Analytical and Numerical Solutions of Riesz Space Fractional Advection-Dispersion Equations with Delay

Mahdi Saedshoar Heris[1] , Mohammad Javidi[1] and Bashir Ahmad[2, *]

Abstract: In this paper, we propose numerical methods for the Riesz space fractional advection-dispersion equations with delay (RFADED). We utilize the fractional backward differential formulas method of second order (FBDF2) and weighted shifted Grünwald difference (WSGD) operators to approximate the Riesz fractional derivative and present the finite difference method for the RFADED. Firstly, the FBDF2 and the shifted Grünwald methods are introduced. Secondly, based on the FBDF2 method and the WSGD operators, the finite difference method is applied to the problem. We also show that our numerical schemes are conditionally stable and convergent with the accuracy of $\mathrm{O}(\kappa + h^2)$ and $\mathrm{O}(\kappa^2 + h^2)$ respectively. Thirdly we find the analytical solution for RFDED in terms Mittag-Leffler type functions. Finally, some numerical examples are given to show the efficacy of the numerical methods and the results are found to be in complete agreement with the analytical solution.

Keywords: Riesz fractional derivative, shifted Grünwald difference operators, fractional advection-dispersion equation, delay differential equations, FBDF method.

1 Introduction

Fractional calculus finds its applications in diverse areas of science, engineering, economics and finance [Bagley and Calico (1991); Weaver Jr, Timoshenko and Young (1990); Marks and Hall (1981); Simo and Woafo (2016); Yang (2019); Cattani (2018); Yang, Abdel-Aty and Cattani (2019); Feng (2017)]. In most of the cases, fractional differential equations (FDEs) cannot be solved exactly, so one needs to seek approximate and numerical techniques to solve these equations. Various numerical methods for solving FDEs are discussed in Galeone et al. [Galeone and Garrappa (2006); Garrappa (2015); Gorenflo (1997); Lubich (1986); Sulaiman, Yavuz, Bulut et al. (2019); Yang, Han, Li et al. (2016)]. However, there are fewer works dealing with numerical methods for delay fractional differential equation [Heris and Javidi (2017b,a, 2018a, 2019); Javidi and Heris

[1]Department of Applied Mathematics, University of Tabriz, Tabriz, Iran.
[2]Nonlinear Analysis and Applied Mathematics (NAAM) Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah 21589, Saudi Arabia.
*Corresponding Author: Bashir Ahmad. Email: bashirahmad_qau@yahoo.com.

(2019)]. In recent years, some authors investigated fractional order partial differential equations [Momani, Odibat and Erturk (2007); Momani and Odibat (2008a); Meerschaert and Tadjeran (2006)], and delay fractional partial differential equations [Zubik-Kowal (2000); Jackiewicz and Zubik-Kowal (2006); Tanthanuch (2012)]. Such equations appear in mathematical modeling of several phenomena occurring in biology, medicine, population ecology, control systems and climate models [Wu (2012)]. For details on numerical method for fractional partial differential equations, for instance, see Tadjeran et al. [Tadjeran, Meerschaert and Scheffler (2006); Liu, Zeng and Li (2015); Ding and Li (2013)].

The fractional advection-dispersion equation (FADE) is an important tool of groundwater hydrology to deal with the transport of passive tracers carried by fluid flow in a porous medium [Momani and Odibat (2008b); Liu, Anh, Turner et al. (2003); Huang and Liu (2005)]. Meerschaert et al. [Meerschaert and Tadjeran (2004)] presented numerical methods for solving one-dimensional fractional advection-diffusion equation involving a Riemann-Liouville fractional derivative on a finite domain. Liu et al. [Liu, Anh and Turner (2004)] transformed the space fractional advection–diffusion equation into a system of ordinary differential equations and solved the resulting equations by using backward differentiation formulas. Shen et al. [Shen, Liu and Anh (2008)] discussed the fundamental solution and discrete random walk model for a time space fractional diffusion equation of distributed order. Numerical approximations and solution techniques for the space time Riesz-Caputo fractional advection-diffusion equation were studied in Shen et al. [Shen, Liu and Anh (2011)]. For more details and examples, we refer the reader to the articles [Ding, Li and Chen (2015); Sousa (2012); Yang, Liu and Turner (2010); Wu, Baleanu and Xie (2016); Wu, Baleanu, Deng et al. (2015)].

In this paper, we focus on designing a numerical method for solving the following Riesz fractional advection-dispersion equation (RFADE) with time delay:

$$\frac{\partial u(x,t)}{\partial t} = K_\alpha \frac{\partial^\alpha u(x,t)}{\partial |x|^\alpha} + K_\beta \frac{\partial^\beta u(x,t)}{\partial |x|^\beta} + u(x,t-\tau) + f(x,t); \tag{1}$$

subject to the initial and boundary conditions:

$$\begin{aligned} &u(x,t) = g(x,t), \;\; -\tau \le t \le 0, \;\; 0 \le x \le L, \\ &u(0,t) = \mu_1(t), \;\; u(L,t) = \mu_2(t), \;\; 0 \le t \le T, \end{aligned} \tag{2}$$

where $0 < \alpha < 1, \; 1 < \beta \le 2, \; K_\alpha \ge 0, \;\; K_\beta > 0$ and $\tau > 0$. The Riesz space fractional operator on a finite domain $[0, L]$ is defined as [Yang, Liu and Turner (2010)]

$$\frac{\partial^\gamma u(x,t)}{\partial |x|^\gamma} = -(-\Delta)^{\frac{\gamma}{2}} u(x,t) = -c_\gamma [{}_0^{RL}D_x^\gamma u(x,t) + {}_x^{RL}D_L^\gamma u(x,t)], \tag{3}$$

where

$$\begin{aligned}
&c_\gamma = \tfrac{1}{2\cos(\frac{\pi\gamma}{2})}, \quad 0 < \gamma \leq 2, \ \gamma \neq 1, \\
&{}_0^{RL}D_x^\gamma u(x,t) = \tfrac{1}{\Gamma(1-\gamma)} \tfrac{\partial}{\partial x} \int_0^x (x-\eta)^{-\gamma} u(\eta,t) d\eta, \ 0 < \gamma < 1, \\
&{}_x^{RL}D_L^\gamma u(x,t) = \tfrac{-1}{\Gamma(1-\gamma)} \tfrac{\partial}{\partial x} \int_x^L (\eta - x)^{-\gamma} u(\eta,t) d\eta, \ 0 < \gamma < 1, \\
&{}_0^{RL}D_x^\gamma u(x,t) = \tfrac{1}{\Gamma(2-\gamma)} \tfrac{\partial^2}{\partial x^2} \int_0^x (x-\eta)^{1-\gamma} u(\eta,t) d\eta, \ 1 < \gamma \leq 2, \\
&{}_x^{RL}D_L^\gamma u(x,t) = \tfrac{1}{\Gamma(2-\gamma)} \tfrac{\partial^2}{\partial x^2} \int_x^L (\eta - x)^{1-\gamma} u(\eta,t) d\eta, \ 1 < \gamma \leq 2.
\end{aligned} \tag{4}$$

Partial differential equations with delay are more complicated and less studied in the literature. The solutions of PDEs are different from the ones for PDEs with delay. If we take exact solution of our model as initial delay solution, then the solution of our model without delay shifts to the right as

$$u_{delay}(x,t)|_{t\in[k\tau,(k+1)\tau]} = u_{without\ delay}(x,t)|_{t\in[(k-1),k\tau]}, \quad k = 1, 2, \cdots .$$

Thus, we can change initial delay solution to control dynamic properties of solutions.

Partial differential equations with delay have recently been studied by many authors and important aspects such as, existence and stability of solutions for these equations, are presented [Wu (2012)]. In general, the exact solution of these equations cannot be obtained. Moreover, it is difficult to study the long-term dynamic properties of these equations. So one resorts to numerical simulation of such equations. In order to investigate the long-term dynamic properties, partial differential equations with delay are considered. As a typical example in the delay field with derivatives of integer order, we take $K_\alpha = 0, \ \ \beta = 2$ in Eq. (1). This equation was considered as the reaction-diffusion equation with delay in Huang et al. [Huang and Vandewalle (2012)]. For the delay field with fractional derivatives, we take $K_\alpha = 0$ in Eq. (1) and the resulting equation is known as the Riesz space fractional diffusion equation with delay and nonlinear source term, see Yang [Yang (2018)].

We first obtain analytical solution for the problem at hand. Then we apply FBDF2 method for $0 < \alpha < 1$ and shifted Grünwald difference operators for $1 < \beta \leq 2$ to approximate the Riesz space fractional derivative. Furthermore, we propose the finite difference method for the RFADED. We also show that the schemes for all h smaller than

$$\Big(- \frac{3\beta(\beta-1)(2-\beta)(3+\beta)\cos(\frac{\alpha\pi}{2})}{4(\frac{3}{2})^\alpha \alpha(8\alpha-5)\cos(\frac{\beta\pi}{2})} \Big)^{\frac{1}{\beta-\alpha}}$$

are stable and convergent with the accuracy of $\mathrm{O}(\kappa + h^2)$ and $\mathrm{O}(\kappa^2 + h^2)$ respectively.

The paper is organized as follows. Section 2 contains some definitions. In Section 3, analytical solution for the given problem is obtained. FBDF2 method is presented in Section 4, while Section 5 deals with Shifted Grünwald method. In Section 6, numerical methods for the RFADED are presented. The paper concludes with numerical simulations and results.

2 Preliminaries

In this section, we recall some definitions related to our work. Let $\mathbb{C}(\mathbb{J},\mathbb{R})$ denote the Banach space of all continuous functions from $\mathbb{J}=[0,T]$ into $\mathbb{R}$ endowed with the norm

$$\|u\|_{\infty}=\sup\{|u(t)|:t\in\mathbb{J}\},\quad T>0 \tag{5}$$

and $\mathbb{C}^n(\mathbb{J},\mathbb{R})$ denotes the class of all real valued functions possessing derivatives upto order n on $\mathbb{J}=[0,T]$, $T>0$.

Definition 2.1 [Kilbas, Srivastava and Trujillo (2006)]**.** The fractional integral of order $\alpha>0$ of the function $f\in\mathbb{C}(\mathbb{J},\mathbb{R})$ is defined as

$$I^{\alpha}f(t)=\frac{1}{\Gamma(\alpha)}\int\limits_0^t\frac{f(s)}{(t-s)^{1-\alpha}}ds,\quad 0<t<T. \tag{6}$$

Definition 2.2 [Kilbas, Srivastava and Trujillo (2006)]**.** The Riemann-Liouville fractional derivative of order $\alpha>0$ of the function $f\in\mathbb{C}^n(\mathbb{J},\mathbb{R})$ is defined as

$$^{RL}D^{\alpha}f(t)=\begin{cases} D^nI^{n-\alpha}f(t)=\dfrac{1}{\Gamma(n-\alpha)}(\dfrac{d^n}{dt^n})\int\limits_0^t\dfrac{f(s)}{(t-s)^{\alpha-n+1}}ds, \\ \qquad\qquad n-1<\alpha<n,\ n\in N, \\ f^{(n)}(t), \qquad\qquad \alpha=n. \end{cases} \tag{7}$$

Definition 2.3 [Kilbas, Srivastava and Trujillo (2006)]**.** The Caputo fractional derivative of order $\alpha>0$ of the function $f\in\mathbb{C}^n(\mathbb{J},\mathbb{R})$ is defined as

$$^{C}D^{\alpha}f(t)=\begin{cases} I^{n-\alpha}D^nf(t)=\dfrac{1}{\Gamma(n-\alpha)}\int\limits_0^t\dfrac{f^{(n)}(s)}{(t-s)^{\alpha-n+1}}ds, \\ \qquad\qquad n-1<\alpha<n,\ n\in N, \\ f^{(n)}(t), \qquad\qquad \alpha=n. \end{cases} \tag{8}$$

Definition 2.4 [Čermák, Horníček and Kisela (2016)]**.** The generalized delay exponential function (of Mittag-Leffler type) is defined by

$$G^{\lambda,\tau,m}_{\alpha,\beta}(t)=\sum_{j=0}^{\infty}\binom{j+m}{j}\frac{\lambda^j(t-(m+j)\tau)^{\alpha(m+j)+\beta-1}}{\Gamma(\alpha(m+j)+\beta)}H(t-(m+j)\tau),\ t>0, \tag{9}$$

where $\lambda\in\mathbb{C}$, $\alpha,\beta,\tau\in\mathbb{R}$ and $m\in\mathbb{Z}$ and $H(z)$ is the Heaviside step function. If $\lambda\in\mathbb{C}$, $\alpha,\beta,\tau\in\mathbb{R}$ and $m\in\mathbb{Z}$, then the Laplace transform of $G^{\lambda,\tau,m}_{\alpha,\beta}(t)$ is

$$L(G^{\lambda,\tau,m}_{\alpha,\beta}(t))(s)=\frac{s^{\alpha-\beta}\exp\{-ms\tau\}}{(s^{\alpha}-\lambda\exp\{-s\tau\})^{m+1}},\quad s>0. \tag{10}$$

3 Analytical solution of problem

In this section, we derive the analytical solution of the RFADED (1-2).

Lemma 3.1. [Ilic, Liu, Turner et al. (2005)] Suppose that the one-dimensional Laplacian $(-\Delta)$ supplemented with Dirichlet boundary condition at $x = 0$ and $x = L$ has a complete set of orthonormal eigenfunctions φ_n associated with eigenvalues λ_n^2 on a boundary region $\Omega = [0, L]$, that is, $(-\Delta)\varphi_n = \lambda_n^2\varphi_n$ leads to the eigenvalues $\lambda_n^2 = \frac{n^2\pi^2}{L^2}$ for $n = 1, 2, ...$ and the corresponding eigenfunctions $\varphi_n = \sqrt{\frac{2}{L}}\sin(\frac{n\pi}{L})$.

In the initial and boundary conditions (2), it is assumed that $\mu_1(t)$ and $\mu_2(t)$ are nonzero smooth functions with first order continuous derivatives. We firstly transform the nonhomogeneous condition into a homogeneous one. Let

$$u(x,t) = V(x,t) + W(x,t), \tag{11}$$

where

$$V(x,t) = \mu_1(t) + x\frac{\mu_2(t) - \mu_1(t)}{L}. \tag{12}$$

Substituting (11) into (1) leads to the the following problem with homogeneous boundary conditions satisfied by $W(x,t)$:

$$\begin{aligned}
&\frac{\partial W(x,t)}{\partial t} + K_\alpha(-\Delta)^{\frac{\alpha}{2}}W(x,t) + K_\beta(-\Delta)^{\frac{\beta}{2}}W(x,t) + W(x,t-\tau) = f_1(x,t),\ t > 0,\\
&W(x,t) = \phi_1(x,t),\ \ -\tau \le t \le 0,\ \ 0 \le x \le L,\\
&W(0,t) = W(L,t) = 0, \quad t \ge 0,
\end{aligned} \tag{13}$$

where

$$\begin{aligned}
&f_1(x,t) = V(x,t-\tau) + f(x,t) - \frac{\partial V(x,t)}{\partial t} - K_\alpha(-\Delta)^{\frac{\alpha}{2}}V(x,t) - K_\beta(-\Delta)^{\frac{\beta}{2}}V(x,t),\\
&\phi_1(x,t) = g(x,t) - \mu_1(t) - \frac{\mu_2(t) - \mu_1(t)}{L}x.
\end{aligned} \tag{14}$$

Assume that the solution of (13) has the form:

$$W(x,t) = X(x)T(t). \tag{15}$$

Substituting (15) into (13), we obtain the Sturm-Liouville problem:

$$\begin{aligned}
&-K_\alpha(-\Delta)^{\frac{\alpha}{2}}X(x) - K_\beta(-\Delta)^{\frac{\beta}{2}}X(x) + \lambda X(x) = 0,\\
&X(0) = 0, \quad X(L) = 0
\end{aligned} \tag{16}$$

and

$$\begin{aligned}
&\frac{dT(t)}{dt} + T(t-\tau) + \lambda T(t) = 0,\\
&T(t) = \nu(x,t),\ \ -\tau \le t \le 0.
\end{aligned} \tag{17}$$

where $\lambda > 0$ and $\nu(x,t)$ is the initial function arising from (2). By Lemma 3.1, the Sturm-Liouville problem (16) has eigenvalues and corresponding eigenfunctions

$$\lambda_n = \frac{n^2\pi^2}{L^2}, \quad X_n(x) = \sin(\frac{n\pi}{L}x),\ \ n = 1, 2, ...\cdot \tag{18}$$

Therefore we set

$$W(x,t)=\sum_{n=1}^{\infty}A_n(t)\sin(\frac{n\pi}{L}x). \tag{19}$$

Inserting (19) into (13) leads to

$$\begin{aligned}&\frac{dA_n(t)}{dt}+b_nA_n(t)+A_n(t-\tau)=f_n(t),\\&A_n(t)=\varphi(t),\end{aligned} \tag{20}$$

where

$$\begin{aligned}&f_n(t)=\frac{2}{L}\int_0^L f_1(x,t)\sin(\tfrac{n\pi}{L}x)dx,\\&f_1(x,t)=\sum_{n=1}^{\infty}f_n(t)\sin(\tfrac{n\pi}{L}x),\\&\varphi(t)=\tfrac{2}{L}\int_0^L\phi_1(x,t)\sin(\tfrac{n\pi}{L}x)dx,\\&b_n=K_\alpha{\lambda_n}^{\alpha}+K_\beta{\lambda_n}^{\beta}.\end{aligned} \tag{21}$$

Taking Laplace transform of (20), we get

$$\bar{A}_n(s)=\frac{\bar{f}_n(s)}{s+e^{-\tau s}+b_n}+\frac{\varphi(0)}{s+e^{-\tau s}+b_n}-\frac{e^{-\tau s}\int_{-\tau}^0 e^{-s\nu}\varphi(\nu)d\nu}{s+e^{-\tau s}+b_n}, \tag{22}$$

where

$$\bar{A}_n(s)=L(A_n(t)),\ \ \bar{f}_n(s)=L(f_n(t)). \tag{23}$$

Writing

$$\begin{aligned}\tfrac{1}{s+e^{-\tau s}+b_n}&=\tfrac{1}{b_n}\tfrac{b_n}{s+e^{-\tau s}}\tfrac{1}{1+\frac{b_n}{s+e^{-\tau s}}}\\&=\sum_{k=0}^{\infty}(-b_n)^k e^{k\tau s}\tfrac{e^{-k\tau s}}{(s+e^{-\tau s})^{k+1}}\\&=\sum_{k=0}^{\infty}(-b_n)^k\sum_{m=0}^{\infty}\tfrac{(k\tau)^m}{m!}\tfrac{s^{1-(1-m)}e^{-k\tau s}}{(s+e^{-\tau s})^{k+1}},\end{aligned} \tag{24}$$

we obtain

$$L^{-1}(\frac{1}{s+e^{-\tau s}+b_n})=\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-b_n)^k(k\tau)^m}{\Gamma(m+1)}G_{1,1-m}^{-1,\tau,k}(t), \tag{25}$$

Similarly

$$L^{-1}(\frac{\bar{f}_n(s)}{s+e^{-\tau s}+b_n})=\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-b_n)^k(k\tau)^m}{\Gamma(m+1)}\int_0^t G_{1,1-m}^{-1,\tau,k}(p)f_n(t-p)dp. \tag{26}$$

Let

$$Z(s)=e^{-\tau s}\int_{-\tau}^0 e^{-s\nu}\varphi(\nu)d\nu, \tag{27}$$

with

$$L^{-1}(Z(s))=z(t). \tag{28}$$

Then

$$L^{-1}(\frac{Z(s)}{s+e^{-\tau s}+b_n}) = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-b_n)^k(k\tau)^m}{\Gamma(m+1)}\int_0^t G_{1,1-m}^{-1,\tau,k}(p)z(t-p)dp. \tag{29}$$

Thus we have

$$\begin{aligned}
A_n(t) &= \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-b_n)^k(k\tau)^m}{\Gamma(m+1)}[\int_0^t G_{1,1-m}^{-1,\tau,k}(p)(f_n(t-p)-z(t-p))dp \\
&\quad +\varphi(0)G_{1,1-m}^{-1,\tau,k}(t)], \\
W(x,t) &= \sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-b_n)^k(k\tau)^m}{\Gamma(m+1)}[\int_0^t G_{1,1-m}^{-1,\tau,k}(p)(f_n(t-p)-z(t-p))dp \\
&\quad +\varphi(0)G_{1,1-m}^{-1,\tau,k}(t)]sin(\tfrac{n\pi}{L}x),
\end{aligned} \tag{30}$$

which yields the analytical solution of (1)-(2) given by

$$\begin{aligned}
u(x,t) &= \mu_1(t) + \frac{\mu_2(t)-\mu_1(t)}{L}x \\
&\quad + \sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-b_n)^k(k\tau)^m}{\Gamma(m+1)}[\int_0^t G_{1,1-m}^{-1,\tau,k}(p)(f_n(t-p)-z(t-p))dp \\
&\quad +\varphi(0)G_{1,1-m}^{-1,\tau,k}(t)]sin(\tfrac{n\pi}{L}x).
\end{aligned} \tag{31}$$

4 Fractional backward Differential Formulas of second order (FBDF2 method)

We consider the initial value problem

$${}_{t_0}^{C}D_t^{\alpha}y(t) = f(t), \;\; y(t_0) = y_0, \;\; 0 < \alpha < 1, \tag{32}$$

where f is a sufficiently smooth function. We now introduce the FBDF method of second order($FBDF2$) for (32) [Heris and Javidi (2018b)]. For $0 < \alpha < 1$, we have

$$\sum_{k=0}^{j}\varpi_k y_{j-k} - b_j y_0 \approx h^{\alpha} f_j, \tag{33}$$

where

$$\frac{j^{-\alpha}}{\Gamma(1-\alpha)} = b_j, \; y_{j-k} = y(t_j - kh), \; f_j = f(t_j), \tag{34}$$

with the coefficients given by

$$\begin{aligned}
&\varpi_k = (\tfrac{3}{2})^{\alpha}\omega_k, \;\; k = 0,1,\cdots, \omega_0 = 1, \\
&\omega_1 = -\tfrac{4}{3}\alpha, \\
&\omega_k = \tfrac{4}{3}(1-\tfrac{\alpha+1}{k})\omega_{k-1} + \tfrac{1}{3}(\tfrac{2(1+\alpha)}{k}-1)\omega_{k-2}.
\end{aligned} \tag{35}$$

Lemma 4.1. [Heris and Javidi (2018b)] For $0 < \alpha < 1$, the coefficients ${\varpi_j}^{(\alpha)}$ satisfy

$$\begin{aligned}
&{\varpi_0}^{(\alpha)} > 0, \, {\varpi_j}^{(\alpha)} < 0, j = 4,5\cdots. \\
&\left|{\varpi_j}^{(\alpha)}\right| < \left|{\varpi_{j-1}}^{(\alpha)}\right| < {\varpi_0}^{(\alpha)}, j = 4,5,\cdots \\
&\sum_{k=0}^{\infty}{\varpi_k}^{(\alpha)} = 0, \;\; \sum_{k=0}^{m}{\varpi_k}^{(\alpha)} > 0, \; m > 3.
\end{aligned} \tag{36}$$

5 Shifted Grünwald method

The standard Grünwald-Letnikov difference formula was often unstable for $1 < \beta \leq 2$ and for time dependent problems [Meerschaert and Tadjeran (2004)]. On the other hand, the Shifted Grünwald difference operators formula is stable and is given by

$$\begin{aligned} M^{\gamma}_{h,p}u(x) &= h^{-\gamma}\sum_{k=0}^{\infty}\omega_k^{(\gamma)}u(x-(k-p)h), \\ N^{\gamma}_{h,q}u(x) &= h^{-\gamma}\sum_{k=0}^{\infty}\omega_k^{(\gamma)}u(x+(k-q)h). \end{aligned} \tag{37}$$

Observe that

$$\begin{aligned} M^{\gamma}_{h,p}u(x) &= {}_{-\infty}D^{\gamma}_{x}u(x) + \mathrm{O}(h), \\ N^{\gamma}_{h,p}u(x) &= {}_{x}D^{\gamma}_{+\infty}u(x) + \mathrm{O}(h), \end{aligned} \tag{38}$$

where $p, q \in Z$ and $\omega_k^{(\gamma)} = (-1)^k \left(\begin{array}{c}\gamma \\ k\end{array}\right)$.

Lemma 5.1. [Tian, Zhou and Deng (2015)] For $1 < \gamma \leq 2$, the coefficients $\omega_k^{(\gamma)}$ satisfy

$$\begin{aligned} &\omega_0^{(\gamma)} = 1,\ \ \omega_1^{(\gamma)} = -\gamma,\ \ \omega_2^{(\gamma)} = \tfrac{\gamma(\gamma-1)}{2}, \\ &1 \geq \omega_2^{(\gamma)} \geq \omega_3^{(\gamma)} \geq ... \geq 0, \\ &\sum_{k=0}^{\infty}\omega_k^{(\gamma)} = 0,\quad \sum_{k=0}^{m}\omega_k^{(\gamma)} < 0,\ \ m \geq 1. \end{aligned} \tag{39}$$

Theorem 5.1. [Tian, Zhou and Deng (2015)] Let $1 < \gamma \leq 2$ and $u \in L^1(R)$, ${}_{-\infty}D^{\gamma}_{x}u$, ${}_{x}D^{\gamma}_{+\infty}u$ and their Fourier transforms belong to $L^1(R)$ and let the weighted and shifted Grünwald difference operators be defined by

$$\begin{aligned} {}_{L}D^{\gamma}_{h,p,q}u(x) &= \frac{\gamma-2q}{2(p-q)}M^{\gamma}_{h,p}u(x) + \frac{2p-\gamma}{2(p-q)}M^{\gamma}_{h,q}u(x), \\ {}_{R}D^{\gamma}_{h,p,q}u(x) &= \frac{\gamma-2q}{2(p-q)}N^{\gamma}_{h,p}u(x) + \frac{2p-\gamma}{2(p-q)}N^{\gamma}_{h,q}u(x). \end{aligned} \tag{40}$$

Then

$$\begin{aligned} {}_{L}D^{\gamma}_{h,p,q}u(x) &= {}_{-\infty}D^{\gamma}_{x}u(x) + \mathrm{O}(h^2), \\ {}_{R}D^{\gamma}_{h,p,q}u(x) &= {}_{x}D^{\gamma}_{+\infty}u(x) + \mathrm{O}(h^2), \end{aligned} \tag{41}$$

where p and q are integers and $p \neq q$.

Remark 5.1. In case of a well defined function $u(x)$ on the bounded interval $[a, b]$ with $u(a) = 0$ or $u(b) = 0$, it can be extended to zero for $x < a$ or $x > b$. Then the left and right Riemann-Liouville fractional derivatives of u at each point $x \in (a, b)$ can be approximated by the WSGD operators with second order accuracy as

$$\begin{aligned}
{}_aD_x^{\gamma}u(x) &= \eta_1 h^{-\gamma}\sum_{k=0}^{\left[\frac{x-a}{h}\right]+p}\omega_k^{(\gamma)}u(x-(k-p)h)\\
&\quad+\eta_2 h^{-\gamma}\sum_{k=0}^{\left[\frac{x-a}{h}\right]+q}\omega_k^{(\gamma)}u(x-(k-q)h)+\mathrm{O}(h^2),\\
{}_xD_b^{\gamma}u(x) &= \eta_1 h^{-\gamma}\sum_{k=0}^{\left[\frac{b-x}{h}\right]+p}\omega_k^{(\gamma)}u(x+(k-p)h)\\
&\quad+\eta_2 h^{-\gamma}\sum_{k=0}^{\left[\frac{b-x}{h}\right]+q}\omega_k^{(\gamma)}u(x+(k-q)h)+\mathrm{O}(h^2),
\end{aligned}\tag{42}$$

where $\eta_1=\frac{\gamma-2q}{2(p-q)}$, $\eta_2=\frac{2p-\gamma}{2(p-q)}$.

Remark 5.2. For $1<\gamma\le 2$ and $(p,q)=(1,0)$, Eq. (42) on the domain $[0,L]$ can be written as

$$\begin{aligned}
{}_0D_x^{\gamma}u(x_i) &= h^{-\gamma}\sum_{k=0}^{i+1}\vartheta_k^{(\gamma)}u(x_{i-k+1})+\mathrm{O}(h^2),\\
{}_xD_L^{\gamma}u(x_i) &= h^{-\gamma}\sum_{k=0}^{m-i+1}\vartheta_k^{(\gamma)}u(x_{i+k-1})+\mathrm{O}(h^2),
\end{aligned}\tag{43}$$

where

$$\vartheta_0^{(\gamma)}=\frac{\gamma}{2}\omega_0^{(\gamma)},\quad \vartheta_k^{(\gamma)}=\frac{\gamma}{2}\omega_k^{(\gamma)}+\frac{2-\gamma}{2}\omega_{k-1}^{(\gamma)},\quad k\ge 1.\tag{44}$$

Lemma 5.2. [Tian, Zhou and Deng (2015)] For $1<\gamma\le 2$, the coefficients $\vartheta_k^{(\gamma)}$ satisfy

$$\begin{aligned}
&\vartheta_0^{(\gamma)}=\tfrac{\gamma}{2}>0,\quad \vartheta_1^{(\gamma)}=\tfrac{2-\gamma-\gamma^2}{2}<0,\quad \vartheta_2^{(\gamma)}=\tfrac{\gamma(\gamma^2+\gamma-4)}{4},\\
&1\ge\vartheta_0^{(\gamma)}\ge\vartheta_3^{(\gamma)}\ge\vartheta_4^{(\gamma)}\ge\ldots\ge 0,\\
&\sum_{k=0}^{\infty}\vartheta_k^{(\gamma)}=0,\quad \sum_{k=0}^{m}\vartheta_k^{(\gamma)}<0,\quad m\ge 2.
\end{aligned}\tag{45}$$

6 Numerical methods

In this section, we consider two cases. In case 1, we approximate the time derivative with order one and the Riesz space fractional derivative with order two. In case 2, we approximate the Riesz space fractional derivative and derive the Crank-Nicolson scheme for the equation. We partition the interval $[0,L]$ into an uniform mesh with the space step size $h=L/M$ and the time step size $t=T/N$, where M,N are two positive integers. The set of grid points are denoted by $x_i=ih$ and $t_j=j\kappa$ for $i=1,...,M$ and $j=n+1,\cdots$.

Case 1. We apply numerical method to Eq. (1) as follows.

Let $u(x_i, t_j) = u_i^j, \quad f(x_i, t_j) = f_i^j$. Then

$$\begin{aligned} \frac{u_i^j - u_i^{j-1}}{\kappa} &= -c_\alpha K_\alpha h^{-\alpha}(\sum_{k=0}^{i} \varpi_k^{(\alpha)} u_{i-k}^j + \sum_{k=0}^{M-i} \varpi_k^{(\alpha)} u_{i+k}^j) \\ &\quad -c_\beta K_\beta h^{-\beta}(\sum_{k=0}^{i+1} \vartheta_k^{(\beta)} u_{i-k+1}^j + \sum_{k=0}^{M-i+1} \vartheta_k^{(\beta)} u_{i+k-1}^j) + u_i^{j-n} + f_i^j + R_1, \end{aligned} \tag{46}$$

where $i = 1, ..., M, \ j = n+1, \cdots$ and R_1 is a truncation error term. Taking $\lambda_\alpha = c_\alpha \kappa K_\alpha h^{-\alpha}$ and $\lambda_\beta = c_\beta \kappa K_\beta h^{-\beta}$, we have

$$\begin{aligned} & u_i^j + \lambda_\alpha(\sum_{k=0}^{i} \varpi_k^{(\alpha)} u_{i-k}^j + \sum_{k=0}^{M-i} \varpi_k^{(\alpha)} u_{i+k}^j) \\ &\quad +\lambda_\beta(\sum_{k=0}^{i+1} \vartheta_k^{(\beta)} u_{i-k+1}^j + \sum_{k=0}^{M-i+1} \vartheta_k^{(\beta)} u_{i+k-1}^j) = u_i^{j-1} + \kappa u_i^{j-n} + \kappa f_i^j. \end{aligned} \tag{47}$$

Introducing

$$\begin{aligned} A &= \begin{pmatrix} \varpi_0^{(\alpha)} & 0 & 0 & \cdots & 0 \\ \varpi_1^{(\alpha)} & \varpi_0^{(\alpha)} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \varpi_{M-2}^{(\alpha)} & \varpi_{M-1}^{(\alpha)} & \cdots & \cdots & \varpi_0^{(\alpha)} \end{pmatrix}, \\ B &= \begin{pmatrix} \vartheta_1^{(\beta)} & \vartheta_0^{(\beta)} & 0 & 0 & \cdots & 0 \\ \vartheta_2^{(\beta)} & \vartheta_1^{(\beta)} & \vartheta_0^{(\beta)} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \vartheta_{M-1}^{(\beta)} & \vartheta_{M-2}^{(\beta)} & \cdots & & \cdots & \vartheta_1^{(\beta)} \end{pmatrix} \end{aligned} \tag{48}$$

and

$$D = \lambda_\alpha(A + A^T) + \lambda_\beta(B + B^T), \ \ U^j = [u_1^j, \, u_2^j, \, ..., \, u_{M-1}^j]^T, \tag{49}$$

Eq. (47) takes the form:

$$(I + D)U^j = IU^{j-1} + F^j, \tag{50}$$

where

$$F^j = \begin{bmatrix} \kappa u_1^{j-n} + \kappa f_1^j - \lambda_\alpha \varpi_1^{(\alpha)} u_0^j - \lambda_\beta(\vartheta_0^{(\beta)} + \vartheta_2^{(\beta)}) u_0^j - (\lambda_\alpha \varpi_{M-1}^{(\alpha)} + \lambda_\beta \vartheta_M^{(\beta)}) u_M^j \\ \kappa u_2^{j-n} + \kappa f_2^j - \lambda_\alpha \varpi_2^{(\alpha)} u_0^j - \lambda_\beta \vartheta_3^{(\beta)} u_0^j - (\lambda_\alpha \varpi_{M-2}^{(\alpha)} + \lambda_\beta \vartheta_{M-1}^{(\beta)}) u_M^j \\ \kappa u_3^{j-n} + \kappa f_3^j - \lambda_\alpha \varpi_3^{(\alpha)} u_0^j - \lambda_\beta \vartheta_4^{(\beta)} u_0^j - (\lambda_\alpha \varpi_{M-3}^{(\alpha)} + \lambda_\beta \vartheta_{M-2}^{(\beta)}) u_M^j \\ \vdots \\ \kappa u_{M-1}^{j-n} + \kappa f_{M-1}^j - \lambda_\alpha \varpi_{M-1}^{(\alpha)} u_0^j - \lambda_\beta \vartheta_M^{(\beta)} u_0^j - (\lambda_\alpha \varpi_1^{(\alpha)} + \lambda_\beta \vartheta_2^{(\beta)}) u_M^j \end{bmatrix}.$$

Case 2. Here we approximate the Riesz space fractional derivative and derive the Crank-Nicolson scheme for the Eq. (1). Letting $u(x_i, t_j) = u_i^j, \ \ f(x_i, t_j) = f_i^j$, we have

$$u_i^j + \eta_\alpha(\sum_{k=0}^{i} \varpi_k^{(\alpha)} u_{i-k}^j + \sum_{k=0}^{M-i} \varpi_k^{(\alpha)} u_{i+k}^j) + \eta_\beta(\sum_{k=0}^{i+1} \vartheta_k^{(\beta)} u_{i-k+1}^j + \sum_{k=0}^{M-i+1} \vartheta_k^{(\beta)} u_{i+k-1}^j)$$
$$= u_i^{j-1} - \eta_\alpha(\sum_{k=0}^{i} \varpi_k^{(\alpha)} u_{i-k}^{j-1} + \sum_{k=0}^{M-i} \varpi_k^{(\alpha)} u_{i+k}^{j-1}) - \eta_\beta(\sum_{k=0}^{i+1} \vartheta_k^{(\beta)} u_{i-k+1}^{j-1} + \sum_{k=0}^{M-i+1} \vartheta_k^{(\beta)} u_{i+k-1}^{j-1})$$
$$+\frac{\kappa}{2}(u_i^{j-n} + u_i^{j-n-1}) + \frac{\kappa}{2}(f_i^j + f_i^{j-1}) + R_2, \tag{51}$$

where $i = 1, ..., M$, $j = n+1, ...$ and R_2 is a truncation error term. Fixing $\eta_\alpha = \frac{c_\alpha \kappa K_\alpha h^{-\alpha}}{2}$ and $\eta_\beta = \frac{c_\beta \kappa K_\beta h^{-\beta}}{2}$, we write

$$\widetilde{D} = \eta_\alpha(A + A^T) + \eta_\beta(B + B^T), \quad U^j = [u_1^j, u_2^j, ..., u_{M-1}^j]^T, \tag{52}$$

where A and B are defined by (48). Thus Eq. (51) simplifies to the following form:

$$(I + \widetilde{D})U^j = (I - \widetilde{D})U^{j-1} + Q^j, \tag{53}$$

where

$$Q^j = \begin{bmatrix} \frac{\kappa}{2}(u_1^{j-n} + u_1^{j-n-1}) + \frac{\kappa}{2}(f_1^j + f_1^{j-1}) - l \\ \frac{\kappa}{2}(u_2^{j-n} + u_2^{j-n-1}) + \frac{\kappa}{2}(f_2^j + f_2^{j-1}) - l_2 \\ \frac{\kappa}{2}(u_3^{j-n} + u_3^{j-n-1}) + \frac{\kappa}{2}(f_3^j + f_3^{j-1}) - l_3 \\ \vdots \\ \frac{\kappa}{2}(u_{M-1}^{j-n} + u_{M-1}^{j-n-1}) + \frac{\kappa}{2}(f_{M-1}^j + f_{M-1}^{j-1}) - l_{M-1} \end{bmatrix}$$

and

$$l_s = (\eta_\alpha \varpi_s^{(\alpha)} + \eta_\beta \vartheta_{s+1}^{(\beta)})(u_0^j + u_0^{j-1}) + (\eta_\alpha \varpi_{M-s}^{(\alpha)} + \eta_\beta \vartheta_{M-s+1}^{(\beta)})(u_M^j + u_M^{j-1}),$$
$$s = 2, 3, ..., M-1,$$
$$l = (\eta_\alpha \varpi_1^{(\alpha)} + \eta_\beta(\vartheta_0^{(\beta)} + \vartheta_2^{(\beta)}))(u_0^j + u_0^{j-1}) + (\eta_\alpha \varpi_{M-1}^{(\alpha)} + \eta_\beta \vartheta_M^{(\beta)})(u_M^j + u_M^{j-1}).$$

6.1 Stability of methods

Lemma 6.1. [Thomas (2013)] Let A be a positive definite matrix of order $m-1$. Then, for any parameter $\nu \geq 0$, the following inequalities hold:

$$\left\|(I + \nu A)^{-1}\right\|_\infty \leq 1, \quad \left\|(I + \nu A)^{-1}(I - \nu A)\right\|_\infty \leq 1.$$

Theorem 6.1. For $h < (-\frac{3\beta(\beta-1)(2-\beta)(3+\beta)\cos(\frac{\alpha\pi}{2})}{4(\frac{3}{2})^\alpha \alpha(8\alpha-5)\cos(\frac{\beta\pi}{2})})^{\frac{1}{\beta-\alpha}}$, D is a strictly diagonally dominant matrix.

Proof. We have

$$D_{i,j} = \begin{cases} \lambda_\alpha \varpi_{j-i}^{(\alpha)} + \lambda_\beta \vartheta_{j-i+1}^{(\beta)}, & j > i+1, \\ \lambda_\alpha \varpi_1^{(\alpha)} + \lambda_\beta(\vartheta_0^{(\beta)} + \vartheta_2^{(\beta)}), & j = i+1, \\ 2\lambda_\alpha \varpi_0^{(\alpha)} + 2\lambda_\beta \vartheta_1^{(\beta)}, & j = i, \\ \lambda_\alpha \varpi_1^{(\alpha)} + \lambda_\beta(\vartheta_0^{(\beta)} + \vartheta_2^{(\beta)}), & j = i-1, \\ \lambda_\alpha \varpi_{i-j}^{(\alpha)} + \lambda_\beta \vartheta_{i-j+1}^{(\beta)}. & j < i-1, \end{cases}$$

where $\lambda_\alpha > 0$ for $0 < \alpha < 1$ and $\lambda_\beta < 0$ for $1 < \beta < 2$. According to Lemma 4.1, $\lambda_\alpha \varpi_k^{(\alpha)} < 0$ when $k \geq 4$ and according to Lemma 5.2, $\lambda_\beta \vartheta_k^{(\beta)} < 0$ when $k \geq 4$. For $k = 2, 3$, if $0 < \alpha \leq \frac{5}{8}$, $1 < \beta < 2$, then $\lambda_\alpha \varpi_2^{(\alpha)} < 0$ and $\lambda_\alpha \varpi_3^{(\alpha)} < 0$. Therefore, $D_{i,j} < 0$ when $j > i+1$ or $j < i-1$. According to the Lemmas 4.1 and 5.2, we have $\varpi_0^{(\alpha)} > 0,\ \ \vartheta_1^{(\beta)} < 0$, which leads to

$$2\lambda_\alpha \varpi_0^{(\alpha)} + 2\lambda_\beta \vartheta_1^{(\beta)} > 0.$$

Therefore $D_{i,i} > 0$. For $D_{i,i+1}$ and $D_{i,i-1}$, we have

$$\vartheta_0^{(\beta)} + \vartheta_2^{(\beta)} = \frac{\beta}{2} + \frac{\beta(\beta^2+\beta-4)}{4} = \frac{\beta(\beta+2)(\beta-1)}{4} > 0,$$
$$\varpi_1^{(\alpha)} < 0.$$

Since $\lambda_\alpha > 0$ and $\lambda_\beta < 0$, therefore

$$D_{i,i+1} = D_{i,i-1} = \lambda_\alpha \varpi_1^{(\alpha)} + \lambda_\beta(\vartheta_0^{(\beta)} + \vartheta_2^{(\beta)}) < 0.$$

For a given i, we can write

$$\begin{aligned}
\sum_{j=1,\, j\neq i}^{M-1} |D_{i,j}| &= \sum_{j=1}^{i-2} |D_{i,j}| + \sum_{j=i+2}^{M-1} |D_{i,j}| + |D_{i,i-1}| + |D_{i,i+1}| \\
&= -\sum_{j=1}^{i-2} (\lambda_\alpha \varpi_{i-j}^{(\alpha)} + \lambda_\beta \vartheta_{i-j+1}^{(\beta)}) - \sum_{j=i+2}^{M-1} (\lambda_\alpha \varpi_{j-i}^{(\alpha)} + \lambda_\beta \vartheta_{j-i+1}^{(\beta)}) \\
&\quad -2\lambda_\alpha \varpi_1^{(\alpha)} - 2\lambda_\beta(\vartheta_0^{(\beta)} + \vartheta_2^{(\beta)}) \\
&< \sum_{j=-\infty}^{i-2} (\lambda_\alpha \varpi_{i-j}^{(\alpha)} + \lambda_\beta \vartheta_{i-j+1}^{(\beta)}) - \sum_{j=i+2}^{+\infty} (\lambda_\alpha \varpi_{j-i}^{(\alpha)} + \lambda_\beta \vartheta_{j-i+1}^{(\beta)}) \\
&\quad -2\lambda_\alpha \varpi_1^{(\alpha)} - 2\lambda_\beta(\vartheta_0^{(\beta)} + \vartheta_2^{(\beta)}) \\
&= -2\lambda_\alpha \sum_{k=2}^{+\infty} \varpi_k^{(\alpha)} - 2\lambda_\beta \sum_{k=3}^{+\infty} \vartheta_k^{(\beta)} - 2\lambda_\alpha \varpi_1^{(\alpha)} - 2\lambda_\beta(\vartheta_0^{(\beta)} + \vartheta_2^{(\beta)}) \\
&= -2\lambda_\alpha \sum_{k=0}^{+\infty} \varpi_k^{(\alpha)} - 2\lambda_\beta \sum_{k=0}^{+\infty} \vartheta_k^{(\beta)} + 2\lambda_\alpha \varpi_1^{(\alpha)} + 2\lambda_\beta \vartheta_1^{(\beta)} \\
&= 2\lambda_\alpha \varpi_1^{(\alpha)} + 2\lambda_\beta \vartheta_1^{(\beta)} = |D_{i,i}|,
\end{aligned} \tag{54}$$

Therefore

$$\sum_{j=1,\, j\neq i}^{M-1} |D_{i,j}| < |D_{i,i}|.$$

Also, if $\frac{5}{8} < \alpha < 1,\ 1 < \beta < 2$, for $h < (-\frac{3\beta(\beta-1)(2-\beta)(3+\beta)\cos(\frac{\alpha\pi}{2})}{4(\frac{3}{2})^\alpha \alpha(8\alpha-5)\cos(\frac{\beta\pi}{2})})^{\frac{1}{\beta-\alpha}}$, we have

$$\lambda_\alpha \varpi_2^{(\alpha)} + \lambda_\beta \vartheta_3^{(\beta)} < 0,\ \ \lambda_\alpha \varpi_3^{(\alpha)} + \lambda_\beta \vartheta_4^{(\beta)} < 0.$$

Therefore, $D_{i,j} < 0$ when $j > i+1$ or $j < i-1$. Then relation (54) is valid for $\frac{5}{8} < \alpha < 1,\ 1 < \beta \leq 2$. Thus the matrix D is strictly diagonally dominant matrix. ■

Lemma 6.2. For $h < (-\frac{3\beta(\beta-1)(2-\beta)(3+\beta)\cos(\frac{\alpha\pi}{2})}{4(\frac{3}{2})^\alpha \alpha(8\alpha-5)\cos(\frac{\beta\pi}{2})})^{\frac{1}{\beta-\alpha}}$, the matrix D is symmetric and positive definite.

Proof. In view of Eq. (49), the matrix D is clearly symmetric. Let υ_0 be an eigenvalue of the matrix D. Then it follows by the Geršgorin circles Theorem [Varga (2010)] that

$$|\upsilon_0 - D_{i,i}| \le \sum_{j=1,\, j\neq i}^{M-1} |D_{i,j}|\,,$$

or

$$D_{i,i} - \sum_{j=1,\, j\neq i}^{M-1} |D_{i,j}| \le \upsilon_0 \le D_{i,i} + \sum_{j=1,\, j\neq i}^{M-1} |D_{i,j}|.$$

Then, by Theorem 6.1, we have

$$\upsilon_0 \ge D_{i,i} - \sum_{j=1,\, j\neq i}^{M-1} |D_{i,j}| \ge 0,$$

which shows that D is positive definite. ■

Remark 6.1. For $h < (-\frac{3\beta(\beta-1)(2-\beta)(3+\beta)\cos(\frac{\alpha\pi}{2})}{4(\frac{3}{2})^{\alpha}\alpha(8\alpha-5)\cos(\frac{\beta\pi}{2})})^{\frac{1}{\beta-\alpha}}$, $\widetilde{D}$ is strictly diagonally dominant and symmetric positive definite.

Theorem 6.2. The first numerical method (50) for $h < (-\frac{3\beta(\beta-1)(2-\beta)(3+\beta)\cos(\frac{\alpha\pi}{2})}{4(\frac{3}{2})^{\alpha}\alpha(8\alpha-5)\cos(\frac{\beta\pi}{2})})^{\frac{1}{\beta-\alpha}}$ is stable.

Proof. Let U^j be the numerical solution and u^j be an exact solution. Since the matrix (I + D) is invertible, we have

$$\varepsilon^j = P\varepsilon^{j-1},$$

where

$$\varepsilon^j = U^j - u^j,\ P = (I+D)^{-1}. \tag{55}$$

By Lemma 6.1, we have

$$\left\|\varepsilon^j\right\|_\infty = \left\|P^{j-1}\varepsilon^0\right\|_\infty \le \|P\|_\infty^{j-1}\left\|\varepsilon^0\right\|_\infty = \left\|(I+D)^{-1}\right\|_\infty^{j-1}\left\|\varepsilon^0\right\|_\infty \le \left\|\varepsilon^0\right\|_\infty.$$

Thus the numerical method (50) is stable. ■

Theorem 6.3. For $h < (-\frac{3\beta(\beta-1)(2-\beta)(3+\beta)\cos(\frac{\alpha\pi}{2})}{4(\frac{3}{2})^{\alpha}\alpha(8\alpha-5)\cos(\frac{\beta\pi}{2})})^{\frac{1}{\beta-\alpha}}$, the second numerical method (53) is stable.

Proof. Let U^j be the numerical solution and u^j be an exact solution. Since the matrix (I + $\widetilde{D}$) is invertible, we have

$$\varepsilon^j = W\varepsilon^{j-1},$$

where

$$\varepsilon^j = U^j - u^j,\ W = (I+\widetilde{D})^{-1}(I-\widetilde{D}), \tag{56}$$

Then
$\varepsilon^j = W^{j-1}\varepsilon^0$,
which, by Lemma 6.1, yields
$\left\|\varepsilon^j\right\|_\infty = \left\|W^{j-1}\varepsilon^0\right\|_\infty \le \|W\|_\infty^{j-1}\left\|\varepsilon^0\right\|_\infty = \left\|(I+\widetilde{D})^{-1}(I-\widetilde{D})\right\|_\infty^{j-1}\left\|\varepsilon^0\right\|_\infty \le \left\|\varepsilon^0\right\|_\infty.$
This shows that numerical method (53) is stable. ■

6.2 Convergence of methods

This subsection is concerned with convergence of the methods presented in Section 5. For the first method, we can write

$$\begin{aligned}
&\frac{\partial u(x_i,t_j)}{\partial t} = \frac{u(x_i,t_j)-u(x_i,t_{j-1})}{\kappa} + \mathrm{O}(\kappa),\\
&K_\alpha \frac{\partial^\alpha u(x_i,t_j)}{\partial|x|^\alpha} = \lambda_\alpha(\sum_{k=0}^{i}\varpi_k^{(\alpha)}u_{i-k}^j + \sum_{k=0}^{M-i}\varpi_k^{(\alpha)}u_{i+k}^j) + \mathrm{O}(h^2),\\
&K_\beta \frac{\partial^\beta u(x_i,t_j)}{\partial|x|^\beta} = \lambda_\beta(\sum_{k=0}^{i+1}\vartheta_k^{(\beta)}u_{i-k+1}^j + \sum_{k=0}^{M-i+1}\vartheta_k^{(\beta)}u_{i+k-1}^j) + \mathrm{O}(h^2),
\end{aligned} \tag{57}$$

Thus the local truncation error of (47) will be of the form:
$T_{i,j} = \mathrm{O}(\kappa^2 + \kappa h^2)$

Theorem 6.4. Let U^j be the numerical solution and u^j be an exact solution of (50). Then for $h < (-\frac{3\beta(\beta-1)(2-\beta)(3+\beta)\cos(\frac{\alpha\pi}{2})}{4(\frac{3}{2})^\alpha\alpha(8\alpha-5)\cos(\frac{\beta\pi}{2})})^{\frac{1}{\beta-\alpha}}$, we have

$$\left\|U^j - u^j\right\|_\infty \le CO(\kappa + h^2), \tag{58}$$

where C is a positive constant.

Proof. It is easy to find that

$$\begin{aligned}
&u_i^j + \lambda_\alpha(\sum_{k=0}^{i}\varpi_k^{(\alpha)}u_{i-k}^j + \sum_{k=0}^{M-i}\varpi_k^{(\alpha)}u_{i+k}^j)\\
&\qquad +\lambda_\beta(\sum_{k=0}^{i+1}\vartheta_k^{(\beta)}u_{i-k+1}^j + \sum_{k=0}^{M-i+1}\vartheta_k^{(\beta)}u_{i+k-1}^j) = u_i^{j-1} + \kappa u_i^{j-n} + \kappa f_i^j,
\end{aligned} \tag{59}$$

and

$$\begin{aligned}
&U_i^j + \lambda_\alpha(\sum_{k=0}^{i}\varpi_k^{(\alpha)}U_{i-k}^j + \sum_{k=0}^{M-i}\varpi_k^{(\alpha)}U_{i+k}^j)\\
&\qquad +\lambda_\beta(\sum_{k=0}^{i+1}\vartheta_k^{(\beta)}U_{i-k+1}^j + \sum_{k=0}^{M-i+1}\vartheta_k^{(\beta)}U_{i+k-1}^j) = U_i^{j-1} + \kappa U_i^{j-n} + \kappa f_i^j.
\end{aligned} \tag{60}$$

Letting $e_i^j = U_i^j - u_i^j$ and using (59) and (60), we obtain

$$\begin{aligned}
&e_i^j + \lambda_\alpha(\sum_{k=0}^{i}\varpi_k^{(\alpha)}e_{i-k}^j + \sum_{k=0}^{M-i}\varpi_k^{(\alpha)}e_{i+k}^j)\\
&\qquad +\lambda_\beta(\sum_{k=0}^{i+1}\vartheta_k^{(\beta)}e_{i-k+1}^j + \sum_{k=0}^{M-i+1}\vartheta_k^{(\beta)}e_{i+k-1}^j) = e_i^{j-1} + \mathrm{O}(\kappa^2 + \kappa h^2),
\end{aligned} \tag{61}$$

which can equivalently be written in matrix-vector form as

$$(I+D)\varepsilon^j = I\varepsilon^{j-1} + \mathrm{O}(\kappa^2+\kappa h^2)\chi,$$

where

$$\varepsilon^j = [e_1^j, e_2^j, ..., e_n^j]^T,\ \chi = [1,1,...,1]^T,\ D = \lambda_\alpha(A+A^T) + \lambda_\beta(B+B^T).$$

Writing

$$P = (I+D)^{-1},\ \ F = \mathrm{O}(\kappa^2+\kappa h^2)(I+D)^{-1},$$

we get

$$\varepsilon^j = P\varepsilon^{j-1} + F,$$

which, on iterating and using the given initial condition, yields

$$\varepsilon^j = (P^{j-1} + P^{j-2} + ... + I)F.$$

By Lemma 6.1, we can write

$$\begin{aligned}\left\|\varepsilon^j\right\|_\infty &\le (\left\|P^{j-1}\right\|_\infty + \left\|P^{j-2}\right\|_\infty + ... + \|I\|_\infty)\|F\|_\infty \\ &= (\|P\|_\infty^{j-1} + \|P\|_\infty^{j-2} + ... + \|I\|_\infty)\|F\|_\infty \\ &\le (1+1+...+1)\|F\|_\infty \\ &\le j\mathrm{O}(\kappa^2+\kappa h^2) = T\mathrm{O}(\kappa+h^2).\end{aligned}$$

Therefore we have

$$\left\|\varepsilon^j\right\|_\infty \le C\mathrm{O}(\kappa+h^2).$$

■

For the second method, we can write

$$\begin{aligned}\frac{u(x_i,t_j)-u(x_i,t_{j-1})}{\kappa} &= \frac{1}{2}(K_\alpha\tfrac{\partial^\alpha u(x_i,t_j)}{\partial|x|^\alpha} + K_\beta\tfrac{\partial^\beta u(x_i,t_j)}{\partial|x|^\beta}) \\ &\quad+\tfrac{1}{2}(K_\alpha\frac{\partial^\alpha u(x_i,t_{j-1})}{\partial|x|^\alpha} + K_\beta\tfrac{\partial^\beta u(x_i,t_{j-1})}{\partial|x|^\beta}) + \mathrm{O}(\kappa^2) \\ K_\alpha\frac{\partial^\alpha u(x_i,t_j)}{\partial|x|^\alpha} &= \eta_\alpha(\sum_{k=0}^{i}\varpi_k^{(\alpha)}u_{i-k}^j + \sum_{k=0}^{M-i}\varpi_k^{(\alpha)}u_{i+k}^j) + \mathrm{O}(h^2), \\ K_\beta\tfrac{\partial^\beta u(x_i,t_j)}{\partial|x|^\beta} &= \eta_\beta(\sum_{k=0}^{i+1}\vartheta_k^{(\beta)}u_{i-k+1}^j + \sum_{k=0}^{M-i+1}\vartheta_k^{(\beta)}u_{i+k-1}^j) + \mathrm{O}(h^2).\end{aligned} \tag{62}$$

Thus the local truncation error of (51) will be of the following form:

$$T_{i,j} = \mathrm{O}(\kappa^3+\kappa h^2).$$

Theorem 6.5. Let U^j be the numerical solution and u^j be an exact solution of (53). Then, for $h < (-\frac{3\beta(\beta-1)(2-\beta)(3+\beta)\cos(\frac{\alpha\pi}{2})}{4(\frac{3}{2})^\alpha\alpha(8\alpha-5)\cos(\frac{\beta\pi}{2})})^{\frac{1}{\beta-\alpha}}$, we have

$$\left\|U^j - u^j\right\|_\infty \le CO(\kappa^2+h^2), \tag{63}$$

where C is a positive constant.

Proof. Obviously

$$\begin{aligned} &u_i^j + \eta_\alpha(\sum_{k=0}^{i} \varpi_k^{(\alpha)} u_{i-k}^j + \sum_{k=0}^{M-i} \varpi_k^{(\alpha)} u_{i+k}^j) + \eta_\beta(\sum_{k=0}^{i+1} \vartheta_k^{(\beta)} u_{i-k+1}^j \\ &+ \sum_{k=0}^{M-i+1} \vartheta_k^{(\beta)} u_{i+k-1}^j) = u_i^{j-1} - \eta_\alpha(\sum_{k=0}^{i} \varpi_k^{(\alpha)} u_{i-k}^{j-1} + \sum_{k=0}^{M-i} \varpi_k^{(\alpha)} u_{i+k}^{j-1}) \\ &-\eta_\beta(\sum_{k=0}^{i+1} \vartheta_k^{(\beta)} u_{i-k+1}^{j-1} + \sum_{k=0}^{M-i+1} \vartheta_k^{(\beta)} u_{i+k-1}^{j-1}) + \tfrac{\kappa}{2}(u_i^{j-n} + u_i^{j-n-1}) \\ &+\tfrac{\kappa}{2}(f_i^j + f_i^{j-1}), \end{aligned} \tag{64}$$

and

$$\begin{aligned} &U_i^j + \eta_\alpha(\sum_{k=0}^{i} \varpi_k^{(\alpha)} U_{i-k}^j + \sum_{k=0}^{M-i} \varpi_k^{(\alpha)} U_{i+k}^j) + \eta_\beta(\sum_{k=0}^{i+1} \vartheta_k^{(\beta)} U_{i-k+1}^j \\ &+ \sum_{k=0}^{M-i+1} \vartheta_k^{(\beta)} U_{i+k-1}^j) = U_i^{j-1} - \eta_\alpha(\sum_{k=0}^{i} \varpi_k^{(\alpha)} U_{i-k}^{j-1} + \sum_{k=0}^{M-i} \varpi_k^{(\alpha)} U_{i+k}^{j-1}) \\ &-\eta_\beta(\sum_{k=0}^{i+1} \vartheta_k^{(\beta)} U_{i-k+1}^{j-1} + \sum_{k=0}^{M-i+1} \vartheta_k^{(\beta)} U_{i+k-1}^{j-1}) + \tfrac{\kappa}{2}(U_i^{j-n} + U_i^{j-n-1}) \\ &+\tfrac{\kappa}{2}(f_i^j + f_i^{j-1}), \end{aligned} \tag{65}$$

Let us set $e_i^j = U_i^j - u_i^j$ and use (64) and (65) to obtain

$$\begin{aligned} &e_i^j + \eta_\alpha(\sum_{k=0}^{i} \varpi_k^{(\alpha)} e_{i-k}^j + \sum_{k=0}^{M-i} \varpi_k^{(\alpha)} e_{i+k}^j) + \eta_\beta(\sum_{k=0}^{i+1} \vartheta_k^{(\beta)} e_{i-k+1}^j \\ &+ \sum_{k=0}^{M-i+1} \vartheta_k^{(\beta)} e_{i+k-1}^j) = e_i^{j-1} - \eta_\alpha(\sum_{k=0}^{i} \varpi_k^{(\alpha)} e_{i-k}^{j-1} + \sum_{k=0}^{M-i} \varpi_k^{(\alpha)} e_{i+k}^{j-1}) \\ &-\eta_\beta(\sum_{k=0}^{i+1} \vartheta_k^{(\beta)} e_{i-k+1}^{j-1} + \sum_{k=0}^{M-i+1} \vartheta_k^{(\beta)} e_{i+k-1}^{j-1}), \end{aligned} \tag{66}$$

which can be expressed in matrix-vector form as

$$(I + \widetilde{D})\varepsilon^j = (I - \widetilde{D})\varepsilon^{j-1} + \mathrm{O}(\kappa^3 + \kappa h^2)\chi,$$

where

$$\varepsilon^j = [e_1^j, e_2^j, ..., e_n^j]^T, \ \ \chi = [1, 1, ..., 1]^T, \ \ \widetilde{D} = \eta_\alpha(A + A^T) + \eta_\beta(B + B^T).$$

Let us take

$$W = (I + \widetilde{D})^{-1}(I + \widetilde{D}), \ Q = \mathrm{O}(\kappa^3 + \kappa h^2)(I + \widetilde{D})^{-1}$$

so that

$$\varepsilon^j = W\varepsilon^{j-1} + Q,$$

which, on iterating and using the given initial condition, becomes

$$\varepsilon^j = (W^{j-1} + W^{j-2} + ... + I)Q.$$

On the other hand, it follows by Lemma 6.1 that

$$\begin{aligned}\left\|\varepsilon^j\right\|_\infty &\le (\left\|W^{j-1}\right\|_\infty + \left\|W^{j-2}\right\|_\infty + ... + \|I\|_\infty)\|Q\|_\infty \\ &= (\|W\|_\infty^{j-1} + \|W\|_\infty^{j-2} + ... + \|I\|_\infty)\|Q\|_\infty \\ &\le (1+1+...+1)\|Q\|_\infty \\ &\le j\mathrm{O}(\kappa^3 + \kappa h^2) = T\mathrm{O}(\kappa^2 + h^2).\end{aligned}$$

In consequence, we obtain

$$\left\|\varepsilon^j\right\|_\infty \le C\mathrm{O}(\kappa^2 + h^2).$$

■

7 Test examples

In this section, we show the efficacy and accuracy of the proposed methods with the aid of examples.

Example 7.1. We consider the following RFDED

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^\beta u(x,t)}{\partial |x|^\beta} + u(x,t-1) + f(x,t); \tag{67}$$

subject to the initial condition:

$$\begin{aligned}&u(x,t) = x^2(1-x)^2 e^{-t}, \ \ -1 \le t \le 0, \ \ 0 \le x \le 1, \\ &u(0,t) = u(1,t) = 0, \ \ 0 \le t \le T,\end{aligned} \tag{68}$$

where $1 < \beta \le 2$ and

$$\begin{aligned}f(x,t) = &-x^2(1-x)^2 e^{-t} - x^2(1-x)^2 e^{1-t} \\ &+ \tfrac{e^{-t}}{2\cos(\frac{\beta\pi}{2})}\big(\tfrac{24((1-x)^{4-\beta}+x^{4-\beta})}{\Gamma(5-\beta)} - \tfrac{12((1-x)^{3-\beta}+x^{3-\beta})}{\Gamma(4-\beta)} \\ &+ \tfrac{2((1-x)^{2-\beta}+x^{2-\beta})}{\Gamma(3-\beta)}\big),\end{aligned}$$

The exact solution of the problem (67)-(68) is $u(x,t) = x^2(1-x)^2 e^{-t}$. Numerical results for the problem (67)-(68) are given by Tabs. 1-2 and shown in Fig. 1.

Table 1: The absolute errors and the convergence orders of the first method (50) for example 7.1 (67-68)

	$\beta = 1.2$		$\beta = 1.5$		$\beta = 1.8$	
$h = \kappa$	$Error$	$Order$	$Error$	$Order$	$Error$	$Order$
1/16	8.1804e-004	–	8.4754e-004	–	8.6058e-004	–
1/32	2.8459e-004	1.52	2.7131e-004	1.64	2.6043e-004	1.72
1/64	1.1126e-004	1.35	9.7550e-005	1.47	8.6715e-005	1.59
1/128	4.8071e-005	1.21	3.9501e-005	1.30	3.2532e-005	1.41

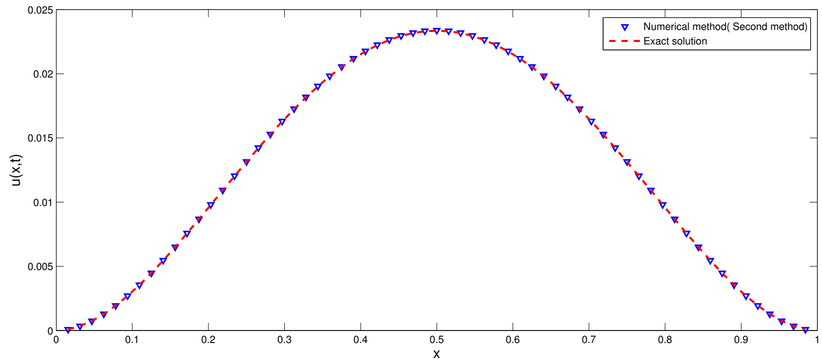

Figure 1: The numerical approximation and exact solution by the second method for example 7.1 (67-68) (RFDED), for $\beta = 1.8$, when T = 1

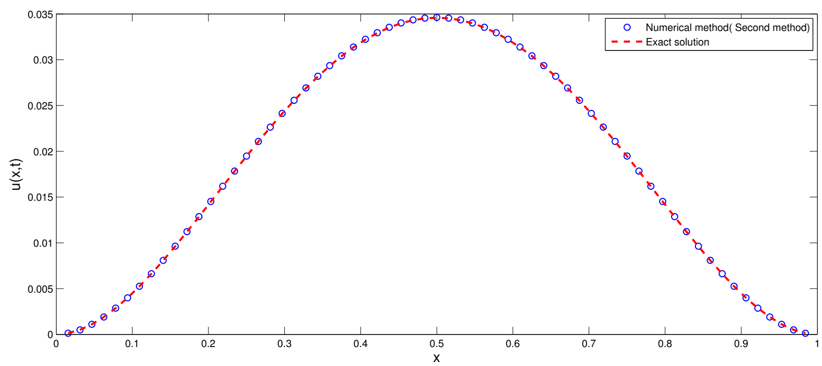

Figure 2: The numerical approximation and exact solution by the second method for example 7.2 (69-70)(RFDED), for $\alpha = 0.1$ and $\beta = 1.8$, when T = 1

Example 7.2. Consider the RFDED

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^{\alpha} u(x,t)}{\partial |x|^{\alpha}} + \frac{\partial^{\beta} u(x,t)}{\partial |x|^{\beta}} + u(x,t-1) + f(x,t); \tag{69}$$

subject to the initial condition:

$$\begin{aligned} &u(x,t) = x^2(1-x)^2\cos(t), \ \ -1 \le t \le 0, \ \ 0 \le x \le 1, \\ &u(0,t) = u(1,t) = 0, \ \ 0 \le t \le T, \end{aligned} \tag{70}$$

where $0 < \alpha < 1, \ 1 < \beta \le 2$ and

$$\begin{aligned} f(x,t) = &-\sin(t)x^2(1-x)^2 - x^2(1-x)^2\cos(t-1) \\ &+\tfrac{\cos(t)}{2\cos(\frac{\alpha\pi}{2})}\Big(\tfrac{\Gamma(5)((1-x)^{4-\alpha}+x^{4-\alpha})}{\Gamma(5-\alpha)} + \tfrac{\Gamma(3)((1-x)^{2-\alpha}+x^{2-\alpha})}{\Gamma(3-\alpha)} - \tfrac{2\Gamma(4)((1-x)^{3-\alpha}+x^{3-\alpha})}{\Gamma(4-\alpha)}\Big)\Big) \\ &+\tfrac{\cos(t)}{2\cos(\frac{\beta\pi}{2})}\Big(\tfrac{\Gamma(5)((1-x)^{4-\beta}+x^{4-\beta})}{\Gamma(5-\beta)} + \tfrac{\Gamma(3)((1-x)^{2-\beta}+x^{2-\beta})}{\Gamma(3-\beta)} - \tfrac{2\Gamma(4)((1-x)^{3-\beta}+x^{3-\beta})}{\Gamma(4-\beta)}\Big)\Big). \end{aligned}$$

Table 2: The absolute errors and the convergence orders of the second method (53) for example 7.1 (67-68)

	$\beta = 1.2$		$\beta = 1.5$		$\beta = 1.8$	
$h = \kappa$	$Error$	$Order$	$Error$	$Order$	$Error$	$Order$
1/16	5.2976e-004	–	6.4753e-004	–	7.4404e-004	–
1/32	1.6155e-004	1.71	1.6133e-004	2.00	1.8018e-004	2.04
1/64	4.5724e-005	1.82	4.5524e-005	1.83	4.3620e-005	2.04
1/128	1.2193e-005	1.91	1.2314e-005	1.89	1.0559e-005	2.04

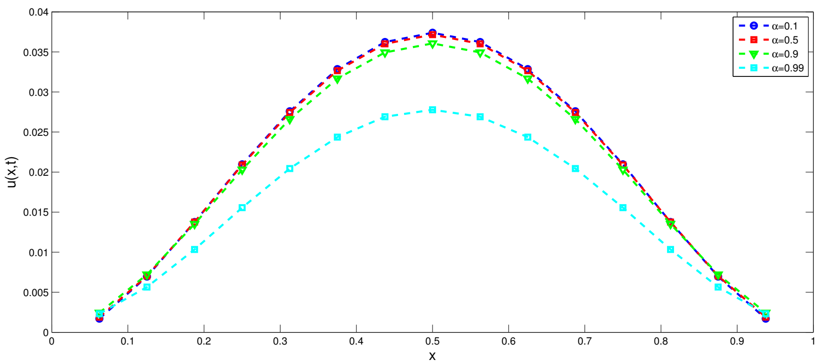

Figure 3: The numerical approximation by the second method for example 7.2 (69-70)(RFDED), for various $\alpha = 0.1, 0.5, 0.9, 0.99$, when T = 1 and $\beta = 1.01$

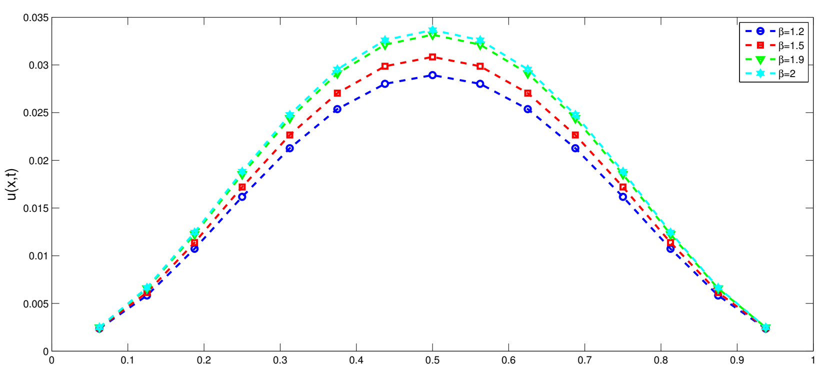

Figure 4: The numerical approximation by the second method for example 7.2 (69-70)(RFDED), for various $\beta = 1.2, 1.5, 1.9, 2$, when T = 1 and $\alpha = 0.99$

Table 3: The absolute errors and the convergence orders of the second method (53) for example 7.2 (69-70)

	$\beta = 1.2$		$\beta = 1.5$		$\beta = 1.8$	
	$Error$	$Order$	$Error$	$Order$	$Error$	$Order$
$\alpha = 0.1, h = \kappa$						
1/16	5.6605e-004	–	7.0613e-004	–	8.1183e-004	–
1/32	1.7032e-004	1.73	1.6957e-004	2.05	1.9726e-004	2.04
1/64	4.8646e-005	1.81	4.5616e-005	1.89	4.7820e-005	2.04
1/128	1.2997e-005	1.90	1.2330e-005	1.90	1.1587e-005	2.04
$\alpha = 0.5, h = \kappa$						
1/16	3.3282e-004	–	5.2206e-004	–	6.7321e-004	–
1/32	1.2341e-004	1.43	1.4371e-004	1.86	1.6488e-004	2.03
1/64	3.8985e-005	1.66	4.2741e-005	1.75	4.0088e-005	2.04
1/128	1.1144e-005	1.81	1.1904e-005	1.84	9.7002e-006	2.05
$\alpha = 0.9, h = \kappa$						
1/16	9.0618e-004	–	5.7238e-004	–	3.7095e-004	–
1/32	2.1508e-004	2.07	9.0954e-005	2.65	7.8189e-005	2.24
1/64	4.6083e-005	2.22	1.9051e-005	2.25	1.6563e-005	2.23
1/128	9.4770e-006	2.28	4.2314e-006	2.17	3.7547e-006	2.14

The exact solution of this problem is $u(x,t) = \cos(t)x^2(1-x)^2$. Numerical results for the given problem are given by Tab. 3 and Figs. 2-4. In Tab. 3, scheme (53) for $h < 0.0952$ is stable and convergent for $\alpha = 0.8,\ \beta = 1.2$.

8 Conclusion

In this paper, we applied the FBDF method of second order and the shifted Grünwald method for solving the Riesz space fractional advection-dispersion equations with delay . The FBDF2 and the shifted Grünwald methods are introduced. Furthermore we find the analytical solution for RFDED in terms of t Mittag-Leffler type functions. The approximation of solution for the Riesz space fractional advection-dispersion equations with delay relies on the FBDF2 method and the WSGD operators, and is obtained by applying the finite difference method. It is shown that the schemes for $h < (-\frac{3\beta(\beta-1)(2-\beta)(3+\beta)\cos(\frac{\alpha\pi}{2})}{4(\frac{3}{2})^{\alpha}\alpha(8\alpha-5)\cos(\frac{\beta\pi}{2})})^{\frac{1}{\beta-\alpha}}$ are stable and convergent with the accuracy of $\mathrm{O}(\kappa + h^2)$ and $\mathrm{O}(\kappa^2 + h^2)$ respectively. Numerical methods presented in this paper are illustrated with the help of the examples. The obtained results clearly demonstrate that our methods are efficient and produce accurate results.

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References

Bagley, R. L.; Calico, R. (1991): Fractional order state equations for the control of viscoelasticallydamped structures. *Journal of Guidance, Control, and Dynamics*, vol. 14, no. 2, pp. 304-311.

Cattani, C. (2018): Sinc-fractional operator on shannon wavelet space. *Frontiers in Physics*, vol. 6, pp. 118-134.

Čermák, J.; Horníček, J.; Kisela, T. (2016): Stability regions for fractional differential systems with a time delay. *Communications in Nonlinear Science and Numerical Simulation*, vol. 31, no. 1, pp. 108-123.

Ding, H.; Li, C. (2013): Numerical algorithms for the fractional diffusion-wave equation with reaction term. *Abstract and Applied Analysis*, vol. 2013.

Ding, H.; Li, C.; Chen, Y. (2015): High-order algorithms for riesz derivative and their applications (ii). *Journal of Computational Physics*, vol. 293, pp. 218-237.

Feng, G. (2017): General fractional calculus in non-singular power-law kernel applied to model anomalous diffusion phenomena in heat transfer problems. *Thermal Science*, vol. 21, no. suppl 1, pp. S11-S18.

Galeone, L.; Garrappa, R. (2006): On multistep methods for differential equations of fractional order. *Mediterranean Journal of Mathematics*, vol. 3, no. 3-4, pp. 565-580.

Garrappa, R. (2015): Trapezoidal methods for fractional differential equations: Theoretical and computational aspects. *Mathematics and Computers in Simulation*, vol. 110, pp. 96-112.

Gorenflo, R. (1997): Fractional calculus: some numerical methods. *Courses and Lectures-International Centre for Mechanical Sciences*, pp. 277-290.

Heris, M. S.; Javidi, M. (2017): On fbdf5 method for delay differential equations of fractional order with periodic and anti-periodic conditions. *Mediterranean Journal of Mathematics*, vol. 14, no. 3, pp. 134.

Heris, M. S.; Javidi, M. (2017): On fractional backward differential formulas for fractional delay differential equations with periodic and anti-periodic conditions. *Applied Numerical Mathematics*, vol. 118, pp. 203-220.

Heris, M. S.; Javidi, M. (2018): On fractional backward differential formulas methods for fractional differential equations with delay. *International Journal of Applied and Computational Mathematics*, vol. 4, no. 2, pp. 72.

Heris, M. S.; Javidi, M. (2018): Second order difference approximation for a class of riesz space fractional advection-dispersion equations with delay. arxiv:1811.10513.

Heris, M. S.; Javidi, M. (2019): Fractional backward differential formulas for the distributed-order differential equation with time delay. *Bulletin of the Iranian Mathematical Society*, vol. 45, no. 4, pp. 1159-1176.

Huang, C.; Vandewalle, S. (2012): Unconditionally stable difference methods for delay partial differential equations. *Numerische Mathematik*, vol. 122, no. 3, pp. 579-601.

Huang, F.; Liu, F. (2005): The fundamental solution of the space-time fractional advection-dispersion equation. *Journal of Applied Mathematics and Computing*, vol. 18, no. 1, pp. 339-350.

Ilic, M.; Liu, F.; Turner, I.; Anh, V. (2005): Numerical approximation of a fractional-in-space diffusion equation, i. *Fractional Calculus and Applied Analysis*, vol. 8, no. 3, pp. 323-341.

Jackiewicz, Z.; Zubik-Kowal, B. (2006): Spectral collocation and waveform relaxation methods for nonlinear delay partial differential equations. *Applied Numerical Mathematics*, vol. 56, no. 3-4, pp. 433-443.

Javidi, M.; Heris, M. S. (2019): Analysis and numerical methods for the riesz space distributed-order advection-diffusion equation with time delay. *SeMA Journal*, pp. 1-19.

Kilbas, A. A. A.; Srivastava, H. M.; Trujillo, J. J. (2006): *Theory and Applications of Fractional Differential Equations*, volume 204. Elsevier Science Limited.

Liu, F.; Anh, V.; Turner, I. (2004): Numerical solution of the space fractional fokker-planck equation. *Journal of Computational and Applied Mathematics*, vol. 166, no. 1, pp. 209-219.

Liu, F.; Anh, V. V.; Turner, I.; Zhuang, P. (2003): Time fractional advection-dispersion equation. *Journal of Applied Mathematics and Computing*, vol. 13, no. 1, pp. 233-245.

Liu, Q.; Zeng, F.; Li, C. (2015): Finite difference method for time-space-fractional schrödinger equation. *International Journal of Computer Mathematics*, vol. 92, no. 7, pp. 1439-1451.

Lubich, C. (1986): Discretized fractional calculus. *SIAM Journal on Mathematical Analysis*, vol. 17, no. 3, pp. 704-719.

Marks, R. J.; Hall, M. W. (1981): Differintegral interpolation from a bandlimited signal's samples. *IEEE Transactions on Acoustics, Speech, and Signal Processing*, vol. 29, no. 4, pp. 872-877.

Meerschaert, M. M.; Tadjeran, C. (2004): Finite difference approximations for fractional advection-dispersion flow equations. *Journal of Computational and Applied Mathematics*, vol. 172, no. 1, pp. 65-77.

Meerschaert, M. M.; Tadjeran, C. (2006): Finite difference approximations for two-sided space-fractional partial differential equations. *Applied Numerical Mathematics*, vol. 56, no. 1, pp. 80-90.

Momani, S.; Odibat, Z. (2008): A novel method for nonlinear fractional partial differential equations: combination of dtm and generalized taylor's formula. *Journal of Computational and Applied Mathematics*, vol. 220, no. 1, pp. 85-95.

Momani, S.; Odibat, Z. (2008): Numerical solutions of the space-time fractional advection-dispersion equation. *Numerical Methods for Partial Differential Equations*, vol. 24, no. 6, pp. 1416-1429.

Momani, S.; Odibat, Z.; Erturk, V. S. (2007): Generalized differential transform method for solving a space-and time-fractional diffusion-wave equation. *Physics Letters A*, vol. 370, no. 5, pp. 379-387.

Shen, S.; Liu, F.; Anh, V. (2008): Fundamental solution and discrete random walk model for a time-space fractional diffusion equation of distributed order. *Journal of Applied Mathematics and Computing*, vol. 28, no. 1, pp. 147-164.

Shen, S.; Liu, F.; Anh, V. (2011): Numerical approximations and solution techniques for the space-time riesz-caputo fractional advection-diffusion equation. *Numerical Algorithms*, vol. 56, no. 3, pp. 383-403.

Simo, H.; Woafo, P. (2016): Effects of asymmetric potentials on bursting oscillations in duffing oscillator. *Optik-International Journal for Light and Electron Optics*, vol. 127, no. 20, pp. 8760-8766.

Sousa, E. (2012): A second order explicit finite difference method for the fractional advection diffusion equation. *Computers & Mathematics with Applications*, vol. 64, no. 10, pp. 3141-3152.

Sulaiman, T. A.; Yavuz, M.; Bulut, H.; Baskonus, H. M. (2019): Investigation of the fractional coupled viscous burgers equation involving mittag-leffler kernel. *Physica A: Statistical Mechanics and Its Applications*, vol. 527, pp. 1-20.

Tadjeran, C.; Meerschaert, M. M.; Scheffler, H. P. (2006): A second-order accurate numerical approximation for the fractional diffusion equation. *Journal of Computational Physics*, vol. 213, no. 1, pp. 205-213.

Tanthanuch, J. (2012): Symmetry analysis of the nonhomogeneous inviscid burgers equation with delay. *Communications in Nonlinear Science and Numerical Simulation*, vol. 17, no. 12, pp. 4978-4987.

Thomas, J. W. (2013): *Numerical Partial Differential Equations: Finite Difference Methods*, volume 22. Springer Science & Business Media.

Tian, W.; Zhou, H.; Deng, W. (2015): A class of second order difference approximations for solving space fractional diffusion equations. *Mathematics of Computation*, vol. 84, no. 294, pp. 1703-1727.

Varga, R. S. (2010): *Geršgorin and His Circles*, volume 36. Springer Science & Business Media.

Weaver Jr, W.; Timoshenko, S. P.; Young, D. H. (1990): *Vibration Problems in Engineering*. John Wiley & Sons.

Wu, G. C.; Baleanu, D.; Deng, Z. G.; Zeng, S. D. (2015): Lattice fractional diffusion equation in terms of a riesz-caputo difference. *Physica A: Statistical Mechanics and Its Applications*, vol. 438, pp. 335-339.

Wu, G. C.; Baleanu, D.; Xie, H. P. (2016): Riesz riemann-liouville difference on discrete domains. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, vol. 26, no. 8, 084308.

Wu, J. (2012): *Theory and Applications of Partial Functional Differential Equations*, volume 119. Springer Science & Business Media.

Yang, A. M.; Han, Y.; Li, J.; Liu, W. X. (2016): On steady heat flow problem involving yang-srivastava-machado fractional derivative without singular kernel. *Thermal Science*, vol. 20, no. suppl 3, pp. S719-S723.

Yang, Q.; Liu, F.; Turner, I. (2010): Numerical methods for fractional partial differential equations with riesz space fractional derivatives. *Applied Mathematical Modelling*, vol. 34, no. 1, pp. 200-218.

Yang, S. (2018): Finite difference method for riesz space fractional diffusion equations with delay and a nonlinear source term. *Journal of Nonlinear Sciences and Applications*, vol. 11, no. 1, pp. 17-25.

Yang, X. J. (2019): *General Fractional Derivatives: Theory, Methods and Applications*. Chapman and Hall/CRC.

Yang, X. J.; Abdel-Aty, M.; Cattani, C. (2019): A new general fractional-order derivative with rabotnov fractional-exponential kernel applied to model the anomalous heat transfer. *Thermal Science*, vol. 23, no. 3.

Zubik-Kowal, B. (2000): Chebyshev pseudospectral method and waveform relaxation for differential and differential-functional parabolic equations. *Applied Numerical Mathematics*, vol. 34, no. 2-3, pp. 309-328.