U) limits and corresponding ILs of 95% ACIs and ST-B CIs with Schemes 1 and 2

(n, m)	ϑ		γ		κ	
	ACI (L, U, IL)	ACI (L, U, IL)	ACI (L, U, IL)	BCI (L, U, IL)	BCI (L, U, IL)	BCI (L, U, IL)
			CS 1 with $(\vartheta = 1.6, \gamma = 1.1, \kappa = 1.2)$ and $(\tau = 0.75, t_0 = 1.0)$			
(60, 35)	1.439, 2.527, 1.048	0.886, 0.937, 0.351	0.982, 0.984, 0.202	0.417, 2.399, 1.006	0.841, 0.89, 0.337	0.941, 0.943, 0.192
(60, 45)	0.673, 2.391, 1.718	0.866, 1.008, 0.442	1.028, 1.036, 0.192	0.64, 2.27, 1.649	0.822, 0.957, 0.424	0.985, 0.993, 0.183
(70, 35)	1.151, 2.139, 0.909	0.819, 1.244, 0.425	1.063, 1.227, 0.169	1.094, 2.031, 0.872	0.778, 1.181, 0.408	1.019, 1.176, 0.161
(70, 45)	0.859, 2.135, 0.976	0.818, 1.25, 0.432	1.062, 1.232, 0.171	0.817, 2.027, 0.937	0.777, 1.187, 0.415	1.018, 1.181, 0.163
(80, 35)	0.872, 2.174, 1.102	0.827, 1.188, 0.361	1.064, 1.188, 0.143	0.829, 2.064, 1.058	0.785, 1.128, 0.346	1.02, 1.139, 0.136
(80, 45)	1.165, 2.265, 0.9	0.82, 1.24, 0.421	1.057, 1.215, 0.167	1.107, 2.056, 0.864	0.779, 1.177, 0.404	1.013, 1.165, 0.159
			CS 1 with $(\vartheta = 1.6, \gamma = 1.1, \kappa = 1.2)$ and $(\tau = 1.0, t_0 = 1.25)$			
(60, 35)	1.444, 2.016, 1.152	0.784, 1.487, 0.725	1.022, 1.438, 0.206	1.373, 1.914, 1.106	0.744, 1.412, 0.696	0.98, 1.378, 0.196
(60, 45)	1.164, 2.132, 0.978	0.817, 1.253, 0.457	1.062, 1.234, 0.176	1.106, 2.024, 0.939	0.776, 1.19, 0.439	1.018, 1.183, 0.167
(70, 35)	1.165, 2.165, 0.9	0.82, 1.24, 0.421	1.057, 1.215, 0.167	1.107, 2.056, 0.864	0.779, 1.177, 0.404	1.013, 1.165, 0.159
(70, 45)	1.151, 2.139, 0.909	0.819, 1.244, 0.425	1.063, 1.227, 0.169	1.094, 2.031, 0.872	0.778, 1.181, 0.408	1.019, 1.176, 0.161
(80, 35)	1.116, 2.156, 1.046	0.824, 1.221, 0.397	1.064, 1.208, 0.145	1.061, 2.047, 1.004	0.782, 1.159, 0.381	1.02, 1.158, 0.138
(80, 45)	1.083, 2.171, 1.083	0.826, 1.2, 0.334	1.063, 1.191, 0.135	1.029, 2.061, 1.039	0.784, 1.139, 0.321	1.019, 1.142, 0.128

(Continued)

Table 4 (continued).

(n, m)	ϑ		γ		κ	
	ACI (L, U, IL)	ACI (L, U, IL)	ACI (L, U, IL)	BCI (L, U, IL)	BCI (L, U, IL)	BCI (L, U, IL)
	CS 1 with ($\vartheta = 1.6, \gamma = 1.1, \kappa = 1.2$) and ($\tau = 1.25, t_0 = 1.5$)					
(60, 35)	1.086, 2.17, 1.084	0.826, 1.202, 0.376	1.064, 1.192, 0.184	1.032, 2.06, 1.04	0.784, 1.141, 0.361	1.02, 1.143, 0.175
(60, 45)	1.089, 2.169, 1.082	0.826, 1.204, 0.378	1.064, 1.194, 0.137	1.035, 2.059, 1.038	0.784, 1.143, 0.363	1.02, 1.145, 0.13
(70, 35)	1.083, 2.171, 1.088	0.827, 1.2, 0.373	1.063, 1.191, 0.128	1.029, 2.061, 1.044	0.785, 1.139, 0.358	1.019, 1.142, 0.122
(70, 45)	1.078, 2.174, 1.096	0.827, 1.198, 0.371	1.064, 1.188, 0.124	1.025, 2.064, 1.052	0.785, 1.137, 0.356	1.02, 1.139, 0.118
(80, 35)	1.078, 2.174, 1.096	0.827, 1.198, 0.371	1.064, 1.188, 0.136	1.025, 2.064, 1.052	0.785, 1.137, 0.356	1.02, 1.139, 0.129
(80, 45)	1.074, 2.176, 1.134	0.827, 1.196, 0.394	1.064, 1.186, 0.127	1.021, 2.066, 1.088	0.785, 1.136, 0.378	1.02, 1.137, 0.121
	CS 2 with ($\vartheta = 1.6, \gamma = 1.1, \kappa = 1.2$) and ($\tau = 0.75, t_0 = 1.0$)					
(60, 35)	1.162, 2.134, 1.297	0.819, 1.251, 0.432	1.062, 1.234, 0.172	1.105, 2.026, 1.245	0.778, 1.188, 0.415	1.018, 1.183, 0.163
(60, 45)	1.078, 2.174, 1.062	0.827, 1.198, 0.371	1.064, 1.188, 0.124	1.025, 2.064, 1.019	0.785, 1.137, 0.356	1.02, 1.139, 0.118
(70, 35)	1.083, 2.171, 1.088	0.826, 1.2, 0.374	1.063, 1.191, 0.128	1.029, 2.061, 1.044	0.784, 1.139, 0.359	1.019, 1.142, 0.122
(70, 45)	1.078, 2.174, 1.096	0.827, 1.198, 0.371	1.064, 1.188, 0.124	1.025, 2.064, 1.052	0.785, 1.137, 0.356	1.02, 1.139, 0.118
(80, 35)	1.074, 2.176, 1.102	0.827, 1.196, 0.369	1.064, 1.186, 0.122	1.021, 2.066, 1.058	0.785, 1.136, 0.354	1.02, 1.137, 0.116
(80, 45)	1.081, 2.173, 1.083	0.827, 1.2, 0.373	1.064, 1.19, 0.126	1.028, 2.063, 1.039	0.785, 1.139, 0.358	1.02, 1.141, 0.12
	CS 2 with ($\vartheta = 1.6, \gamma = 1.1, \kappa = 1.2$) and ($\tau = 1.0, t_0 = 1.25$)					
(60, 35)	1.027, 2.199, 1.172	0.832, 1.168, 0.336	1.063, 1.163, 0.165	0.976, 2.088, 1.125	0.79, 1.109, 0.322	1.019, 1.115, 0.157
(60, 45)	1.162, 2.134, 0.972	0.818, 1.252, 0.434	1.062, 1.234, 0.157	1.105, 2.026, 0.933	0.777, 1.189, 0.417	1.018, 1.183, 0.149
(70, 35)	1.17, 2.13, 0.969	0.818, 1.256, 0.438	1.062, 1.238, 0.172	1.112, 2.022, 0.93	0.777, 1.193, 0.42	1.018, 1.187, 0.163
(70, 45)	1.975, 1.979, 1.108	0.682, 2.515, 0.233	0.587, 2.367, 0.148	1.877, 1.879, 1.063	0.648, 2.388, 0.224	0.563, 2.269, 0.141
(80, 35)	1.072, 2.178, 1.106	0.828, 1.195, 0.367	1.064, 1.186, 0.127	1.019, 2.068, 1.061	0.786, 1.135, 0.352	1.02, 1.137, 0.121
(80, 45)	1.078, 2.174, 1.096	0.827, 1.198, 0.371	1.064, 1.188, 0.124	1.025, 2.064, 1.052	0.785, 1.137, 0.356	1.02, 1.139, 0.118

(Continued)

Table 4 (continued).

(n, m)	ϑ		γ		κ	
	ACI (L, U, IL)	ACI (L, U, IL)	ACI (L, U, IL)	BCI (L, U, IL)	BCI (L, U, IL)	BCI (L, U, IL)
	CS 2 with ($\vartheta = 1.6, \gamma = 1.1, \kappa = 1.2$) and ($\tau = 1.25, t_0 = 1.5$)					
(60, 35)	1.164, 2.132, 0.978	0.817, 1.253, 0.457	1.062, 1.234, 0.176	1.106, 2.024, 0.939	0.776, 1.19, 0.439	1.018, 1.183, 0.167
(60, 45)	1.151, 2.139, 0.909	0.819, 1.244, 0.425	1.063, 1.227, 0.169	1.094, 2.031, 0.872	0.778, 1.181, 0.408	1.019, 1.176, 0.161
(70, 35)	1.116, 2.156, 1.046	0.824, 1.221, 0.397	1.064, 1.208, 0.145	1.061, 2.047, 1.004	0.782, 1.159, 0.381	1.02, 1.158, 0.138
(70, 45)	1.083, 2.171, 1.083	0.826, 1.2, 0.334	1.063, 1.191, 0.135	1.029, 2.061, 1.039	0.784, 1.139, 0.321	1.019, 1.142, 0.128
(80, 35)	1.072, 2.178, 1.117	0.828, 1.195, 0.369	1.064, 1.186, 0.128	1.019, 2.068, 1.072	0.786, 1.135, 0.354	1.02, 1.137, 0.122
(80, 45)	1.078, 2.174, 1.087	0.827, 1.198, 0.372	1.063, 1.189, 0.123	1.025, 2.064, 1.043	0.785, 1.137, 0.357	1.019, 1.14, 0.117

Table 5: Lower (L) and Upper (U) limits and corresponding ILs of 95% ACIs and ST-B CIs with Schemes 3 and 4

(n, m)	ϑ		γ		κ	
	ACI (L, U, IL)	BCI (L, U, IL)	ACI (L, U, IL)	BCI (L, U, IL)	ACI (L, U, IL)	BCI (L, U, IL)
	CS 3 with ($\vartheta = 1.6, \gamma = 1.1, \kappa = 1.2$) and ($\tau = 0.75, t_0 = 1.0$)					
(60, 35)	1.261, 2.04, 1.141	0.785, 1.483, 0.718	1.016, 1.424, 0.204	1.389, 1.937, 1.095	0.745, 1.408, 0.689	0.974, 1.365, 0.194
(60, 45)	1.178, 2.158, 0.968	0.818, 1.249, 0.453	1.056, 1.222, 0.174	1.12, 2.049, 0.929	0.777, 1.186, 0.435	1.012, 1.171, 0.165
(70, 35)	1.165, 2.765, 0.9	0.82, 1.24, 0.421	1.057, 1.215, 0.167	1.107, 2.056, 0.864	0.779, 1.177, 0.404	1.013, 1.165, 0.159
(70, 45)	1.13, 2.182, 1.036	0.825, 1.217, 0.393	1.058, 1.196, 0.144	1.074, 2.072, 0.994	0.783, 1.156, 0.377	1.014, 1.146, 0.137
(80, 35)	1.168, 2.164, 0.988	0.821, 1.241, 0.394	1.057, 1.217, 0.164	1.11, 2.055, 0.948	0.78, 1.178, 0.378	1.013, 1.167, 0.156
(80, 45)	1.173, 2.161, 0.969	0.819, 1.246, 0.415	1.056, 1.22, 0.166	1.115, 2.052, 0.93	0.778, 1.183, 0.398	1.012, 1.169, 0.158
	CS 3 with ($\vartheta = 1.6, \gamma = 1.1, \kappa = 1.2$) and ($\tau = 1.0, t_0 = 1.25$)					
(60, 35)	0.738, 2.388, 1.623	0.863, 1.025, 0.264	1.03, 1.042, 0.214	0.702, 2.267, 1.558	0.819, 0.973, 0.253	0.987, 0.999, 0.203
(60, 45)	1.085, 2.204, 1.117	0.829, 1.191, 0.32	1.058, 1.174, 0.199	1.031, 2.093, 1.072	0.787, 1.131, 0.307	1.014, 1.125, 0.189
(70, 35)	1.082, 2.205, 1.12	0.829, 1.189, 0.353	1.057, 1.173, 0.194	1.029, 2.094, 1.075	0.787, 1.129, 0.339	1.013, 1.124, 0.184
(70, 45)	1.027, 2.199, 1.172	0.832, 1.168, 0.336	1.063, 1.163, 0.165	0.976, 2.088, 1.125	0.79, 1.109, 0.322	1.019, 1.115, 0.157

(Continued)

Table 5 (continued).

(n, m)	ϑ		γ		κ	
	ACI (L, U, IL)	BCI (L, U, IL)	ACI (L, U, IL)	BCI (L, U, IL)	ACI (L, U, IL)	BCI (L, U, IL)
CS 3 with ($\vartheta = 1.6, \gamma = 1.1, \kappa = 1.2$) and ($\tau = 1.0, t_0 = 1.25$)						
(80, 35)	1.162, 2.134, 0.972	0.818, 1.252, 0.434	1.062, 1.234, 0.157	1.105, 2.026, 0.933	0.777, 1.189, 0.417	1.018, 1.183, 0.149
(80, 45)	1.17, 2.13, 0.969	0.818, 1.256, 0.438	1.062, 1.238, 0.172	1.112, 2.022, 0.93	0.777, 1.193, 0.42	1.018, 1.187, 0.163
CS 3 with ($\vartheta = 1.6, \gamma = 1.1, \kappa = 1.2$) and ($\tau = 1.25, t_0 = 1.5$)						
(60, 35)	1.082, 2.205, 1.12	0.829, 1.189, 0.353	1.057, 1.173, 0.194	1.029, 2.094, 1.075	0.787, 1.129, 0.339	1.013, 1.124, 0.184
(60, 45)	1.162, 2.134, 0.972	0.818, 1.252, 0.434	1.062, 1.234, 0.157	1.105, 2.026, 0.933	0.777, 1.189, 0.417	1.018, 1.183, 0.149
(70, 35)	1.17, 2.13, 0.969	0.818, 1.256, 0.438	1.062, 1.238, 0.172	1.112, 2.022, 0.93	0.777, 1.193, 0.42	1.018, 1.187, 0.163
(70, 45)	1.168, 2.164, 0.988	0.821, 1.241, 0.394	1.057, 1.217, 0.164	1.11, 2.055, 0.948	0.78, 1.178, 0.378	1.013, 1.167, 0.156
(80, 35)	1.151, 2.139, 0.909	0.819, 1.244, 0.425	1.063, 1.227, 0.169	1.094, 2.031, 0.872	0.778, 1.181, 0.408	1.019, 1.176, 0.161
(80, 45)	1.085, 2.204, 1.095	0.829, 1.191, 0.364	1.058, 1.174, 0.121	1.031, 2.093, 1.051	0.787, 1.131, 0.349	1.014, 1.125, 0.115
CS 4 with ($\vartheta = 1.6, \gamma = 1.1, \kappa = 1.2$) and ($\tau = 0.75, t_0 = 1.0$)						
(60, 35)	1.085, 2.204, 1.117	0.829, 1.191, 0.32	1.058, 1.174, 0.199	1.031, 2.093, 1.072	0.787, 1.131, 0.307	1.014, 1.125, 0.189
(60, 45)	1.178, 2.158, 0.968	0.818, 1.249, 0.453	1.056, 1.222, 0.174	1.12, 2.049, 0.929	0.777, 1.186, 0.435	1.012, 1.171, 0.165
(70, 35)	1.165, 2.133, 0.996	0.818, 1.254, 0.426	1.063, 1.235, 0.163	1.107, 2.025, 0.956	0.777, 1.191, 0.409	1.019, 1.184, 0.155
(70, 45)	0.929, 2.359, 1.63	0.862, 1.028, 0.266	1.036, 1.052, 0.156	0.883, 2.24, 1.564	0.818, 0.976, 0.255	0.993, 1.008, 0.148
(80, 35)	1.074, 2.176, 1.12	0.827, 1.196, 0.369	1.064, 1.186, 0.132	1.021, 2.066, 1.075	0.785, 1.136, 0.354	1.02, 1.137, 0.125
(80, 45)	1.075, 2.177, 1.113	0.828, 1.197, 0.369	1.065, 1.187, 0.147	1.022, 2.067, 1.068	0.786, 1.137, 0.354	1.021, 1.138, 0.14
CS 4 with ($\vartheta = 1.6, \gamma = 1.1, \kappa = 1.2$) and ($\tau = 1.0, t_0 = 1.25$)						
(60, 35)	1.725, 2.165, 0.97	0.82, 1.24, 0.421	1.057, 1.215, 0.167	1.107, 2.056, 0.864	0.779, 1.177, 0.404	1.013, 1.165, 0.179
(60, 45)	1.192, 2.152, 1.185	0.818, 1.257, 0.446	1.055, 1.231, 0.19	1.133, 2.043, 1.137	0.777, 1.194, 0.428	1.011, 1.18, 0.181
(70, 35)	1.173, 2.161, 0.969	0.819, 1.246, 0.415	1.056, 1.22, 0.166	1.115, 2.052, 0.93	0.778, 1.183, 0.398	1.012, 1.169, 0.158
(70, 45)	1.074, 2.176, 1.12	0.827, 1.196, 0.369	1.064, 1.186, 0.132	1.021, 2.066, 1.075	0.785, 1.136, 0.354	1.02, 1.137, 0.125
(80, 35)	0.725, 2.361, 1.102	0.861, 1.027, 0.369	1.036, 1.05, 0.129	0.689, 2.242, 1.058	0.818, 0.975, 0.354	0.993, 1.007, 0.123
(80, 45)	1.069, 2.179, 1.11	0.829, 1.192, 0.363	1.064, 1.184, 0.126	1.016, 2.069, 1.065	0.787, 1.132, 0.348	1.02, 1.135, 0.12

(Continued)

Table 5 (continued).

(n, m)	ϑ		γ		κ	
	ACI (L, U, IL)	BCI (L, U, IL)	ACI (L, U, IL)	BCI (L, U, IL)	ACI (L, U, IL)	BCI (L, U, IL)
CS 4 with ($\vartheta = 1.6, \gamma = 1.1, \kappa = 1.2$) and ($\tau = 1.25, t_0 = 1.5$)						
(60, 35)	1.178, 2.158, 0.968	0.818, 1.249, 0.453	1.056, 1.222, 0.174	1.12, 2.049, 0.929	0.777, 1.186, 0.435	1.012, 1.171, 0.165
(60, 45)	1.17, 2.13, 0.969	0.818, 1.256, 0.438	1.062, 1.238, 0.172	1.112, 2.022, 0.93	0.777, 1.193, 0.42	1.018, 1.187, 0.163
(70, 35)	1.162, 2.134, 0.972	0.818, 1.252, 0.434	1.062, 1.234, 0.157	1.105, 2.026, 0.933	0.777, 1.189, 0.417	1.018, 1.183, 0.149
(70, 45)	1.083, 2.171, 1.068	0.826, 1.2, 0.374	1.063, 1.191, 0.128	1.029, 2.061, 1.025	0.784, 1.139, 0.359	1.019, 1.142, 0.122
(80, 35)	1.081, 2.173, 1.092	0.827, 1.2, 0.373	1.064, 1.19, 0.126	1.028, 2.063, 1.048	0.785, 1.139, 0.358	1.02, 1.141, 0.12
(80, 45)	1.074, 2.176, 1.136	0.827, 1.196, 0.466	1.064, 1.186, 0.114	1.021, 2.066, 1.57	0.785, 1.136, 0.447	1.02, 1.137, 0.108

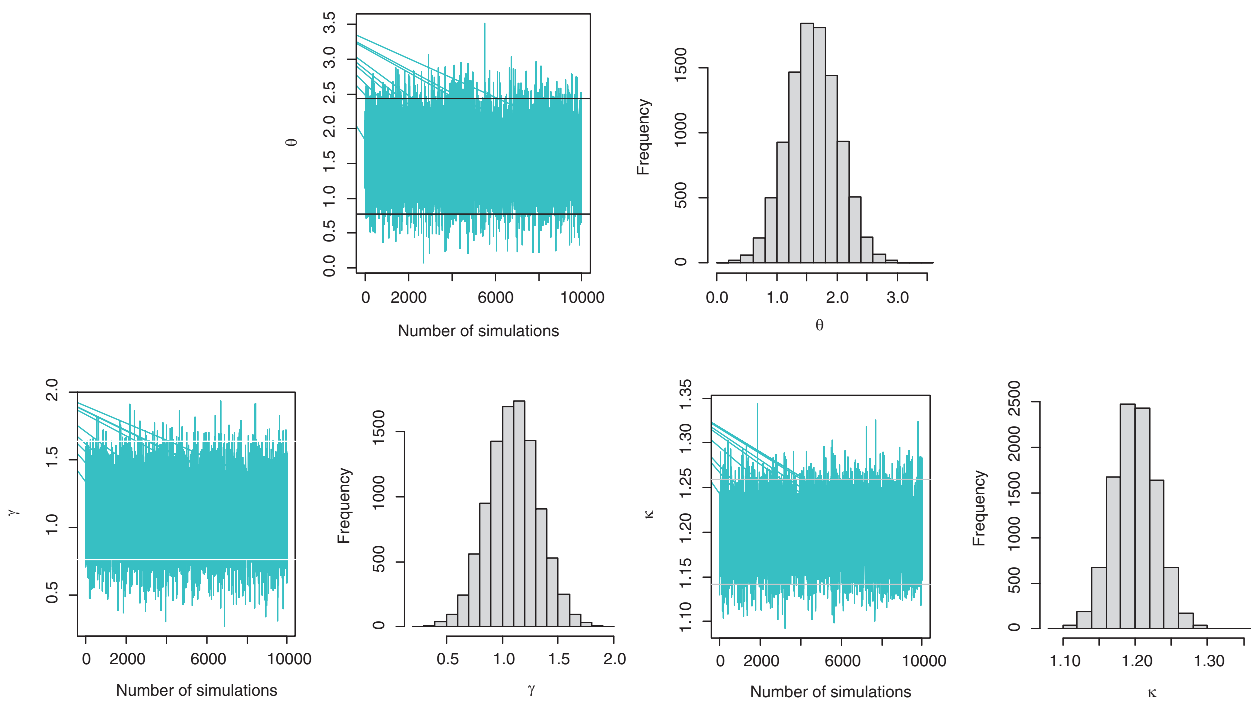

Figure 4: The plot of simulated samples and histogram of the parameters $(\vartheta, \gamma, \kappa)$ respectively with Scheme 1

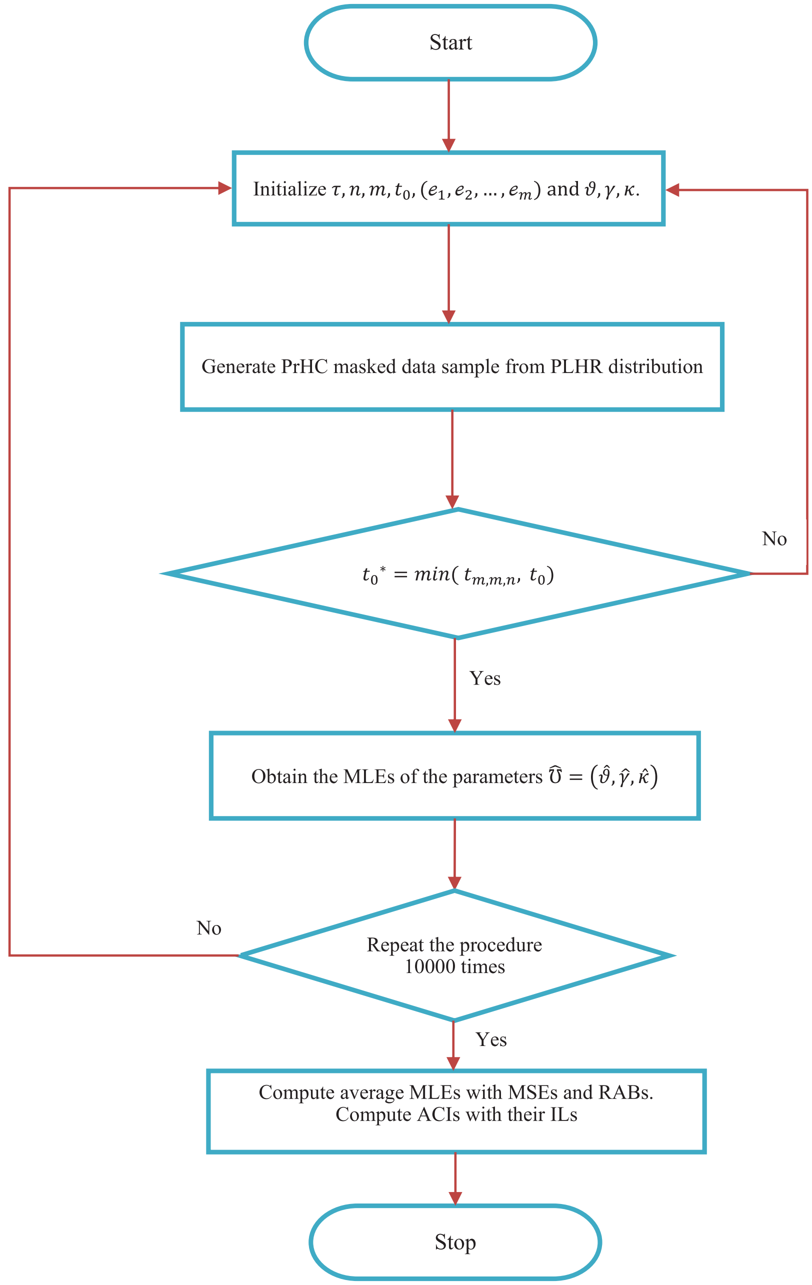

Figure 5: The flow chart to demonstrate the proposed estimation method

From the results listed in Tables 2 and 3, it can be observed that, for fixed τ and t_0, the MSEs and RABs of the model parameters are decreasing in most of the test schemes as a result of an increase in the values of n and m and MLEs $\hat{\vartheta}, \hat{\gamma}$ and $\hat{\kappa}$ of the model parameters ϑ, γ and κ are getting more closer to their respective real values and this is reasonable because when sample data is large, we can expect more precise estimates. A decreasing pattern in the values of MSEs and RABs can also be observed as n and m increases simultaneously in all cases for fixed values of n, τ and t_0. For fixed n and m, the MSEs and RABs are also getting smaller as the values of τ and t_0 getting larger and MLEs are approaching to their respective real values and this is also very much possible because with a larger experimental time, we can expect more failures and hence sample size will be larger in that case. Results reported in Tables 4 and 5 shows that expected lengths of both 95% ACIs and ST-B CIs are getting shorter with increasing values of n and m, for fixed values of τ and t_0 in almost all cases except for ϑ which is quite acceptable since ϑ is not a parameter of the distribution under study instead it is acceleration factor. It is also observed that both 95% ACIs and ST-B CIs are getting shorter when the values of n and m increases simultaneously. As a comparison between 95% ACIs and ST-B CIs, it is observed that the ST-B CIs provides more narrower expected ILs in almost every case. Therefore, now based on the above findings, we can conclude that the proposed model and methods of estimation in this paper have performed very well, and hence all the statistical assumptions for fitting the model and regarding the estimation are satisfactory.

5 Real Data Application

In this section, an actual data set is utilized to further illustrate the performance of the suggested estimation technique and to demonstrate the application of the PLHR distribution in practice in the field of reliability engineering. The R statistical programming language is used for computation. The dataset reported in Table 6 is an uncensored dataset consisting of real failure times of an airplane's air conditioning systems (in hours) was first discussed by [68].

Table 6: Failure times of the air conditioning system of an airplane (in hours)

1	3	5	7	11	11	11	12	14	14	14	16	16	20	21
23	42	47	52	62	71	71	87	90	95	120	120	225	246	261

The Kolmogorov-Smirnov (K-S) goodness of fit test is used to fit the PLHR distribution to real data. The K-S test is used to compare a given data sample with a reference continuous probability distribution. It is based on the K-S distance, which is the absolute maximum distance between the sample's empirical distribution function and the reference distribution's cumulative distribution function and its corresponding p-value. In this current example, the K-S distance was determined to be 0.14146 with a p-value of 0.5856, which is greater than 0.05. Fig. 6 displays the plots of the empirical CDF *vs.* fitted CDF of PLHR distribution and histogram of data *vs.* fitted PDF of PLHR distribution. Consequently, it is evident from the K-S distance, p-value and Fig. 6 that the PHLR distribution and considered sample data tabulated in Table 6 both have the same probability distribution. Therefore, given data can be used as an illustration for our model.

Now under SSPALT, for illustrative purposes, let's consider the stress change time τ to be 24, the test termination time t_0 to be 130, and the sample size m to be 20 with respect to the progressive censoring scheme ($e_1 = e_2 = \ldots = e_6 = 0,\ e_7 = 1,\ e_8 = 0,\ e_9 = e_{10} = 1,\ e_{11} = 0,\ e_{12} = e_{13} = 1,\ e_{14} = 0,\ e_{15} = 1,\ e_{16} = e_{17} = 0,\ e_{18} = e_{19} = 0,\ e_{20} = 1$). So, utilizing the data provided in Table 6 with 0% masking, we have the following PrHC data reported in Table 7 on normal and accelerated stress.

For PrHC data in Table 7 with 0% masking under SSPALT, the MLEs of the parameters with initial values $\gamma = 0.042003$ and $\kappa = -0.3054115$ which are the estimates of the parameters based on the complete data. And the initial value of the acceleration factor ϑ is set to be 1.2. The MLEs with their corresponding standard errors are reported in Table 9.

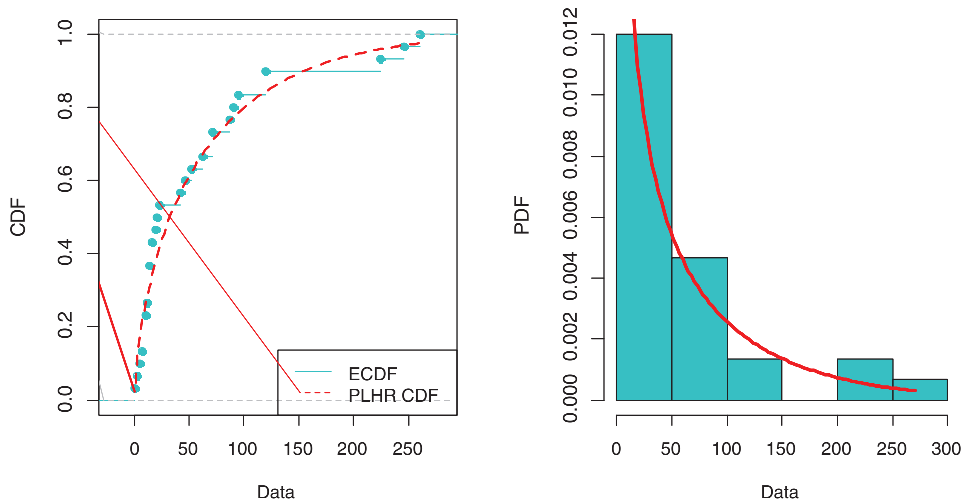

Figure 6: Plot of empirical CDF *vs.* PLHR CDF and histogram of data *vs.* fitted PDF of the PLHR distribution

Table 7: PrHC data of air conditioning systems with 0% masking at each stress level

Normal Stress level:	1	3	5	7	11	11	12	14	16	20	23
Accelerated Stress level:	42	47	52	71	71	87	90	95	120	120	

Now, to demonstrate the effect of masking, at each stress level, 20% of failed systems are chosen at random to be masked. So, utilizing the data provided in Table 7 with 20% masking, the data obtained after masking with the PrHC scheme under SSPALT at each stress level is reported in the Table 8 as follows:

Table 8: PrHC data of air conditioning systems with 20% masking at each stress level

Normal Stress level:	1	3	5	7	11	12	14	16	23
Accelerated Stress level:	42	47	52	71	71	87	90	95	

For PrHC data in Table 8 with 20% masking under SSPALT, we choose the same initial values of γ, κ and ϑ as it was in the case of 0% masking. The MLEs with their corresponding standard errors are obtained and reported in Table 9.

From Table 9, it can be observed that the ML estimates are more accurate with smaller standard errors when the sample size is large or the masking proportion is 0%, than the estimates under a smaller sample size or the masking level is 20%.

Table 9: MLEs of parameters with their corresponding standard errors based on real data set

Data type	γ		κ		ϑ	
	Estimate	Standard error	Estimate	Standard error	Estimate	Standard error
PrHC with 0% masking	0.035045	0.0011069	−0.254216	0.0041549	1.245230	0.0037102
PrHC with 20% masking	0.0371531	0.0275630	−0.3201362	0.0102856	1.2712872	0.0210763

6 Conclusions

In this article, the SSPALT model has been developed for PrHC data to analyse the lifetime of a hybrid system of tree components under mask causes of failure. Assuming that the failure of components independently follows PLHR distribution, estimates of the parameters of PLHR distribution and the acceleration factor are then obtained using the MLE technique. The performance of MLEs is investigated through their respective MSEs and RABs. 95% ACIs and ST-B CIs are also constructed and their performance is investigated in terms of their respective ILs. A simulation study has also been conducted to investigate and compare the performance of estimates for the hybrid system under SSPALT for PrHC masked data by utilizing the Monte Carlo simulation technique. Additionally, a real-world data application for an airplane's air conditioning systems was utilized to demonstrate the proposed approach. As a comparison between 95% ACIs and ST-B CIs, it is observed that the ST-B CIs provide narrower expected ILs in almost every case. Based on the findings, it can be concluded that the proposed model and method of estimation performed well. Hence all the statistical assumptions for fitting the model and regarding the estimation are satisfactory. As a future research project, the present study may be extended to more complex systems using the Bayesian estimation technique with different censored data.

Data Availability: The data used in this paper is available in the paper.

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Conflicts of Interest: The authors declare that they have no conflicts of interest to report regarding the present study.

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Appendix A

Elements of F are:

$$\frac{\partial^2 \ell}{\partial \vartheta^2} = -r^*(r-n_u)(t_0-\tau)^2\left(1+\kappa\gamma\Psi_0^{\kappa-1}\right) - (t_0-\tau)\left\{\frac{2r^*(r-n_u)(t_0-\tau)\left(\Psi_0+\gamma\Psi_0^{\kappa}\right)(1-\phi_0)\phi_0}{\left(1-(1-\phi_0)^2\right)}\right\}$$

$$\times\left\{\frac{\left(1+\kappa\gamma\Psi_0^{\kappa-1}\right)}{\left(\Psi_0+\gamma\Psi_0^{\kappa}\right)} + \frac{\left(\Psi_0+\gamma\Psi_0^{\kappa}\right)\phi_0}{(1-\phi_0)} - \left(\Psi_0+\gamma\Psi_0^{\kappa}\right) + \frac{2\left(\Psi_0+\gamma\Psi_0^{\kappa}\right)(1-\phi_0)\phi_0}{\left(1-(1-\phi_0)^2\right)}\right\}$$

$$-\frac{(r-n_u)}{\vartheta^2} + \sum_{i=n_u+1}^{r}(1+r_i)(t_i-\tau)^2\left(1+\kappa\gamma\Psi_2^{\kappa-1}\right) + \sum_{i=n_u+1}^{r}(t_i-\tau)\left\{\frac{\left(1+\kappa\gamma\Psi_2^{\kappa-1}\right)}{\left(\Psi_2+\gamma\Psi_2^{\kappa}\right)}\right\}$$

$$\times\left\{\frac{\gamma\kappa(\kappa-1)\Psi_2^{\kappa-2}}{\left(1+\kappa\gamma\Psi_2^{\kappa-1}\right)} - \frac{\left(1+\kappa\gamma\Psi_2^{\kappa-1}\right)}{\left(\Psi_2+\gamma\Psi_2^{\kappa}\right)}\right\} - \sum_{i=n_u+1}^{r}(t_i-\tau)\left\{\frac{2(1+r_i)\left(\Psi_2+\gamma\Psi_2^{\kappa}\right)(1-\phi_1)\phi_1}{\left(1-(1-\phi_1)^2\right)}\right\}$$

$$\times\left\{\frac{\left(1+\kappa\gamma\Psi_2^{\kappa-1}\right)}{\left(\Psi_2+\gamma\Psi_2^{\kappa}\right)} + \frac{\phi_1\left(\Psi_2+\gamma\Psi_2^{\kappa}\right)}{(1-\phi_1)} - \left(\Psi_2+\gamma\Psi_2^{\kappa}\right) + \frac{2\left(\Psi_2+\gamma\Psi_2^{\kappa}\right)(1-\phi_1)\phi_1}{\left(1-(1-\phi_1)^2\right)}\right\}$$

$$\frac{\partial^2 \ell}{\partial \gamma^2} = \frac{\phi_0\Psi_0^{\kappa+1}}{(\kappa+1)(1-\phi_0)} - \frac{\Psi_0^{\kappa+1}}{(\kappa+1)} + \frac{2(1-\phi_0)\phi_0\Psi_0^{\kappa+1}}{(\kappa+1)\left(1-(1-\phi_0)^2\right)} - \sum_{i=1}^{n_u}\left\{\frac{t_i^{\kappa}}{\left(t_i+\gamma t_i^{\kappa}\right)}\right\}^2 - \sum_{i=n_u+1}^{r}\left\{\frac{\Psi_2^{\kappa}}{\Psi_2+\gamma\Psi_2^{\kappa}}\right\}^2$$

$$+\sum_{i=1}^{n_u}\left\{\frac{t_i^{\kappa+1}\Psi_1}{(\kappa+1)\{1-\Psi_1\}} - \frac{t_i^{\kappa+1}}{(\kappa+1)} + \frac{2t_i^{\kappa+1}(1-\Psi_1)\Psi_1}{(\kappa+1)\left\{1-(1-\Psi_1)^2\right\}}\right\}$$

$$+\sum_{i=n_u+1}^{r}\left\{\frac{\Psi_2^{\kappa+1}\phi_1}{(\kappa+1)\{1-\phi_1\}} - \frac{\Psi_2^{\kappa+1}}{(\kappa+1)} + \frac{2\Psi_2^{\kappa+1}(1-\phi_1)\phi_1}{(\kappa+1)\left\{1-(1-\phi_1)^2\right\}}\right\}$$

$$\frac{\partial^2\ell}{\partial\kappa^2}=r^*(r-n_u)\gamma\left\{\frac{\Psi_0^{\kappa+1}\Lambda_0}{(\kappa+1)^2}\right\}\left\{log\Psi_0-\frac{log\Psi_0}{\Lambda_0}-\frac{2}{(\kappa+1)}\right\}-\left\{r^*(r-n_u)\gamma\frac{2\Psi_0^{\kappa+1}\Lambda_0(1-\phi_0)}{(\kappa+1)^2\left(1-(1-\phi_0)^2\right)}\right\}$$

$$\times\left\{log\Psi_0-\frac{log\Psi_0}{\Lambda_0}-\frac{\Psi_0^{\kappa+1}\phi_0\gamma\Lambda_0}{(\kappa+1)^2(1-\phi_0)}-\frac{2}{(\kappa+1)}-\frac{2\Psi_0^{\kappa+1}(1-\phi_0)\phi_0\gamma\Lambda_0}{(\kappa+1)^2\left(1-(1-\phi_0)^2\right)}\right\}$$

$$+\sum_{i=1}^{n_u}\frac{\gamma t_i^{\kappa}logt_i}{\left(t_i+\gamma t_i^{\kappa}\right)}\left\{logt_i-\frac{\gamma t_i^{\kappa}logt_i}{\left(t_i+\gamma t_i^{\kappa}\right)}\right\}+\sum_{i=1}^{n_u}(1+r_i)\gamma\left\{\frac{t_i^{\kappa+1}\Lambda_1}{(\kappa+1)^2}\right\}\left\{logt_i-\frac{logt_i}{\Lambda_1}-\frac{2}{(\kappa+1)}\right\}$$

$$-\sum_{i=1}^{n_u}(1+r_i)\gamma\frac{2t_i^{\kappa+1}\Lambda_1(1-\Psi_1)}{(\kappa+1)^2\left(1-(1-\Psi_1)^2\right)}$$

$$\times\left\{logt_i-\frac{logt_i}{\Lambda_1}-\frac{\gamma t_i^{\kappa+1}\Lambda_1\Psi_1}{(\kappa+1)^2(1-\Psi_1)}-\frac{2}{(\kappa+1)}-\frac{2\gamma t_i^{\kappa+1}\Lambda_1(1-\Psi_1)\Psi_1}{(\kappa+1)^2\left(1-(1-\Psi_1)^2\right)}\right\}$$

$$+\sum_{i=n_u+1}^{r}\frac{\gamma\Psi_2^{\kappa+1}log\Psi_2}{\left(\Psi_2+\gamma\Psi_2^{\kappa}\right)}\left\{log\Psi_2-\frac{\gamma\Psi_2^{\kappa}log\Psi_2}{\left(\Psi_2+\gamma\Psi_2^{\kappa}\right)}\right\}+\sum_{i=n_u+1}^{r}(1+r_i)\gamma\left\{\frac{\Psi_2^{\kappa+1}\Lambda_2}{(\kappa+1)^2}\right\}$$

$$\times\left\{log\Psi_2-\frac{log\Psi_2}{\Lambda_2}-\frac{2}{(\kappa+1)}\right\}-\sum_{i=n_u+1}^{r}(1+r_i)\gamma\frac{2\Psi_2^{\kappa+1}\Lambda_2(1-\phi_1)}{(\kappa+1)^2\left(1-(1-\phi_1)^2\right)}$$

$$\times\left\{log\Psi_2-\frac{log\Psi_2}{\Lambda_2}-\frac{\gamma\Psi_2^{\kappa+1}\Lambda_2\phi_1}{(\kappa+1)^2(1-\phi_1)}-\frac{2}{(\kappa+1)}-\frac{2\gamma\Psi_2^{\kappa+1}\Lambda_2(1-\phi_1)\phi_1}{(\kappa+1)^2\left(1-(1-\phi_1)^2\right)}\right\}$$

$$\frac{\partial^2\ell}{\partial\vartheta\partial\gamma}=\frac{\partial^2\ell}{\partial\gamma\partial\vartheta}=-r^*(r-n_u)(t_0-\tau)\Psi_0^{\kappa}-\sum_{i=n_u+1}^{r}(1+r_i)(t_i-\tau)\Psi_2^{\kappa}$$

$$+\left\{\frac{2r^*(r-n_u)(t_0-\tau)\Psi_0^{\kappa+1}(1-\phi_0)\phi_0}{(\kappa+1)\left(1-(1-\phi_0)^2\right)}\left(\frac{(\kappa+1)}{\Psi_0}+\frac{\left(\Psi_0+\gamma\Psi_0^{\kappa}\right)\phi_0}{(1-\phi_0)}\right.\right.$$

$$\left.\left.+\frac{2\left(\Psi_0+\gamma\Psi_0^{\kappa}\right)(1-\phi_0)\phi_0}{\left(1-(1-\phi_0)^2\right)}\right)\right\}+\sum_{i=n_u+1}^{r}\left\{\frac{\Psi_2^{\kappa}(t_i-\tau)}{\left(\Psi_2+\gamma\Psi_2^{\kappa}\right)}\left(\frac{\kappa}{\Psi_2}-\frac{\left(1+\gamma\kappa\Psi_2^{\kappa-1}\right)}{\left(\Psi_2+\gamma\Psi_2^{\kappa}\right)}\right)\right\}$$

$$+\sum_{i=n_u+1}^{r}\left\{\frac{2(1+r_i)(t_i-\tau)\Psi_2^{\kappa+1}(1-\phi_1)\phi_1}{(\kappa+1)\left(1-(1-\phi_1)^2\right)}\right.$$

$$\left.\times\left(\frac{(\kappa+1)}{\Psi_2}+\frac{\left(\Psi_2+\gamma\Psi_2^{\kappa}\right)\phi_1}{(1-\phi_1)}+\frac{2\left(\Psi_2+\gamma\Psi_2^{\kappa}\right)(1-\phi_1)\phi_1}{\left(1-(1-\phi_1)^2\right)}\right)\right\}$$

$$\frac{\partial^2 \ell}{\partial \vartheta \partial \kappa} = \frac{\partial^2 \ell}{\partial \kappa \partial \vartheta} = r^* (r - n_u) \gamma \left\{ \frac{(\kappa+1)(t_0-\tau)\Lambda_0 \Psi_0^{\kappa}}{(\kappa+1)^2} \right\} \left\{ 1 - \frac{1}{log \Lambda_0} \right\}$$

$$- r^* (r - n_u) \gamma \left\{ \frac{2(t_0-\tau)\Psi_0^{\kappa+1}\Lambda_0 (1-\phi_0)}{(\kappa+1)^2 \left(1-(1-\phi_0)^2\right)} \right\} \left\{ \frac{(\kappa+1)}{\Psi_0} \left(1 - \frac{1}{log\Lambda_0} \right) + \left(\frac{\left(\Psi_0 + \gamma \Psi_0^{\kappa}\right)\phi_0}{(1-\phi_0)} \right) \right.$$

$$\left. + \left(\frac{2\left(\Psi_0 + \gamma\Psi_0^{\kappa}\right)(1-\phi_0)\phi_0}{\left(1-(1-\phi_0)^2\right)} \right) \right\}$$

$$+ \sum_{i=n_u+1}^{r} \left\{ \frac{\gamma (t_i - \tau)\Psi_2^{\kappa} log\Psi_2}{\Psi_2\left(\Psi_2 + \gamma\Psi_2^{\kappa}\right)} \left(\kappa + \frac{1}{log\Psi_2} - \frac{\left(1 + \kappa\gamma\Psi_2^{\kappa}\right)}{\left(\Psi_2 + \gamma\Psi_2^{\kappa}\right)} \right) \right\}$$

$$+ \sum_{i=n_u+1}^{r} \left\{ \frac{(1+r_i)(t_i-\tau)\gamma\Psi_2^{\kappa+1}\Lambda_2}{(\kappa+1)\Psi_2} \left(1 - \frac{1}{log\Lambda_2} \right) \right\}$$

$$- \sum_{i=n_u+1}^{r} \left\{ \frac{2(1+r_i)(t_i-\tau)\gamma\Psi_2^{\kappa+1}\Lambda_2(1-\phi_1)}{(\kappa+1)^2\left(1-(1-\phi_1)^2\right)} \left(\frac{(\kappa+1)}{\Psi_2} - \frac{(\kappa+1)}{\Psi_2 log\Lambda_2} \right. \right.$$

$$\left. \left. + \frac{\left(\Psi_2 + \gamma\Psi_2^{\kappa}\right)\phi_1}{(1-\phi_1)} + \frac{2\left(\Psi_2 + \gamma\Psi_2^{\kappa}\right)(1-\phi_1)\phi_1}{\left(1-(1-\phi_1)^2\right)} \right) \right\}$$

$$\frac{\partial^2 \ell}{\partial \gamma \partial \kappa} = \frac{\partial^2 \ell}{\partial \kappa \partial \gamma} = \left\{ \frac{r^*(r-n_u)\Psi_0^{\kappa+1}\Lambda_0}{(\kappa+1)^2} \right\}$$

$$- \left\{ \frac{2r^*(r-n_u)\gamma\Psi_0^{\kappa+1}\Lambda_0(1-\phi_0)}{(\kappa+1)^2\left(1-(1-\phi_0)^2\right)} \left(\frac{1}{\gamma} + \frac{\Psi_0^{\kappa+1}\phi_0}{(\kappa+1)(1-\phi_0)} + \frac{2\Psi_0^{\kappa+1}(1-\phi_0)\phi_0}{(\kappa+1)\left\{1-(1-\phi_0)^2\right\}} \right) \right\}$$

$$+ \sum_{i=1}^{n_u} \left\{ \frac{\gamma t_i^{\kappa} log t_i}{\left(t_i + \gamma t_i^{\kappa}\right)} \left(\frac{1}{\gamma} - \frac{t_i^{\kappa}}{\left(t_i + \gamma t_i^{\kappa}\right)} \right) \right\} + \sum_{i=1}^{n_u} \left\{ \frac{(1+r_i) t_i^{\kappa+1}\Lambda_1}{(\kappa+1)^2} \right\}$$

$$- \sum_{i=1}^{n_u} \left\{ \frac{2\gamma(1+r_i)t_i^{\kappa+1}\Lambda_1(1-\Psi_0)}{(\kappa+1)^2\left(1-(1-\Psi_0)^2\right)} \left(\frac{1}{\gamma} + \frac{\Psi_0 t_i^{\kappa+1}}{(\kappa+1)(1-\Psi_0)} + \frac{2\{1-\Psi_0\}\Psi_0 t_i^{\kappa+1}}{(\kappa+1)\left(1-(1-\Psi_0)^2\right)} \right) \right\}$$

$$+ \sum_{i=n_u+1}^{r} \left\{ \frac{\gamma\Psi_2^{\kappa} log\Psi_2}{\left(\Psi_2 + \gamma\Psi_2^{\kappa}\right)} \left(\frac{1}{\gamma} - \frac{\Psi_2^{\kappa}}{\left(\Psi_2 + \gamma\Psi_2^{\kappa}\right)} \right) \right\} + \sum_{i=n_u+1}^{r} \left\{ \frac{(1+r_i)\Psi_2^{\kappa+1}\Lambda_2}{(\kappa+1)^2} \right\}$$

$$- \sum_{i=n_u+1}^{r} \left\{ \frac{2(1+r_i)\gamma\Psi_2^{\kappa+1}\Lambda_2(1-\phi_1)}{(\kappa+1)^2\left(1-(1-\phi_1)^2\right)} \left(\frac{1}{\gamma} + \frac{\Psi_2^{\kappa+1}\phi_1}{(\kappa+1)(1-\phi_1)} + \frac{2\Psi_2^{\kappa+1}(1-\phi_1)\phi_1}{(\kappa+1)\left(1-(1-\phi_1)^2\right)} \right) \right\}$$

where, $(1-(\kappa+1)\,log\Psi_0) = \Lambda_0$; $(1-(\kappa+1)\,log t_i) = \Lambda_1$; $(1-(\kappa+1)\,log\Psi_2) = \Lambda_2$.