

Coherence Based Sufficient Condition for Support Recovery Using Generalized Orthogonal Matching Pursuit

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Abstract: In an underdetermined system, compressive sensing can be used to recover the support vector. Greedy algorithms will recover the support vector indices in an iterative manner. Generalized Orthogonal Matching Pursuit (GOMP) is the generalized form of the Orthogonal Matching Pursuit (OMP) algorithm where a number of indices selected per iteration will be greater than or equal to 1. To recover the support vector of unknown signal 'x' from the compressed measurements, the restricted isometric property should be satisfied as a sufficient condition. Finding the restricted isometric constant is a non-deterministic polynomial-time hardness problem due to that the coherence of the sensing matrix can be used to derive the sufficient condition for support recovery. In this paper a sufficient condition based on the coherence parameter to recover the support vector indices of an unknown sparse signal 'x' using GOMP has been derived. The derived sufficient condition will recover support vectors of P -sparse signal within ' P ' iterations. The recovery guarantee for GOMP is less restrictive, and applies to OMP when the number of selection elements equals one. Simulation shows the superior performance of the GOMP algorithm compared with other greedy algorithms.

Keywords: Compressed sensing; restricted isometric constant; generalized orthogonal matching pursuit; support recovery; recovery guarantee; coherence

1 Introduction

Compressed sensing will aid to recover a P -sparse (P -non zero elements) unknown signal $x \in R^n$ using the linear equation

$$y = Hx + v \quad (1)$$

where $y \in R^m$ is an observation vector, $H \in R^{m \times n}$ ($m \ll n$) is a sensing matrix, v is a noise vector. Though the above equation is under determined, P -sparse x can be recovered from under determined measurements y by imposing condition on matrix H using Restricted Isometric Property (RIP). Greedy algorithms are popular to recover the support vector indices for example Orthogonal Matching Pursuit (OMP), Compressive Sampling Matching Pursuit (CoSAMP), Basis Pursuit. The condition to recover any P -sparse vector x ,



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Restricted Isometric Constant (RIC) $\delta \in (0, 1)$ is the smallest constant that satisfy the relation given below

$$(1 - \delta_p) \|x\|_2^2 \leq \|Hx\|_2^2 \leq (1 + \delta_p) \|x\|_2^2 \quad (2)$$

The smallest constant δ satisfying Eq. (2) is referred as RIC. Greedy approach is most popular and lesser computational complexity than the state-of-the-art algorithms like OMP, and CoSAMP. Most of the recovery guarantee has been proposed based on the RIC which accurately recovers all the support indices of the sparse signal with the help of the RIP [1–4]. In general, it is Non-deterministic Polynomial-time (NP)-hard to evaluate RIC for a given matrix H in the expression to satisfy the stringent conditions. Recovery conditions based on the mutual coherence have been proposed in the literature earlier for the OMP algorithm (special case where a number of support elements selected in this algorithm is equal to one ($Q=1$)) [5,6]. Coherence statistics will be helpful in solving many signal processing problems including support detection and vector quantization [7–10]. In this paper, a sufficient condition has been derived to guarantee the recovery performance based on coherence for Generalized Orthogonal Matching Pursuit (GOMP). In recent years recovery conditions based on the RIC have been proposed to ensure recovery of the accurate P -sparse signal with GOMP [11–15]. Furthermore in, the author [11,16] proposes sufficient condition has been improved to $\delta_{NK+1} < \frac{1}{\sqrt{K/N+1}}$. Though the conditions are tighter bounds it is NP-hard to find out the RIC of the matrix H [11, 17–19]. Sufficient condition based on the mutual coherence to accurately recover the support indices of the P -sparse signal with GOMP in noisy case has been proposed. GOMP algorithm has the choice to choose number elements to be selected in each iteration $Q \leq (m - 1)/P$. As a generalized case if the number of selection element is equal to 1, it converges to OMP whose performance is inferior to the GOMP.

1.1 Contributions

Sufficient condition to guarantee the support recovery for GOMP algorithm has been derived using the coherence of the sensing matrix H .

GOMP can select $Q \leq (m - 1)/P$ elements in each iteration. When $Q = (m - 1)/P$, GOMP algorithm recovery performance is better than the OMP where $Q=1$.

$\mu \leq \frac{1}{\sqrt{\frac{|\omega|}{Q} + 1(Q|\omega|)}}$ is a sufficient condition based on coherence statistics to recover the support vectors of

P -sparse signal in ‘ P ’ iterations using GOMP for noisy case. Simulations have been carried out to validate the theoretical results and compare with the other greedy algorithms.

GOMP Algorithm:

Step 1: Input: $H \in R^{m \times n}$, $y \in R^n$, P , $Q \leq (m - 1)/P$

Step 2: Initialize: $r_t = y = 0$

Step 3: For Each $t \leq P$

Step 4: Compute the correlation and find the first Q indices where correlation is $i_t = \arg \max_{n \in S_t} |H_n r_{t-1}|$

Step 5: Select the first ‘ Q ’ largest value indices $i_t = \{i_1 \dots i_Q\}$

Step 6: Update sub band support indices $S_t = S_{t-1} \cup i_t$

Step 7: Estimate the signal vector $\hat{Z}_t = \arg \min_{z \sup p(z)} \|y - H_{S_t} z\|_2^2$

Step 8: Update residual $r_t = y - H_{S_t} \hat{Z}_t$

Step 9: $t = t + 1$;

Step 10: Go to step 2

- Step 11: End For Return index set of sub band support indices $\ddot{S} = S_t$;
- Step 12: Output: Support vector indices \widehat{S}
- Step 13: end

1.2 Preliminaries: Generalized Orthogonal Matching Pursuit (GOMP)

Algorithm for generalized orthogonal matching pursuit shown in GOMP algorithm has three parts identification, augmentation and residual update. In the step 4, magnitude of the correlation between each column of matrix H and residual is found and the values are arranged in descending order. Step 5: First Q largest column indices $i_t = \{i_1 \dots i_Q\}$ has been chosen. Step 6: Active support indices is concatenated with the previous iteration support sets.

Line 7 finding the least square estimate of the signal vector, to find the residual r_t is updated after eliminating the active column indices from the measurements is shown in line 8. In other words, least square will ensure projection of y on to the complement space $P_{S_t}^L$, this will ensure that the indices in S_t in the current iteration cannot be selected repetitively in the upcoming iterations. The selected support indices in the current iteration is projected to the orthogonal space in that way we can avoid the same support indices will be chosen in the upcoming iterations.

2 Sufficient Condition

Coherence of sensing matrix H is maximum of absolute correlation between the two columns of H and it is given by $\mu(H) \hat{=} \max_{p \neq q} |h_p^* h_q|$, where h^* is conjugate of h . To analyze the performance of the support recovery algorithm some of the useful lemmas have been derived in the literature [12].

Lemma 1: Let $H \in R^{m \times n}$ satisfies the RIP of orders P_1 and P_2 with $P_1 < P_2$, then $\delta_{P_1} \leq \delta_{P_2}$. This is the monotonicity property of the RIC.

Lemma 2: For any matrix H , the RIP constant δ_P bounded by $\delta_P \leq \mu(P - 1)$ where P is the sparsity of the signal $x(t)$ and μ is the mutual coherence of matrix H .

Lemma 3: Suppose a matrix H satisfy RIP of order P and $Q \leq P$, then for any signal

$$x \in R^n ||H_Q^T x||_2^2 \leq (1 + \delta_P) ||x||_2^2$$

In this lemma 3 lower bound can be defined with the help of coherence μ and sparsity of the signal P

$$\max_{i \in \omega \setminus S} H_i^T P_S^L H_{\omega \setminus S} - \frac{1}{N} \sum_{j \in W} |H_j^T P_S^L H_{\omega \setminus S} x_{\omega \setminus S}|. \tag{3}$$

The above expression plays a key role in proving the theorem 1. It holds good for generalized case of OMP.

Remark 1: The condition $N(k + 1) + |\omega| - k \leq m$ in lemma 1 is to ensure the assumption that H satisfies the RIP of order $N(k + 1) + |\omega| - 1$ makes sense.

Theorem 1: Let H satisfy the coherence with $\mu \leq \frac{1}{1 - \sqrt{\frac{|\omega|}{Q} + 1} [(Q(P+1) + |\omega| - P) - 1] \mu}$ for an integer N with $1 \leq Q \leq (m - 1)P$. Then GOMP either identifies at least t_0 indices in ω if GOMP terminates after performing t_0 iterations with $1 \leq t_0 \leq P$ or recovers ω in P iterations provided that

$$\min_{i \in S} \|x[i]\|_2 > \frac{2s}{1 - \sqrt{\frac{|\omega|}{Q} + 1[(Q(P + 1) + |\omega| - P) - 1]\mu}}$$

The proof for the special case $\omega = P - 1$ using Lemma 3 is as follows:

First we need to prove that the selection of Q elements contains at least one support index in iterations will be considered as success. The proof for the first iteration should be carried at first. Let us assume

$$V = \{j_1, j_2 \dots, j_N\} \tag{4}$$

Instead of proving $P_{t+1} \setminus P_t \cap \omega \neq \phi$ we will show

$$|H_{j_1}^T r_t| \geq \dots \geq |H_{j_N}^T r_t| \geq |H_{j \in \omega^c \setminus V}^T r_t| \tag{5}$$

$$\max_{j \in \omega} |H_i^T r_t| > |H_{j_N}^T r_t| \tag{6}$$

By expression (5), $H_{j_Q}^T r_t \leq \frac{1}{Q} \max_{j \in \omega} |H_i^T r_t|$ Thus, to show $P_{t+1} \setminus P_t \cap \omega = \varphi$, it suffices to show

$$\max_{i \in W} |H_i^T r_t| > \frac{1}{Q} \max_{j \in \omega} |H_j^T r_t| \tag{7}$$

residual in the algorithm at t^{th} iteration can be expressed as

$$\begin{aligned} r_t &= y - H_{S_t} x_{S_t} = (I - H_{S_t} (H_{S_t}^T H_{S_t})^{-1} H_{S_t}^T) y = P_{S_t}^L (H_{\omega} x_{\omega} + v) \\ &= P_{S_t}^L (H_{\omega \cap S_t} + H_{\omega \setminus S_t} x_{\omega \setminus S_t} + v) \\ &= P_{S_t}^L H_{\omega \setminus S_t} x_{\omega \setminus S_t} + P_{S_t}^L v \end{aligned} \tag{8}$$

Thus, by Eq. (8) and the triangular inequality

$$\max_{i \in \omega} |H_i^T r_t| \geq \max_{i \in \omega \setminus S_t} (|H_i^T P_{S_t}^L H_{\omega \setminus S_t} x_{\omega \setminus S_t}| - |H_i^T P_{S_t}^L v|) \tag{9}$$

$$\frac{1}{N} \sum_{j \in W} |H_j^T r_t| \leq \frac{1}{N} \sum_{j \in W} |H_j^T P_{S_t}^L H_{\omega \setminus S_t} x_{\omega \setminus S_t}| + \max_{j \in W} |H_i^T P_{S_t}^L v| \tag{10}$$

It is a two-step process to find the lower bound. To find the lower bound on maximum of I_T is a two step process to find the lower bound on $\max_i \omega |H_i^T r_t|$. A lower bound on $\max_{i \in \omega} |H_i^T r_t|$ will be derived, and it requires $Q \leq P$. Thus, to show (10), it suffices to show

$$\beta_1 > \beta_2 \tag{11}$$

$$\beta_1 = \max_{i \in \omega \setminus S_t} |H_i^T P_{S_t}^L H_{\omega \setminus S_t}| - \frac{1}{N} \sum_{j \in W} |H_j^T P_{S_t}^L H_{\omega \setminus S_t} x_{\omega \setminus S_t}| \tag{12}$$

$$\beta_2 = \max_{i \in \omega \setminus S_t} |H_i^T P_{S_t}^L v| + \max_{j \in \omega} |H_j^T P_{S_t}^L v| \tag{13}$$

$$\max_{i \in \omega \setminus S_t} |H_i^T P_{S_t}^L v| = |H_{i_0}^T P_{S_t}^L v| \tag{14}$$

$$\max_{j \in W} |H_i^T P_{S_t}^L v| + \max_{j \in \omega} |H_{j_0}^T P_{S_t}^L v| \tag{15}$$

$$\begin{aligned}
 \beta_2 &= \|H_{i_0 \cap j_0}^T P_{S_t}^L v\| \\
 &\leq \sqrt{2 \|H_{i_0 \cup j_0}^T P_{S_t}^L v\|_2} \\
 &\leq \sqrt{2(1 + \delta_Q(P + 1) + |\omega| - p) \|v\|_2}
 \end{aligned} \tag{16}$$

Lower bound on β_1 will be derived

$$0 \leq t \leq |\omega \cap S_t| = l \leq |\omega| - 1$$

$$\beta_1 \geq \frac{1 - \sqrt{\frac{|\omega|}{Q} + 1\delta_Q(P + 1) + |\omega| - l} \|x_{\omega \setminus S_t}\|_2}{\sqrt{|\omega| - l}} \tag{17}$$

$$\|x_{\omega \setminus S_t}\|_2 \geq \sqrt{\frac{|\omega| - 1}{P(1 + \delta_Q(P + 1) + |\omega| - P)}} \sqrt{MAVR.SNR} \|v\|_2 \tag{18}$$

$$\|x_{\omega \setminus S_t}\|_2 \geq \sqrt{|\omega| - l \min_{i \in \omega} |x_i|} \tag{19}$$

$$\begin{aligned}
 &= \sqrt{|\omega| - l (\sqrt{MAVR} \|x\|_2 / \sqrt{K})} \\
 &\geq \sqrt{\frac{|\omega| - 1}{P(1 + \delta_Q(P + 1) + |\omega| - P)}} \sqrt{MAVR} \|Hx\|_2 \\
 &\geq \sqrt{\frac{|\omega| - 1}{P(1 + \delta_Q(P + 1) + |\omega| - P)}} \sqrt{MAVR.SNR} \|v\|_2
 \end{aligned}$$

$$\|Hx\| = \|H_\omega x_\omega\|_2 \tag{20}$$

$$\begin{aligned}
 &\leq \sqrt{(1 + \delta_Q(P + 1) + |\omega| - P)} \\
 \beta_1 &\geq \frac{\left(1 - \sqrt{\frac{|\omega|}{Q} + 1\delta_Q(P + 1) + |\omega| - P}\right) \sqrt{MAVR.SNR} \|v\|_2}{\sqrt{P(1 + \delta_Q(P + 1) + |\omega| - P)}}
 \end{aligned} \tag{21}$$

$$\frac{\left(1 - \sqrt{\frac{|\omega|}{Q} + 1\delta_Q(P + 1) + |\omega| - P}\right) \sqrt{MAVR.SNR} \|v\|_2}{\sqrt{P(1 + \delta_Q(P + 1) + |\omega| - P)}} > \sqrt{2(1 + \delta_Q(P + 1) + |\omega| - P) \|v\|_2} \tag{22}$$

$$\min_{i \in S} \|x[i]\|_2 > \frac{2s}{1 - \sqrt{\frac{|\omega|}{Q} + 1\delta_Q(P + 1) + |\omega| - P}} \tag{23}$$

Hence the proof.

Remark: from the expression (23) it has been proved that if elements of the sparse signal satisfy the constraint, then the GOMP algorithm will identify at least one support index in t iterations. Figs. 2 and 3 show the performance comparison of GOMP and OMP algorithms at various Signal to Noise Ratio (SNR) level for different sparsity. It shows that GOMP has superior performance over the OMP at every SNR levels for the same sparsity. A sufficient condition has been derived in the previous section. Simulations have been carried out to validate the performance of the GOMP algorithm.

3 Numerical and Simulations

In the simulation, the unknown sparse signal is defined as

$$x(t) = \sum_{j=1}^L \sqrt{E_j W_j} \sin c(W_j(t - t_j) \cos W_j(t - t_j)) \quad (24)$$

where L is the number of occupied bands, E_j is the energy coefficient, W_j is the j^{th} sub-band in the wide bandwidth, t_j is the time offset between t_j and $t=0$, and ω_j is the carrier frequency. Simulations have been carried out in MATLAB 2019a. In all the trials t_j in W_j is randomly chosen in MHz, and E_j is randomly chosen. Elements of sensing matrix are identical, independent Gaussian distribution. Non zero elements in the P -sparse vector x are evenly distributed in all possible locations. Support recovery performance of the GOMP algorithm has been evaluated over 500 independent trials for different sparsity levels and Signal to Noise ratio (SNR). We evaluate the performance for support recovery with GOMP over 500 independent trails for different sparsity levels and SNR. The measurement matrices involved in this paper are all Gauss random matrices while the reconstruction algorithms considered for the comparison are OMP, Subspace Pursuit (SP), and CoSAMP algorithms.

Number of support indices selected in each iteration should satisfy the constraint $Q \leq (m - 1)/P$. Support recovery performance of GOMP algorithm has been plotted for different number selection elements in every iteration range from 1 to 7 at various SNR levels, sparsity $P=6$. In Fig. 1 number of selections varied and the performance of the GOMP algorithm has been studied. The number selection element $Q=7$ has the superior support recovery performance than that of the traditional OMP ($Q=1$). In Fig. 4 computation time has been plotted for different sparsity. It is obvious that computational time is slightly higher than the OMP, CoSAMP.

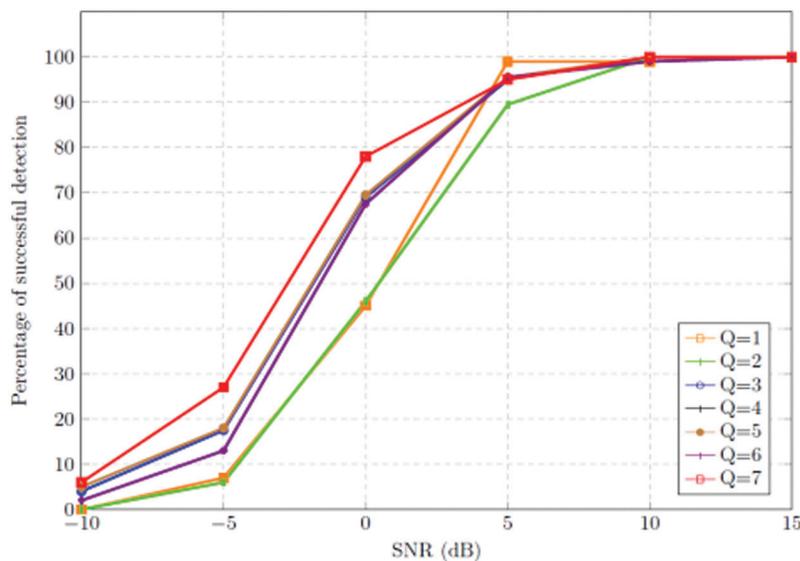


Figure 1: Detection performance of GOMP algorithm for different number of selection indices in each iteration at various signal to noise ratio levels

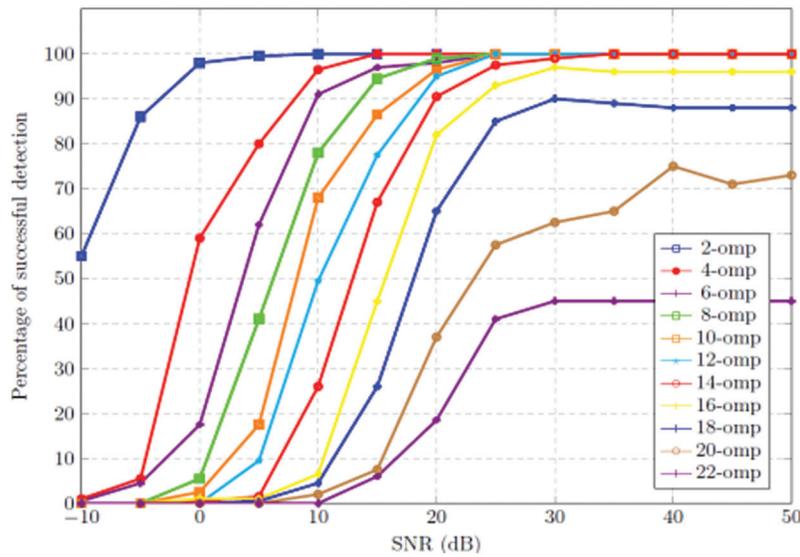


Figure 2: Detection performance of the OMP (Special case where number of selection element per iteration $Q = 1$) algorithm for different sparsity P varies from 2 to 22 at various SNR levels

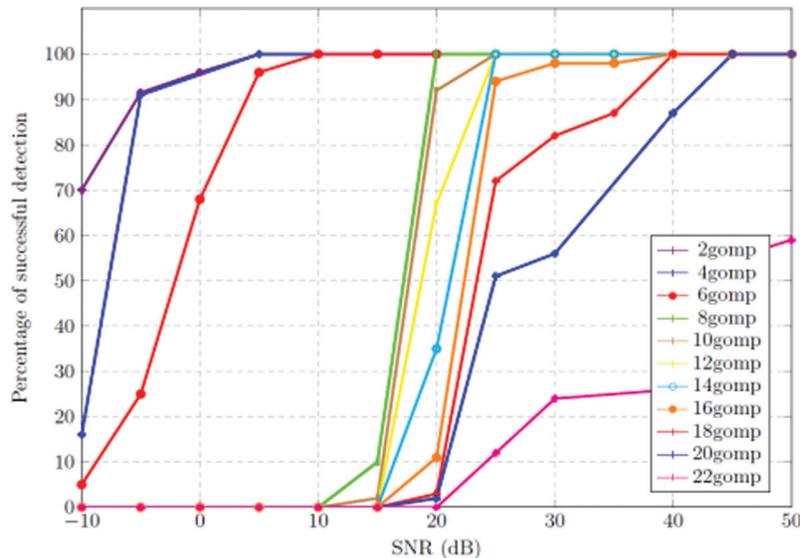


Figure 3: Detection performance of the GOMP algorithm for different sparsity P varies from 2 to 22 at various SNR levels

In this paper, successful recovery of support vector indices with respect to SNR for given sparsity of P using generalized orthogonal matching pursuit is shown in the Fig. 2. Sparsity of the signal x will be varied between 2 and 22. It is evident from the Fig. 3 that the algorithm can able to recover the signal support successfully only when there are few non-zero elements.

We note that the precise reconstruction probability and the computation time of the sparse signal vary with the level of sparsity. Reconstruction results for the different algorithms are presented in Fig. 5, where the x and y axes represent the level of sparsity and probability of exact recovery, respectively. Percentage of

successful detection of support vectors at lower SNR range in GOMP is better compared to the other algorithm because GOMP recovers at least one true support vector in each iteration.

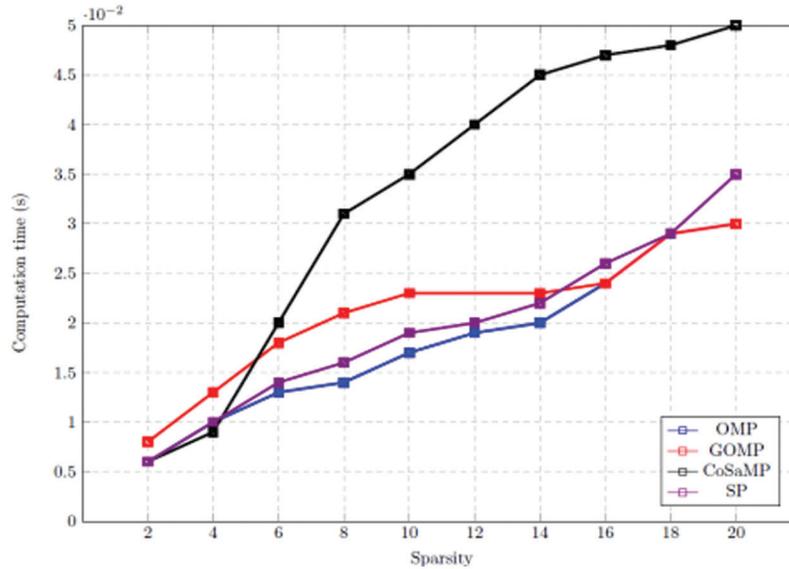


Figure 4: Graph between different sparsity level and computation time for different greedy algorithms

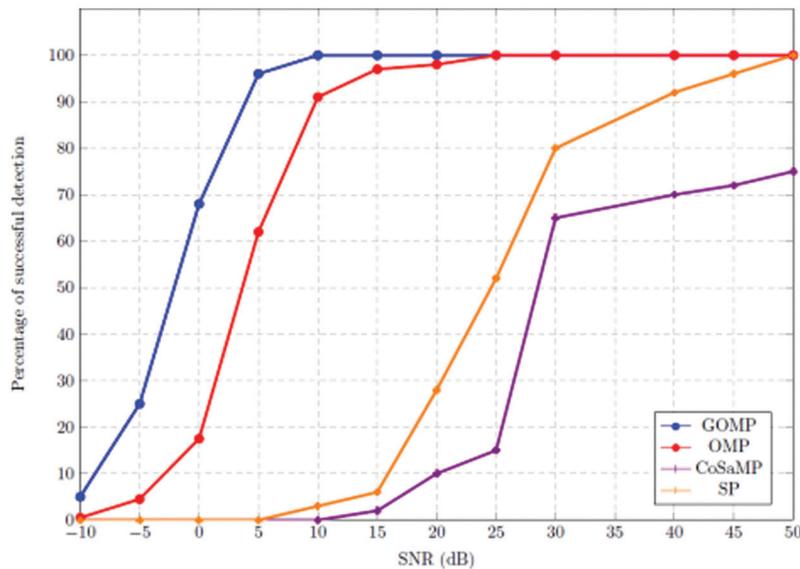


Figure 5: Detection performance of various greedy algorithm plotted between signal to noise ratio in deciBel and percentage of successful detection

4 Conclusions

In this paper, it has been proved that $\mu \leq \frac{1}{\sqrt{\left(\frac{|\omega|}{Q}+1\right)(Q(P+1)+|\omega|-P-1)}}$ with a condition on signal amplitude for the accurate support recovery of P-sparse signal based on the coherent statistics. Further number of support index chosen in each iteration is $Q \leq (m - 1)/P$. This condition is more flexible than the

$Q \leq P$. When the number of elements chosen is equal to 1, the GOMP converges to OMP. In the noisy case, it is a sufficient condition based on the coherence statistics to recover the support vector accurately in t iterations. Simulation has been carried out GOMP algorithm support recovery performance superior to the other methods at different SNR and sparsity levels. It is evident from the simulation that the GOMP has superior detection performance over other greedy algorithms like OMP, CoSAMP, and SP at different sparsity and SNR.

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