

**ARTICLE**

Damped Mathieu Equation with a Modulation Property of the Homotopy Perturbation Method

Yusry O. El-Dib and Nasser S. Elgazery*

Department of Mathematics, Faculty of Education, Ain Shams University, Roxy, Cairo, Egypt

*Corresponding Author: Nasser S. Elgazery. Email: nasser522000@gmail.com

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ABSTRACT

In this article, the main objective is to employ the homotopy perturbation method (HPM) as an alternative to classical perturbation methods for solving nonlinear equations having periodic coefficients. As a simple example, the nonlinear damping Mathieu equation has been investigated. In this investigation, two nonlinear solvability conditions are imposed. One of them was imposed in the first-order homotopy perturbation and used to study the stability behavior at resonance and non-resonance cases. The next level of the perturbation approaches another solvability condition and is applied to obtain the unknowns become clear in the solution for the first-order solvability condition. The approach assumed here is so significant for solving many parametric nonlinear equations that arise within the engineering and nonlinear science.

KEYWORDS

Damped Mathieu Equation; parametric nonlinear oscillator; resonance instability; homotopy perturbation method (HPM)

1 Introduction

In wide engineering and physical applications, the nonlinear oscillators exist. Also, the parametric excitation takes place when a modifying physical parameter, such as a moment of stiffness or inertia, acts in a forcing model. This excitement yields a variable time coefficient, commonly an oscillation, in the governing system of motion. On the other hand, an external excitation outcome acting as an inhomogeneous part in the model of motion. Furthermore, minor parametric excitement produces a major response when the frequency of the excitement is far from the fundamental resonance, as shown in [1–5].

A classical example of parametric excitation is the swinging pendulum with oscillating support. The equation of motion describing the model is the well-known Mathieu equation. In 1868 Mathieu studied the vibration of elliptical membranes [6]. Consequently, he introduced the Mathieu equation that is an example of a linear differential equation (LDE) with parametric excitation. The Mathieu equation has application to the dynamics of passive towed arrays in submarines, as well as serving as a useful model for many interesting problems in physics, biology, applied mathematics, and engineering mechanics fields [7]. For some of the non-linear variations of the Mathieu, the equation has been presented in [8,9]. Moreover, the oscillations of the mechanical systems under the action of an oscillatory external force may reveal a Duffing problem, for instance, see references [10–14]. Recently, Moatimid [15] attempted to



study the stability analysis of a parametric Duffing oscillator. In this investigation Moatimid showed that the cubic stiffness parameter and the damped parameter have a destabilizing influence, however, the parametric and natural frequencies are of stabilizing influences.

The main target in the present work is how to achieve accurate approximate solutions of the nonlinear oscillators with highly strong nonlinearity. In recent centuries, many analytical approaches were developed to work out the periodic motion of nonlinear oscillators, such as the averaging method, perturbation methods, harmonic balance method, and the generalized harmonic method. The classical perturbation procedure depends on small parameters and chooses unsuitable small parameters that can lead to wrong solutions [16]. Therefore, a new perturbation technique was first proposed by He et al. [17–20]. This technique is named as the HPM, which represents a combination of the Homotopy analysis and classical perturbation methods. It has a full promise of the traditional perturbation techniques. The major property of the HPM is in its ability and flexibility to deal with many types of linear and nonlinear differential equations conveniently and accurately. Further, the HPM provides us with an appropriate direction to calculate an approximate or an analytic solution to several models arising in different fields. He [21] was built the most two considerable steps in the criteria of the HPM with a suitable initial guess and suggested an alternative approach to the construction of the Homotopy equation. Hence, He applied HPM to solve the Lighthill equation [17], Duffing equation [22], and Blasius equation [23], then the idea goes through and has been applied to solve nonlinear wave equations [17], boundary value problems [20]. Babolian et al. [24] applied the homotopy perturbation method to solve the Burgers, the modified Korteweg-de Vries, and regularized long-wave equations.

On the other hand, the HPM has more improved and developed by many engineers and scientists, for instance, a couple of the Laplace transforms and Homotopy perturbation method was implemented by El-Dib et al. [25]. The HPM with two expanding parameters that efficient for some partial nonlinear equations was suggested by He [26] and El-Dib [27]. Also, El-Dib [28] introduced a modified version of the HPM via the multiple scales technique. This new modification works particularly well for the nonlinear oscillators. Furthermore, away from the traditional HPM, Ren et al. [29] made another couple of the HPM and multiple time scales to become a powerful mathematical tool for many nonlinear equations. They displayed that the present procedure may be further afflicted by incorporating several known technologies. It provides solutions to nonlinear equations, whilst the classical perturbation technique became unsuccessful. Moreover, Rabbani [30] introduced a new homotopy perturbation approach for solving main non-linear models through the projection method. A new homotopy perturbation technique for solving linear and nonlinear Schrödinger equations has been addressed by Ayati et al. [31]. Further, by utilizing the HPM, a novel approach in examining the nonlinear Rayleigh-Taylor instability is conducted by El-Dib et al. [32]. Recently, a periodic solution of the cubic nonlinear Klein–Gordon equation using the He-multiple-scales method has been investigated by El-Dib [33]. Also, El-Dib et al. [34] investigated the impact of fractional derivative properties on the periodic analytic solution of the nonlinear oscillations using the HPM. Moreover, He's-multiple-scale scale to analyze the cubic-quintic Duffing equation has been analyzed by El-Dib et al. [35]. El-Dib [36] presented a stability approach of a fractional-delayed Duffing oscillator. A Nonlinear Instability of a Cylindrical Interface between two MHD Darcian flows has been studied by Moatimid et al. [37]. Further, El-Dib [38] introduced a modified multiple scale technique for the stability of the fractional delayed nonlinear oscillator. Besides, a periodic solution of the fractional sine-Gordon equation has been studied by Shen et al. [39]. Elgazery [40] applied the HPM to give a periodic solution of the Newell-Whitehead-Segel model. Further, for more very useful modification of the homotopy perturbation approach, Yu et al. [41] introduced HPM with an auxiliary parameter for nonlinear oscillators. Also, HPM for Fangzhu oscillator has been used by He et al. [42]. Finally, for more very useful modification of the homotopy perturbation approach, see [43–45].

Motivated by possibility applications in engineering, biology, and physics, which is based on studying the solution of the damped Mathieu equation. Hence, in the present work, our objective is to apply the HPM to linear or nonlinear equations having periodic coefficients such Mathieu equation which has been of great importance among researchers. For the presentation of this article; the rest of the manuscript is systematized as follows: [Section 2](#) is introducing the HPM for the mathematical formulation. The modulation procedure, in detail, is displayed in [Section 3](#). The non-resonance case, stability analysis of the non-resonance case, the resonance case of Ω is near ω , stability analysis of the linear Mathieu equation, and stability analysis for the nonlinear case are represented through [Sections 4 to 8](#), respectively. Finally, the main obtained outcomes are summarized as concluding remarks in [Section 9](#).

2 Mathematical Formulation

To explain the proposed technique, consider the following parametric pendulum equation as an illustrative example:

$$\frac{d^2y}{dt^2} + \mu \frac{dy}{dt} + (\omega^2 + 2q \cos 2\Omega t) \sin y = 0, \quad (1)$$

where μ , ω , q and Ω are real physical constants. μ is the damping coefficient parameter, ω^2 is the square of the natural frequency of the unforced system, q is the amplitude of the parametric excitation, and Ω is the frequency of the parametric excitation. By using the Taylor expansion up to cubic power, the above parametric pendulum equation becomes the following damping cubic nonlinear Mathieu equation:

$$\frac{d^2y}{dt^2} + \mu \frac{dy}{dt} + (\omega^2 + 2q \cos 2\Omega t)y = \left(\frac{\omega^2}{3!} + \frac{2q}{3!} \cos 2\Omega t\right)y^3 + \dots \quad (2)$$

The homotopy perturbation method can be considered as a combination of the classical perturbation technique and the homotopy (whose origin is in the topology [46]), but not restricted to the limitations of traditional perturbation methods. For example, this method does not require neither small parameter nor linearization and only requires little iteration to obtain accurate solutions [17] and [18].

We define the two parts of [Eq. \(2\)](#) as $L(y)$ and $N(y)$, where

$$L(y) = \frac{d^2y}{dt^2} + \omega^2 y, \text{ and } N(y) = \mu \frac{dy}{dt} + 2qy \cos 2\Omega t - ky^3, \quad (3)$$

where, $k = \frac{1}{3!}(\omega^2 + 2q \cos 2\Omega t)$.

Construct the homotopy statement as

$$H(y, \rho) = L(y) + \rho N(y) = 0; \quad \rho \in [0, 1]. \quad (4)$$

As in He's a homotopy perturbation method, it is obvious that when $\rho = 0$, [Eq. \(4\)](#) becomes the harmonic equation; $L(y) = 0$. Thus,

$$\frac{d^2y(t)}{dt^2} + \omega^2 y(t) = 0. \quad (5)$$

According to linear differential equations theory, the general solution of [Eq. \(5\)](#) is expressed in terms of two linearly independent solutions, say, $e^{i\omega t}$ and $e^{-i\omega t}$. Thus, the composite solutions may be in the form

$$y(t) = Ae^{i\omega t} + \bar{A}e^{-i\omega t}, \quad (6)$$

where A and its complex conjugate \bar{A} are arbitrary constants of integration. [Eq. \(4\)](#) becomes the original nonlinear Mathieu [Eq. \(2\)](#) as $\rho = 1$. For arbitrary the small parameter ρ , the solution of [Eq. \(4\)](#) can be

sought in terms of ρ so that the function $y(t)$ becomes $y(t, \rho)$. Accordingly, Eq. (4) can rewrite as

$$H(y, \rho) = \left(\frac{d^2}{dt^2} + \omega^2 \right) y(t, \rho) + \rho \left(\mu \frac{d}{dt} + 2q \cos 2\Omega t - ky^2(t, \rho) \right) y(t, \rho) = 0. \quad (7)$$

It can be noticed that the homotopy function (7) is essentially the same as (4), except for the function $y(t, \rho)$, which contains embedded the homotopy parameter ρ . The introduction of that parameter within the differential equation is a strategy to redistribute the periodic part between the successive iterations of the homotopy method, and thus increase the probabilities of finding the sought solution. Thus, as ρ moves from 0 to 1, the function $y(t, \rho)$ moves from $y_0(t)$ to $y_{app}(t)$. Expand the function $y(t, \rho)$ as a power series in the small parameter ρ such that

$$y(t, \rho) = y_0(t) + \rho y_1(t) + \rho^2 y_2(t) + \dots, \quad (8)$$

where $y_n(t)$; $n = 1, 2, 3, \dots$ are unknowns in needs to determine. Substituting this expansion into the homotopy Eq. (7) and equating terms with identical powers of ρ , leads to

$$\rho^0 : \frac{d^2 y_0}{dt^2} + \omega^2 y_0 = 0, \quad (9)$$

$$\rho^1 : \frac{d^2 y_1}{dt^2} + \omega^2 y_1 = -\mu \frac{dy_0}{dt} - 2q \cos 2\Omega t y_0 + ky_0^3, \quad (10)$$

$$\rho^2 : \frac{d^2 y_2}{dt^2} + \omega^2 y_2 = -\mu \frac{dy_1}{dt} - 2q \cos 2\Omega t y_1 + 3ky_0^2 y_1, \dots \quad (11)$$

Eq. (9) can be satisfied by

$$y_0(t) = Ae^{i\omega t} + \bar{A}e^{-i\omega t}. \quad (12)$$

Substituting (12) into Eq. (10) gets the form

$$\begin{aligned} \frac{d^2 y_1}{dt^2} + \omega^2 y_1 = & -i\omega\mu(Ae^{i\omega t} - \bar{A}e^{-i\omega t}) - qA(e^{i(\omega+2\Omega)t} + e^{i(\omega-2\Omega)t}) - q\bar{A}(e^{-i(\omega+2\Omega)t} + e^{-i(\omega-2\Omega)t}) \\ & + k(A^3 e^{3i\omega t} + 3A^2 \bar{A}e^{i\omega t} + 3\bar{A}^2 A e^{-i\omega t} + \bar{A}^3 e^{-3i\omega t}). \end{aligned} \quad (13)$$

Before analyzing the first-order problem, we must distinguish between the two cases. The case of the frequency Ω doesn't equal the nature frequency ω (which is known as the non-resonance case). The second one is the specific case when Ω approaches ω (which is known as the resonance case).

For arbitrary frequency Ω , there are secular terms that appear in the Eq. (13). Elimination such secular terms require that the arbitrary constant A be zero. This means that the expansion (8) cannot be successful to obtain a valid solution for exciting homotopy Eq. (7).

3 The Modulation Procedure

To obtain uniform expansions for problems of this kind, the expansion (8) needs to be modified. If we modulate the initial solution (6) so that the constant A becomes $A(\tau)$ with $\tau = \rho t$, such that

$$\frac{dA}{dt} = \rho \frac{dA}{d\tau} \quad \text{and} \quad \frac{d^2 A}{dt^2} = \rho^2 \frac{d^2 A}{d\tau^2}. \quad (14)$$

Then Eq. (12) in the modulate case becomes

$$Y_0(t, \tau) = A(\tau)u_0(t) + \bar{A}(\tau)\bar{u}_0(t), \quad (15)$$

where

$$u_0(t) = e^{i\omega t} \text{ and } \bar{u}_0(t) = e^{-i\omega t}. \quad (16)$$

Consequently, the homotopy state, Eq. (7), in the modulated form becomes

$$\left(\frac{d^2}{dt^2} + \omega^2\right)Y(t, \tau, \rho) + \rho\left(\mu\frac{d}{dt} + 2q\cos 2\Omega t\right)Y(t, \tau, \rho) = \rho kY^3(t, \tau, \rho). \quad (17)$$

It is convenient to choose the modulated function $Y(t, \tau, \rho)$ in separated variables as

$$Y(t, \tau, \rho) = A(\tau)u(t, \rho) + \bar{A}(\tau)\bar{u}(t, \rho). \quad (18)$$

The function $u(t, \rho)$ can be expanded as a power series in the small parameter ρ such that

$$u(t, \rho) = u_0(t) + \rho u_1(t) + \rho^2 u_2(t) + \dots, \quad (19)$$

where $u_n(t)$; $n = 1, 2, 3, \dots$ are unknowns to be evaluated. If the expansion (19) is substituted into Eq. (18) then gets

$$\begin{aligned} Y(t, \tau, \rho) &= A(\tau)(u_0(t) + \rho u_1(t) + \rho^2 u_2(t) + \dots) + cc. \\ &= Y_0(t, \tau) + \rho Y_1(t, \tau) + \rho^2 Y_2(t, \tau) + \dots, \end{aligned} \quad (20)$$

where $cc.$ indicates to the complex conjugate for the preceding terms and

$$Y_n(t, \tau) = A(\tau)u_n(t) + \bar{A}(\tau)\bar{u}_n(t). \quad (21)$$

It is noted that:

$$\frac{d}{dt}Y(t, \tau, \rho) = \frac{d}{dt}[A(\tau)u(t, \rho) + cc] = A(\tau)\dot{u}(t, \rho) + \rho u(t, \rho)A'(\tau) + cc, \quad (22)$$

and

$$\frac{d^2}{dt^2}Y(t, \tau, \rho) = A(\tau)\ddot{u}(t, \rho) + 2\rho A'(\tau)\dot{u}(t, \rho) + \rho^2 u(t, \rho)A''(\tau) + cc, \quad (23)$$

where dots indicate differentiation concerning the time t , while dashes refer to the derivative for the time modulate τ . Substituting (18) into Eq. (17) using (22) and (23) gives

$$A(\ddot{u} + \omega^2 u) + \rho(2\dot{u}A' + \mu A\dot{u} + 2qAu \cos 2\Omega t) + \rho^2(A'' + \mu A')u - \rho k(A^3 u^3 + 3A^2 \bar{A}u^2 \bar{u}) + cc. = 0. \quad (24)$$

Eq. (24) remains to obey the same homotopy concept because it's become the same harmonic Eq. (5) as $\rho \rightarrow 0$. $\lim_{\rho \rightarrow 1} A' = \frac{d}{d\tau} \left(\lim_{\rho \rightarrow 1} A \right) = 0$, besides, consequently the original Eq. (2) is found.

In the light of Eq. (19), the modulate homotopy Eq. (24) will be expanded as a power series in ρ so that the following non-homogenous harmonic equations are imposed

$$\rho^0 : A(\ddot{u}_0 + \omega^2 u_0) + \bar{A}(\ddot{\bar{u}}_0 + \omega^2 \bar{u}_0) = 0, \quad (25)$$

$$\rho^1 : A(\ddot{u}_1 + \omega^2 u_1) + 2\dot{u}_0 A' + \mu A \dot{u}_0 + 2qA u_0 \cos 2\Omega t - kA^2 (Au_0^3 + 3\bar{A}u_0^2 \bar{u}_0) + cc. = 0, \quad (26)$$

$$\rho^2 : A(\ddot{u}_2 + \omega^2 u_2) + 2\dot{u}_1 A' + \mu A \dot{u}_1 + 2qA u_1 \cos 2\Omega t + u_0 A'' + \mu A' u_0 - 3kA^2 [Au_0^2 u_1 + \bar{A}(u_0^2 \bar{u}_1 + 2u_0 \bar{u}_0 u_1)] + cc. = 0. \quad (27)$$

It is noted that Eq. (25) has been satisfied by Eq. (16) and the zero-order solution for Eq. (17) as approved in Eq. (15). Substituting Eq. (16) into Eq. (26) becomes

$$A(\ddot{u}_1 + \omega^2 u_1) + [i\omega(2A' + \mu A) - 3kA^2 \bar{A}] e^{i\omega t} + qA (e^{i(\omega+2\Omega)t} + e^{i(\omega-2\Omega)t}) - kA^3 e^{3i\omega t} + cc. = 0. \quad (28)$$

This equation contains secular terms at the non-resonance case and other secular terms when the applied frequency Ω approaches the natural frequency ω .

4 The Non-Resonance Case

The analysis in this case concerned with the arbitrary chosen for the applied frequency Ω , in Eq. (28). At this stage, secular terms are removed when

$$A' + \frac{1}{2}\mu A + \frac{3ik}{2\omega} A^2 \bar{A} = 0, \quad (29)$$

with its complex conjugate one. This leads to obtaining the valid function $u_1(t)$ as

$$u_1(t) = \frac{q}{4\Omega} \left(\frac{e^{i(\omega+2\Omega)t}}{(\Omega + \omega)} + \frac{e^{i(\omega-2\Omega)t}}{(\Omega - \omega)} \right) + \frac{kA^2}{8\omega^2} e^{3i\omega t}. \quad (30)$$

Consequently, the solution of the first-order problem is formulated as

$$Y_1(t, \tau) = \frac{qA(\tau)}{4\Omega} \left(\frac{e^{i(\omega+2\Omega)t}}{(\omega + \Omega)} + \frac{e^{i(\omega-2\Omega)t}}{(\omega - \Omega)} \right) + \frac{q\bar{A}(\tau)}{4\Omega} \left(\frac{e^{-i(\omega+2\Omega)t}}{(\omega + \Omega)} + \frac{e^{-i(\omega-2\Omega)t}}{(\omega - \Omega)} \right) + \frac{k}{8\omega^2} (A^3(\tau)e^{3i\omega t} + \bar{A}^3(\tau)e^{-3i\omega t}). \quad (31)$$

Substituting Eqs. (16) and (30) into Eq. (27), using Eq. (29), yields

$$A(\ddot{u}_2 + \omega^2 u_2) + \left[A'' + \mu A' + \frac{q^2 A}{2(\Omega^2 - \omega^2)} - \frac{3k^2}{8\omega^2} A^3 \bar{A}^2 \right] e^{i\omega t} + \frac{3k^2 A^4 \bar{A}}{8\omega^2} e^{3i\omega t} - \frac{3k^2 A^5}{8\omega^2} e^{5i\omega t} + \frac{q^2 A}{4\Omega(\Omega^2 - \omega^2)} \left[(\Omega - \omega)e^{i(\omega+4\Omega)t} + (\Omega + \omega)e^{i(\omega-4\Omega)t} \right] + \frac{3kqA^2 \bar{A}}{2\omega(\Omega^2 - \omega^2)} \left[(\Omega - 2\omega)e^{i(\omega+2\Omega)t} - (\Omega + 2\omega)e^{i(\omega-2\Omega)t} \right] - \frac{3kqA^3}{4\Omega(\Omega^2 - \omega^2)} \left[(\Omega - \omega)e^{i(3\omega+2\Omega)t} + (\Omega + \omega)e^{i(3\omega-2\Omega)t} \right] + cc. = 0. \quad (32)$$

The valid solution requires to be removed the terms that producing unbounded solution. These terms imply the following nonlinear solvability condition:

$$A'' + \mu A' + \frac{q^2 A}{2(\Omega^2 - \omega^2)} - \frac{3k^2}{8\omega^2} A^3 \bar{A}^2 = 0. \quad (33)$$

The second-order solution is found to be

$$\begin{aligned}
Y_2(t, \tau) = & \frac{q^2 A(\tau)}{32\Omega^2(\Omega^2 - \omega^2)} \left[\frac{(\Omega - \omega)}{(\omega + 2\Omega)} e^{i(\omega+4\Omega)t} + \frac{(\Omega + \omega)}{(\omega - 2\Omega)} e^{i(\omega-4\Omega)t} \right] \\
& - \frac{3kqA^3(\tau)}{16\Omega(\Omega^2 - \omega^2)^2} \left[\frac{(\Omega - \omega)^2}{(2\omega + \Omega)} e^{i(3\omega+2\Omega)t} + \frac{(\Omega + \omega)^2}{(\Omega - 2\omega)} e^{i(3\omega-2\Omega)t} \right] \\
& + \frac{3kqA^2(\tau)\bar{A}(\tau)}{8\Omega^2\omega(\Omega^2 - \omega^2)} \left[\frac{(\Omega - 2\omega)}{(\omega + \Omega)} e^{i(\omega+2\Omega)t} + \frac{(\Omega + 2\omega)}{(\Omega - \omega)} e^{i(\omega-2\Omega)t} \right] \\
& - \frac{k^2 A^5(\tau)}{64\omega^2} e^{5i\omega t} + \frac{3k^2 A^4(\tau)\bar{A}(\tau)}{64\omega^2} e^{3i\omega t} + cc.
\end{aligned} \tag{34}$$

If the accuracy to the second-order perturbation is enough, then the approximate solution at the non-resonance case is formulated by substituting Eqs. (15), (16), (31) and (34) into Eq. (20), and setting $\rho = 1$, gets

$$\begin{aligned}
Y(t) = & \lim_{\substack{\rho \rightarrow 1, \\ \tau \rightarrow t}} Y(t, \tau, \rho) \\
= & A(t)e^{i\omega t} + \frac{k}{8\omega^2} A^3(t)e^{3i\omega t} - \frac{k^2 A^5(t)}{64\omega^2} e^{5i\omega t} + \frac{3k^2 A^4(t)\bar{A}(t)}{64\omega^2} e^{3i\omega t} \\
& + \frac{qA(t)}{4\Omega} \left(\frac{e^{i(\omega+2\Omega)t}}{(\omega + \Omega)} + \frac{e^{i(\omega-2\Omega)t}}{(\omega - \Omega)} \right) + \frac{3kqA^2(t)\bar{A}(t)}{8\Omega^2\omega(\Omega^2 - \omega^2)} \left[\frac{(\Omega - 2\omega)}{(\omega + \Omega)} e^{i(\omega+2\Omega)t} + \frac{(\Omega + 2\omega)}{(\Omega - \omega)} e^{i(\omega-2\Omega)t} \right] \\
& + \frac{q^2 A(t)}{32\Omega^2(\Omega^2 - \omega^2)} \left[\frac{(\Omega - \omega)}{(\omega + 2\Omega)} e^{i(\omega+4\Omega)t} + \frac{(\Omega + \omega)}{(\omega - 2\Omega)} e^{i(\omega-4\Omega)t} \right] \\
& - \frac{3kqA^3(t)}{16\Omega(\Omega^2 - \omega^2)^2} \left[\frac{(\Omega - \omega)^2}{(2\omega + \Omega)} e^{i(3\omega+2\Omega)t} + \frac{(\Omega + \omega)^2}{(\Omega - 2\omega)} e^{i(3\omega-2\Omega)t} \right] + cc.
\end{aligned} \tag{35}$$

5 Stability Analysis for the Non-Resonance Case

The stability criteria in the non-resonance case can be obtained from solving Eq. (29). One may use the following polar form [16]:

$$A(\tau) = \frac{1}{2}\xi(\tau)e^{i\eta(\tau)}, \tag{36}$$

with real the unknown functions $\xi(\tau)$ and $\eta(\tau)$. Insert Eq. (36) into the first-order solvability condition (29) which will separate into real and imaginary parts and gives

$$\xi(\tau) = \xi_0 e^{-\frac{1}{2}\mu\tau} \text{ and } \eta(\tau) = \frac{3k}{2\mu\omega} \xi_0 e^{-\frac{1}{2}\mu\tau} + \eta_0, \tag{37}$$

where, ξ_0 and η_0 are integration constants. The stability criteria in the non-resonance case require that $\mu > 0$.

6 The Resonance Case Ω is Near ω

Return to the first-order problem Eq. (28) and re-analyzed it because of the nearness of Ω to ω . We express this approach by introducing the detuning parameter σ [16] such that

$$\Omega = \omega + \rho\sigma. \tag{38}$$

Accordingly, we have

$$-i(\omega - 2\Omega)t = i\omega t + 2i\sigma\tau. \quad (39)$$

Elimination of secular terms from Eq. (28), because of Eqs. (38) and (39) yields

$$A' + \frac{1}{2}\mu A - \frac{iq}{2\omega}\bar{A}e^{2i\sigma\tau} + \frac{3ik}{2\omega}A^2\bar{A} = 0. \quad (40)$$

The first-order solution in this case is

$$Y_1(t, \tau) = \frac{q}{4\Omega(\omega + \Omega)} \left(Ae^{i(\omega+2\Omega)t} + \bar{A}e^{-i(\omega+2\Omega)t} \right) + \frac{k}{8\omega^2} \left(A^3 e^{3i\omega t} + \bar{A}^3 e^{-3i\omega t} \right). \quad (41)$$

Using Eq. (41) with Eq. (27), we obtain the uniform solution for the second-order problem, and the following solvability is presented:

$$A'' + \mu A' + \frac{q^2}{4\Omega(\omega + \Omega)} A + \frac{kq}{8\omega^2} A^3 e^{-2i\sigma\tau} - \frac{3kq}{4\Omega(\omega + \Omega)} A\bar{A}^2 e^{2i\sigma\tau} - \frac{3k^2}{8\omega^2} A^3 \bar{A}^2 = 0, \quad (42)$$

with its complex conjugate. The valid function $Y_2(t, \tau)$ is given by

$$\begin{aligned} Y_2(t, \tau) = & \frac{3ik}{64\omega^3} (2A' + \mu A + 2ikA^2\bar{A}) A^2 e^{3i\omega t} - \frac{3k^2}{192\omega^4} A^5 e^{5i\omega t} + \frac{q^2}{32\Omega^2(\omega + \Omega)(\omega + 2\Omega)} A e^{i(\omega+4\Omega)t} \\ & + i \frac{q}{8\Omega^2(\omega + \Omega)^2} \left[(A' + \frac{1}{2}\mu A)(\omega + 2\Omega) + 3ikA^2\bar{A} \right] e^{i(\omega+2\Omega)t} \\ & - \frac{kq}{16(2\omega + \Omega)(\omega + \Omega)} \left[\frac{3}{\Omega(\omega + \Omega)} - \frac{1}{2\omega^2} \right] A^3 e^{i(3\omega+2\Omega)t} + cc. \end{aligned} \quad (43)$$

The approximate solution up to the second-order is formulated by substituting from Eqs. (15), (16), (41) and (43) into Eq. (16) gets

$$\begin{aligned} Y(t) = & \lim_{\substack{\rho \rightarrow 1 \\ \tau \rightarrow t}} (Y_0 + \rho Y_1 + \rho^2 Y_2) \\ = & A e^{i\omega t} + \bar{A} e^{-i\omega t} + \frac{k}{8\omega^2} \left[A + \frac{3i}{8\omega} (2A' + \mu A + 2ikA^2\bar{A}) \right] A^2 e^{3i\omega t} - \frac{3k^2}{192\omega^4} A^5 e^{5i\omega t} \\ & + \frac{q}{4\Omega(\omega + \Omega)} \left\{ A + i \frac{q}{2\Omega(\omega + \Omega)} \left[(A' + \frac{1}{2}\mu A)(\omega + 2\Omega) + 3ikA^2\bar{A} \right] \right\} e^{i(\omega+2\Omega)t} \\ & + \frac{q^2}{32\Omega^2(\omega + \Omega)(\omega + 2\Omega)} A e^{i(\omega+4\Omega)t} - \frac{kq}{16(2\omega + \Omega)(\omega + \Omega)} \left[\frac{3}{\Omega(\omega + \Omega)} - \frac{1}{2\omega^2} \right] A^3 e^{i(3\omega+2\Omega)t} + cc. \end{aligned} \quad (44)$$

7 Stability Analysis of the Linear Mathieu Equation

In the limiting case as $k \rightarrow 0$ into Eq. (2), linear damping Mathieu equation arrived. In this case, the two solvability conditions (40) and (42) that produced at the resonance case of Ω is near ω having the following limit case:

$$A' + \frac{1}{2}\mu A - \frac{iq}{2\omega}\bar{A}e^{2i\sigma\tau} = 0, \quad (45)$$

$$A'' + \mu A' + \frac{q^2}{4\Omega(\omega + \Omega)} A = 0. \quad (46)$$

The first-order solvability condition (45) can be used to find the stability picture at the resonance case. The second-order solvability condition (46) can be used to find the value of the detuning parameter σ .

It is easy to show that the Eq. (45) can be satisfied by the form

$$A(\tau) = \left[\left(\sigma + \frac{q}{2\omega} \right) \sin \Theta \tau + i \Theta \cos \Theta \tau \right] e^{(i\sigma - \frac{1}{2}\mu)\tau}, \quad (47)$$

where, the parameter μ must be positive, to find a damping solution. The argument Θ is given by the following characteristic equation:

$$\Theta^2 = \sigma^2 - \frac{q^2}{4\omega^2}. \quad (48)$$

The parameter σ can be evaluated by substituting Eq. (47) into the second-order solvability condition (46) to gets

$$\sigma = -\frac{q}{2\omega} \text{ or } \sigma = \pm \frac{\omega}{4q} \left(\mu^2 - \frac{q^2}{\omega^2} - \frac{q^2}{\Omega(\Omega + \omega)} \right). \quad (49)$$

The use of the first value of σ , Eq. (48) yields a zero solution for Eq. (45). For a non-zero solution, the other values σ are conforming. Inserting Eq. (49) into Eq. (48) gets

$$\Theta^2 = \frac{\omega^2}{16q^2} \left(\mu^2 - \frac{q^2}{\omega^2} - \frac{q^2}{\Omega(\Omega + \omega)} \right)^2 - \frac{q^2}{4\omega^2}. \quad (50)$$

The stability criteria require that the right-hand-side of Eq. (50) be positive, which implies that

$$\left(\mu^2 - \frac{q^2}{\omega^2} - \frac{q^2}{\Omega(\Omega + \omega)} \right)^2 - \frac{4q^4}{\omega^4} > 0. \quad (51)$$

Stability condition (51) can be rearranged in powers of the applied frequency Ω as

$$\Omega^2(\mu^2\omega^2 - 3q^2) + \Omega\omega(\mu^2\omega^2 - 3q^2) - q^2\omega^2 > 0, \quad (52)$$

and

$$\Omega^2(\mu^2\omega^2 + q^2) + \Omega\omega(\mu^2\omega^2 + q^2) - q^2\omega^2 < 0. \quad (53)$$

The transition curves separating stable state from an unstable state corresponding to

$$\Omega_1 = \frac{-\omega(\mu^2\omega^2 - 3q^2) - \omega\sqrt{\mu^4\omega^4 - 2\mu^2\omega^2q^2 - 3q^4}}{2(\mu^2\omega^2 - 3q^2)}, \quad (54)$$

and

$$\Omega_2 = \frac{-\omega(\mu^2\omega^2 + q^2) - \omega\sqrt{(\mu^2\omega^2 + q^2)(\mu^2\omega^2 + 5q^2)}}{2(\mu^2\omega^2 + q^2)}. \quad (55)$$

8 Stability Analysis for the Nonlinear Case

The first-order solvability condition (40) can be used to find the stability picture at the resonance case. The second-order solvability condition (42) can be used to find the value of the detuning parameter σ .

To relax the periodic term into the Eq. (40) we let

$$A(\tau) = [\alpha(\tau) + i\beta(\tau)]e^{i\sigma\tau}, \quad (56)$$

with real functions α and β . Insert Eq. (56) into Eq. (40), separating real and imaginary parts yields:

$$\alpha' + \frac{1}{2}\mu\alpha - \left(\sigma + \frac{q}{2\omega} + \frac{3k}{2\omega}(\alpha^2 + \beta^2) \right) \beta = 0, \quad (57)$$

$$\beta' + \frac{1}{2}\mu\beta + \left(\sigma - \frac{q}{2\omega} - \frac{3k}{2\omega}(\alpha^2 + \beta^2) \right) \alpha = 0. \quad (58)$$

In order to solve the above coupled nonlinear Eqs. (57) and (58), we may discuss the behavior at the steady-state response. This case is corresponding to the case of $\frac{d..}{d\tau} = 0$. If the solutions of Eqs. (57) and (58), at the steady-state, are represented by α_0 and β_0 , which are given by

$$\frac{1}{2}\mu\alpha_0 - \left(\sigma + \frac{q}{2\omega} + \frac{3kr^2}{2\omega} \right) \beta_0 = 0, \quad (59)$$

$$\frac{1}{2}\mu\beta_0 + \left(\sigma - \frac{q}{2\omega} - \frac{3kr^2}{2\omega} \right) \alpha_0 = 0, \quad (60)$$

where $r^2 = \alpha_0^2 + \beta_0^2$ is used. Eqs. (59) and (60) are two coupled algebraic equations in α_0 and β_0 . For nontrivial solutions in α_0 and β_0 , we obtain

$$\sigma^2 = \frac{(q + 3kr^2)^2}{4\omega^2} - \frac{1}{4}\mu^2. \quad (61)$$

Besides, the constants α_0 and β_0 may be chosen as

$$\alpha_0 = \left(\sigma + \frac{q + 3kr^2}{2\omega} \right), \text{ and } \beta_0 = \frac{1}{2}\mu. \quad (62)$$

Squaring both equations in (62) and adding we get

$$\left(\sigma + \frac{q + 3kr^2}{2\omega} \right)^2 = r^2 - \frac{1}{4}\mu^2. \quad (63)$$

Combing Eq. (61) with Eq. (63) yields

$$\sigma = \frac{2\omega^2 r^2 - (q + 3kr^2)^2}{2\omega(q + 3kr^2)}. \quad (64)$$

In order to find a constrain for a bounded solution we may modulate the functions α and β as

$$\alpha(\tau) = \alpha_0 + \alpha_1(\tau) \quad \text{and} \quad \beta(\tau) = \beta_0 + \beta_1(\tau), \quad (65)$$

where the functions $\alpha_1(\tau)$ and $\beta_1(\tau)$ refer to a small deviation from the steady-state solution α_0 and β_0 . Then the system of Eqs. (57) and (58) in the linearizing form becomes

$$\alpha_1' + \left(\frac{1}{2}\mu - \frac{3k}{\omega}\alpha_0\beta_0\right)\alpha_1 - \left(\sigma + \frac{q + 3kr^2}{2\omega} + \frac{3k}{\omega}\beta_0^2\right)\beta_1 = 0, \quad (66)$$

$$\beta_1' + \left(\frac{1}{2}\mu - \frac{3k}{\omega}\alpha_0\beta_0\right)\beta_1 + \left(\sigma - \frac{q + 3kr^2}{2\omega} - \frac{3k}{\omega}\alpha_0^2\right)\alpha_1 = 0. \quad (67)$$

The above system is two coupled linear differential equations of first-order in the two functions α_1 and β_1 . This system can be satisfied by

$$\alpha_1(\tau) = \left(\sigma + \frac{q + 3kr^2}{2\omega} + \frac{3k}{\omega}\beta_0^2\right) e^{-\left(\frac{1}{2}\mu - \frac{3k}{\omega}\alpha_0\beta_0\right)\tau} \sin \Theta\tau, \quad (68)$$

$$\beta_1(\tau) = e^{-\left(\frac{1}{2}\mu - \frac{3k}{\omega}\alpha_0\beta_0\right)\tau} \Theta \cos \Theta\tau, \quad (69)$$

where Θ is given by the following characteristic equation:

$$\Theta^2 = \left(\sigma - \frac{q + 9kr^2}{2\omega} + \frac{3k}{4\omega}\mu^2\right) \left(\sigma + \frac{q + 3kr^2}{2\omega} + \frac{3k}{4\omega}\mu^2\right). \quad (70)$$

where relations (62) are used. This characteristic equation depends on the two related parameters σ and r^2 . This relation between them is given in Eq. (61) or in Eq. (64).

With the help of the second-order solvability condition (42) one can find an expression for both the unknowns σ and r^2 in terms of the frequency Ω . To accomplish this, one may substitute the steady-state solution

$$A(\tau) = (\alpha_0 + i\beta_0)e^{i\sigma\tau}, \quad (71)$$

into the second-order solvability condition (42). Separating the real and imaginary parts, produces the following relations, between the parameters σ , Ω and r^2 :

$$\sigma^2 - \frac{q^2}{4\Omega(\omega + \Omega)} + \frac{1}{16}kq\mu^2 \left(\frac{\Omega^2 + \omega\Omega - 6\omega^2}{\omega^2\Omega(\omega + \Omega)}\right) - \left[\frac{kq}{8} \left(\frac{\Omega^2 + \omega\Omega - 6\omega^2}{\omega^2\Omega(\omega + \Omega)}\right) - \frac{3k^2}{8\omega^2} \left(\sigma + \frac{q + 3kr^2}{2\omega}\right)\right] r^2 = 0, \quad (72)$$

$$\sigma + \frac{kq}{8} \left(\frac{\Omega^2 + \omega\Omega + 6\omega^2}{\omega^2\Omega(\omega + \Omega)}\right) \left(\sigma + \frac{q + 3kr^2}{2\omega}\right) + \frac{3k^2}{2\omega^2} r^2 = 0, \quad (73)$$

where relations (62) are used. Removing the parameter σ from Eq. (73), by using its equivalent in Eq. (64), gives a polynomial of second-order in r^2 :

$$r^4 + \frac{\Omega(\omega + \Omega)(12k^2q + 8\omega^3 - 23kq\omega) + 6kq\omega^3}{36k^2(k - \omega)\Omega(\omega + \Omega)} r^2 - \frac{q^2\omega}{9k^2(k - \omega)} = 0; \quad k \neq \omega. \quad (74)$$

Replacing r^4 and r^2 into Eq. (72) with their equivalents in Eqs. (64) and (73) leads to the following quadratic equation in the detuning parameter σ :

$$\begin{aligned}
& 6k\omega\Omega(\omega + \Omega)(8\omega^2 - 1) [8\omega\Omega(\omega + \Omega) + kq(\Omega^2 + \omega\Omega + 6\omega^2)] \sigma^2 \\
& - \left[\Omega^2(\omega + \Omega)^2 [2\omega(8\omega + kq)(3k^2q - 3kq + 4\omega^2 - 2\omega q) + 3k^2q^2] \right. \\
& \quad \left. + 6\omega^2\Omega(\omega + \Omega) [3k^2q^2 + 8\omega^4 + 4kq^2\omega^2 + 2kq\omega(3k^2q - 3kq + 4\omega^2 - 2\omega q)] + 144kq^2\omega^6 \right] \sigma \quad (75) \\
& + \Omega^2(\omega + \Omega)^2 [3k^2q(\omega\mu^2 - q)(8\omega + kq) + kq^2(3kq - 4\omega^2 + 2\omega q)] - 36k\omega^5(3k^2\mu^2 + 2kq + 2q^3) \\
& + 6k\omega^2\Omega(\omega + \Omega) [3k^2q^2(\omega\mu^2 - q) - q\omega(3k\mu^2 + 2q)(8\omega + kq) + q^2(3kq - 4\omega^2 + 2\omega q) - 2q^2\omega] = 0.
\end{aligned}$$

This equation gives two values σ_1 and σ_2 for the detuning parameter σ which makes the solution (56) without unknowns.

The stabilization for the problem requires that the right-hand side of Eq. (70) be positive provided that the exponential in Eqs. (68) and (69) has positive values. It is noted that the stability reveals as the coefficient of the periodic term in Eq. (2) tends to zero. The instability arrived as the parameter q going away the zero value. Thus, the stability conditions are found as

$$\mu > 0, \quad \frac{3k}{\omega}\sigma + \frac{3kq + 9k^2r^2}{2\omega^2} - 1 < 0, \quad (76)$$

$$\sigma > \frac{q + 6kr^2}{2\omega} - \frac{3k}{4\omega}\mu^2, \quad (77)$$

$$\sigma + \frac{q + 3kr^2}{2\omega} + \frac{3k}{4\omega}\mu^2 < 0. \quad (78)$$

Removing the parameter r^2 from the above stability conditions by using Eq. (73) yields the following two conditions for stability:

$$\mu > 0, \quad \omega(k - 1)\sigma < \frac{1}{6}(2\omega^2 - 3kq) + \frac{2kq^2(\Omega^2 + \omega\Omega + 6\omega^2)}{8\omega\Omega(\omega + \Omega) + kq(\Omega^2 + \omega\Omega + 6\omega^2)}, \quad (79)$$

$$\frac{3}{k}\sigma > \frac{q}{\omega} - \frac{3q^2(\Omega^2 + \omega\Omega + 6\omega^2)}{2\omega[8\omega\Omega(\omega + \Omega) + kq(\Omega^2 + \omega\Omega + 6\omega^2)]}. \quad (80)$$

The transition curves separating the stable state of unstable one are corresponding to

$$\sigma = \frac{(2\omega^2 - 3kq)[8\omega\Omega(\omega + \Omega) + kq(\Omega^2 + \omega\Omega + 6\omega^2)] + 12kq^2(\Omega^2 + \omega\Omega + 6\omega^2)}{6\omega(k - 1)[8\omega\Omega(\omega + \Omega) + kq(\Omega^2 + \omega\Omega + 6\omega^2)]}, \quad (81)$$

and

$$\sigma = \frac{2kq[8\omega\Omega(\omega + \Omega) + kq(\Omega^2 + \omega\Omega + 6\omega^2)] - 3kq^2(\Omega^2 + \omega\Omega + 6\omega^2)}{6\omega[8\omega\Omega(\omega + \Omega) + kq(\Omega^2 + \omega\Omega + 6\omega^2)]}. \quad (82)$$

Using the definition (38) the above transition curves can be sought within the parameter ρ as

$$\Omega = \omega + \rho \frac{(2\omega^2 - 3kq)[8\omega\Omega(\omega + \Omega) + kq(\Omega^2 + \omega\Omega + 6\omega^2)] + 12kq^2(\Omega^2 + \omega\Omega + 6\omega^2)}{6\omega(k - 1)[8\omega\Omega(\omega + \Omega) + kq(\Omega^2 + \omega\Omega + 6\omega^2)]}, \quad (83)$$

$$\Omega = \omega + \rho \frac{2kq[8\omega\Omega(\omega + \Omega) + kq(\Omega^2 + \omega\Omega + 6\omega^2)] - 3kq^2(\Omega^2 + \omega\Omega + 6\omega^2)}{6\omega[8\omega\Omega(\omega + \Omega) + kq(\Omega^2 + \omega\Omega + 6\omega^2)]}. \quad (84)$$

To obtain the transition curves, independent of the parameter ρ , we may be inserting Eq. (81) as well as Eq. (82), into the relation (75), then the following transition curves are imposed

$$a_3\Omega^3(\Omega + \omega)^3 + a_2\omega^2\Omega^2(\Omega + \omega)^2 + 36a_1\omega^4\Omega(\Omega + \omega) + a_0 = 0, \quad (85)$$

$$b_3\Omega^3(\Omega + \omega)^3 + b_2\omega^2\Omega^2(\Omega + \omega)^2 + 36b_1\omega^4\Omega(\Omega + \omega) + b_0 = 0. \quad (86)$$

It is noted that the instability state lies between the above transition curves. The constant coefficients a_j and $b_j, j = 0, 1, 2, 3$ are given below:

$$\begin{aligned} a_3 = & k(8\omega^2 - 1) [(2\omega^2 - 3kq)(8\omega + kq) + 12kq^2]^2 \\ & - (k - 1) [(2\omega^2 - 3kq)(8\omega + kq) + 12kq^2] [2\omega(8\omega + kq)(3k^2q - 3kq + 4\omega^2 - 2\omega q) + 3k^2q^2] \\ & + 6\omega(k - 1)^2(8\omega + kq) [3k^2q(\omega\mu^2 - q)(8\omega + kq) + kq^2(3kq - 4\omega^2 + 2\omega q)], \end{aligned}$$

$$\begin{aligned} a_2 = & 12k^2q(8\omega^2 - 1)(2\omega^2 - 3kq + 12q) [(2\omega^2 - 3kq)(8\omega + kq) + 12kq^2] \\ & - 6(k - 1) [(2\omega^2 - 3kq)(8\omega + kq) + 12kq^2] [3k^2q^2 + 8\omega^4 + 4kq^2\omega^2 + 2kq\omega(3k^2q - 3kq + 4\omega^2 - 2\omega q)] \\ & - 6kq(k - 1)(2\omega^2 - 3kq + 12q) [2\omega(8\omega + kq)(3k^2q - 3kq + 4\omega^2 - 2\omega q) + 3k^2q^2] \\ & + 6 \times 36k\omega(k - 1)^2k^2q^2(\omega\mu^2 - q)(8\omega + kq) + 36k\omega(k - 1)^2q^2(3kq - 4\omega^2 + 2\omega q)(8\omega + 2kq) \\ & - 36k\omega(k - 1)^2q\omega(3k\mu^2 + 2q)(8\omega + kq)^2 - 72k\omega(k - 1)^2q^2\omega(8\omega + kq), \end{aligned}$$

$$\begin{aligned} a_1 = & + k^3q^2(8\omega^2 - 1)(2\omega^2 - 3kq + 12q)^2 - 4kq^2\omega^2(k - 1) [(2\omega^2 - 3kq)(8\omega + kq) + 12kq^2] \\ & - kq(k - 1)(2\omega^2 - 3kq + 12q) [3k^2q^2 + 8\omega^4 + 4kq^2\omega^2 + 2kq\omega(3k^2q - 3kq + 4\omega^2 - 2\omega q)] \\ & + 6k^2q\omega(k - 1)^2 [3k^2q^2(\omega\mu^2 - q) - q\omega(3k\mu^2 + 2q)(8\omega + kq) + q^2(3kq - 4\omega^2 + 2\omega q) - 2q^2\omega] \\ & - 6k\omega^2(k - 1)^2(8\omega + kq)(3k^2\mu^2 + 2kq + 2q^3), \end{aligned}$$

$$a_0 = -144kq^2\omega^5 6kq\omega^3(k - 1)(2\omega^2 - 3kq + 12q) - 36 \times 36k^2q\omega^8(k - 1)^2(3k^2\mu^2 + 2kq + 2q^3),$$

$$\begin{aligned} b_3 = & k^2q^2(8\omega^2 - 1)[2(8\omega + kq) - 3q]^2 + 6q\omega(8\omega + kq) [3k(\omega\mu^2 - q)(8\omega + kq) + q(3kq - 4\omega^2 + 2\omega q)] \\ & - [2q(8\omega + kq) - 3q^2] [2\omega(8\omega + kq)(3k^2q - 3kq + 4\omega^2 - 2\omega q) + 3k^2q^2], \end{aligned}$$

$$\begin{aligned} b_2 = & 12k^2q^3(8\omega^2 - 1)(2k - 3)[2(8\omega + kq) - 3q] + 6kq^2 [3k(\omega\mu^2 - q)(8\omega + kq) + q(3kq - 4\omega^2 + 2\omega q)] \\ & - 6q[2(8\omega + kq) - 3q] [3k^2q^2 + 8\omega^4 + 4kq^2\omega^2 + 2kq\omega(3k^2q - 3kq + 4\omega^2 - 2\omega q)] \\ & - 6q^2(2k - 3) [2\omega(8\omega + kq)(3k^2q - 3kq + 4\omega^2 - 2\omega q) + 3k^2q^2] \\ & + 36\omega(8\omega + kq) [3k^2q^2(\omega\mu^2 - q) - q\omega(3k\mu^2 + 2q)(8\omega + kq) + q^2(3kq - 4\omega^2 + 2\omega q) - 2q^2\omega], \end{aligned}$$

$$\begin{aligned} b_1 = & k^2q^4(8\omega^2 - 1)(2k - 3)^2 - 4q^2\omega^2 [2kq(8\omega + kq) - 3kq^2] - 6\omega^2(3k^2\mu^2 + 2kq + 2q^3)(8\omega + kq) \\ & - q(2k - 3) [3k^2q^2 + 8\omega^4 + 4kq^2\omega^2 + 2kq\omega(3k^2q - 3kq + 4\omega^2 - 2\omega q)] \\ & + 6kq\omega [3k^2q^2(\omega\mu^2 - q) - q\omega(3k\mu^2 + 2q)(8\omega + kq) + q^2(3kq - 4\omega^2 + 2\omega q) - 2q^2\omega], \end{aligned}$$

$$b_0 = -72 \times 6\omega^8kq(9k^2\mu^2 + 6kq + 4kq^3).$$

9 Conclusion

The homotopy perturbation method (HPM) is one of a easy, powerful, efficient, and accurate approach for evaluating solutions of a large class of nonlinear equations without the need of a discretization or

linearization process. HPM is a combination of the homotopy and perturbation methods. That can take the advantages of the conventional perturbation method and eliminating its restrictions. It yields a rapid convergence of the solution series with a few iterations leading to accurate solutions, and the round-off errors are avoided. In general, this method has been successfully used to solve different kinds of linear and nonlinear problems in engineering and science. So, in the present work, we propose a variation of the homotopy perturbation approach *via* a modulation method that allows finding analytic solutions for ordinary differential models with periodic coefficients. This article is prepared to analyze a parametrically excited oscillator in the presence of strong cubic nonlinearity. The simplest model of this kind is the Mathieu equation that contains a small parameter [47,48]. The present analysis that employs the homotopy perturbation approach [17], has no dependence on models having a small parameter. Due to the present modulation approach, at each level of perturbation, a solvability condition is enjoined. By solving these solvability conditions drives to examining the stability behavior. In each resonance/non-resonant cases stability conditions, are obtained.

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