

## Axisymmetric longitudinal wave propagation in a finite pre-strained compound circular cylinder made from compressible materials

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**Abstract:** The axisymmetric longitudinal wave propagation in a finite pre-strained compound (composite) cylinder is investigated within the scope of a piecewise homogeneous body model utilizing three-dimensional linearized theory wave propagation in an initially stressed body. The materials of the inner and outer cylinder are assumed to be compressible. The elasticity relations for those are given through the harmonic potential. The algorithm for constructing of the computer programmes and obtaining numerical results is discussed. The numerical results regarding the influences of the initial strains in the inner and outer cylinders on the wave dispersion are presented and analysed. These results are obtained for the case where the material of the inner solid cylinder is stiffer than that of the outer hollow cylinder. In particular, it is established that the initial stretching of the cylinders causes the wave propagation velocity to increase.

**Keywords:** Compound cylinder, finite initial strain, wave dispersion, compressible material

### 1 Introduction

Elastodynamic problems are continuously arising in almost all areas of natural sciences and engineering. As time goes by, these problems have increasingly attracted the attention of various fundamental and applied areas of science. In the case discussed herein, the intensive development of some fields of the dynamics of the deformed bodies was stimulated by the engineering requirements of certain key

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industries. Accordingly, in the second half of the 20<sup>th</sup> century, the study of the nonlinear elastodynamic problems became urgent.

An interesting and urgent problem, which also applies to the nonlinear dynamical effects in the elastic medium, is the elastodynamic problems for initially stressed bodies. Initial stresses occur in structural elements during their manufacture and assembly, in the Earth's crust under the action of geostatic and geodynamic forces, in composite materials, etc.

The initial strains (stresses) can be also classified as one of the reference particularities of the body within which the various types wave propagations occur. At the same time, there are also other types reference particularities, such as the existence of the crack in the mentioned body which also influence significantly its dynamics: see, for example Guz, Menshikov, Zozulya and Guz (2007), Guz and Zozulya (2007).

At present, the theory of elastodynamics for initially stressed bodies is currently meant the linearized theory of the elastodynamics for the initially stressed bodies constructed using the linearization principle from the general nonlinear theory of elasticity or its simplified modifications. Under certain conditions, linearized equations make it possible to investigate all kinds of dynamical problems for initially stressed bodies. In this case it is necessary to distinguish the so called approximate and exact approaches. The approximate approaches are based on the Bernoulli, Kirchoff-Love and Timoshenko hypotheses and other methods of reducing three-dimensional (two-dimensional) problems to two-dimensional (one-dimensional) ones. It is evident that the approximate approaches simplify the mathematics involved in finding a solution. However, in many cases the results obtained by employing these approaches may not be acceptable in the qualitative and quantitative senses. For example, the applied theories of rods, plates and shells describe only a few propagating waves (modes). Moreover, within the framework of these approaches the near-surface dynamical processes for the initially stressed bodies cannot be described. Therefore the use of the exact approach is preferable; i.e., the so called Three-dimensional Linearized Theory of Elastic Waves in Initially Stressed Bodies (TLTEWISB) for investigations of the dynamical problems of elastic bodies with initial stresses. The general problems of the TLTEWISB have been elaborated in many investigations such as Biot (1965), Green, Rivlin and Shield (1952), Eringen and Suhubi (1975a), Guz (1986a, 1986b, 2004), Truestell (1961) and etc.

Almost all the investigations carried out by employing TLTEWISB, (except Akbarov (2006a, 2006b, 2006c, 2006d, 2007a), Yahnioğlu (2007) and some others listed therein) refer to the influence of the initial stresses on the speed and the dispersion of various types of waves; see, for example, papers Hayes and Rivlin

(1961), Chadwick and Jarvis(1979a, 1979b), Dowaik and Ogden (1991), Ogden and Sotiropoulos (1998), Fu and Mielke (2002), Daniel (2008), Akbarov and Guz (2004), Akbarov and Ozisik (2003, 2004), Rogerson and Sandiford (2000), Zhuk and Guz (2007), Guz, Rushchitsky and Guz (2007), Guz, Rushchitsky and Guz (2008) and the papers listed therein. Reviews of these investigations were considered in papers Akbarov (2007b), Guz (2002), Guz and Makhort (2000); moreover, a systematic analysis of these investigations was given in monographs Guz (1986a, 1986b, 2004). It follows from these references that a considerable part of the investigations refer to layered composite materials. Also there are a considerable number of investigations on the wave propagation in the pre-stressed cylinders Belward (1976), Demiray and Suhubi (1970), Green (1961), Guz, Kushnir and Makhort (1975), Kushnir (1979) and others. However, in these investigations the subject of research was a homogeneous circular cylinder. Consequently, up to now, investigations on the wave propagation in the pre-stressed compound (composite) cylinder are almost completely absent. One notable exception was an investigation on the axisymmetric longitudinal wave propagation in the compound cylinder was made in Akbarov and Guz (2004). Yet, in this investigation it was assumed that the materials of the cylinders are moderately rigid and the initial strains in them are small; i.e. these strains can be neglected with respect to unity in the corresponding equations and relations of the TLTEWISB. As a result of the foregoing assumptions in paper Akbarov and Guz (2004) it was concluded that the effect of the initial stresses (i.e. the initial uniaxial homogeneous stresses the values of which are less than the corresponding yield stresses) on the wave propagation velocity in compound cylinders is insignificant. The mentioned differences between the present and the other (Akbarov and Guz (2004)) investigations will also be noted and detailed in the subsequent sections.

It should be noted that within elastic deformations, a considerable effect of the initial strains or stresses on the wave propagation in the body can be attained under finite initial strains which usually takes place in rubber-like high elastic materials. Taking this statement into account in the present paper, the axisymmetric longitudinal wave propagation in a compound cylinder with finite initial strains is studied. It is assumed that the materials of the constituents are high elastic compressible ones and the elasticity relations of those are given through the harmonic potential. The numerical results regarding the influence of the initial strains on the wave dispersion are presented and analyzed.

## **2 Formulation of the problem**

We consider the compound (composite) circular cylinder shown in Fig. 1 and assume that in the natural state the radius of the inner solid cylinder is  $R$ , the thickness

of the external hollow cylinder is  $h$ .

In the natural state we determine the position of the points of the cylinders by the Lagrangian coordinates in the Cartesian system of coordinates  $Oy_1y_2y_3$  as well as in the cylindrical system of coordinates  $Or\theta y_3$ .

Assume that the cylinders have infinite length in the direction of the  $Oy_3$  axis and the initial stress state in each component of the considered body is axisymmetric with respect to this axis and homogeneous. Such a stress field may be present with stretching of the considered body along the  $Oy_3$  axis.

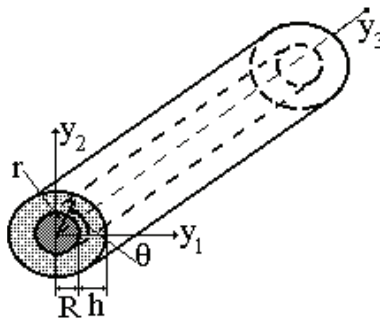


Figure 1: The geometry of the compound cylinder

The stretching may be conducted for the inner solid cylinder and the external hollow cylinder separately before they are compounded. However, the results which will be discussed below can also be related to the case where the inner solid and hollow external cylinders are stretched together after the compounding. In this case as a result of the difference of Poisson's coefficients of the inner and external cylinders' materials the inhomogeneous initial stresses acting on the areas which are parallel to the  $Oy_3$  axis arise. Nevertheless, according to the well known mechanical consideration, the mentioned inhomogeneous initial stresses can be neglected under consideration, because these stresses are less significant than those acting on the areas which are perpendicular to the  $Oy_3$  axis.

With the initial state of the cylinders we associate the Lagrangian cylindrical system of coordinates  $O'r'\theta'y'_3$  and the Cartesian system of coordinates  $O'y'_1y'_2y'_3$ . The values related to the inner solid cylinder and external hollow cylinder will be denoted by the upper indices (2) and (1), respectively. Furthermore, we denote the values related to the initial state by an additional upper index, 0. Thus, the initial strain state in the inner solid cylinder and external hollow cylinder can be

determined as follows.

$$u_{\mu}^{(k),0} = (\lambda_{\mu}^{(k)} - 1)y_m, \quad \lambda_1^{(k)} = \lambda_2^{(k)} \neq \lambda_3^{(k)}, \quad \lambda_{\mu}^{(k)} = \text{const}, \quad m = 1, 2, 3; \quad k = 1, 2, \quad (1)$$

where  $u_{\mu}^{(k),0}$  is a displacement and  $\lambda_{\mu}^{(k)}$  is the elongation along the  $Oy_m$  axis. We introduce the following notation

$$y'_i = \lambda_i^{(k)} y_i, \quad \rho' = \lambda_1^{(k)} \rho, \quad R' = \lambda_1^{(2)} R. \quad (2)$$

The values related to the system of the coordinates associated with the initial state below, i.e. with  $O'y'_1y'_2y'_3$ , will be denoted by upper prime.

Within this framework, let us investigate the axisymmetric wave propagation along the  $O'y'_3$  axis in the considered body. We make this investigation by the use of coordinates  $\rho'$  and  $y'_3$  in the framework of the TLTEWISB. We will follow the style and notation used in the monograph Guz (2004).

Thus, we write the basic relations of the TLTEWISB for the compressible body under an axisymmetrical state. These relations are satisfied within each constituent of the considered body because we use the piecewise homogeneous body model.

The equations of motion are

$$\begin{aligned} \frac{\partial}{\partial \rho'} Q'_{\rho'\rho'}^{(k)} + \frac{\partial}{\partial y'_3} Q'_{r'3}^{(k)} + \frac{1}{\rho'} \left( Q'_{\rho'\rho'}^{(k)} - Q'_{\theta'\theta'}^{(k)} \right) &= \rho'^{(k)} \frac{\partial^2}{\partial t^2} u'_{r'}^{(k)}, \\ \frac{\partial}{\partial \rho'} Q'_{3\rho'}^{(k)} + \frac{\partial}{\partial y'_3} Q'_{33}^{(k)} + \frac{1}{\rho'} Q'_{3\rho'}^{(k)} &= \rho'^{(k)} \frac{\partial^2}{\partial t^2} u'_3{}^{(k)}. \end{aligned} \quad (3)$$

The mechanical relations are

$$\begin{aligned} Q'_{\rho'\rho'}^{(k)} &= \omega'_{1111} \frac{\partial u'_{\rho'}^{(k)}}{\partial \rho'} + \omega'_{1122} \frac{u'_{\rho'}^{(k)}}{\rho'} + \omega'_{1133} \frac{\partial u'_3{}^{(k)}}{\partial y'_3}, \\ Q'_{\theta'\theta'}^{(k)} &= \omega'_{2211} \frac{\partial u'_{\rho'}^{(k)}}{\partial \rho'} + \omega'_{2222} \frac{u'_{\rho'}^{(k)}}{\rho'} + \omega'_{2233} \frac{\partial u'_3{}^{(k)}}{\partial y'_3}, \\ Q'_{33} &= \omega'_{3311} \frac{\partial u'_{\rho'}^{(k)}}{\partial \rho'} + \omega'_{3322} \frac{u'_{\rho'}^{(k)}}{\rho'} + \omega'_{3333} \frac{\partial u'_3{}^{(k)}}{\partial y'_3}, \\ Q'_{\rho'3} &= \omega'_{1313} \frac{\partial u'_{\rho'}^{(k)}}{\partial y'_3} + \omega'_{1331} \frac{\partial u'_3{}^{(k)}}{\partial \rho'}, \quad Q'_{3\rho'} = \omega'_{3113} \frac{\partial u'_{\rho'}^{(k)}}{\partial y'_3} + \omega'_{3131} \frac{\partial u'_3{}^{(k)}}{\partial \rho'}. \end{aligned} \quad (4)$$

In (3) and (4) through the  $Q'_{\rho'\rho'}^{(k)}, \dots, Q'_{3\rho'}^{(k)}$  the perturbation of the components of Kirchoff stress tensor are denoted. The notation  $u'_{\rho'}^{(k)}, u'_{3}^{(k)}$  shows the perturbations of the components of the displacement vector. The constants  $\omega'_{1111}^{(k)}, \dots, \omega'_{3333}^{(k)}$  in (3), (4) are determined through the mechanical constants of the inner and outer cylinders' materials and through the initial stress state.  $\rho'^{(k)}$  is a density of the  $k$ -th material.

As it has been noted above, in the present investigation we assume that the elasticity relations of the cylinders' materials are described by harmonic potential. This potential is given as follows:

$$\Phi = \frac{1}{2}\lambda s_1^2 + \mu s_2 \tag{5}$$

where

$$s_1 = \sqrt{1+2\varepsilon_1} + \sqrt{1+2\varepsilon_2} + \sqrt{1+2\varepsilon_3} - 3,$$

$$s_2 = \left(\sqrt{1+2\varepsilon_1} - 1\right)^2 + \left(\sqrt{1+2\varepsilon_2} - 1\right)^2 + \left(\sqrt{1+2\varepsilon_3} - 1\right)^2. \tag{6}$$

In relation (6)  $\lambda, \mu$  are material constants,  $\varepsilon_i (i = 1, 2, 3)$  are the principal values of the Green's strain tensor. The expressions (5) and (6) are supplied by the corresponding indices under solution procedure.

For the considered axisymmetric case the components of the Green's strain tensor are determined through the components of the displacement vector by the following expressions:

$$\varepsilon_{\rho\rho} = \frac{\partial u_\rho}{\partial \rho} + \frac{1}{2} \left(\frac{\partial u_\rho}{\partial \rho}\right)^2 + \frac{1}{2} \left(\frac{\partial u_3}{\partial \rho}\right)^2, \quad \varepsilon_{\theta\theta} = \frac{u_\rho}{\rho} + \frac{1}{2} \left(\frac{u_\rho}{\rho}\right)^2,$$

$$\varepsilon_{r3} = \frac{1}{2} \left(\frac{\partial u_3}{\partial \rho} + \frac{\partial u_\rho}{\partial y_3} + \frac{\partial u_\rho}{\partial \rho} \frac{\partial u_\rho}{\partial y_3} + \frac{\partial u_3}{\partial \rho} \frac{\partial u_3}{\partial y_3}\right),$$

$$\varepsilon_{33} = \frac{\partial u_3}{\partial y_3} + \frac{1}{2} \left(\frac{\partial u_3}{\partial \rho}\right)^2 + \frac{1}{2} \left(\frac{\partial u_3}{\partial y_3}\right)^2. \tag{7}$$

In this case the components  $S_{ij}$  of the Lagrange stress tensor are determined as follows:

$$S_{\rho\rho} = \frac{\partial \Phi}{\partial \varepsilon_{\rho\rho}}, \quad S_{\theta\theta} = \frac{\partial \Phi}{\partial \varepsilon_{\theta\theta}}, \quad S_{33} = \frac{\partial \Phi}{\partial \varepsilon_{33}}, \quad S_{\rho 3} = \frac{\partial \Phi}{\partial \varepsilon_{\rho 3}}, \quad S_{\rho 3} = S_{3\rho}. \tag{8}$$

Note that the expressions (6)-(8) are written in the arbitrary system of cylindrical coordinate system without any restriction related to the association of this system to the natural or initial state of the considered compound cylinders.

For the considered case the relations between the perturbation of the Kirchoff stress tensor and the perturbation of the components of the Lagrange stress tensor can be written as follows:

$$\begin{aligned} Q'_{\rho'\rho'} &= \lambda_1^{(k)} S_{\rho'\rho'}^{(k)}, & Q'_{\theta'\theta'} &= \lambda_1^{(k)} S_{\theta'\theta'}^{(k)}, & Q'_{33} &= \left(\lambda_3^{(k)}\right)^2 S_{33}^{(k)} + \lambda_3^{(k)} S_{33}^{(k),0} \frac{\partial u_3^{(k)}}{\partial y_3}, \\ Q'_{\rho'3} &= \left(\lambda_1^{(k)}\right)^{-1} S_{\rho'3}^{(k)}, & Q'_{3\rho'} &= \left(\lambda_1^{(k)}\right)^{-1} S_{3\rho'}^{(k)} + \lambda_3^{(k)} S_{33}^{(k),0} \frac{\partial u_{\rho'}^{(k)}}{\partial y_3}. \end{aligned} \quad (9)$$

According to Guz (2004), by linearization of equation (8) and taking (9) and (1) into account, we obtain the following expressions for the stress  $S_{33}^{(k),0}$  and for the constants  $\lambda_2^{(k)}$ ,  $\lambda_1^{(k)}$ ,  $\omega'_{1111}$ ,  $\dots$ ,  $\omega'_{3333}$  in (4) for the potential (5):

$$\begin{aligned} S_{33}^{(k),0} &= \left[ \lambda^{(k)} \left( 2\lambda_1^{(k)} + \lambda_3^{(k)} - 3 \right) + 2\mu^{(k)} \left( \lambda_3^{(k)} - 1 \right) \right] \left( \lambda_3^{(k)} \right)^{-1}, \\ \lambda_2^{(k)} = \lambda_1^{(k)} &= \left[ 2 - \frac{\lambda^{(k)}}{\mu^{(k)}} \left( \lambda_3^{(k)} - 3 \right) \right] \left[ 2 \left( \frac{\lambda^{(k)}}{\mu^{(k)}} + 1 \right) \right]^{-1}, \\ \omega'_{1111} &= \left( \lambda_3^{(k)} \right)^{-1} \left( \lambda^{(k)} + 2\mu^{(k)} \right), & \omega'_{3333} &= \left( \frac{\lambda_3^{(k)}}{\lambda_1^{(k)}} \right)^2 \left( \lambda^{(k)} + 2\mu^{(k)} \right), \\ \omega'_{1122} &= \left( \lambda_3^{(k)} \right)^{-1} \lambda^{(k)}, & \omega'_{1133} &= \left( \lambda_1^{(k)} \right)^{-1} \lambda^{(k)}, & \omega'_{1221} &= \left( \lambda_3^{(k)} \right)^{-1} \mu^{(k)}, \\ \omega'_{1313} &= 2\mu^{(k)} \left( \lambda_1^{(k)} + \lambda_3^{(k)} \right)^{-1}, & \omega'_{3113} &= 2\mu^{(k)} \left( \lambda_1^{(k)} \right)^{-2} \left( \lambda_3^{(k)} \right)^2 \left( \lambda_1^{(k)} + \lambda_3^{(k)} \right)^{-1}. \end{aligned} \quad (10)$$

Consequently, according to the relations (10), in the present investigations the initial stress-strain state is determined within the scope of the nonlinear theory of elasticity, but in Akbarov and Guz (2004) the initial strain state was determined within the scope of the classical linear theory of elasticity. Namely this statement causes the main distinguish between the results of the present and Akbarov and Guz (2004) investigations.

Thus, the wave propagation in the considered body will be investigated by the use of the equations (3), (4) and (10). In this case we will assume that the following complete contact conditions are satisfied.

$$Q'_{\rho'\rho'}|_{\rho'=R'} = Q'_{\rho'\rho'}|_{\rho'=R'}, \quad Q'_{\rho'3}|_{\rho'=R'} = Q'_{\rho'3}|_{\rho'=R'},$$

$$u'_{\rho'}|_{\rho'=R'} = u'_{\rho'}|_{\rho'=R'}, \quad u'_{3}|_{\rho'=R'} = u'_{3}|_{\rho'=R'}. \tag{11}$$

With this we exhaust the formulation of the problem. It should be noted that in the case where  $\lambda_3^{(k)} = \lambda_1^{(k)} = 1.0, (k = 1, 2)$  the above described formulation transforms to the corresponding one of the classical linear theory of the elastodynamics for the compressible body.

### 3 Solution procedure

Up to now, various types of numerical and semi-analytical methods have been developed to solve the dynamical problems of deformable solid body mechanics. The present level of these methods are described, for example, in papers by Yoda and Kodama (2006), Lu and Zhu (2007), Chen, Fu and Zhang (2007), Gato and Shie (2008), Liu, Chen, Li and Cen (2008), Lin, Lee, Tsai, Chen, Wang and Lee (2008), Wang and Wang (2008) and in many others. Note that all of these methods are realized by employing modern computer modelling. However, there are also other methods, so called analytical + numerical methods, according to which, up to a certain stage of the solution procedure, analytical expressions are obtained for the sought values, but after this stage procedures based on visual numerical results arrived at with modern PC modelling are also employed. In the present paper, the latest version of computer modelling is employed.

Thus, according to Guz (2004), we use the following representation for the displacement:

$$u'_{\rho'} = -\frac{\partial^2}{\partial \rho' \partial y_3} X^{(k)},$$

$$u'_{3} = \frac{1}{\omega'_{1133} + \omega'_{1313}} \left( \omega'_{1111} \Delta'_1 + \omega'_{3113} \frac{\partial^2}{\partial y_3^2} - \rho'^{(k)} \frac{\partial^2}{\partial t^2} \right) X^{(k)}, \tag{12}$$



where  $X^{(k)}$  satisfies the following equation:

$$\left[ \left( \Delta'_1 + \left( \xi'_{2,3} \right)^2 \frac{\partial^2}{\partial y_3'^2} \right) \left( \Delta'_1 + \left( \xi'_3 \right)^2 \frac{\partial^2}{\partial y_3'^2} \right) - \rho'^{(k)} \left( \frac{\omega'_{1111} + \omega'_{1331}}{\omega'_{1111} \omega'_{1331}} \Delta'_1 + \frac{\omega'_{3333} + \omega'_{3113}}{\omega'_{1111} \omega'_{1331}} \frac{\partial^2}{\partial y_3'^2} \right) \frac{\partial^2}{\partial t^2} + \frac{\rho'^{(k)}}{\omega'_{1111} \omega'_{1331}} \frac{\partial^4}{\partial t^4} \right] X^{(k)} = 0. \quad (13)$$

In (12) and (13) the following notation is used.

$$\Delta'_1 = \frac{d^2}{d\rho'^2} + \frac{1}{\rho'} \frac{d}{d\rho'},$$

$$\left( \xi'_{2,3} \right)^2 = d^{(k)} \pm \left[ \left( d^{(k)} \right)^2 - \omega'_{3333} \omega'_{3113} \left( \omega'_{1111} \omega'_{1331} \right)^{-1} \right]^{\frac{1}{2}}$$

$$d^{(k)} = \left( 2\omega'_{1111} \omega'_{1331} \right)^{-1} \left[ \omega'_{1111} \omega'_{3333} + \omega'_{1331} \omega'_{3113} - \left( \omega'_{1133} + \omega'_{1313} \right) \right]. \quad (14)$$

We represent the function  $X^{(m)} = X^{(m)}(\rho', y_3', t)$  as

$$X^{(m)} = X_1^{(m)}(\rho') \cos(ky_3' - \omega t), \quad m = 1, 2. \quad (15)$$

Substituting (15) in (13) and doing some mathematical manipulations we obtain the following equation for  $X_1^{(m)}(\rho')$ :

$$\left( \Delta'_1 + \left( \zeta'^{(m)}_{2,3} \right)^2 \right) \left( \Delta'_1 + \left( \zeta'^{(m)}_3 \right)^2 \right) X_1^{(m)}(\rho') = 0. \quad (16)$$

The constants  $\zeta'^{(k)}_{2,3}$  are determined from the following equation:

$$\begin{aligned} & \omega'_{1111} \omega'_{1331} \left( \zeta'^{(m)} \right)^4 - k^2 \left( \zeta'^{(m)} \right)^2 \left[ \omega'_{1111} \left( \rho^{(m)} c^2 - \omega'_{3333} \right) + \right. \\ & \left. \omega'_{1331} \left( \rho^{(m)} c^2 - \omega'_{3113} \right) + \left( \omega'_{1133} + \omega'_{1313} \right)^2 \right] + \\ & k^4 \left( \rho^{(m)} c^2 - \omega'_{3333} \right) \left( \rho^{(m)} c^2 - \omega'_{3113} \right) = 0, \end{aligned} \quad (17)$$

where  $c = \omega/k$ , i.e.  $c$  is the phase velocity of the propagating wave. We determine the following expression for  $X_1^{(m)}(\rho')$  from equations (16) and (17).

$$X_1^{(1)}(\rho') = B_2^{(1)} E_0^{(1)}(k\rho' \zeta_2^{(1)}) + B_3^{(1)} E_0^{(1)}(k\rho' \zeta_3^{(1)}),$$

$$X_1^{(2)}(\rho') = A_2^{(2)} G_0^{(2)}(k\rho' \zeta_2^{(2)}) + A_3^{(2)} G_0^{(2)}(k\rho' \zeta_3^{(2)}) + B_2^{(2)} E_0^{(2)}(k\rho' \zeta_2^{(2)}) + B_3^{(2)} E_0^{(2)}(k\rho' \zeta_3^{(2)}), \quad (18)$$

where

$$E_0^{(1)}(k\rho' \zeta_\mu^{(1)}) = \begin{cases} J_0(k\rho' \zeta_\mu^{(1)}) & \text{if } (\zeta_\mu^{(1)})^2 > 0, \\ I_0(k\rho' |\zeta_\mu^{(1)}|) & \text{if } (\zeta_\mu^{(1)})^2 < 0, \end{cases} \quad (19)$$

$$G_0^{(2)}(k\rho' \zeta_\mu^{(2)}) = \begin{cases} J_0(k\rho' \zeta_\mu^{(2)}) & \text{if } (\zeta_\mu^{(2)})^2 > 0, \\ I_0(k\rho' |\zeta_\mu^{(2)}|) & \text{if } (\zeta_\mu^{(2)})^2 < 0. \end{cases}$$

$$E_0^{(2)}(k\rho' \zeta_\mu^{(2)}) = \begin{cases} Y_0(k\rho' \zeta_\mu^{(2)}) & \text{if } (\zeta_\mu^{(2)})^2 > 0, \\ K_0(k\rho' |\zeta_\mu^{(2)}|) & \text{if } (\zeta_\mu^{(2)})^2 < 0. \end{cases} \quad (20)$$

In (19) and (20)  $J_0(x)$  and  $Y_0(x)$  are Bessel functions of the first and second kind of order zero;  $I_0(x)$  and  $K_0(x)$  are Bessel function of a purely imaginary argument in order zero and Macdonald function in order zero, in turn.

Thus, using the expressions (15), (18)-(20), (12), (10), (9), (4) we obtain the dispersion equation

$$\det \|\alpha_{ij}\| = 0, \quad i, j = 1, 2, 3, 4, 5, 6 \quad (21)$$

from (11), where

$$\alpha_{ij} = \alpha_{ij}(c/c_2^{(2)}, kR, \mu^{(2)}/\mu^{(1)}, \lambda^{(2)}/\mu^{(2)}, \lambda^{(1)}/\mu^{(1)}, \lambda_3^{(2)}, \lambda_3^{(1)}) \quad (22)$$

To reduce the size of the article we do not give here the explicit expressions of  $\alpha_{ij}$ . Thus the dispersion equation for the considered wave propagation problem has been derived in the form (21) and (22).

#### 4 Numerical results and discussions

In the present section instead of the upper indices (1) and (2) we will use the upper indices (m) and (f) respectively. Assume that  $\mu^{(f)}/\mu^{(m)} = 2.0$ ;  $\rho^{(f)}/\rho^{(m)} = 1.0$ ,  $\lambda^{(f)}/\mu^{(f)} = \lambda^{(m)}/\mu^{(m)} = 1.0$ ,  $h/R = 1.0$  and consider the dispersion curves  $c = c(kR)$  and analyze the influence of the elongation parameters  $\lambda_3^{(f)}$  and  $\lambda_3^{(m)}$  on these curves.

Note that under obtaining numerical results the dispersion equation (21) is solved numerically by employing the well known “bisection” method. The corresponding PC programmes of the algorithm are composed by the authors. For the considered problem the solution procedure of the equation (21) is carried out in the following manner. For each value of  $kR$  ( $k$  is a wave number,  $R$  is a radius of the inner solid cylinder) the  $n$  ( $n \geq 5$ ) subsequent roots (denoted by  $(c/c_2^{(f)0})_1 < \dots < (c/c_2^{(f)0})_n$ , where  $c_2^{(f)0} = \sqrt{\mu^{(f)}/\rho^{(f)}}$ ) of the equation (21) are found. In this case the values of  $\lambda_3^{(f)}$  and  $\lambda_3^{(m)}$  are fixed. Note that for the values of  $kR$  which are very near to zero, the first three and fifth roots are

$$\left(\frac{c}{c_2^{(f)0}}\right)_1 \left(= \sqrt{\frac{\rho^{(f)}\omega'_{3113}{}^{(m)}}{\rho^{(m)}\mu^{(f)}}}\right) < \left(\frac{c}{c_2^{(f)0}}\right)_2 \left(= \sqrt{\frac{\rho^{(f)}\omega'_{3333}{}^{(m)}}{\rho^{(m)}\mu^{(f)}}}\right) < \left(\frac{c}{c_2^{(f)0}}\right)_3 \left(= \sqrt{\frac{\omega'_{3113}{}^{(f)}}{\mu^{(f)}}}\right) < \left(\frac{c}{c_2^{(f)0}}\right)_5 \left(= \sqrt{\frac{\omega'_{3333}{}^{(f)}}{\mu^{(f)}}}\right) \quad (23)$$

These roots caused by the expression of the equation (17) and determine the nondispersive wave speeds. For  $kR \rightarrow 0$ , the fourth root regards the first dispersive mode. Moreover, the roots which are between the roots (23) and the roots which are smaller than  $(c/c_2^{(f)0})_1$  or are greater than  $(c/c_2^{(f)0})_5$  determine the dispersive wave speeds and form the dispersion curves.

According to the relations given in equation (10), the roots (23) depend on the parameter  $\lambda_3^{(f)}$  and  $\lambda_3^{(m)}$ , i.e. on the initial strains. For denoting this dependence we introduce the following notation

$$\frac{c_k^{(m)0}}{c_2^{(f)0}} = \left(\frac{c_k^{(m)}}{c_2^{(f)0}}\right) \Big|_{\lambda_3^{(m)}=1.0}, \quad \frac{c_k^{(m)\alpha}}{c_2^{(f)0}} = \left(\frac{c_k^{(m)}}{c_2^{(f)0}}\right) \Big|_{\lambda_3^{(m)}=\alpha \neq 1.0},$$

$$\frac{c_k^{(f)0}}{c_2^{(f)0}} = \left(\frac{c_k^{(f)}}{c_2^{(f)0}}\right) \Big|_{\lambda_3^{(f)}=1.0}, \quad \frac{c_k^{(f)\alpha}}{c_2^{(f)0}} = \left(\frac{c_k^{(f)}}{c_2^{(f)0}}\right) \Big|_{\lambda_3^{(f)}=\alpha \neq 1.0}, \quad k = 1, 2.$$

$$c_2^{(m)}(\lambda_3^{(m)}) = \sqrt{\frac{\omega'_{3113}{}^{(m)}}{\rho^{(m)}}}, \quad c_2^{(f)}(\lambda_3^{(f)}) = \sqrt{\frac{\omega'_{3113}{}^{(f)}}{\rho^{(f)}}},$$

$$c_1^{(m)}(\lambda_3^{(m)}) = \sqrt{\frac{\omega'_{3333}{}^{(m)}}{\rho^{(m)}}}, \quad c_1^{(f)}(\lambda_3^{(f)}) = \sqrt{\frac{\omega'_{3333}{}^{(f)}}{\rho^{(f)}}}. \tag{24}$$

For consideration the trustiness and correctness of the used algorithm and programmes, as well as for consideration of some basic particularities of the influence of the initial strains in the components of the compound cylinder on the wave dispersion, first, we analyze the dispersion curves regarding the first mode, i.e. the fundamental mode.

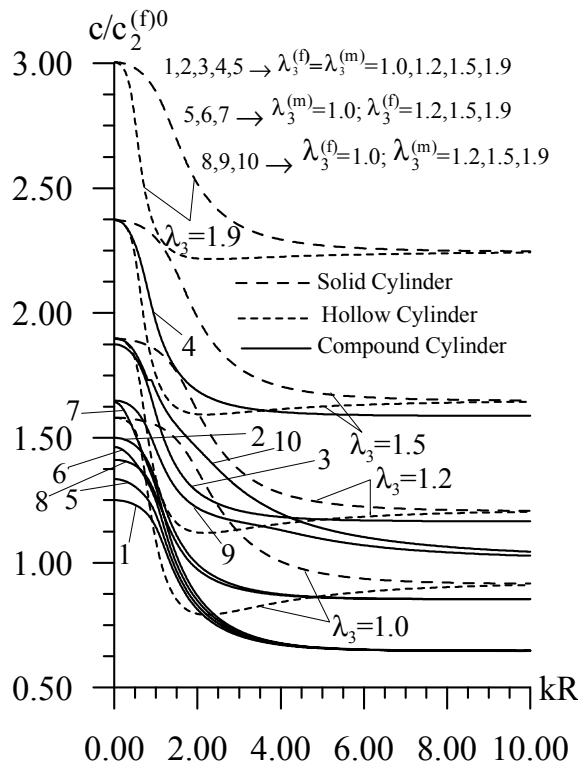


Figure 2: Dispersion curves corresponding the first mode

These curves are given in Fig. 2 for various values of  $\lambda_3^{(m)}$  and  $\lambda_3^{(f)}$ . Note that in Fig. 2 the dispersion curves corresponding to the wave propagation which takes place separately in the solid and hollow cylinders are also given. In this case it

is assumed that the materials of the hollow and solid cylinders are the same, i.e. it is assumed that the hollow cylinder is also made from the material of the solid cylinder.

We denote the wave propagation velocity for solid, hollow and compound cylinders through the notation  $c^{sc}$ ,  $c^{hc}$  and  $c^{cc}$  respectively. It follows from the graphs given in Fig. 2 that

$$\begin{aligned}
 c^{sc} &\rightarrow c_b^{(f)}(\lambda_3^{(f)}), & c^{hc} &\rightarrow c_b^{(f)}(\lambda_3^{(f)}) \text{ as } kR \rightarrow 0, \\
 c^{sc} &\rightarrow c_\rho^{(f)}(\lambda_3^{(f)}), & c^{hc} &\rightarrow c_\rho^{(f)}(\lambda_3^{(f)}) \text{ as } kR \rightarrow \infty
 \end{aligned}
 \tag{25}$$

where  $c_b^{(f)}(\lambda_3^{(f)})$  is a ‘‘bar’’ velocity,  $c_\rho^{(f)}(\lambda_3^{(f)})$  is a Rayleigh wave velocity in the pre-strained cylinder material. In this case by using the corresponding asymptotic analyses we obtain that

$$\begin{aligned}
 c_b^{(f)}(\lambda_3^{(f)}) &= \lambda_3^{(f)} c_b^{(f)0}, \\
 c_b^{(f)0} &= \sqrt{2(1 + \lambda^{(f)} / (2(\lambda^{(f)} + \mu^{(f)}))) \mu^{(f)} / \rho^{(f)}}
 \end{aligned}
 \tag{26}$$

However, we cannot write such simple analytic expression for the calculation of the values of  $c_\rho^{(f)}(\lambda_3^{(f)})$ ; the calculation of the values  $c_\rho^{(f)}(\lambda_3^{(f)})$  is made numerically through the corresponding equation given, for example, in monograph Guz (2004) and elsewhere. Consequently, the results obtained for the solid and hollow cylinders coincide with the known results Eringen and Suhubi (1975), Guz (2004) and agree with the well-known mechanical considerations.

The analyses of the dispersion curves regarding the compound cylinder (Fig. 2) show that the following asymptotic estimations for the wave propagation velocity occur.

$$\frac{c^{cc}}{c_2^{(f)0}} \rightarrow \frac{c_b^{cc}(\lambda_3^{(f)}, \lambda_3^{(m)})}{c_2^{(f)0}} = \left[ \frac{\left( e^{(f)}(\lambda_3^{(f)})^2 \eta^{(f)} + e^{(m)}(\lambda_3^{(m)})^2 \mu^{(m)} \eta^{(m)} / \mu^{(f)} \right)}{(\eta^{(f)} + \eta^{(m)} \rho^{(m)} / \rho^{(f)})} \right]^{\frac{1}{2}}$$

as  $kR \rightarrow 0$  (27)

$$\frac{c^{cc}}{c_2^{(f)0}} \rightarrow \frac{c_\rho^{(m)}(\lambda_3^{(m)})}{c_2^{(f)0}} \Big|_{\lambda_3^{(m)}=1.0} \text{ as } kR \rightarrow \infty \text{ for the case where } \lambda_3^{(m)} = 1.0, \lambda_3^{(f)} \geq 1.0$$

(28)

$$\frac{c^{cc}}{c_2^{(f)0}} \rightarrow \min \left\{ 1.0, \frac{c_\rho^{(m)}(\lambda_3^{(m)})}{c_2^{(f)0}} \right\} \text{ as } kR \rightarrow \infty \text{ for the case where } \lambda_3^{(m)} \geq 1.0, \quad \lambda_3^{(f)} = 1.0 \quad (29)$$

$$\frac{c^{cc}}{c_2^{(f)0}} \rightarrow \frac{c_\rho^{(m)}(\lambda_3^{(m)})}{c_2^{(f)0}} \text{ as } kR \rightarrow \infty \text{ for the case where } \lambda_3^{(m)} = \lambda_3^{(f)} \geq 1.0 \quad (30)$$

In relation (27) the following notation is used.

$$e^{(f)} = 2 \left( 1. + \lambda^{(f)} / \left( 2 \left( \lambda^{(f)} + \mu^{(f)} \right) \right) \right), \quad e^{(m)} = 2 \left( 1. + \lambda^{(m)} / \left( 2 \left( \lambda^{(m)} + \mu^{(m)} \right) \right) \right), \quad \eta^{(f)} = \left( 1 + \frac{h}{R} \right)^{-2}, \quad \eta^{(m)} = \left( 2 \frac{h}{R} + \left( \frac{h}{R} \right)^2 \right) \left( 1 + \frac{h}{R} \right)^{-2} \quad (31)$$

Note that the expression (27) is obtained by the following manner. First from the dispersion relation of the axisymmetric longitudinal wave propagation in the solid cylinder we determine the asymptotic root as  $kR \rightarrow 0$ . This root is determined by relation (26). Taking the expression  $(\lambda_3^{(f)})^2 e^{(f)} \mu^{(f)}$  as a “modulus of elasticity” for the pre-stretched cylinder, we determine the effective (normalized) “modulus of elasticity” for the compound cylinder by the use of the well-known expression  $\left[ (\lambda_3^{(f)})^2 e^{(f)} \mu^{(f)} \eta^{(f)} + (\lambda_3^{(m)})^2 e^{(m)} \mu^{(m)} \eta^{(m)} \right]$ . Dividing this expression into the averaged density  $(\rho^{(f)} \eta^{(f)} + \rho^{(m)} \eta^{(m)})$ , we determine the expressions (27), (31). In this case between the asymptotic (limit) values (27)-(30) the behaviour of the dispersion curves related to the compound cylinder are similar to those obtained for the solid cylinder, i.e., the values  $c^{cc}/c_2^{(f)0}$  (as the values of  $c^{sc}/c_2^{(f)0}$ ) decrease monotonically with  $kR$ . However, the dependence between  $c^{hc}/c_2^{(f)0}$  and  $kR$  is non-monotonic. In this case the existence of an initial tensional strain in the components of the compound cylinder causes the wave propagation velocity to increase.

Note that the aforementioned similarity of the dispersion curves attained for the compound cylinders with those attained for the solid cylinder can be explained with the fact that the stiffness of the inner cylinder material is greater than that for outer hollow cylinder material, i.e.  $\mu^{(f)}/\mu^{(m)} > 1.0$ . The numerical results which are not given here show that in the cases where  $\mu^{(f)}/\mu^{(m)} < 1.0$  the character of the dependencies between  $c^{cc}/c_2^{(f)0}$  and  $kR$  is similar to that attained for the hollow cylinder.

The analyses of the numerical results related with the case where  $\mu^{(f)}/\mu^{(m)} < 1.0$  will be the subject of the other paper of the authors. Moreover we note that in the qualitative sense the foregoing results agree with the results attained in Akbarov and Guz (2004). However, in Akbarov and Guz (2004), according to the corresponding problem statement, for initial strain  $\epsilon_{33}^{(m)0}$ ,  $\epsilon_{33}^{(f)0}$  the values 0.004, 0.008 and 0.01 are selected. But in the present investigation, as it has been noted above, for the considered values of  $\lambda_3^{(m)}$ ,  $\lambda_3^{(f)}$ , i.e. for  $\lambda_3^{(m)}$ ;  $\lambda_3^{(f)} = 1.2, 1.5$  and  $1.9$ , it is obtained from equations (1) and (7)  $\epsilon_{33}^{(m)0}$ ;  $\epsilon_{33}^{(f)0} = 0.220, 0.625$  and  $1.305$ . Namely this statement allows us to disclose the significant effect of the initial strains on the considered wave propagation velocity. At the same time, this statement forces us to establish the analytical expressions (27) – (31) for asymptotic-limit values of the wave propagation velocity; such expressions have not been attained in Akbarov and Guz (2004).

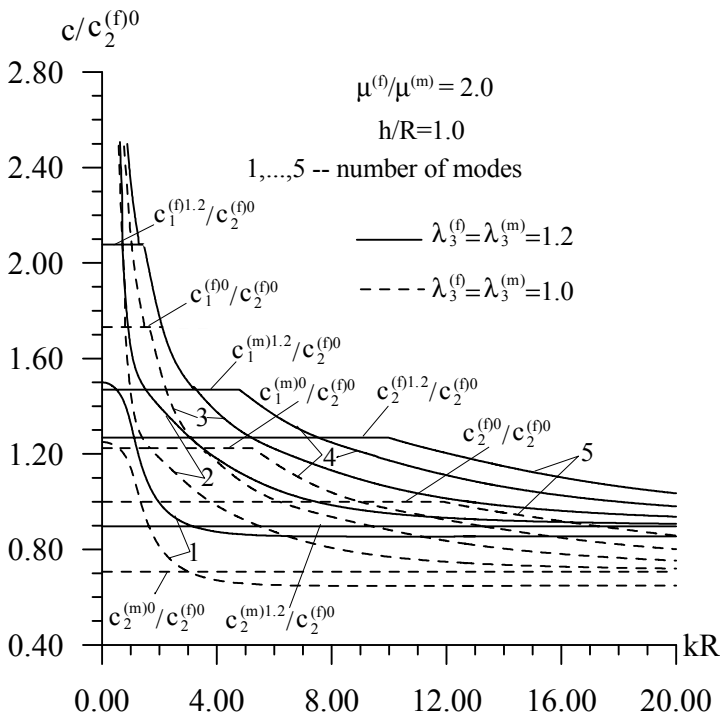


Figure 3: Dispersion curves corresponding the cases where  $\lambda_3^{(m)} = \lambda_3^{(f)} = 1.0$ (dashed lines) and  $\lambda_3^{(m)} = \lambda_3^{(f)} = 1.2$ (solid lines)

Now we consider the general appearance of the graphs of the dependencies be-

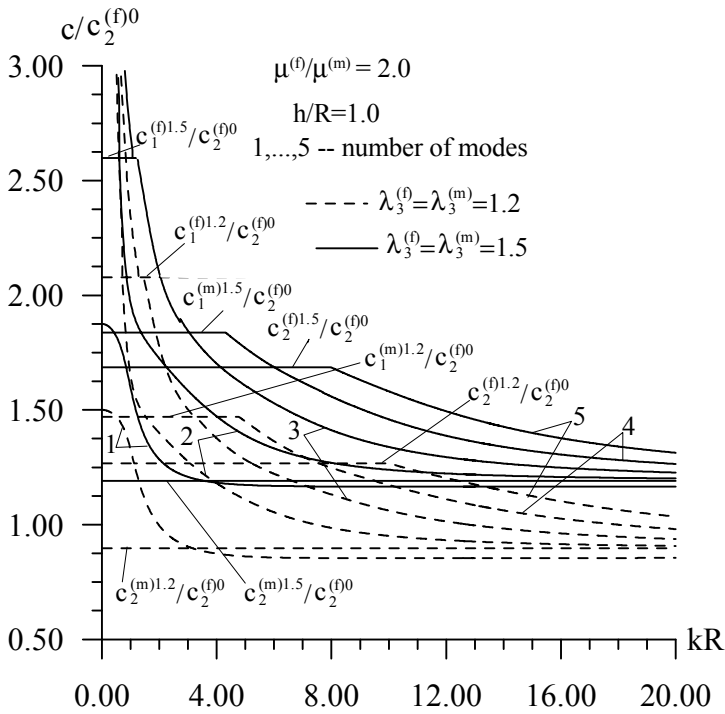


Figure 4: Dispersion curves corresponding to the cases where  $\lambda_3^{(m)} = \lambda_3^{(f)} = 1.2$ (dashed lines) and  $\lambda_3^{(m)} = \lambda_3^{(f)} = 1.5$ (solid lines)

tween  $c^{cc}/c_2^{(f)0}$  and  $kR$  which are formed with the first six roots of the dispersion equation (21). These graphs are illustrated in Figs. 3, 4 and 5 in the case where  $\lambda_3^{(m)} = \lambda_3^{(f)} = \lambda_3$ . Note that in these figures the lines corresponding to the aforementioned nondispersive wave velocities determined by the expressions given in equation (23) are also drawn. Moreover, note that in each of these figures the graphs are constructed for two selected subsequent values of  $\lambda_3$ : in Figs. 3, 4 and 5 the graphs drawn by dashed (solid) lines correspond to the values of  $\lambda_3 = 1.0, 1.2$  and  $1.5$  ( $\lambda_3 = 1.2, 1.5$  and  $1.9$ ). Such an illustration of the dispersion curves allows us to demonstrate clearly the influence of the initial stretching of the components of the compound cylinder on the wave propagation velocity in the second and subsequent modes.

It follows from Figs. 3, 4 and 5 that in the considered case using the aforementioned six roots of the equation (21) we obtain the dispersion curves for the first five, four, three and two modes of waves the propagation velocity  $c (= c^{cc})$  which satisfies the inequalities



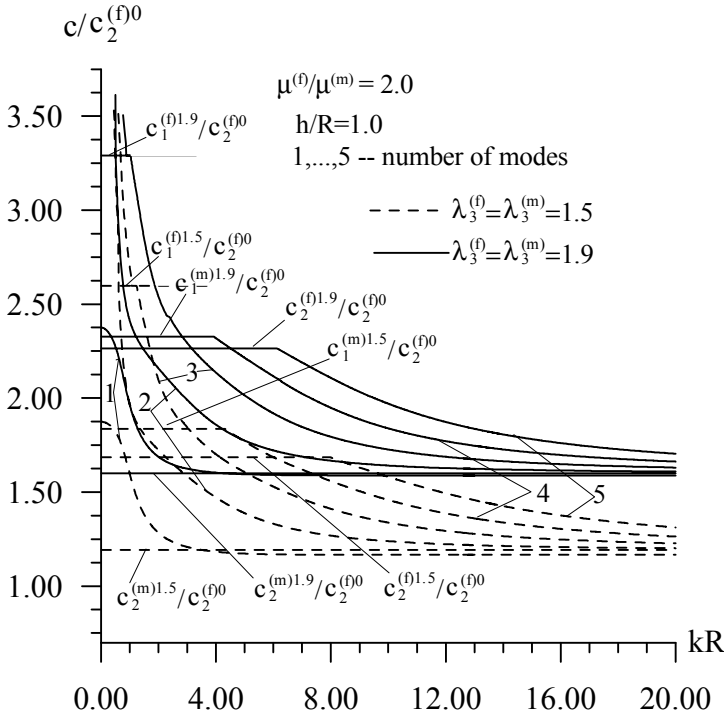


Figure 5: Dispersion curves corresponding to the cases where  $\lambda_3^{(m)} = \lambda_3^{(f)} = 1.5$ (dashed lines) and  $\lambda_3^{(m)} = \lambda_3^{(f)} = 1.9$ (solid lines)

$c_1^{(m)}(\lambda_3^{(m)}) < c < c_1^{(f)}(\lambda_3^{(f)})$  (iii) and  $c > c_1^{(f)}(\lambda_3^{(f)})$  (iiii), respectively. Consequently, the dispersion curves of the second and subsequent modes are divided into four parts: the parts I, II, III and IV correspond to the cases (i), (ii), (iii) and (iiii), respectively. In the figures these parts are separated from each other by the straight lines indicating the values of  $c_1^{(m)}$ ,  $c_2^{(f)}$  and  $c_1^{(f)}$  attained at selected values of the parameter  $\lambda_3 (= \lambda_3^{(m)} = \lambda_3^{(f)})$ . In this case for each of these parts the dispersion curve  $c = c(kR)$  and its derivative  $dc/d(kR)$  are continuous. Such a continuity occurs also at the points connecting the parts I and II, as well as II and III. At the point contacting parts III and IV the dispersion curve  $c = c(kR)$  and its first order derivative is also continuous for second mode, but at a point the mentioned continuity is violated for the third mode.

It follows from the graphs given in Figs. 3, 4 and 5 that the second and subsequent modes do not have a finite limit as  $kR \rightarrow 0$ . But these modes do have a finite limit as  $kR \rightarrow \infty$  and this limit increases with  $\lambda_3$ . In the second mode the noted limit is

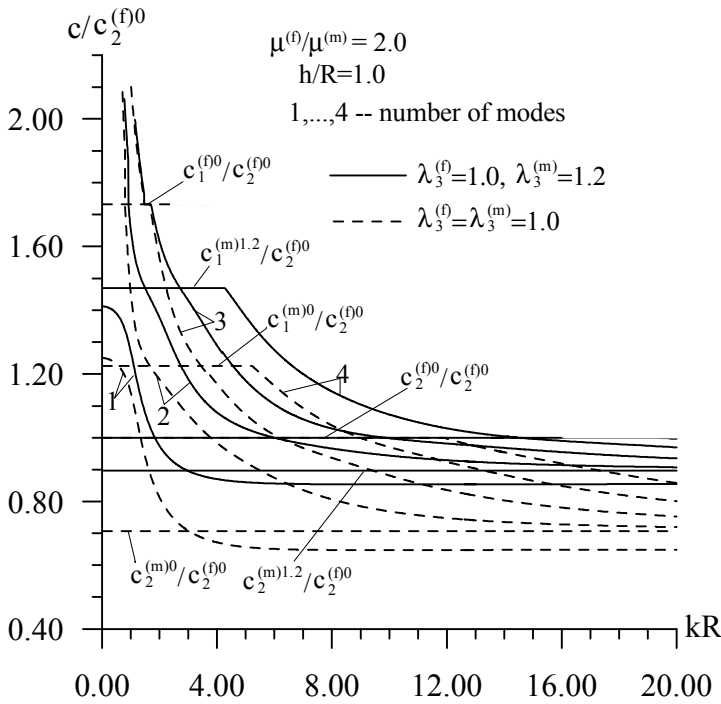


Figure 6: Dispersion curves corresponding to the cases where  $\lambda_3^{(m)} = \lambda_3^{(f)} = 1.0$ (dashed lines) and  $\lambda_3^{(m)} = 1.2, \lambda_3^{(f)} = 1.0$ (solid lines)

equal to  $c_2^{(m)\alpha}/c_2^{(f)0}$  under  $\lambda_3 = \alpha$ .

The foregoing figures show that the dispersion curves corresponding to the first mode arise within the first three parts. Moreover these figures show that the values of  $kR$  corresponding to the relations  $c/c_2^{(f)0} = c_2^{(m)\alpha}/c_2^{(f)0}, c/c_2^{(f)0} = c_2^{(f)\alpha}/c_2^{(f)0}$  and  $c/c_2^{(f)0} = c_1^{(m)\alpha}/c_2^{(f)0}$  for the first mode, as well as the values of  $kR$  corresponding to the relations  $c/c_2^{(f)0} = c_2^{(f)\alpha}/c_2^{(f)0}, c/c_2^{(f)0} = c_1^{(m)\alpha}/c_2^{(f)0}$  and  $c/c_2^{(f)0} = c_1^{(f)\alpha}/c_2^{(f)0}$  for the second and subsequent modes increase with  $\lambda_3 (= \alpha)$ .

At the same time, within each of the foregoing parts, the wave propagation velocity increases with initial stretching of the components of the compound cylinder. This conclusion holds also in cases where the initial strains occur only in one component of the cylinder. As an example, in Fig. 6 the dispersion curves corresponding to the case where  $\lambda_3^{(f)} = 1.0, \lambda_3^{(m)} = 1.2$  are given. These curves prove the noted statement.

## 5 Conclusions

From the results analyzed above the following conclusions were reached:

The initial stretching of the components of the compound cylinder causes the axisymmetric wave propagation velocity to increase.

As for numerical investigations the case where the material of the inner cylinder is stiffer than that of the outer hollow cylinder material, therefore the character of the dispersion curves attained for the compound cylinder is similar to that attained for the solid cylinder, i.e., the values of the wave propagation velocity  $c/c_2^{(f)0}$  decrease monotonically with  $kR$ .

As for numerical investigations the case where the material of the inner cylinder is stiffer than that of the outer hollow cylinder material, therefore the character of the dispersion curves attained for the compound cylinder is similar to that attained for the solid cylinder, i.e., the values of the wave propagation velocity  $c/c_2^{(f)0}$  decrease monotonically with  $kR$ .

In the first mode the values of  $c^{cc}/c_2^{(f)0}$  have a finite limit as  $kR \rightarrow 0$ . This limit is a “bar” velocity for the pre-strained compound cylinder and determined by the expressions (27), (31).

According to the expression (28)-(30), the lower limit of the wave propagation velocity in the first mode can be determined as  $\min \left\{ c_2^{(f)\alpha}/c_2^{(f)0}; c_\rho^{(m)} \left( \lambda_3^{(m)} \right) / c_2^{(f)0} \right\}$  where  $\alpha = \lambda_3^{(f)}$ .

The lower limits of all the modes increase with tensional initial elongation of the components of the cylinder along the wave propagation direction.

The dispersion curves attained for the second and subsequent modes are divided into four parts by the velocities determined by the expressions in equation (23).

The dispersion curves regarding the first mode arise within the foregoing first three parts.

Under the existence of the initial tension in the components of the cylinder the aforementioned parts “move” wholly up with increasing  $\lambda_3$ . According to the expressions given in equations (23) and (10), the length of the intervals corresponding to these parts, i.e. the values of  $L_I = \left( c_\rho^{(m)} \left( \lambda_3^{(m)} \right) - c_2^{(f)\alpha} \right) / c_2^{(f)0}$ ,  $L_{II} = \left( c_1^{(m)\alpha} - c_2^{(f)\alpha} \right) / c_2^{(f)0}$ ,  $L_{III} = \left( c_1^{(f)\alpha} - c_1^{(m)\alpha} \right) / c_2^{(f)0}$  depends on the initial strains in the components of the compound cylinder. For example, the values of  $L_I$  and  $L_{III}$  increase, but the values of  $L_{II}$  decrease with  $\lambda_3 \left( = \lambda_3^{(f)} = \lambda_3^{(m)} \right)$ .

In Akbarov and Guz (2004), according to the corresponding problem statement, for initial strains  $\varepsilon_{33}^{(m)0}$ ,  $\varepsilon_{33}^{(f)0}$  the values 0.004, 0.008 and 0.01 are selected. But

in the present investigation, according to the selected values of  $\lambda_3^{(m)}$ ,  $\lambda_3^{(f)}$ , i.e. for  $\lambda_3^{(m)}$ ;  $\lambda_3^{(f)} = 1.2, 1.5$  and  $1.9$ , it is obtained from equations (1) and (7) that  $\varepsilon_{33}^{(m)0}$ ;  $\varepsilon_{33}^{(f)0} = 0.220, 0.625$  and  $1.305$ . Namely this statement causes the significant effect of the initial strains on the considered wave propagation velocity. At the same time, this statement forces us to establish the analytical expressions (27) – (31) for asymptotic-limit values of the wave propagation velocity; such expressions have not been attained in Akbarov and Guz (2004).

Although the discussed numerical results are obtained for the particular selected cases, but they have also a general meaning for the wave propagation problems for the finite pre-strained compound cylinders made from high-elastic materials.

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