

**ARTICLE**

An Alternative Algorithm for the Symmetry Classification of Ordinary Differential Equations

Yi Tian^{1,2,*} and Jing Pang^{3,4}¹College of Data Science and Application, Inner Mongolia University of Technology, Hohhot, 010080, China²Inner Mongolia Autonomous Region Engineering and Technology Research Center of Big Data Based Software Service, Hohhot, 010080, China³College of Science, Inner Mongolia University of Technology, Hohhot, 010051, China⁴Inner Mongolia Key Laboratory of Statistical Analysis Theory for Life Data and Neural Network Modeling, Inner Mongolia University of Technology, Hohhot, 010051, China

*Corresponding Author: Yi Tian. Email: ttxsun@163.com

Received: 07 October 2020 Accepted: 10 June 2021

ABSTRACT

This is the first paper on symmetry classification for ordinary differential equations (ODEs) based on Wu's method. We carry out symmetry classification of two ODEs, named the generalizations of the Kummer-Schwarz equations which involving arbitrary function. First, Lie algorithm is used to give the determining equations of symmetry for the given equations, which involving arbitrary functions. Next, differential form Wu's method is used to decompose determining equations into a union of a series of zero sets of differential characteristic sets, which are easy to be solved relatively. Each branch of the decomposition yields a class of symmetries and associated parameters. The algorithm makes the classification become direct and systematic. Yuri Dimitrov Bozhkov, and Pammela Ramos da Conceição have used the Lie algorithm to give the symmetry classifications of the equations talked in this paper in 2020. From this paper, we can find that the differential form Wu's method for symmetry classification of ODEs with arbitrary function (parameter) is effective, and is an alternative method.

KEYWORDS

Kummer-Schwarz equation; ordinary differential equations (ODEs); differential form Wu's method

1 Introduction

In the past decades, a wealth of methods have been developed to deal with exact solutions of differential equations (DEs), which include partial differential equations (PDEs) and ordinary differential equations (ODEs). Some of the most important methods are the homotopy perturbation method [1–3], variational iteration method [4–6], Taylor series method [7], and the exp-function method [8–10], etc. At present, symmetries of PDEs are widely used in mechanics, mathematics and physics fields, from symmetries of a PDEs, one can obtain more important information on solving PDEs, such as exact solutions, conservation laws, and integral factors, etc. Hence the topics finding symmetries of a PDEs have being brought about the interest of more and more people. We have done some work on symmetries of PDEs, in reference [11] the Wu's method is used to complete symmetry classification of PDEs, in reference [12] the



traditional Lie algorithm is used to complete symmetry classification of the diffusion-convection equation, in reference [13] the Wu's method is used to simplify the symmetry computation of PDEs and a special symmetry reduction approach is used for a class of wave equations.

In my memory, researchers always pay attention to PDEs, well there are not enough research results on ODEs, especially on the symmetries of DEs in literature. In this paper, we mainly consider a class of ODEs appeared in the fields of physics and engineering, named the generalizations of the Kummer-Schwarz equations [14]:

$$\frac{y'''}{y'} + n \left(\frac{y''}{y'} \right)^2 = f(y)(y')^2, \quad (1)$$

and

$$\frac{y'''}{y'} + n \left(\frac{y''}{y'} \right)^2 = f(y), \quad (2)$$

where $n \in R$ is a constant, not necessarily integer.

In symmetry analysis of DEs, the first step is determining symmetries of the given equations, this process can be come down to solving a determining equations, which is however sometimes large and not easy to solve directly, in this paper, differential form Wu's method [11,15] is used to decompose the determining equations into a series of equations, which are easy to be solved relatively.

The Wu's method (also named characteristic set algorithm) [16] established by the Chinese mathematician Wu Wen Tsun in the 1970s. It also has become a fundamental algorithmic theory in algebraic geometry together with the Gröbner base algorithm [17]. The method has been applied in a wide range of science fields, such as mechanical theorem proving [18], optimization problems, surface-fitting problems in CAGD, Bar Linkage Design, ... , etc. [16]. The differential analogue of Wu's method was proposed in the 1980s [19]. The method is more especially on target to deal with the zero set of a differential polynomial system (dps) and efficient differential elimination without directly involving the concept of an algebra ideal. As far as i know, this is the first paper on symmetry classification of ODEs based on Wu's method.

2 Lie Point Symmetries of Eq. (1)

To begin with, we note that Eq. (1) can be written as

$$y'''y' + n(y'')^2 - f(y)(y')^4 = 0, \quad (3)$$

The point symmetry

$$\begin{aligned} x^* &= x + \varepsilon \zeta(x, y) + O(\varepsilon^2), \\ y^* &= y + \varepsilon \eta(x, y) + O(\varepsilon^2), \end{aligned} \quad (4)$$

is admitted by Eq. (3) if and only if it satisfies the determining equation

$$X^{(3)}(y'''y' + n(y'')^2 - f(y)(y')^4) = 0, \quad (5)$$

for any y that solves Eq. (3).

$$X = \zeta(x, y) \frac{\partial}{\partial x} + \eta(x, y) \frac{\partial}{\partial y}, \quad (6)$$

is the infinitesimal generator of the point symmetry (4).

$$X^{(3)} = X + \eta^{(1)} \frac{\partial}{\partial y'} + \eta^{(2)} \frac{\partial}{\partial y''} + \eta^{(3)} \frac{\partial}{\partial y'''} \quad (7)$$

with

$$\eta^{(i)} = D_x \eta^{(i-1)} - y^{(i)} D_x \xi, \quad i = 1, 2, 3, \quad \eta^{(0)} = \eta, \quad (8)$$

is the three-order extension (prolongation) of X .

$$D_x = \partial_x + y' \partial_y + y'' \partial_{y'} + \dots + y^{(k)} \partial_{y^{(k-1)}} \dots \quad (9)$$

is the total derivative operator.

The determining Eq. (5) simplify to

$$DTE = \begin{cases} \eta_x = 0, & -3\xi_y - n\xi_y = 0, & -6\xi_{yy} - 2n\xi_{yy} = 0, & -f(y)\xi_y - \xi_{yyy} = 0, \\ -3\xi_{xx} - 2n\xi_{xx} = 0, & -3\xi_{xy} = 0, & -\xi_{xxx} = 0, \\ -9\xi_{xy} - 4n\xi_{xy} + 3\eta_{yy} + 2n\eta_{yy} = 0, \\ -3\xi_{xy} - 2f(y)\eta_y + \eta_{yyy} - \eta f'(y) = 0, \end{cases} \quad (10)$$

taking left hand side for each equation, we have following corresponding differential polynomial system

$$DPS = \begin{cases} \eta_x, & -3\xi_y - n\xi_y, & -6\xi_{yy} - 2n\xi_{yy}, & -f(y)\xi_y - \xi_{yyy}, \\ -3\xi_{xx} - 2n\xi_{xx}, & -3\xi_{xy}, & -\xi_{xxx}, \\ -9\xi_{xy} - 4n\xi_{xy} + 3\eta_{yy} + 2n\eta_{yy}, \\ -3\xi_{xy} - 2f(y)\eta_y + \eta_{yyy} - \eta f'(y), \end{cases} \quad (11)$$

Step 1 Compute $Zero(DPS)$.

Under the rank $x \prec y \prec \xi \prec \eta$, using the differential form Wu's method, we obtain characteristic set

$$DCS_1 = \{\xi_y, \xi_{xx}, \eta\}, \quad (12)$$

with IS products $I_1 * I_2 * I_3 \neq 0$, and

$$I_1 = 3f'(y)^2 - 2f(y)f''(y), \quad (13)$$

$$I_2 = 3 + n, \quad (14)$$

$$I_3 = 3 + 2n, \quad (15)$$

then, we get

$$Zero(DPS) = Zero(DCS_1 / I_1 * I_2 * I_3) + Zero(DPS, I_1) + Zero(DPS, I_2) + Zero(DPS, I_3), \quad (16)$$

in which

$$Zero(DCS_1 / I_1 * I_2 * I_3) = \{\xi = c_1 x + c_2, \eta = 0\}, \quad (17)$$

and c_1, c_2 are arbitrary constants.

Step 2 $I_1 = 0$, compute $Zero(DPS, I_1)$.

Obtain characteristic set

$$DCS_2 = \{\xi_y, \xi_{xx}, \eta_x, 2f(y)\eta_y + \eta f'(y)\}, \quad (18)$$

with IS products $I_2 * I_3 \neq 0$, then, we get

$$\text{Zero}(DPS, I_1) = \text{Zero}(DCS_2/I_2 * I_3) \quad (19)$$

in which

Case 1. $\text{Zero}(DCS_2/I_2 * I_3) = \left\{ \xi = c_3x + c_4, \eta = \frac{c_5(e^{\sqrt{6}y})^{1/4}}{\sqrt{c_1e^{\sqrt{6}y} + c_2}} \right\}$, with the corresponding function $f(y) = c_1e^{\frac{\sqrt{6}y}{2}} + c_2e^{-\frac{\sqrt{6}y}{2}}$, and c_1, c_2, \dots, c_5 are arbitrary constants.

Case 2. $\text{Zero}(DCS_2/I_2 * I_3) = \{ \xi = c_1x + c_2, \eta = F(y) \}$, with the corresponding function $f(y) = 0, F(y)$ is an arbitrary function of y , and c_1, c_2 are arbitrary constants.

Step 3 $I_2 = 0$, compute $\text{Zero}(DPS, I_2)$.

The determining Eq. (5) simplify to

$$DTE_1 = \begin{cases} \eta_x = 0, \xi_{xx} = 0, \\ -f(y)\xi_y - \xi_{yyy} = 0, 3\xi_{xy} - 3\eta_{yy} = 0, \\ -3\xi_{xyy} - 2f(y)\eta_y + \eta_{yyy} - \eta f'(y) = 0, \end{cases} \quad (20)$$

taking left hand side for each equation, we have following corresponding differential polynomial system:

$$DPS_1 = \begin{cases} \eta_x, \xi_{xx}, \\ -f(y)\xi_y - \xi_{yyy}, 3\xi_{xy} - 3\eta_{yy}, \\ -3\xi_{xyy} - 2f(y)\eta_y + \eta_{yyy} - \eta f'(y), \end{cases} \quad (21)$$

under the rank $x < y < \xi < \eta$, using the differential form Wu's method, we obtain characteristic set

$$DCS_3 = \{f(y)\xi_y + \xi_{yyy}, \xi_{xy}, \xi_{xx}, \eta\}, \quad (22)$$

with IS products $I_4 \neq 0$, and

$$I_4 = 27f'(y)^4 - 18f(y)f'(y)^2f''(y) - 56f''(y)^3 + 72f'(y)f''(y)f^{(3)}(y) - 18f'(y)^2f^{(4)}(y), \quad (23)$$

then, we get

$$\text{Zero}(DPS, I_2) = \text{Zero}(DCS_3/I_4) + \text{Zero}(DPS, I_2, I_4), \quad (24)$$

in which

Case 1. $\text{Zero}(DCS_3/I_4) = \{ \xi = c_1x + c_2, \eta = 0 \}$, with the corresponding function $f(y)$ is an arbitrary function of y , and c_1, c_2 are arbitrary constants.

Case 2. $\text{Zero}(DCS_3/I_4) = \{ \xi = c_1x + \frac{1}{2}c_2u^2 + c_3u + c_4, \eta = 0 \}$, with the corresponding function $f(y)$ is an arbitrary function of y , and c_1, c_2, \dots, c_4 are arbitrary constants.

Case 3. $\text{Zero}(DCS_3/I_4) = \{ \xi = c_1x + F(y), \eta = 0 \}$, with the corresponding function $f(y) = 0, F(y)$ is an arbitrary function of y , and c_1 is an arbitrary constant.

Step 3.1 $I_4 = 0$, compute $\text{Zero}(DPS, I_2, I_4)$.

Obtain characteristic set

$$DCS_4 = \{f(y)\xi_y + \xi_{yyy}, \xi_{xx}, \eta_x, 3\eta_y f'(y) + \eta f''(y), 9\xi_{xy} f'(y)^2 - 4\eta f''(y)^2 + 3\eta f'(y) f^{(3)}(y)\}, \quad (25)$$

with IS products is empty set, then, we get

$$\text{Zero}(DPS, I_2, I_4) = \text{Zero}(DCS_4), \quad (26)$$

in which

Case 1. $\text{Zero}(DCS_4) = \{\xi = c_2x + c_3 + c_4 \sin(\sqrt{c_1}y) + c_5 \cos(\sqrt{c_1}y), \eta = 0\}$, with the corresponding function $f(y) = c_1 \geq 0$, and c_1, c_2, \dots, c_5 are arbitrary constants.

Case 2. $\text{Zero}(DCS_4) = \{\xi = c_1x + c_2, \eta = F(y)\}$, and with the corresponding function $f(y) = c_3$, $F(y)$ is an arbitrary function of y , and c_1, c_2, c_3 are arbitrary constants.

Case 3. $\text{Zero}(DCS_4) = \{\xi = (c_4x + c_6)e^{-c_1y} + (c_3x + c_7)e^{c_1y} + c_2x + c_5, \eta = F(y)\}$, with the corresponding function $f(y) = -c_1^2$, $F(y)$ is an arbitrary function of y , and c_1, c_2, \dots, c_7 are arbitrary constants.

Case 4. $\text{Zero}(DCS_4) = \{\xi = c_1x + c_2, \eta = 0\}$, with the corresponding function $f(y) = c_3$, and c_1, c_2, c_3 are arbitrary constants.

Case 5. $\text{Zero}(DCS_4) = \{\xi = (c_4x + c_6)e^{-c_1y} + (c_3x + c_7)e^{c_1y} + c_2x + c_5, \eta = 0\}$, with the corresponding function $f(y) = -c_1^2$, and c_1, c_2, \dots, c_7 are arbitrary constants.

Case 6. $\text{Zero}(DCS_4) = \{\xi = c_1x + c_2, \eta = c_3y + c_4\}$, with the corresponding function $f(y) = \frac{c_5}{(c_3y + c_4)^2}$, and c_1, c_2, \dots, c_5 are arbitrary constants.

Case 7. $\text{Zero}(DCS_4) = \{\xi = c_1x + \frac{1}{2}c_2y^2 + c_3y + c_4, \eta = c_5y + c_6\}$, with the corresponding function $f(y) = 0$, and c_1, c_2, \dots, c_6 are arbitrary constants.

Step 4 $I_3 = 0$, compute $\text{Zero}(DPS, I_3)$.

The determining Eq. (5) simplify to

$$DTE_2 = \begin{cases} \eta_x = 0, \xi_y = 0, -2\xi_{xxx} = 0, \\ -4f(y)\eta_y + 2\eta_{yyy} - 2\eta f'(y) = 0, \end{cases} \quad (27)$$

taking left hand side for each equation, we have following corresponding differential polynomial system

$$DPS_2 = \begin{cases} \eta_x, \xi_y, -2\xi_{xxx}, \\ -4f(y)\eta_y + 2\eta_{yyy} - 2\eta f'(y), \end{cases} \quad (28)$$

under the rank $x \prec y \prec \xi \prec \eta$, using the differential form Wu's method, we obtain characteristic set

$$DCS_5 = \{\xi_y, \xi_{xxx}, \eta_x, 2f(y)\eta_y - \eta_{yyy} + \eta f'(y)\}, \quad (29)$$

with IS products is empty set, and we get

$$\text{Zero}(DPS, I_3) = \text{Zero}(DCS_5), \quad (30)$$

in which

Case 1. $\text{Zero}(DCS_5) = \{\xi = \frac{1}{2}c_1x^2 + c_2x + c_3, \eta = 0\}$, with the corresponding function $f(y)$ is an arbitrary function of y , and c_1, c_2, c_3 are arbitrary constants.

Case 2. $\text{Zero}(DCS_5) = \{\xi = \frac{1}{2}c_1x^2 + c_2x + c_3, \eta = F(y)\}$, with the corresponding function $f(y) = \frac{\frac{F'(y)^2}{2} + F(y)F''(y) + c_4}{F(y)^2}$, and c_1, c_2, \dots, c_4 are arbitrary constants.

At the end of this section, we give the symmetry classification for Eq. (1) with $n = 0$ based on differential form Wu's method.

As stated before, the determining Eq. (5) simplify to

$$DTE_3 = \begin{cases} \xi_y = 0, \eta_{yy} = 0, -3\xi_{xx} + 3\eta_{xy} = 0, \\ -3f(y)\eta_x + 3\eta_{xyy} = 0, -\xi_{xxx} + 3\eta_{xxy} = 0, \\ \eta_{xxx} = 0, -2f(y)\eta_y + \eta_{yyy} - \eta f'(y) = 0, \end{cases} \quad (31)$$

taking left hand side for each equation, we have following corresponding differential polynomial system:

$$DPS_3 = \begin{cases} \xi_y, \eta_{yy}, -3\xi_{xx} + 3\eta_{xy}, \\ -3f(y)\eta_x + 3\eta_{xyy}, -\xi_{xxx} + 3\eta_{xxy}, \\ \eta_{xxx}, -2f(y)\eta_y + \eta_{yyy} - \eta f'(y), \end{cases} \quad (32)$$

Step 1 Compute $Zero(DPS_3)$.

Under the rank $x \prec y \prec \xi \prec \eta$, using the differential form Wu's method, we obtain characteristic set

$$DCS_6 = \{\xi_y, \xi_{xx}, \eta\}, \quad (33)$$

with IS products $I_5 \neq 0$, and

$$I_5 = 15f'(y)^3 - 18f(y)f'(y)f''(y) + 4f(y)^2f^{(3)}(y), \quad (34)$$

then, we get

$$Zero(DPS_3) = Zero(DCS_6/I_5) + Zero(DPS_3, I_5), \quad (35)$$

in which

$$Zero(DCS_6/I_5) = \{\xi = c_1x + c_2, \eta = 0\}, \quad (36)$$

and c_1, c_2 are arbitrary constants.

Step 2 $I_5 = 0$, compute $Zero(DPS_3, I_5)$.

Obtain characteristic set DCS_6 with IS products $I_6 \neq 0$, and

$$I_6 = 3f'(y)^2 - 2f(y)f''(y), \quad (37)$$

then, we get

$$Zero(DPS_3, I_5) = Zero(DCS_6/I_6) + Zero(DPS_3, I_5, I_6), \quad (38)$$

in which

$$Zero(DCS_6/I_6) = Zero(DCS_6/I_5), \quad (39)$$

Step 2.1 $I_6 = 0$, compute $Zero(DPS_3, I_5, I_6)$.

Obtain characteristic set

$$DCS_7 = \{\xi_y, \xi_{xx}, \eta_x, 2f(y)\eta_y + \eta f'(y)\} \quad (40)$$

with IS products is empty set, and we get

$$Zero(DPS_3, I_5, I_6) = Zero(DCS_7), \quad (41)$$

in which

Case 1. $Zero(DCS_7) = \{\xi = c_1x + c_2, \eta = c_5(c_3y + c_4)\}$, with the corresponding function $f(y) = \frac{4}{(c_3y + c_4)^2}$, and c_1, c_2, \dots, c_5 are arbitrary constants.

Case 2. $Zero(DCS_7) = \{\xi = c_1x + c_2, \eta = F(y)\}$, with the corresponding function $f(y) = 0$, $F(y)$ is an arbitrary function of y , and c_1, c_2 are arbitrary constants.

3 Lie Point Symmetries of Eq. (2)

The second generalized Kummer-Schwarz equation Eq. (2) can be written as

$$y'''y' + n(y'')^2 - f(y)(y')^2 = 0. \quad (42)$$

Consider the third-order prolongation

$$X^{(3)} = X + \eta^{(1)} \frac{\partial}{\partial y'} + \eta^{(2)} \frac{\partial}{\partial y''} + \eta^{(3)} \frac{\partial}{\partial y'''}, \quad (43)$$

of a Lie point symmetry generator

$$X = \zeta(x, y) \frac{\partial}{\partial x} + \eta(x, y) \frac{\partial}{\partial y}, \quad (44)$$

of Eq. (42), then the invariance condition is

$$X^{(3)}(y'''y' + n(y'')^2 - f(y)(y')^2)|_{(3)} = 0, \quad (45)$$

the determining equations from (45) simplify to

$$\overline{DTE} = \begin{cases} \eta_x = 0, & -3\xi_y - n\xi_y = 0, & -6\xi_{yy} - 2n\xi_{yy} = 0, \\ -\xi_{yyy} = 0, & -3\xi_{xx} - 2n\xi_{xx} = 0, & -3f(y)\xi_y - 3\xi_{xy} = 0, \\ -9\xi_{xy} - 4n\xi_{xy} + 3\eta_{yy} + 2n\eta_{yy} = 0, \\ -3\xi_{xy} + \eta_{yyy} = 0, \\ -2f(y)\xi_x - \xi_{xxx} - \eta f'(y) = 0, \end{cases} \quad (46)$$

taking left hand side for each equation, we have following corresponding differential polynomial system

$$\overline{DPS} = \begin{cases} \eta_x, & -3\xi_y - n\xi_y, & -6\xi_{yy} - 2n\xi_{yy}, \\ -\xi_{yyy}, & -3\xi_{xx} - 2n\xi_{xx}, & -3f(y)\xi_y - 3\xi_{xy}, \\ -9\xi_{xy} - 4n\xi_{xy} + 3\eta_{yy} + 2n\eta_{yy}, \\ -3\xi_{xy} + \eta_{yyy} = 0, \\ -2f(y)\xi_x - \xi_{xxx} - \eta f'(y) = 0, \end{cases} \quad (47)$$

Step 1 Compute $Zero(\overline{DPS})$.

Under the rank $x \prec y \prec \xi \prec \eta$, using the differential form Wu's method, we obtain characteristic set

$$\overline{DCS}_1 = \{\xi_y, \xi_x, \eta\}, \quad (48)$$

with IS products $\overline{I}_1 * \overline{I}_2 * \overline{I}_3 \neq 0$, and

$$\overline{I}_1 = f'(y)^2 f''(y) - 2f(y) f''(y)^2 + f(y) f'(y) f^{(3)}(y), \quad (49)$$

$$\overline{I}_2 = 3 + n, \quad (50)$$

$$\overline{I}_3 = 3 + 2n, \quad (51)$$

then, we get

$$Zero(\overline{DPS}) = Zero(\overline{DCS}_1 / \overline{I}_1 * \overline{I}_2 * \overline{I}_3) + Zero(\overline{DPS}, \overline{I}_1) + Zero(\overline{DPS}, \overline{I}_2) + Zero(\overline{DPS}, \overline{I}_3), \quad (52)$$

in which

$$\text{Zero}(\overline{DCS_1}/\overline{I_1} * \overline{I_2} * \overline{I_3}) = \{\xi = c_1, \eta = 0\}, \quad (53)$$

and c_1 is an arbitrary constant.

Step 2 $\overline{I_1} = 0$, compute $\text{Zero}(\overline{DPS}, \overline{I_1})$.

Obtain characteristic set

$$\overline{DCS_2} = \{\xi_y, \eta_x, 2f(y)\xi_x + \eta f'(y), \eta_y f(y) f'(y) - \eta f'(y)^2 + \eta f(y) f''(y)\}, \quad (54)$$

with IS products $\overline{I_2} * \overline{I_3} \neq 0$, then, we get

$$\text{Zero}(\overline{DPS}, \overline{I_1}) = \text{Zero}(\overline{DCS_2}/\overline{I_2} * \overline{I_3}), \quad (55)$$

in which

Case 1. $\text{Zero}(\overline{DCS_2}/\overline{I_2} * \overline{I_3}) = \{\xi = c_3x + c_4, \eta = c_1y + c_2\}$, with the corresponding function $f(y) = c_5(c_1y + c_2)^{\frac{2c_3}{c_1}}$, and c_1, c_2, \dots, c_5 are arbitrary constants.

Case 2. $\text{Zero}(\overline{DCS_2}/\overline{I_2} * \overline{I_3}) = \{\xi = c_1, \eta = 0\}$, with the corresponding function $f(y) = (c_3 - u)^{c_2} c_4$, and c_1, c_2, \dots, c_4 are arbitrary constants.

Case 3. $\text{Zero}(\overline{DCS_2}/\overline{I_2} * \overline{I_3}) = \{\xi = c_1, \eta = F(y)\}$, with the corresponding function $f(y) = c_2$, $F(y)$ is an arbitrary function of y , and c_1, c_2 are arbitrary constants.

Case 4. $\text{Zero}(\overline{DCS_2}/\overline{I_2} * \overline{I_3}) = \{\xi = F(x), \eta = G(y)\}$, with the corresponding function $f(y) = 0$, $F(x)$ is an arbitrary function of x , $G(y)$ is an arbitrary function of y .

Step 3 $\overline{I_2} = 0$, compute $\text{Zero}(\overline{DPS}, \overline{I_2})$.

The determining Eq. (45) simplify to

$$\overline{DTE_1} = \begin{cases} \eta_x = 0, & -\xi_{yyy} = 0, & 3\xi_{xx} = 0, \\ -3f(y)\xi_y - 3\xi_{xxy} = 0, & 3\xi_{xy} - 3\eta_{yy} = 0, \\ -3\xi_{xyy} + \eta_{yyy} = 0, \\ -2f(y)\xi_x - \xi_{xxx} - \eta f'(y) = 0, \end{cases} \quad (56)$$

taking left hand side for each equation, we have following corresponding differential polynomial system

$$\overline{DPS_1} = \begin{cases} \eta_x, & -\xi_{yyy}, & 3\xi_{xx}, \\ -3f(y)\xi_y - 3\xi_{xxy}, & 3\xi_{xy} - 3\eta_{yy}, \\ -3\xi_{xyy} + \eta_{yyy}, \\ -2f(y)\xi_x - \xi_{xxx} - \eta f'(y), \end{cases} \quad (57)$$

under the rank $x \prec y \prec \xi \prec \eta$, using the differential form Wu's method, we obtain characteristic set

$$\overline{DCS_3} = \{\xi_x, \xi_y, \eta\}, \quad (58)$$

with IS products $\overline{I_1} \neq 0$, then, we get

$$\text{Zero}(\overline{DPS}, \overline{I_2}) = \text{Zero}(\overline{DCS_3}/\overline{I_1}) + \text{Zero}(\overline{DPS}, \overline{I_2}, \overline{I_1}), \quad (59)$$

in which

$$\text{Zero}(\overline{DCS_3}/\overline{I_1}) = \{\xi = c_1, \eta = 0\}, \quad (60)$$

and c_1 is an arbitrary constant.

Step 3.1 $\overline{I_1} = 0$, compute $\text{Zero}(\overline{DPS}, \overline{I_2}, \overline{I_1})$.

Obtain characteristic set $\overline{DCS_2}$ with IS products is empty set, then, we get

$$\text{Zero}(\overline{DPS}, \overline{I_2}, \overline{I_1}) = \text{Zero}(\overline{DCS_2}) = \text{Zero}(\overline{DCS_2}/\overline{I_2} * \overline{I_3}), \quad (61)$$

Step 4 $\overline{I_3} = 0$, compute $\text{Zero}(\overline{DPS}, \overline{I_3})$.

The determining Eq. (45) simplify to

$$\overline{DTE_2} = \begin{cases} \eta_x = 0, \xi_y = 0, 2\eta_{yyy} = 0, \\ -4f(y)\xi_x - 2\xi_{xxx} - 2\eta f'(y) = 0, \end{cases} \quad (62)$$

taking left hand side for each equation, we have following corresponding differential polynomial system

$$\overline{DPS_2} = \begin{cases} \eta_x, \xi_y, 2\eta_{yyy}, \\ -4f(y)\xi_x - 2\xi_{xxx} - 2\eta f'(y), \end{cases} \quad (63)$$

under the rank $x \prec y \prec \xi \prec \eta$, using the differential form Wu's method, we obtain characteristic set

$$\overline{DCS_4} = \{\xi_y, \xi_x, \eta\}, \quad (64)$$

with IS products $\overline{I_4} \neq 0$, and

$$\begin{aligned} \overline{I_4} = & -3f'(y)^2 f''(y)^2 + 6f(y) f''(y)^3 + 2f'(y)^3 f^{(3)}(y), \\ & -6f(y) f'(y) f''(y) f^{(3)}(y) + f(y) f'(y)^2 f^{(4)}(y), \end{aligned} \quad (65)$$

then, we get

$$\text{Zero}(\overline{DPS}, \overline{I_3}) = \text{Zero}(\overline{DCS_4}/\overline{I_4}) + \text{Zero}(\overline{DPS}, \overline{I_3}, \overline{I_4}), \quad (66)$$

in which

$$\text{Zero}(\overline{DCS_4}/\overline{I_4}) = \{\xi = c_1, \eta = 0\}, \quad (67)$$

and c_1 is an arbitrary constant.

Step 4.1 $\overline{I_4} = 0$, compute $\text{Zero}(\overline{DPS}, \overline{I_3}, \overline{I_4})$.

Obtain characteristic set $\overline{DCS_2}$ with IS products $\overline{I_5} \neq 0$, and

$$\overline{I_5} = -3f''(y)^2 + 2f'(y) f^{(3)}(y), \quad (68)$$

then, we get

$$\text{Zero}(\overline{DPS}, \overline{I_3}, \overline{I_4}) = \text{Zero}(\overline{DCS_2}/\overline{I_5}) + \text{Zero}(\overline{DPS}, \overline{I_3}, \overline{I_4}, \overline{I_5}), \quad (69)$$

in which

Case 1. $\text{Zero}(\overline{DCS_2}/\overline{I_5}) = \{\xi = c_4x + c_5, \eta = \frac{1}{2}c_1y^2 + c_2y + c_3\}$, with the corresponding function

$$f(y) = e^{\frac{4c_4 \arctan\left(\frac{c_1y+c_2}{\sqrt{2c_1c_3-c_2^2}}\right)}{\sqrt{2c_1c_3-c_2^2}}} c_6, \text{ and } c_1, c_2, \dots, c_6 \text{ are arbitrary constants.}$$

Case 2. $Zero(\overline{DCS_2}/\overline{I_5}) = \{\xi = c_1, \eta = F(y)\}$, with the corresponding function $f(y) = c_2$, $F(y)$ is an arbitrary function of y , and c_1, c_2 are arbitrary constants.

Case 3. $Zero(\overline{DCS_2}/\overline{I_5}) = \{\xi = c_1, \eta = 0\}$, with the corresponding function $f(y) = e^{\frac{c_5 \sqrt{c_2 c_3} + 2 \arctan h \left(\frac{c_3 (c_4 + y)}{\sqrt{c_2 c_3}} \right)}{\sqrt{c_2 c_3}}}$, and c_1, c_2, \dots, c_5 are arbitrary constants.

Case 4. $Zero(\overline{DCS_2}/\overline{I_5}) = \{\xi = c_1, \eta = 0\}$, with the corresponding function $f(y) = c_2$, and c_1, c_2 are arbitrary constants.

Case 5. $Zero(\overline{DCS_2}/\overline{I_5}) = \{\xi = F(x), \eta = G(y)\}$, with the corresponding function $f(y) = 0$, and $F(x)$ is an arbitrary function of $x, G(y)$ is an arbitrary function of y .

Step 4.2 $\overline{I_5} = 0$, compute $Zero(\overline{DPS}, \overline{I_3}, \overline{I_4}, \overline{I_5})$.

Obtain characteristic set $\overline{DCS_2}$ with IS products is empty set, and we get

$$Zero(\overline{DPS}, \overline{I_3}, \overline{I_4}, \overline{I_5}) = Zero(\overline{DCS_2}) = Zero(\overline{DCS_2}/\overline{I_5}). \tag{70}$$

At the end of this section, we give the symmetry classification for Eq. (2) with $n = 0$ based on differential form Wu's method.

As stated before, the determining Eq. (45) simplify to

$$\overline{DTE_3} = \begin{cases} \xi_y = 0, \eta_{yy} = 0, -3\xi_{xx} + 3\eta_{xy} = 0, -f(y)\eta_x + \eta_{xxx} = 0, \\ -2f(y)\xi_x - \xi_{xxx} + 3\eta_{xy} - \eta f'(y) = 0, \end{cases} \tag{71}$$

taking left hand side for each equation, we have following corresponding differential polynomial system:

$$\overline{DPS_3} = \begin{cases} \xi_y, \eta_{yy}, -3\xi_{xx} + 3\eta_{xy}, -f(y)\eta_x + \eta_{xxx}, \\ -2f(y)\xi_x - \xi_{xxx} + 3\eta_{xy} - \eta f'(y), \end{cases} \tag{72}$$

Step 1 Compute $Zero(\overline{DPS_3})$.

Under the rank $x \prec y \prec \xi \prec \eta$, using the differential form Wu's method, we obtain characteristic set

$$\overline{DCS_5} = \{\xi_y, \xi_x, \eta\}, \tag{73}$$

with IS products $\overline{I_1} \neq 0$, then, we get

$$Zero(\overline{DPS_3}) = Zero(\overline{DCS_5}/\overline{I_1}) + Zero(\overline{DPS_3}, \overline{I_1}), \tag{74}$$

in which

$$Zero(\overline{DCS_5}/\overline{I_1}) = \{\xi = c_1, \eta = 0\}, \tag{75}$$

and c_1 is an arbitrary constant.

Step 2 $\overline{I_1} = 0$, compute $Zero(\overline{DPS_3}, \overline{I_1})$.

Obtain characteristic set $\overline{DCS_2}$ with IS products is empty set, then, we get

$$Zero(\overline{DPS_3}, \overline{I_1}) = Zero(\overline{DCS_2}) = Zero(\overline{DCS_2}/\overline{I_2} * \overline{I_3}). \tag{76}$$

4 Conclusion

In this paper, Lie algorithm combined with differential form Wu's method is used to complete the symmetry classification of ODEs containing arbitrary parameter. This process can be reduced to solve a system of determining equations, then the differential form Wu's method is used to decompose the determining equations into a series of equations, which are easy to solve relatively. To illustrate the

usefulness of this method, we apply it to the generalizations of the Kummer-Schwarz equations, and the results show the performance of the present work.

In addition, the second-order nonlinear ODE [20]

$$2(K(y)y')' + xy' = 0, \quad (77)$$

and the following ODE [21]:

$$y'' = f(x)y^2, \quad (78)$$

are studied by many researchers. The algorithm of Wu's method is performed on computer in this paper, we try to give the symmetry classification of Eqs. (77) and (78), after four days, the program is still running, only part of the results are obtained. In the next, we will make some improvements to the differential form Wu's method, and expect to get the complete symmetry classification results.

Acknowledgement: The author is very thankful to Professor Jihuan He, who is work in national engineering laboratory for modern silk, college of textile and engineering, Soochow University, for his help.

Funding Statement: The authors received no specific funding for this study.

Conflicts of Interest: The authors declare that they have no conflicts of interest to report regarding the present study.

References

1. He, J. H. (2012). Homotopy perturbation method with an auxiliary term. *Abstract and Applied Analysis*, 2012, 1–7. DOI 10.1155/2012/857612.
2. He, J. H. (2014). Homotopy perturbation method with two expanding parameters. *Indian Journal of Physics*, 88(2), 193–196. DOI 10.1007/s12648-013-0378-1.
3. Wu, Y., He, J. H. (2018). Homotopy perturbation method for nonlinear oscillators with coordinate-dependent mass. *Results in Physics*, 10, 270–271. DOI 10.1016/j.rinp.2018.06.015.
4. Anjum, N., He, J. H. (2019). Laplace transform: Making the variational iteration method easier. *Applied Mathematics Letters*, 92, 134–138. DOI 10.1016/j.aml.2019.01.016.
5. He, J. H. (2006). Some asymptotic methods for strongly nonlinear equations. *International Journal of Modern Physics B*, 20(10), 1141–1199. DOI 10.1142/S0217979206033796.
6. He, J. H., Kong, H. Y., Chen, R. X., Hu, M. S., Chen, Q. L. (2014). Variational iteration method for Bratu-like equation arising in electrospinning. *Carbohydrate Polymers*, 105, 229–230. DOI 10.1016/j.carbpol.2014.01.044.
7. He, J. H., Ji, F. Y. (2019). Taylor series solution for Lane-Emden equation. *Journal of Mathematical Chemistry*, 57(8), 1932–1934. DOI 10.1007/s10910-019-01048-7.
8. He, J. H. (2013). Exp-function method for fractional differential equations. *International Journal of Nonlinear Sciences and Numerical Simulation*, 14(6), 363–366. DOI 10.1515/ijnsns-2011-0132.
9. He, J. H. (2012). Asymptotic methods for solitary solutions and compactons. *Abstract and Applied Analysis*, 2012, 1–130. DOI 10.1155/2012/916793.
10. He, J. H., Wu, X. H. (2006). Exp-function method for nonlinear wave equations. *Chaos Solitons & Fractals*, 30(3), 700–708. DOI 10.1016/j.chaos.2006.03.020.
11. Tian, Y., Wang, K. L. (2020). Polynomial characteristic method: An easy approach to lie symmetry. *Thermal Science*, 24(4), 2629–2635. DOI 10.2298/TSCI2004629T.
12. Tian, Y. (2019). Diffusion-convection equations and classical symmetry classification. *Thermal Science*, 23(4), 2151–2156. DOI 10.2298/TSCI1904151T.
13. Tian, Y. (2019). Symmetry reduction-A promising method for heat conduction equations. *Thermal Science*, 23(4), 2219–2227. DOI 10.2298/TSCI1904219T.

14. Bozhkov, Y. D., Conceição, P. R. (2020). On the generalizations of the Kummer–Schwarz equation. *Nonlinear Analysis*, 192, 111691. DOI 10.1016/j.na.2019.111691.
15. Tian, Y., Wang, K. L. (2020). Conservation laws for partial differential equations based on the polynomial characteristic method. *Thermal Science*, 24(4), 2529–2534. DOI 10.2298/TSCI2004529T.
16. Wu, W. T. (2000). *Mathematics mechanization*. Beijing: Science Press.
17. Cox, D. A., Little, J., Oshea, D. (1992). *Ideals, varieties, and algorithms: An introduction to computational algebraic geometry and commutative algebra*. New York: Springer-Verlag.
18. Wu, W. T. (1984). Basic principles of mechanical theorem–proving in elementary geometry. *Systems Science and Mathematical Sciences*, 4(3), 207–235. DOI 10.12341/jssms08900.
19. Wu, W. T. (1989). On the foundation of algebraic differential geometry. *Systems Science and Mathematical Sciences*, 2(4), 289–312.
20. Bluman, G., Hoskins, J. G. (2016). Higher order symmetries and integrating factors for ordinary differential equations. *Journal of Mathematical Analysis and Applications*, 435(1), 133–161. DOI 10.1016/j.jmaa.2015.09.064.
21. Wafo Soh, C., Mahomed, F. M. (1999). Preliminary group classification of $y''' = f(x)y^2$. *Mathematical and Computational Applications*, 4(3), 289–295. DOI 10.3390/mca4030289.