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Article

Computer Oriented Numerical Scheme for Solving Engineering Problems

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Abstract: In this study, we construct a family of single root finding method of optimal order four and then generalize this family for estimating of all roots of non-linear equation simultaneously. Convergence analysis proves that the local order of convergence is four in case of single root finding iterative method and six for simultaneous determination of all roots of non-linear equation. Some non-linear equations are taken from physics, chemistry and engineering to present the performance and efficiency of the newly constructed method. Some real world applications are taken from fluid mechanics, i.e., fluid permeability in biogels and biomedical engineering which includes blood Rheology-Model which as an intermediate result give some nonlinear equations. These non-linear equations are then solved using newly developed simultaneous iterative schemes. Newly developed simultaneous iterative schemes reach to exact values on initial guessed values within given tolerance, using very less number of function evaluations in each step. Local convergence order of single root finding method is computed using CAS-Maple. Local computational order of convergence, CPU-time, absolute residuals errors are calculated to elaborate the efficiency, robustness and authentication of the iterative simultaneous method in its domain.

Keywords: Biomedical engineering; convergence order; iterative method; CPU-time; simultaneous method

1 Introduction

Finding roots of non-linear equation

$$f(x) = 0, \tag{1}$$

Is the one of the primal problems of science and engineering. Non-linear equation arise almost in all fields of science. To approximate the root of Eq. (1), researchers and engineers look towards numerical iterative techniques, which are further classified to approximate single [1–7] and all roots of Eq. (1). In this research paper, we work on both types of iterative methods. The most popular method among single root finding method is classical Newton method having locally quadratic convergence:

$$v^{(i)} = x^{(i)} - \left(\frac{f(x^{(i)})}{f'(x^{(i)})}\right). \quad i = 0,\ 1,\ 2,\ \ldots \tag{2}$$

Engineers and mathematician are interested in simultaneous methods due to their global convergence region and implemented for parallel computing as well. More detail on simultaneous iterative methods can be seen in [8–17] and reference cited there in.

The main aim of this paper is to propose a modified family of Noor et al. method and generalize it into numerical simultaneous technique for parallel estimation of all roots of Eq. (1).

This paper is organized in five sections. In Section 2, we construct optimal fourth-order family of single root finding method and generalize it to simultaneous method of order six. In Section 3, computational aspect of the newly constructed simultaneous method is discussed and the method is compared with existing method of the same convergence order existing in the literature. In Section 4, we illustrate some engineering applications as numerical test examples to show the performance and efficiency of the simultaneous method. Conclusion is described in Section 5.

2 Construction of Simultaneous Method

Noor et al. [18] present a two-step 4^{th} order method (abbreviated as AS):

$$u^{(i)} = y^{(i)} - \left(\frac{f(y^{(i)})}{f'(y^{(i)}) - \beta f(y^{(i)})}\right), \tag{3}$$

where $y^{(i)} = x^{(i)} - \left(\frac{f(x^{(i)})}{f'(x^{(i)}) - \beta f(x^{(i)})}\right)$ and $\beta \in \Re$.

According to Kung and Traub [19] conjecture, the iterative method (AS) is not optimal as it requires 2 evaluations of functions and 2 of its first derivatives. To make iterative method (AS) optimal, we use the following approximation [20]:

$$f'(y^{(i)}) \cong 2\left(\frac{f(y^{(i)}) - f(x^{(i)})}{y^{(i)} - x^{(i)}}\right) - f'(x^{(i)}), \tag{4}$$

in Eq. (3).

$$\begin{cases} u^{(i)} = y^{(i)} - \left(\dfrac{f(y^{(i)})}{2\left(\frac{f(y^{(i)}) - f(x^{(i)})}{y^{(i)} - x^{(i)}}\right) - f'(x^{(i)}) - \beta f(y^{(i)})}\right), \\ where \\ y^{(i)} = x^{(i)} - \left(\frac{f(x^{(i)})}{f'(x^{(i)}) - \beta f(x^{(i)})}\right). \end{cases} \tag{5}$$

The method Eq. (5) is now optimal and the convergence order of Eq. (5) is 4 if ζ is simple root of Eq. (1). Let $\varepsilon = x - \zeta$, then by using Maple-18, we find error equation as:

$$\frac{u^{(i)} - \zeta}{(x^{(i)} - \zeta)^4} \to (-\alpha^3 + 3\alpha^2 C_2 - 3\alpha C_2^2 + C_2^3 + \alpha C_3 - C_2 C_3); \qquad C_k(x) = \frac{f^{(k)}(x)}{k! f'(x)}, \tag{6}$$

$k = 2, 3, \ldots$ or

$$u^{(i)} - \zeta = O(\in^4). \tag{7}$$

Suppose, Eq. (1) has n simple roots. Then $f(x)$ and $f'(x)$ can be written as:

$$f(x) = (x - x_1)(x - x_2)\ldots(x - x_n) = \prod_{j=1}^{n}(x - x_j) \text{ and} \tag{8}$$

$$\begin{aligned} f'(x) &= (x - x_2)(x - x_3)\ldots(x - x_n) + (x - x_1)(x - x_3)\ldots(x - x_n) + \ldots \\ &\quad + (x - x_1)(x - x_2)\ldots(x - x_{n-1}) \\ f'(x) &= \sum_{k=1}^{n}\prod_{\substack{j\neq k \\ j=1}}^{n}(x - x_j). \end{aligned} \tag{9}$$

This implies,

$$\frac{f'(x)}{f(x)} = \sum_{j=1}^{n}\left(\frac{1}{(x - x_j)}\right), \quad \text{or} \tag{10}$$

$$\frac{f(x)}{f'(x)} = \sum_{j=1}^{n}\left(\frac{1}{(x - x_j)}\right)^{-1} = \frac{1}{\frac{1}{x - x_k} + \sum\limits_{\substack{j=1 \\ j\neq k}}^{n}\left(\frac{1}{(x - x_j)}\right)}, \tag{11}$$

or

$$x - x_k = \frac{1}{\frac{f'(x)}{f(x)} - \sum\limits_{\substack{j=1 \\ j\neq k}}^{n}\left(\frac{1}{(x - x_j)}\right)}. \tag{12}$$

This gives, Albert Ehrlich method (see [21]).

$$v_k^{(i)} = x_k^{(i)} - \frac{1}{\frac{1}{N(x_k^{(i)})} - \sum\limits_{\substack{j=1 \\ j\neq k}}^{n}\left(\frac{1}{(x_k^{(i)} - x_j^{(i)})}\right)}, \qquad \text{where } N(x_k^{(i)}) = \frac{f(x_k^{(i)})}{f'(x_k^{(i)})}. \tag{13}$$

Now from Eq. (11), an estimation of $\frac{f(x_k^{(i)})}{f'(x_k^{(i)})}$ is formed by replacing $x_j^{(i)}$ with $u_j^{(i)}$ from Eq. (5) as follows:

$$\frac{f(x_k^{(i)})}{f'(x_k^{(i)})} = \frac{1}{\frac{1}{N(x_k^{(i)})} - \sum\limits_{\substack{j=1 \\ j\neq k}}^{n}\left(\frac{1}{(x_k^{(i)} - u_j^{(i)})}\right)}, \tag{14}$$

Using Eq. (14) in Eq. (2), we have new family of simultaneous method (abbreviated as MS):

$$v_k^{(i)} = x_k^{(i)} - \frac{1}{\frac{1}{N(x_k^{(i)})} - \sum\limits_{\substack{j=1 \\ j\neq k}}^{n}\left(\frac{1}{(x_k^{(i)} - u_j^{(i)})}\right)}, \qquad (k,\ j = 1,\ \ldots,\ n). \tag{15}$$

In case of multiple roots:

$$v_k^{(i)} = x_k^{(i)} - \frac{\sigma_k}{\frac{1}{N(x_k^{(i)})} - \sum_{\substack{j=1 \\ j \neq k}}^{n} \left(\frac{\sigma_j}{(x_k^{(i)} - u_j^{(i)})} \right)}, \tag{16}$$

where $u_j^{(i)} = y_j^{(i)} - \left(\frac{f(y_j^{(i)})}{2\left(\frac{f(y_j^{(i)}) - f(x_j^{(i)})}{y_j^{(i)} - x_j^{(i)}} \right) - f'(x_j^{(i)}) - \beta f(y_j^{(i)})} \right)$ and $y_j^{(i)} = x_j^{(i)} - \left(\frac{f(x_j^{(i)})}{f'(x_j^{(i)}) - \beta f(x_j^{(i)})} \right)$.

Convergence Analysis

Here, we discuss the convergence of simultaneous schemes (MS) as:

Theorem: *Let $\zeta_1, \ldots, \zeta_n$ be simple roots with multiplicity $\sigma_1, \ldots, \sigma_n$ of* Eq. (1). *If $x_1^{(0)}, \ldots, x_n^{(0)}$ be the initial calculations of the roots respectively and sufficiently close to actual roots, then MS has convergence order six.*

Proof: Let $\varepsilon_k = x_k - \zeta_k$ and $\varepsilon_k' = v_k - \zeta_k$ be the errors in x_k and v_k estimations respectively. For simplification, we omit iteration index i. Considering method MS, we have:

$$v_k = x_k - \frac{\sigma_k}{\frac{\sigma_k}{N(x_k)} - \sum_{\substack{j \neq k \\ j=1}}^{n} \left(\frac{\sigma_j}{(x_k - u_j)} \right)}, \tag{17}$$

where $N(x_k) = \left(\frac{f(x_k)}{f'(x_k)} \right)$. Then, obviously for distinct roots:

$$\frac{1}{N(x_k)} = \left(\frac{f'(x_k)}{f(x_k)} \right) = \sum_{j=1}^{n} \left(\frac{1}{(x_k - \zeta_j)} \right) = \frac{1}{(x_k - \zeta_k)} + \sum_{\substack{j \neq k \\ j=1}}^{n} \left(\frac{1}{(x_k - \zeta_j)} \right). \tag{18}$$

Thus, for multiple roots we have from MS:

$$v_k - \zeta_k = x_k - \zeta_k - \frac{\sigma_i}{\frac{\sigma_k}{(x_k - \zeta_k)} + \sum_{\substack{j \neq i \\ j=1}}^{n} \left(\frac{\sigma_j (x_k - u_j - x_k + \zeta_j)}{(x_k - \zeta_j)(x_k - u_j)} \right)}, \tag{19}$$

$$\varepsilon_k' = \varepsilon_k - \frac{\sigma_i}{\frac{\sigma_k}{\varepsilon_k} + \sum_{\substack{j \neq i \\ j=1}}^{n} \left(\frac{-\sigma_j (u_j - \zeta_j)}{(x_k - \zeta_j)(x_k - u_j)} \right)} = \varepsilon_k - \frac{\sigma_i \varepsilon_i}{\sigma_k + \varepsilon_k \sum_{\substack{j \neq i \\ j=1}}^{n} \left(\frac{-\sigma_j (u_j - \zeta_j)}{(x_k - \zeta_j)(x_k - u_j)} \right)}, \tag{20}$$

$$= \varepsilon_k - \frac{\sigma_k . \varepsilon_k}{\sigma_k + \varepsilon_k \sum_{\substack{j \neq k \\ j=1}}^{n} (E_k \varepsilon_j^4)}, \tag{21}$$

where $u_j - \zeta_j = O(\varepsilon_j^4)$ from Eq. (7) and $E_i = \left(\frac{-\sigma_j}{(x_k - \zeta_j)(x_k - u_j)} \right)$.

Thus,

$$\varepsilon_k' = \frac{\varepsilon_k^2 \sum_{\substack{j \neq k \\ j=1}}^{n} (E_k \varepsilon_j^4)}{\sigma_k + \varepsilon_k \sum_{\substack{j \neq k \\ j=1}}^{n} (E_k \varepsilon_j^4)}. \tag{22}$$

If we assume $|\varepsilon_j| = O|\varepsilon|$, then from Eq. (22), we have:

$\varepsilon_k' = O(\varepsilon)^6$.

Hence the theorem.

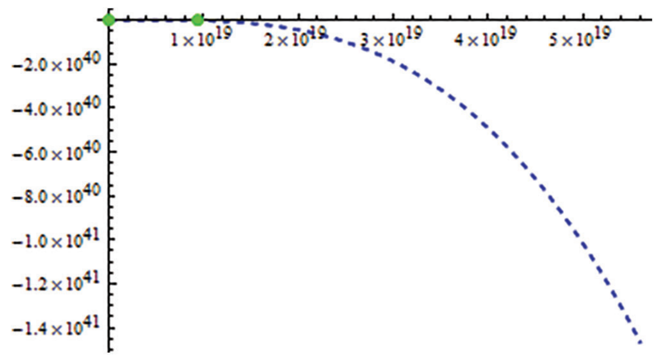

Figure 1: Shows location of exact real roots of $f_2(x)$ on x-axis

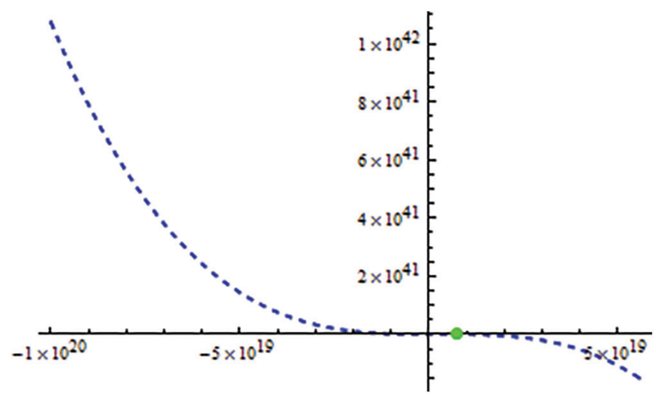

Figure 2: Shows location of exact real roots of $f_3(x)$ on x-axis

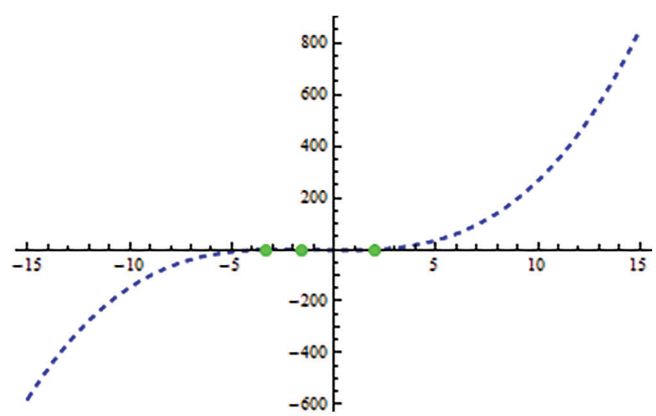

Figure 3: Shows location of exact real roots of $f_4(x)$ on x-axis

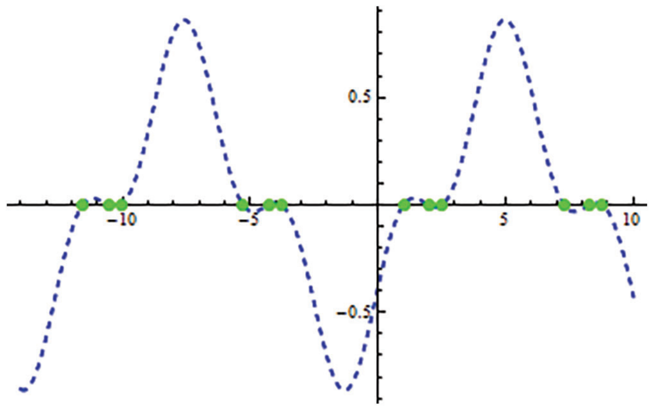

Figure 4: Shows location of exact real roots of $f_5(x)$ on x-axis

3 Computational Aspect

In this section, computational efficiency of the Petkovic et al. [22] method (abbreviated as MP)

$$x_k^{(i+1)} = x_k^{(i)} - \frac{1}{\frac{1}{N_k\left(x_k^{(i)}\right)} - \sum\limits_{\substack{j=1 \\ j\neq k}}^{n} \frac{1}{\left(x_k^{(i)} - Z_j^{(i)}\right)}}, \tag{23}$$

where $Z_j^{(i)} = x_j^{(i)} - u\left(x_j^{(i)}\right)\left(\frac{\beta_j + \gamma_j t\left(x_j^{(i)}\right)}{1 - \delta_j t\left(x_j^{(i)}\right)}\right)$, $t\left(x_j^{(i)}\right) = \left(\frac{f\left(x_j^{(i)} - \theta_j u\left(x_j^{(i)}\right)\right)}{f'\left(x_j^{(i)}\right)}\right)$, $\theta_j = \frac{2\sigma_j}{\sigma_j + 2}$, $\beta_j = -\frac{(\sigma_j)^2}{2}$, $\delta_j = \left(\frac{\sigma_j + 2}{\sigma_j}\right)^{\sigma_j}$, $\gamma_j = \frac{\sigma_j(\sigma_j + 2)}{2}\delta_j$ and the new method is presented as MS. Efficiency of iterative method is given by

$$EL(m) = \frac{\log \mathbf{r}}{\mathbf{D}}, \tag{24}$$

where $\mathbf{r}$ is the convergence order and $\mathbf{D}$ is the computational cost:

$$\mathbf{D} = \mathbf{D}(m) = w_{aq}Aq_m + w_m M_m + w_d D_m. \tag{25}$$

Thus, Eq. (24) becomes:

$$EL(m) = \left(\frac{\log \mathbf{r}}{w_{aq}Aq_m + w_m M_m + w_d D_m}\right). \tag{26}$$

Using data given by in Tab. 1 and Eq. (26), we calculate $\rho((MS), (X))$ as follows:

$$\begin{cases} \rho((MS),\ (MP)) = \left(\frac{EL(MS)}{EL(MP)} - 1\right) \times 100 \\ \rho((MP),\ MS)) = \left(\frac{EL(MS)}{EL(MP)} - 1\right) \times 100 \end{cases} \tag{27}$$

Figs. 5–6, graphically illustrates these percentage ratios. It is evident from Figs. 5–6 that MS method has dominating efficiency as compared to MP method.

Table 1: The number of basic operations

Method	AS_m	M_m	D_m
MS	$9m^2 + O(m)$	$1m^2 + O(m)$	$2m^2 + O(m)$
MP	$8m^2 + O(m)$	$6m^2 + O(m)$	$2m^2 + O(m)$

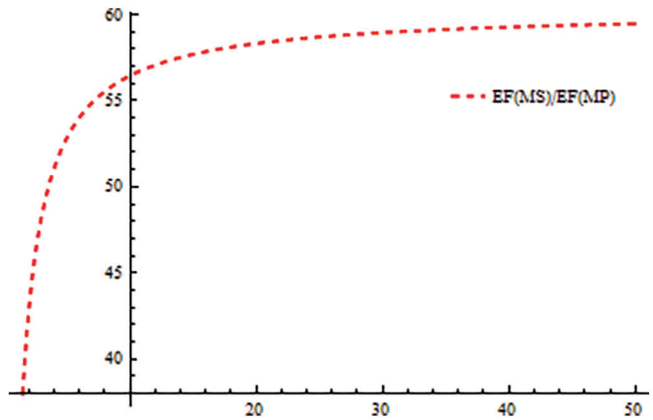

Figure 5: Shows computational efficiency of methods MS w.r.t method MP

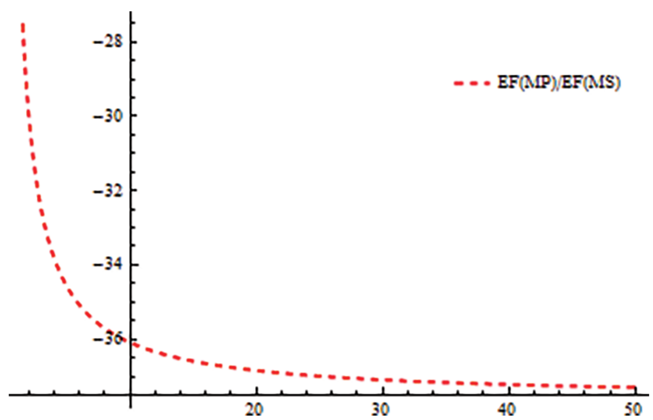

Figure 6: Shows computational efficiency of methods MP w.r.t method MS

4 Numerical Results

Here, we compare numerical results of our newly constructed method MS with Petković et al. method MP of convergence order 6. All numerical computations are performed using CAS Maple 18 with 64 digits floating point arithmetic with stopping criteria as follows.

(i) $\quad e_k = |f(x^{(i)})| < \varepsilon$

where e_k represents the absolute error. Take $\varepsilon = 10^{-30}$ as tolerance for simultaneous methods. In all Tables stopping criteria (i) is used, CPU represents computational time in seconds and $\rho_k^{(i)}$ represents local computational order of convergence [23].

Applications in Engineering

In this section, we discuss some applications in engineering.

Example 1: [24] Blood Rheology Model

Blood, which is a non-Newtonian fluid is modeled as a Caisson Fluid. Caisson fluid model predicts that simple fluid like water, blood will flow through a tube in such a way that the central core of the fluids will move as a plug with little deformation and velocity gradient occurs near the wall.

To elaborate the plug flow of Caisson fluid flow, we used the following non-linear equation:

$$G = 1 - \frac{16}{7}\sqrt{x} + \frac{4}{3}x - \frac{1}{21}x^4, \tag{28}$$

where reduction in flow rate is measured by G. Using $G = 0.40$ in Eq. (28), we have:

$$f_1(x) = \frac{1}{441}x^8 - \frac{8}{63}x^5 - 0.05714285714x^4 + \frac{16}{9}x^2 - 3.624489796x + 0.36 \tag{29}$$

The exact solutions of Eq. (29) are graphed using maple command smartplot3d [$f_1(x)$], shown in Fig. 7.

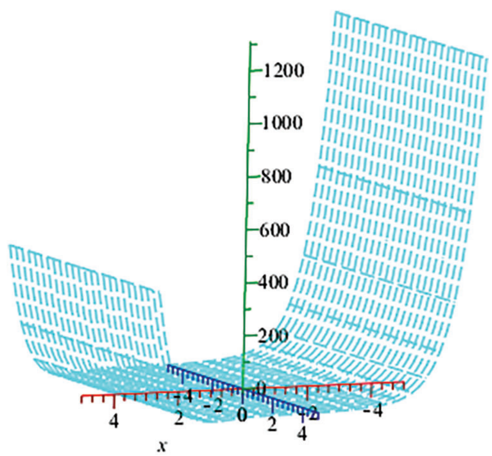

Figure 7: Shows the analytical solution of $f_1(x)$ graphically

The exact roots of Eq. (29) up to ten decimal place are:

$\zeta_1 = 0.1046986515,\ \zeta_2 = 3.822389235,\ \zeta_3 = 1.553919850 + .9404149899i,$
$\zeta_4 = -1.238769105 + 3.408523568i,\ \zeta_5 = -2.278694688 + 1.987476450i$
$\zeta_6 = -2.278694688 - 1.987476450i,\ \zeta_7 = -1.238769105 - 3.408523568i,$
$\zeta_8 = 1.553919850 - .9404149899i.$

and

$$\overset{(0)}{x_1} = 0.1,\ \overset{(0)}{x_2} = 3.8,\ \overset{(0)}{x_3} = 1.5 + 0.9i,\ \overset{(0)}{x_4} = 1.2 + 3.4i.$$

$$\overset{(0)}{x_5} = -2.2 + 1.9i,\ \overset{(0)}{x_6} = -2.2 - 1.9i,\ \overset{(0)}{x_7} = -1.2 - 3.4i,\ \overset{(0)}{x_8} = 1.5 - 0.9i.$$

are chosen as initial guessed values. Tab. 2, clearly shows the dominance behavior of MS over MP iterative method in terms of CPU time and absolute error. Roots of $f_1(x)$ are calculated at third iteration.

Table 2: Simultaneous finding of all distinct roots of non-linear function $f_1(x)$

Method	CPU	$e_1^{(3)}$	$e_2^{(3)}$	$e_3^{(3)}$	$e_4^{(3)}$	$e_5^{(3)}$	$e_6^{(3)}$	$e_7^{(3)}$	$e_8^{(3)}$	$\rho_k^{(2)}$
MP	0.407	9.8e–20	1.4e–15	7.2e–14	6.8e–16	2.0e–13	4.0e–14	2.7e–15	2.4e–14	5.98
MS	0.235	1.8e–39	1.6e–31	1.6e–27	0.0	1.5e–26	0.0	0.0	1.3e–31	6.35

Example 2: [25] Fluid Permeability in Biogels

Specific Hydraulic Permeability relates the pressure gradient to fluid velocity in porous medium (agarose gel or extracellular Fiber matrix) results the following non-linear polynomial equation:

$$\mathbf{k} = \frac{R_e x^3}{20(1-x)^2}, \tag{30}$$

$$\text{or} \quad R_e x^3 - 20\mathbf{k}(1-x)^2 = 0 \tag{31}$$

where $\mathbf{k}$ is specific hydraulic permeability, R_e radius of the fiber and $0 \le x \le 1$ is the porosity.

Using $\mathbf{k} = 0.4655$ and $R_e = 100*10^{-9}$, we have:

$$f_2(x) = -100 * 10^{-9}x^3 + 9.3100 * x^2 - 18.6200 * x + 9.3100 \tag{32}$$

The exact solution are 3D-plot for different values of R_e and k are graphed using maple command smartplot3d [$f_2(x)$ and $f_3(x)$] shown in Fig. 8 for $f_2(x)$ and Fig. 9 for $f_3(x)$ respectively. Fig. 10, shows combined graph of $f_2(x)$ and $f_3(x)$ for $-5 \le \mathbf{k} \le 5,\ -5 \le R_e \le 5,\ -400 \le x \le 400$.

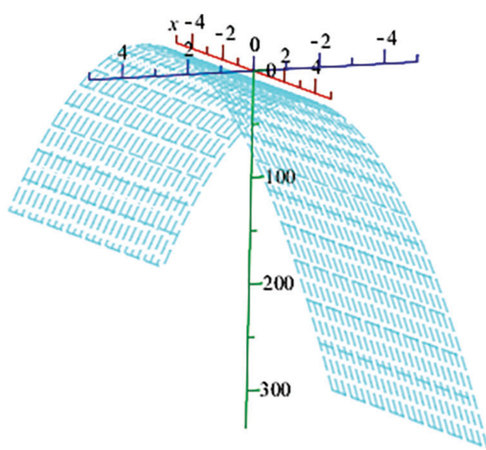

Figure 8: Shows graphically the analytical solution of $f_2(x)$ using maple command "smartplot3d [$f_2(x)$] for $\mathbf{k} = 0.4655,\ \mathrm{R}_e = 100$

The exact roots of Eq. (32) are

$\zeta_1 = 0.9999999997,\ \zeta_2 = 1.000000000,\ \zeta_3 = 9.31*10^{18}$. The locations of exact root of Eq. (31) on x-axis as shown in Fig. 1.

We choose the following initial estimates for simultaneous determination of all roots of Eq. (32):

$\overset{(0)}{x_1} = 0.9,\ \overset{(0)}{x_2} = 1.1,\ \overset{(0)}{x_3} = 9.3 * 10^{17}.$

Using $\mathbf{k} = 0.3655$ and $R_e = 10*10^{-9}$, we have:

$$f_3(x) = -100 * 10^{-9}x^3 + 9.3100 * x^2 - 18.6200 * x + 7.3100 \tag{33}$$

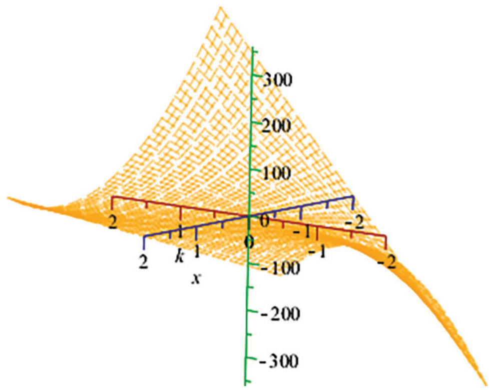

Figure 9: Shows the analytical solution of $f_3(x)$ in maple using command "smartplot3d [$f_3(x)$]" for $-5 \le \mathbf{k} \le 5,\ R_e = 10$

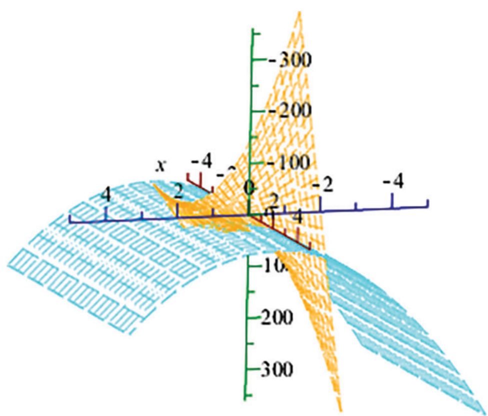

Figure 10: Shows graphically the analytical solution of $f_1(x)$ using maple command "smartplot3d [$f_2(x)$ or $f_3(x)$]." for $-5 \le \mathbf{k} \le 5, -400 \le x \le 400$

The exact roots of Eq. (33) are

$\zeta_1 = 0.9999999997,\ \zeta_2 = 1.000000000,\ \zeta_3 = 7.31*10^{18}$. The locations of exact root of Eq. (33) on x-axis are shown in Fig. 2.

We choose the following initial estimates for simultaneous determination of all roots of Eq. (33):

$\overset{(0)}{x_1} = 0.9,\ \overset{(0)}{x_2} = 1.1,\ \overset{(0)}{x_3} = 7.3 * 10^{17}.$

Tab. 3, clearly shows the dominance behavior of MS over MP iterative method in terms of CPU time and absolute error. Roots of $f_2(x)$ are calculated at third iteration.

Table 3: Simultaneous determination of all distinct roots of $f_2(x)$

Method	CPU	$e_1^{(3)}$	$e_2^{(3)}$	$e_3^{(3)}$	$\rho_k^{(2)}$
MP	0.018	0.002	0.002	3.7e–30	5.64
MS	0.015	1.2 e–7	2.5e–7	4.5e–45	6.01

Tab. 4, clearly shows the dominance behavior of MS over MP iterative method in terms of CPU time and absolute error. Roots of $f_3(x)$ are calculated at the third iteration. Figs. 7–10, shows analytical approximate solution of $f_1(x) - f_3(x)$ using maple command smartplot3d. Figs. 1–10, clearly show that analytical approximate and exact solutions match.

Table 4: Simultaneous determination of all distinct roots of $f_3(x)$

Method	CPU	$e_1^{(3)}$	$e_2^{(3)}$	$e_3^{(3)}$	$\rho_k^{(2)}$
MP	0.019	0.002	0.002	3.7e–35	5.01
MS	0.012	1.2 e–4	2.5e–4	4.5e–45	5.91

Example 3: [26] Beam Designing Model (An Engineering Problem)

An engineer considers a problem of embedment x of a sheet-pile wall resulting a non-linear function given as:

$$f_4(x) = \frac{x^3 + 2.87x^2 - 10.28}{4.62} - x. \tag{34}$$

The exact roots of Eq. (34) are represented in Fig. 3 on x-axis and $\zeta_1 = 2.0021$, $\zeta_2 = -3.3304$, $\zeta_3 = -1.5417$. The initial guessed values are taken as:

$$\overset{(0)}{x_1} = 1.17, \ \overset{(0)}{x_2} = -7.4641, \ \overset{(0)}{x_3} = -0.5359.$$

Tab. 5, clearly shows the dominance behavior of MS over MP iterative method in terms of CPU time and absolute error. Roots of $f_4(x)$ are calculated at the third iteration.

Table 5: Simultaneous determination of all distinct roots of $f_4(x)$

Method	CPU	$e_1^{(3)}$	$e_2^{(3)}$	$e_3^{(3)}$	$\rho_k^{(2)}$
MP	0.016	5.3e–21	5.2e–20	2.2e–20	5.41
MS	0.015	0.0	0.0	0.0	6.38

Example 4: [27]

Consider

$$f_5(x) = \sin^3\left(\frac{x-1}{2}\right) \sin^3\left(\frac{x-2}{2}\right) \sin^3\left(\frac{x-2.5}{2}\right), \tag{35}$$

with multiple exact roots of Eq. (35) as represented in Fig. 4 are $\zeta_1 = 1$, $\zeta_2 = 2$, $\zeta_3 = 2.5$. The initial guessed values of the exact roots have been taken as:

$\overset{(0)}{x_1} = -0.2,\ \overset{(0)}{x_2} = 1.7,\ \overset{(0)}{x_3} = 3.$

For distinct roots, we have:

$$f_{5-1*}(x) = \sin\left(\frac{x-1}{2}\right)\ \sin\left(\frac{x-2}{2}\right)\ \sin\left(\frac{x-2.5}{2}\right). \tag{36}$$

Tab. 6, clearly shows the dominance behavior of MS over MP iterative method in terms of CPU time and absolute error. Roots of $f_{5-1*}(x)$ and $f_5(x)$ are calculated at the third iteration.

Table 6: Simultaneous determination of all roots of $f_{5-1*}(x)$ and $f_5(x)$

Method	CPU	$e_1^{(3)}$	$e_2^{(3)}$	$e_3^{(3)}$	$\rho_k^{(2)}$
MP	0.047	0.019	4.6e–3	9.9e–10	4.91
MS	0.031	3.5e–6	1.8e–4	0.0	5.32
MP	0.094	5.0e–5	1.7e–2	3.3e–5	4.01
MS	0.062	1.9e–6	2.4e–23	0.0	5.78

5 Conclusion

In this research article, we developed an optimal family of single root finding method of convergence order 4 and then extended this family to an efficient numerical algorithm of convergence order 6 for approximating all roots of Eq. (1). The computational efficiency of our method MS is very large as compared to MP as given in Tab. 1, which is also obvious from Figs. 5–6. From all Figs. 1–10, Tabs. 1–6, residual error and CPU time clearly show the dominance efficiency of iterative scheme MS as compared to MP on same number of iterations.

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