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ARTICLE

On Fractional Integro-Differential Equation with Nonlinear Time Varying Delay

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ABSTRACT

In this manuscript, we analyze the solution for class of linear and nonlinear Caputo fractional Volterra Fredholm integro-differential equations with nonlinear time varying delay. Also, we demonstrate the stability analysis for these equations. Our paper provides a convergence of semi-analytical approximate method for these equations. It would be desirable to point out approximate results.

KEYWORDS

Convergence; stability; fractional integro-differential equation

1 Introduction

Nowadays, the study of differential equations with nonlinear time varying delay has attracted a wide-ranging interest [1–4]. Meanwhile, few research texts have been based on the theoretical aspects. It is worth to construct an accurate strategy that will facilitate the task of many scholars, followers and experts later. In mathematics, stability of the system is a mathematical property which deals with the behavior of the dynamic systems and corresponds to the convergence of solutions of differential, integro differential and fractional equations [5–12].

It considers as a suitable central role in the study of fractional system [13–15]. This work is perhaps the best straightforward technique to generalize the related papers [16–21].

In [21], Soliman et al. carried out a detailed and comprehensive analysis of fractional Volterra Fred-holm integro-differential equation with constant delay. Likewise, our present study motivated, constructed, continuously developed thus far parallels and properly cited (inspired) by the results of the above work [22–26]. It also includes analysis extensively, but differs significantly about the aforementioned work. It is an attempt to generalize different precedent works and in order to achieve a notable contribution with its counterparts. For more details, this context relied on the appropriate analysis of some linear and nonlinear problems described by fractional integro-differential equations with variable delays. More precisely, we receive a noticeable attention to the nonlinear integro-differential equations with nonlinear time varying delay as in [17]. So far, the strategy of the current research contains four beneficial items. At the first, existence and uniqueness of the solution of the proposed problem will be examined. Secondly, we will provide the readers with the stability and the convergence of this solution. At the end, we shall suggest a method and its convergence therein. This literature consists of five sections. Some basic definitions and theorems are introduced in the first section. Indeed, the suggested problem is discussed and analytical explanation of the proposed problem is detailed in section two. Some Important properties of the solution is mentioned in section three. A modified method is explained in section four. Finally, An experimental example is illustrated in section five.

1.1 Mathematical Tools and Theoretical Background

The most commonly notations, definitions and theorems are mentioned. The presented preliminaries are related to our paper.

Definition 1.1. The Banach space $U = C([a, b], \mathbb{R}^4)$ is the space of all real-valued continuous functions from $:[a, b] \to \mathbb{R}^4$, let $U(t) = \{u(t): u(t) \in C([a, b], \mathbb{R}^4$ and $D^s u \in C([a, b], \mathbb{R}^4,\ s \in (0, 1]\}$ endowed with the norm $\|\ .\ \|$; $\|\ u\ \| = \max\{|u(t)| + |D^s u(t)| : t \in [a, b]\}$, $D^s u$ denotes the Caputo derivative of fractional order s [27,28].

Definition 1.2. Let $\sigma: U \to U$ be a mapping on a Banach space $(U, \|\ .\ \|)$. The point $u \in U$ is called a fixed point of σ with $\sigma u = u$.

Definition 1.3. The mapping σ on a Banach space $(U, \|\ .\ \|)$ is called contractive if there exists $C \in (0, 1)$, such that

$$\|\sigma u - \sigma v\| \leq C\|u - v\|,\ \forall u,\ v \in U. \tag{1}$$

Definition 1.4. For $w \in C([a, b], \mathbb{R}^4)$, the q−th Caputo fractional derivative of a function is defined by

$$D^\alpha w(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{1+\alpha-n} \frac{d^n w}{ds^n} ds, \tag{2}$$

where $n = [q] + 1,\ m \in \mathbb{N},\ \ [q]$ is Euler Gamma function for q and $[q]$ denotes the integer part of the real number q.

Definition 1.5. The Mittag-Leffler type is defined as

$$E_{a,b}(t) = \sum_{k=1}^{\infty} \frac{t^k}{\Gamma(ak+b)}, \tag{3}$$

at b = 1, a > 0 the previous equation becomes the classical Mittag-Leffler.

Definition 1.6. [27] The fractional equation

$$D^\alpha u(t) = f(u(t)) \tag{4}$$

is Ulam-Hyers stable if there exists C_f such that for each $\varepsilon > 0$ and $v \in C([a, b], \mathbb{R}^4)$

$$\|D^\alpha v(t) - f(v(t))\| \leq \varepsilon,\ \forall t \in [a,\ b], \tag{5}$$

$\exists$ a solution $v \in C([a, b], \mathbb{R}^4)$ of Eq. (1.7) with

$$\|v(t) - u(t)\| \leq C_f \varepsilon,\ \ t \in [a,\ b], \tag{6}$$

t is the independent variable.

Definition 1.7. [27] The fractional integro-equation

$$D^{\alpha}u(t) = f(u(t)) + \int_a^b G(t,\ s,\ u(s))ds \tag{7}$$

is Ulam-Hyers stable Rassias stable with respect to v if there exists a positive constant

L > 0 with the following property: For each $u(t)$ satisfying

$$\left| D^{\alpha}u(t) - f(u(t)) - \int_a^b G(t,\ s,\ u(s))ds \right| \leq v(t) \tag{8}$$

then ∃ some solution $u_0(t)$ of the above equation such that

$$\|u(t) - u_0(t)\| \leq L\, v(t). \tag{9}$$

Definition 1.8. The sequence $u_{n_{n=1}}^{\infty}$ of functions converges uniformly on a set D if there exists ε, N > 0 such that $\forall t \in D,\ n \in N,\ \|u_n(t) - u(t)\| \leq \varepsilon$.

Lemma 1.9. [26] (Gronwall'lemma)

Let u and v be nonnegative continuous functions on some interval $t \in [a, b]$. Also, let the function f(t) be positive, continuous and monotonically non-decreasing on [a, b] and u satisfies the inequality

$$u(t) \leq f(t) + \int_a^b u(s)v(s)ds, \tag{10}$$

Then, there holds the inequality

$$u(t) \leq f(t)\ \exp(\int_a^b v(s)ds). \tag{11}$$

Lemma 1.10. [26] (Pachpatte'inequality) Let $u,\ v$ and w be nonnegative continuous functions on $\mathbb{R}^+$ and f(t) be a positive and non-decreasing continuous function, the inequality

$$u(t) \leq f(t) + \int_0^t v(s) \left[u(s) + \int_a^b w(p)u(p)dp \right] ds, \tag{12}$$

holds, then

$$u(t) \leq f(t) \left[1 + \int_0^t v(s)\ \exp\left(\int_0^s (v(s) + w(p))dp \right) \right]. \tag{13}$$

Theorem 1.11. [26] (Banach contraction mapping theorem)

Let $(U, \|\ .\ \|)$ be a Banach space, $\sigma: U \to U$ is an operator. If σ is contraction (contractive) mapping. Then σ has exactly one fixed point.

Theorem 1.12. [27] (Schauder fixed point theorem)

Let $(U, \| . \|)$ be a closed, convex and nonempty subset of a Banach space C[a, b], suppose that $\sigma: U \rightarrow U$ is a continuous mapping such that $\sigma(U)$ is a relatively compact subset of C[a, b]. Then σ has at least one fixed point in U.

2 Analytic Explanation of the Problem

Throughout this paper, we will consider Caputo fractional integro-differential equation of the form:

$$D^{\alpha}u(t) = f\left(t,\ u(t-\tau(t)),\ \int_a^t G(t,\ x,\ u(x))dx,\ \int_a^b H(t,\ x,\ u(x))dx\right), \tag{14}$$

through the initial condition:

$$u(a) = \chi(u) \tag{15}$$

where D^{α} refers to the $\alpha-$ th fractional derivative of the anonymous function $u(t) \in U = C([a, b], \mathbb{R}^4)$ which characterized by the Caputo operator, $\tau(t)$ is the time varying delay (continuous delay function),

$0 < \alpha < 1$, $0 < \tau < \mathrm{b}$, $f:[a, b] \times U \times U \times U \rightarrow U$ is a continuous function, $G, H:[a, b]^2 \times U \rightarrow U \rightarrow U$ are nonlinear Lipschitz continuous functions of $u(t)$ and $\chi: U \rightarrow \mathbb{R}^4$ is a continuous function.

Let us assume the following conditions:

(1)There exists a constant $C_f > 0$, for each $u_1,\ u_2,\ v_1,\ v_2,\ w_1,\ w_2 \in U$

$$|f(t,\ u_1,\ v_1,\ w_1) - f(t,\ u_2,\ v_2,\ w_2)| \le C_f[|u_1 - u_2| + |v_1 - v_2| + |w_1 - w_2|] \tag{16}$$

(2)There exists a constant $C_G > 0$

$$\left|\int_a^t G(t,\ x,\ u(x))dx - \int_a^t G(t,\ x,\ v(x))dx\right| \le C_G|u - v| \tag{17}$$

(3)There exists a constant $C_H > 0$

$$\left|\int_a^b H(t,\ x,\ u(x))dx - \int_a^t H(t,\ x,\ v(x))dx\right| \le C_H|u - v| \tag{18}$$

(4)There exists a constant $C_\tau > 0$

$$|u(t-\tau) - v(t-\tau)| \le C_\tau|u - v| \tag{19}$$

(5)There exists a constant $C_\chi > 0$

$$|\chi(u) - \chi(v)| \le C_\chi|u - v| \tag{20}$$

Theorem 2.1. The Eq. (14) is equivalent to

$$u(t) = I^{\alpha}f\left(t,\ u(t-\tau(t)),\ \int_a^t G(t,\ x,\ u(x))dx,\ \int_a^b H(t,\ x,\ u(x))dx\right) + \chi(u) \tag{21}$$

Proof. Integrating two both sides of Eq. (14), we get

$$I\,D^{\alpha}u(t) = If\left(t,\; u(t-\tau(t))\int_{a}^{t} G(t,\; x,\; u(x))dx,\; \int_{a}^{b} H(t,\; x,\; u(x))dx\right).$$

This leads to

$$I^{1-\alpha}u(t) - \vartheta = If\left(t,\; u(t-\tau(t)),\; \int_{a}^{t} G(t,\; x,\; u(x))dx,\; \int_{a}^{b} H(t,\; x,\; u(x))dx\right),$$

Operate with I^{α}, we have

$$I\,u(t) = I^{\alpha+1}f\left(t,\; u(t-\tau(t)),\; \int_{a}^{t} G(t,\; x,\; u(x))dx,\; \int_{a}^{b} H(t,\; x,\; u(x))dx\right) + \frac{\vartheta}{\Gamma(\alpha)}\int_{0}^{t}\frac{ds}{(t-s)^{1-\alpha}}. \tag{22}$$

Then,

$$I\,u(t) = I^{\alpha+1}f\left(t,\; u(t-\tau(t)),\; \int_{a}^{t} G(t,\; x,\; u(x))dx,\; \int_{a}^{b} H(t,\; x,\; u(x))dx\right) + \frac{\vartheta t^{\alpha}}{\Gamma(\alpha+1)}. \tag{23}$$

Differentiating we obtain

$$u(t) = I^{\alpha}f\left(t,\; u(t-\tau(t)),\; \int_{a}^{t} G(t,\; x,\; u(x))dx,\; \int_{a}^{b} H(t,\; x,\; u(x))dx\right) + \frac{\vartheta t^{\alpha-1}}{\Gamma(\alpha)}.$$

where ϑ is a constant, then at t = a we deduce that (14) is equivalent to (21).

Let $\sigma{:}U \to U$, for any $u \in U$. Now, we establish the following theorem for the fixed point σ.

Theorem 2.2. The operator σ maps U into itself and it is also continuous on $[a, b]$.

Proof.

$$\begin{aligned}\|\sigma u(t)\| &= \left\|\chi(u) + \frac{1}{\Gamma(\alpha)}\int_{0}^{t}(t-s)^{\alpha-1}\left[f\left(t,\; u(t-\tau(t)),\; \int_{a}^{t} G(t,\; x,\; u(x))dx,\; \int_{a}^{b} H(t,\; x,\; u(x))dx\right)\right]ds\right\| \\ &\le c\|u\| + C_G C_H\frac{f_{\max}}{\Gamma(\alpha+1)}t^{\alpha}\|u(s-\tau(s))\|\|u\|^{2} \\ &\le d\|u\| \\ &\le C\end{aligned} \tag{24}$$

That is, σ maps U into itself. Also, A becomes uniformly bounded. Suppose a sufficiently small number $n > 0$,

$$\begin{aligned}&\|\sigma u(t+n)-\sigma u(t)\| = \\ &\frac{1}{\Gamma(\alpha)}\left[\left\|\int_a^t (t-s)^{\alpha-1}\left[f\left(s,\ u((s+n)-\tau(s+n)),\ \int_a^t G(s+n,\ x,\ u(x))dx,\ \int_a^b H(s+n,\ x,\ u(x))dx\right)\right]ds\right\|\right] \\ &-\frac{c}{\Gamma(\alpha)}\left[\left\|\int_a^t (t-s)^{\alpha-1}\left[f\left(s,\ u((s)-\tau(s+n)),\ \int_a^t G(s,\ x,\ u(x))dx,\ \int_a^b H(s,\ x,\ u(x))dx\right)\right]ds\right\|\right]\end{aligned} \tag{25}$$

In short,

$$\begin{aligned}\|\sigma u(t+n)-\sigma u(t)\| &\leq \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\|u((s+n)-\tau(s+n))-u((s)-\tau(s))\| \\ &+\left\|\int_a^s (G(s+n,\ x,\ u(x))-G(s,\ x,\ u(x)))dx\right\|ds \\ &+\left\|\int_a^b (H(s+n,\ x,\ u(x))-H(s,\ x,\ u(x)))dx\right\|ds \\ &\leq \frac{C_f t^{\alpha}}{\Gamma(\alpha+1)}\|u((s+n)-\tau(s+n))-u((s)-\tau(s))\| \\ &+C_G\|u(s+n)-u(s)\|+C_H\|u(s+n)-u(s)\|.\end{aligned} \tag{26}$$

Consequently, we thus conclude that

$$\begin{aligned}\|\sigma u(t+n)-\sigma u(t)\| &\leq \frac{C_f t^{\alpha}}{\Gamma(\alpha+1)}[C_\tau+C_G+C_H]\|u(s+n)-u(s)\| \\ &\leq F\|u(s+n)-u(s)\|\end{aligned} \tag{27}$$

where $t\in[a,\ b]$, $F=\max\left\{\frac{C_f t^{\alpha}}{\Gamma(\alpha+1)}[C_\tau+C_G+C_H]\right\}$, $0<F<1$. It follows that, $\|\sigma u(t+n)-\sigma u(t)\|\to 0$ *as* $n\to\infty$.

Then, σu(t) is continuous on [a, b]. Our approach for proving that σ is continuous, we suppose that un converge to u, $\forall n\in N$. Then

$$\begin{aligned}\|\sigma u_n(t)-\sigma u(t)\| \leq \|\chi(u_n)-\chi(u)\| &+ I^{\alpha}f\left(t,\ u_n(t-\tau(t)),\ \int_a^t G(t,\ x,\ u_n(x))dx,\ \int_a^b H(t,\ x,\ u_n(x))dx\right) \\ &- I^{\alpha}f\left(t,\ u(t-\tau(t)),\ \int_a^t G(t,\ x,\ u(x))dx,\ \int_a^b H(t,\ x,\ u(x))dx\right),\end{aligned}$$

If we follow the conditions Eqs. (16)–(20), we reach

$$\| \sigma u_n(t) - \sigma u(t)\| \leq C_\chi \|u_n - u\| + \frac{C_f}{\Gamma(\alpha)}(C_\tau \|u_n - u\| + C_G\|u_n - u\| + C_H\|u_n - u\|)$$

This is equivalent to

$$\| \sigma u_n(t) - \sigma u(t)\| \leq \left[C_\chi + \frac{C_f}{\Gamma(\alpha)}(C_\tau + C_G + C_H)\right]\|u_n - u\|$$

This shows that $\sigma u_n \to \sigma u$.

2.1 Existence and uniqueness of the solution for Eq. (14)

In what follows, we will investigate the existence and uniqueness of solution for the fractional integro-differential equation with time-varying delay (variable delay).

Theorem 2.3. Suppose that the conditions Eqs. (16)–(20) hold, then the non-linear fractional integro-differential Eq. (14) has at least a unique solution $u \in U$.

Proof. By analogous proof to the continuity of σ operator.

$$\| \sigma u(t) - \sigma v(t)\| \leq \|\chi(u) - \chi(v)\| + I^\alpha f\left(t,\ u(t-\tau(t)),\ \int_a^t G(t,\ x,\ u(x))dx,\ \int_a^b H(t,\ x,\ u(x))dx\right)$$
$$- I^\alpha f\left(t,\ v(t-\tau(t)),\ \int_a^t G(t,\ x,\ v(x))dx,\ \int_a^b H(t,\ x,\ v(x))dx\right),$$

For brevity,

$$\| \sigma u(t) - \sigma v(t)\| \leq \|\chi(u) - \chi(v)\| + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[\|u(s-\tau(s)) - v(t-\tau(s))\|$$
$$, + \left\|\int_a^s (G(t,\ x,\ u(x)) - G(t,\ x,\ u(x)))dx\right\| + \left\|\int_a^b (H(t,\ x,\ u(x)) - H(t,\ x,\ u(x)))dx\right\|]ds \tag{28}$$
$$\leq C_\chi\|u - v\| + \frac{C_f t^\alpha}{\Gamma(\alpha+1)}[\|u(s-\tau(s)) - v(t-\tau(s))\| + C_G\|u(s) - v(t)\| + C_H\|u(s) - v(t)\|.$$

In consequent, we have $\| \sigma u(t) - \sigma v(t)\| \leq Y\|u(s) - v(t)\|$, where $t \in [a,b]$, $Y = \max\left\{\frac{C_f t^\alpha}{\Gamma(\alpha+1)}[C_\tau + C_G + C_H] + C_\chi\right\}$, $0 < Y < 1$.

We conclude that σ is Lipschitz on U with Lipschitz constant Y. It is well known that σ is a fixed point as a consequence of Theorem 1.11., i.e., σ is a contractive mapping. Eq. (14) has immediately at least a unique solution $u \in U$.

Lemma 2.1. Assume that {u(t)} is a continuous function on [a, b], it satisfies

$$\begin{cases} D^{\alpha}u(t) = f\left(t,\ u(t-\tau(t)),\ \int_a^t G(t,\ x,\ u(x))dx,\ \int_a^b H(t,\ x,\ u(x))dx\right),\ \alpha \in (0,\ 1), \\ u(a) = \chi(u) \end{cases}$$

Further, $\|u(t_1) - u(t_2)\| \le q$. Then {u(t)} is equicontinuous on [a, b].

Proof. Without loss of generality, for $t_1,\ t_2 \in [a, b]$ such that $t_1 < t_2$, we get

$$\begin{aligned} &\|\sigma u(t_1) - \sigma u(t_2)\| \le \|\chi(u(t_1)) - \chi(u(t_2))\| \\ &+\frac{1}{\Gamma(\alpha)}\int_0^{t_1} (t_1-s)^{\alpha-1}\left[\left|f(s, u(s-\tau(s)), \int_a^s (G(x, w, u(w))dw, \int_a^b (H(x, w, u(w))dw\right|\right] ds \\ &-\frac{1}{\Gamma(\alpha)}\int_0^{t_2} (t_2-s)^{\alpha-1}\left[\left|f(s, u(s-\tau(s)), \int_a^s (G(x, w, u(w))dw, \int_a^b (H(x, w, u(w))dw\right|\right] ds \\ &\le\frac{1}{\Gamma(\alpha)}\int_0^{t_1} [(t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1}]\left[\left|f(s, u(s-\tau(s)), \int_a^s (G(x, w, u(w))dw, \int_a^b (H(x, w, u(w))dw\right|\right] ds \\ &+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2} (t_2-s)^{\alpha-1}\left[\left|f(s, u(s-\tau(s)), \int_a^s (G(x, w, u(w))dw, \int_a^b (H(x, w, u(w))dw\right|\right] ds + C_{\chi}\|u-v\| \\ &\le qC_{\chi} + \frac{\|f\|_{\infty}}{\Gamma(\alpha+1)}[t_1^{\alpha} - t_2^{\alpha} + 2(t_2-t_1)^{\alpha}] \\ &\qquad \to 0. \end{aligned} \tag{29}$$

whenever $t_1 \to t_2,\ \ q > 0$, where $\|f\|_{\infty} = \sup_{t\in[a,b]} |f(t, \ldots)|$. Thus, σu(t) is equicontinuous function in U. This means that σ is relatively compact. Hence, σ is compact. In view of Theorem 1.12, σ has at least one fixed point (solution of (14)) in U.

Lemma 2.2. If the conditions Eqs. (16)–(20) satisfied, then the non-linear Eq. (14) has a unique solution provided

$$\max\left\{\frac{C_f b^{\alpha}}{\Gamma(\alpha+1)}[C_{\tau} + C_G + C_H] + C_{\chi}\right\} < 1. \tag{30}$$

Our following attention is focused on checking the stability of the solution u(t) for Eq. (14) in the frame of Ulam-Hyers and Ulam-Hyers-Rassias.

2.2 Stability Analysis of the Solution for Eq. (14)

Theorem 2.4. Assume that the conditions Eqs. (16)–(20) hold. Then the non-linear fractional integro-differential Eq. (14) is Ulam-Hyers stable.

Proof. Let $u \in U$ be a solution of Eq. (14), D(s) is a continuous and non negative function such that

$$\sup\left\{\int_0^t (t-s)^{\alpha-1}[R(s)]ds\right\} < \infty,\ \left|D^\alpha u(t) - f\left(t,\ u(t-\tau(t)),\ \int_a^t G(t,x,u(x))dx,\ \int_a^b H(t,x,u(x))dx\right)\right| \le \varepsilon.$$

Now, we are going to apply the integral operator I^α to both sides of above equation, we arrive at

$$\left|u(t) - \chi(u) - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left(f(s,\ u(s-\tau(s)),\ \int_a^s G(s,\ p,\ u(p))dp,\ \int_a^b H(s,\ p,\ u(p))dp\right)ds\right|$$
$$\le \frac{\varepsilon}{\Gamma(\alpha)}t^\alpha \int_0^t (t-s)^{\alpha-1}ds.$$

Equivalently

$$\begin{aligned}&\left|u(t) - \chi(u) - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left(f(s,\ u(s-\tau(s)),\ \int_a^s G(s,\ p,\ u(p))dp,\ \int_a^b H(s,\ p,\ u(p))dp\right)ds\right|\\ &\quad\le \frac{\varepsilon t^\alpha}{\Gamma(\alpha+1)}\\ &\quad\le \varepsilon E_{1,1}(t).\end{aligned} \tag{31}$$

for $v(t) \in U$, it can be written as

$$v(t) = \chi(u) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left(f(s,\ v(s-\tau(s)),\ \int_a^s G(s,\ p,\ v(p))dp,\ \int_a^b H(s,\ p,\ v(p))dp\right)ds,$$

The difference |u(t) − v(t)| is given as

$$\begin{aligned}|u(t) - v(t)| &= |u(t) - u(t) + u(t) - v(t)|\\ &\le \left|u(t) - \chi(u) - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left(f(s,\ u(s-\tau(s)),\ \int_a^s G(s,\ p,\ u(p))dp,\ \int_a^b H(s,\ p,\ u(p))dp\right)ds\right|\\ &+ \left|u(t) - \chi(v) - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left(f(s,\ v(s-\tau(s)),\ \int_a^s G(s,\ p,\ v(p))dp,\ \int_a^b H(s,\ p,\ v(p))dp\right)ds\right|\end{aligned}$$

or equivalent to

$$
\begin{aligned}
|u(t)-v(t)| \leq & \frac{\varepsilon b^{\alpha}}{\Gamma(\alpha+1)}+|\chi(u)-\chi(v)| \\
& +\left|\frac{1}{\Gamma(\alpha)} \int_{0}^{t}(t-s)^{\alpha-1}\left(\int_{a}^{s} G(s, p, u(p)) dp-\int_{a}^{s} G(s, p, v(p)) dp\right) ds\right| \\
& +\frac{1}{\Gamma(\alpha)} \int_{0}^{t}(t-s)^{\alpha-1}\left(\int_{a}^{b} H(s, p, u(p)) dp-\int_{a}^{b} H(s, p, v(p)) dp\right) ds
\end{aligned}
$$

$$
\begin{aligned}
|u(t)-v(t)| & \leq \frac{\varepsilon b^{\alpha}}{\Gamma(\alpha+1)}+\frac{C_{f}}{\Gamma(\alpha)} \int_{0}^{t}(t-s)^{\alpha-1}\left(C_{G}+C_{H}\right)|u-v|\, ds+C_{\chi}|u-v| \\
& \leq \frac{\varepsilon b^{\alpha}}{\Gamma(\alpha+1)}+\frac{R}{\Gamma(\alpha)} \int_{0}^{t}(t-s)^{\alpha-1}|u-v|\, ds+C_{\chi}|u-v|
\end{aligned}
$$

From Gronwall's lemma Eqs. (10), (11) yields

$$
\begin{aligned}
|u(t)-v(t)| & \leq \frac{\varepsilon b^{\alpha}}{\Gamma(\alpha+1)}+\frac{R}{\Gamma(\alpha)} \int_{0}^{t}(t-s)^{\alpha-1}|u-v|\, ds+C_{\chi}|u-v| \\
& \leq \frac{\varepsilon b^{\alpha}}{\Gamma(\alpha+1)} \exp \left(\frac{R}{\Gamma(\alpha)} \int_{0}^{t}(t-s)^{\alpha-1}|u-v|\, ds\right)+C_{\chi}|u-v| \\
& \leq \varepsilon K .
\end{aligned} \tag{32}
$$

where $K>0$, $R=C_f(C_G+C_H)$ (CG + CH) such that

$$|u(t)-v(t)| \leq \varepsilon K. \tag{33}$$

In consequence, the problem (14) is stable in the sense of Ulam-Hyers. This completes the proof.

Theorem 2.5. Suppose that the conditions Eqs. (16)–(20) satisfied, $P(t) \in U$ is an increasing function and $\exists C_p>0$ such that $I^{\alpha} \leq C_p P(t)$ for any $t \in [a, b]$. Then the non-linear fractional Eq. (14) is Ulam-Hyers-Rassias stable.

Proof. Let $w \in U$ be a solution of the following inequality

$$\left\| D^{\alpha} u(t)=f\left(t, u(t-\tau(t)), \int_{a}^{t} G(t, x, u(x)) dx, \int_{a}^{b} H(t, x, u(x)) dx\right)\right\| \leq \varepsilon P(t). \tag{34}$$

Further, for any $t \in [a, b]$, $\varepsilon > 0$. Assume that $u \in U$ is the solution of (14). Now, integrate (14), that is

$$\left| w(t) - \chi(w) - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left(f(s,\ w(s-\tau(s)),\ \int_a^s G(s,\ p,\ w(p))dp,\ \int_a^b H(s,\ p,\ w(p))dp \right) ds \right|$$

$$\begin{aligned} &\leq \frac{\varepsilon}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} P(t) ds \\ &\leq \frac{\varepsilon}{\Gamma(\alpha)} I^\alpha P(t) \\ &\leq \varepsilon C_p P(t). \end{aligned}$$

It can be easily noticed that

$$\begin{aligned} &|w(t) - u(t)| = |w(t) - w(t) + w(t) - u(t)| \\ &\leq \left| w(t) - \chi(w) - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left(f(s,\ w(s-\tau(s)),\ \int_a^s G(s,\ p,\ w(p))dp,\ \int_a^b H(s,\ p,\ w(p))dp \right) ds \right| \\ &+ \left| w(t) - \chi(u) - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left(f(s,\ u(s-\tau(s)),\ \int_a^s G(s,\ p,\ u(p))dp,\ \int_a^b H(s,\ p,\ u(p))dp \right) ds \right| \end{aligned}$$

Hence,

$$\|w(t) - u(t)\| \leq \varepsilon C_p P(t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} E(s) |w-u| ds.$$

It directly follows from Pachpatte's lemma Eqs. (12)–(13) that

$$\|w(t) - u(t)\| \leq \varepsilon C P(t), \tag{35}$$

for C > 0 which ends the proof.

Let us extend our results to asymptotically stable solution. For that, we shall perform the absolute value for the solution of (14)

$$|u(t)| \leq \left| \chi(u) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left(f(s,\ u(s-\tau(s)),\ \int_a^s G(s,\ p,\ u(p))dp,\ \int_a^b H(s,\ p,\ u(p))dp \right) ds \right|$$

In fact, by means of Cauchy Schwartz inequality, we deduce

$$|u(t)| \leq |\chi(u)| + \frac{\sqrt{TV}}{\Gamma(\alpha)}.$$

where

$$T = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{2\alpha-2} ds,\ T^{\frac{1}{2}} = \frac{t^{\alpha-0.5}}{\Gamma(1-2\alpha)},$$

$$V = \int_0^t \left| f(s,\ u(s-\tau(s)),\ \int_a^s G(s,\ p,\ u(p))dp,\ \int_a^b H(s,\ p,\ u(p))dp) \right|^2 ds$$

Now, we observe that $|u(t)| \to 0$ whenever $t \to \infty$. Therefore, the zero solution of (14) is said to be asymptotically stable.

3 Some Important Properties of the Solution

There is no doubt that there are various properties that characterize solutions. So, we will look at two features, namely continuous dependence of solution and estimates on the solution.

Theorem 3.1. For the two solutions $u_1(t)$, $u_2(t)$ of Eq. (14).

$|u_1(t) - u_2(t)| \leq \int_0^t A(t)ds + |\chi_1(u) - \chi_2(u)|$, where $A(t) = |u_1(t) - u_2(t)|\ R(t-s)^{\alpha-1}|$. Then Eq. (14) depends continuously on the solution.

Proof. See proof of Theorem 4.1.

Theorem 3.2. Assume that the function f in Eq. (14) is Lipschitz function. If u(t) is a solution of Eq. (14), then

$|u(t)| \leq \int_0^t B(t)ds + |\chi(u)|$ where $B(t) = |u(t)|\ R(t-s)^{\alpha-1}$.

Proof. Also, the proof is similar to the proof of Theorem 4.1.

4 Modified Variational Iteration Method with Adomain Decomposition Method

The ongoing method is distinct and the results improve quickly. We will create the correct functional in the following form:

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(s) \left[D^\alpha u(s) - f\left(s,\ E(u(s-\tau(s)),\ \int_a^s G(s,\ x,\ u(x))dx,\ \int_a^b H(s,\ x,\ u(x))dx \right) \right] ds, \quad (36)$$

where λ is a Lagrange multiplier. The solution is defined by the infinite series

$$u(t) = \sum_{i=1}^{\infty} u^{(i)}(t). \quad (37)$$

The nonlinear function can be written as [16,24]

$$A_j(t) = \frac{1}{\Gamma(k+1)} \frac{d^k}{d\lambda^k} E(\sum_{j=1}^{\infty} \lambda^j u_j), \quad (38)$$

$$B_j(t) = \frac{1}{\Gamma(k+1)} \frac{d^k}{d\lambda^k} G(\sum_{j=1}^{\infty} \lambda^j u_j), \quad (39)$$

$$C_j(t) = \frac{1}{\Gamma(k+1)} \frac{d^k}{d\lambda^k} H(\sum_{j=1}^{\infty} \lambda^j u_j), \quad (40)$$

Since A_n, B_n and C_n are the Adomain polynomials of $u_0, u_1, \ldots, u_n$. Substitute (37)–(40) in (36), we have

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(s)\left[D^\alpha u_n(s) - f\left(s,\ \sum_{n=0}^{\infty} A_n,\ \int_a^s \sum_{n=0}^{\infty} B_n dx,\ \int_a^b \sum_{n=0}^{\infty} C_n dx\right)\right] ds, \tag{41}$$

For $\lambda = -1$, with another formula

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(s)\left[I^\alpha D^\alpha u_n(s) - I^\alpha f\left(s,\ \sum_{n=0}^{\infty} A_n,\ \int_a^s \sum_{n=0}^{\infty} B_n dx,\ \int_a^b \sum_{n=0}^{\infty} C_n dx\right)\right] ds, \tag{42}$$

5 Experimental and Numerical Examples

Here, we give examples (application situations for the applied fractional equation) which clarifying the gained results.

5.1 Illustrative Examples

In this subsection, we shall present the numerical results gained by employing iterative methods namely modified variational iteration method with Adomain decomposition method

Example 1

$$\begin{cases} D^\alpha u(t) = u^2(t - \tau(t)) + \dfrac{2t^{2-\alpha}}{\Gamma(3-\alpha)} - \dfrac{1}{4}\displaystyle\int_1^t (u(s))^{2-\alpha} ds - \int_{-1}^1 (u(s))^\alpha ds, \\ u(0) = 0,\ \tau(t) = 0.5\, t. \end{cases} \tag{43}$$

Solution

$$u_{n+1}(t) = I^\alpha u^2(t - \tau(t)) + I^\alpha \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} - \frac{1}{4} I^\alpha \int_1^t (u(s))^{2-\alpha} ds - I^\alpha \int_{-1}^1 (u(s))^\alpha ds \tag{44}$$

$$u_{n+1}(t) = I^\alpha \sum_{n=0}^{\infty} A_n + I^\alpha \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} - \frac{1}{4} I^\alpha \int_1^t \sum_{n=0}^{\infty} B_n ds - I^\alpha \int_{-1}^1 \sum_{n=0}^{\infty} C_n ds \tag{45}$$

For n = 0

$$u_1(t) = I^\alpha A_0 + I^\alpha \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} - \frac{1}{4} I^\alpha \int_1^t B_0 ds - I^\alpha \int_{-1}^1 C_0 ds. \tag{46}$$

By Adomain decomposition method, $\alpha = 0.5$, we get $A_0 = B_0 = C_0 = 0$. Hence $u_1 = t^2$ is also the exact (analytical) solution. Exactly, $u_0 = u_2 = \ldots = 0$.

Example 2

$$\begin{cases} D^\alpha u(t) = u(t - \tau(t)) + \dfrac{t^{1-\alpha}}{\Gamma(2-\alpha)} - \displaystyle\int_0^t \frac{u(s)}{t} ds - \int_{-1}^1 \frac{u(s) ds}{4}, \\ u(0) = 0,\ \tau(t) = 0.5\, t. \end{cases} \tag{47}$$

Solution

$$u_{n+1}(t) = I^{\alpha}u(t-\tau(t)) + I^{\alpha}\frac{t^{1-\alpha}}{\Gamma(2-\alpha)} - I^{\alpha}\int_{0}^{t}\frac{u(s)}{t}ds - I^{\alpha}\int_{-1}^{1}\frac{u(s)ds}{4} \tag{48}$$

By the same procedures of the previous example, we have $u_1 = t$.

5.2 Graphical Representation of Solution for Eq. (43)

Firstly, let α approach to 0.5 and n = 1, the obtained solution for this case represent graphically in Fig. 1.

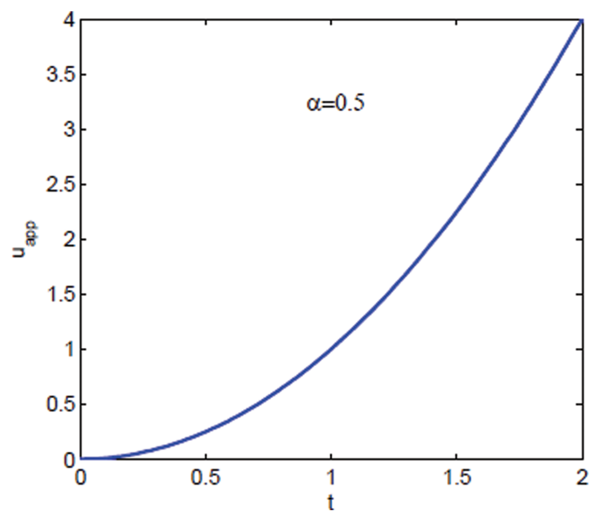

Figure 1: Approximate solution at α = 0.5

Finally, let α approach to 1 and n = 1, 2, 3. Approximate solutions for this case is obtained in Fig. 2.

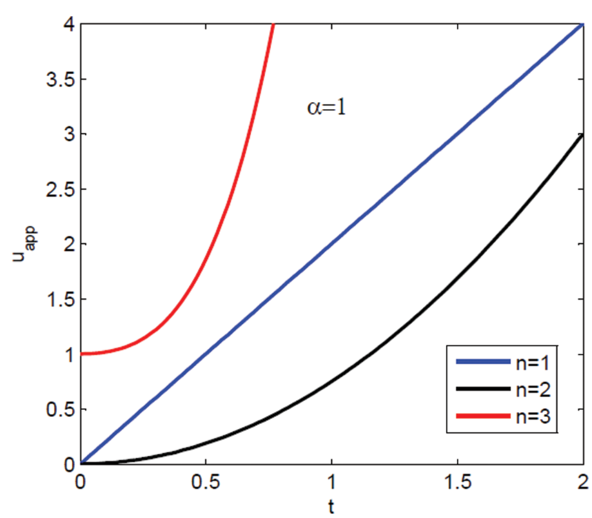

Figure 2: Approximate solutions at α = 1

Table 1 shows the analysis results for Eq. (43).

Table 1: Results analysis for Eq. (43)

Values of t	α = 0.5			α = 1		
	$u_{app} = t^2$ n = 1	$u_{app} = 0$ n = 2	$u_{app} = 0$ n = 3	$u_{app} = 2t$ n = 1	$u_{app} = \frac{3}{4}t^2$ n = 2	$u_{app} = \frac{9}{4}t^4 + \frac{13}{4}t^3 + \frac{5}{4}t^2 + 1$ n = 3
0	0	0	0	0	0	1
0.5	0.25	0	0	1	0.1875	1.859375
1	1	0	0	2	0.75	7.75
1.5	2.25	0	0	3	1.6875	26.171875
2	4	0	0	4	3	68

Figs. 1–2 indicate the difference to u_{app} at different values of n.

Fig. 1 is the relationship between t and the approximate solution u_{app} at $\alpha = 0.5$, n = 1 only, otherwise, at other values of n we find that the approximate solution approaches zero.

Fig. 2 is the relationship between t and the approximate solution u_{app} at $\alpha = 1$, n = 1, 2, 3.

6 Conclusion

We emphasize that the analysis of fractional integro-differential equations with delay attracts considerable attention by many scientists [29,30]. The beneficial contribution of this work was the discovery much of the tools in the analysis process for many equations. It should be noted that the choice of α plays a vital role in the results of the suggested problem and this was evident in the examples that were listed in our research. We find that when the approximate solution and the exact solution apply when a value of α is at a certain value of n, and other than this value of n, the approximate solution is 0.

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