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Random Loads Fatigue and Dynamic Simulation: a New Procedure to Evaluate the Behaviour of Non-Linear Systems

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Abstract: In this paper the problem of the correct evaluation of the stress state of mechanical components of non linear systems in the frequency domain was analysed. This is one of the most important steps in the frequency domain evaluation of the fatigue behaviour of components submitted to random loads. A new methodology to obtain an accurate representation in frequency domain of the non-linear behaviour of the system as well as of the stress state of the components both in terms of power spectral density (PSD) function and of frequency response function (FRF) was proposed and validated. This methodology is useful in multibody simulation and modelling environment (MBS) and fits well into a damage evaluation scenario that also involves the non Gaussianity of the process.

Nomenclature

σ	Universe standard deviation
σ_c	Generic sample standard deviation
n	Generic sample numerousness
dt	Time sampling interval
df	Frequency discretization interval
ω	Angular frequency
j	Imaginary unit
x	System state variable (state space model)
$\dot{x}$	First derivative of system state variable (state space model)
u	Input variable (state space model)
y	Output variable (state space model)
$\mathbf{A}$	State to state matrix (state space model)

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$\mathbf{B}$	Input to state matrix (state space model)
$\mathbf{C}$	State to output matrix (state space model)
$\mathbf{D}$	Input to output matrix (state space model)
$\mathbf{I}$	Identity matrix
q	Generalised or Lagrangian coordinate of the generic flexible body
$\mathbf{H}_q$	Matrix of the frequency response functions obtained starting with generic inputs and with outputs being the Lagrangian coordinates
$\mathbf{G}_x$	Power spectral density matrix of inputs
$\mathbf{X}$	Reconstructed set of input time processes
$\mathbf{G}_q$	Power spectral density matrix of the Lagrangian coordinates
$\mathbf{\Phi}^{\sigma}$	Modal matrix of the i-th element or node of the model, expressed in terms of stress
$\mathbf{S}$	Power spectral density matrix of the stress tensor of the i-th element or node of the model
N	Numerousness of the state space matrices sample
$\bar{\mathbf{G}}_q$	Output PSD matrix of the proposed procedure. Lagrangian coordinates
$\bar{\mathbf{S}}$	Output PSD matrix of the proposed procedure. Stress state of the i-th element or node of the model
$\bar{\mathbf{H}}_q$	Output matrix of the frequency response functions. Lagrangian coordinates
m	Mass
M	Mass
k	Linear stiffness
K	Non linear stiffness
c	Linear damping
C	Non linear damping
r	Relative displacement
$\dot{r}$	Relative velocity
λ_{ng}	Corrective factor of damage obtained in the hypothesis of Gaussianity
D_g	Damage obtained in the hypothesis of Gaussianity

1 Introduction

The virtual evaluation of the fatigue behaviour of mechanical components is typically conducted using multibody models (MB) of the system [Schiehlen (1997); Braccesi and Cianetti (2005); Braccesi et al. (2006); Bel Knani et al. (2007)] (which allow complete simulation of the dynamic behaviour), flexible models of the components (for example finite element or modal models [Bishop and Sherratt (2000); Preumont and Piefort (1994)]) and tools for the stress recovery and the evaluation of the fatigue behaviour [Braccesi and Cianetti (2005); Braccesi et al.

(2006); Bel Knani et al. (2007); Bishop and Sherratt (2000); Preumont and Piefort (1994); Dirlik (1985)].

Such a procedure can be conducted both in the classical time domain and in the frequency domain. In the latter it is possible both to conduct the dynamic analysis and the evaluation of the stress state and to obtain the load *spectra* and the consequent evaluation of the damage.

In particular, this activity falls under the sector of research concerned with the analysis of the fatigue behaviour of mechanical components subject to random loads and which has as its objective the development of the potential of the so-called frequency domain approach; this approach calls for obtaining the stress state (represented using the matrix of the power spectral density functions of the stress tensor components) through the dynamic frequency analysis of the system [Braccesi and Cianetti (2003); Braccesi et al. (2005a)] (i.e. MB simulation and modelling environment) and the evaluation either of the load spectrum or directly of the fatigue damage of the component through reconstruction techniques, also developed in the frequency domain, beginning either with the power spectral density (PSD) matrix [Braccesi et al. (2005b)] or with a single spectral density function representing the stress tensor [Preumont and Piefort (1994); Pitoiset et al. (1998); Pitoiset and Preumont (2000); Pitoiset et al. (2001)].

This type of approach, which can be adopted in random loads conditions, typically Gaussian and stationary, has proven to be much more robust and rapid than the classic time domain approach [Lori (2005)], allowing a response on the reliability of a component to be obtained as quickly as the designer needs in the phase of defining and evaluating the various design solutions.

The restrictive hypotheses, or in any event those subject to criticism on the part of a significant sector of the scientific community, regard the linear behaviour of the mechanical system and the Gaussianity of the stress state of the component.

The first hypothesis concerns the methods followed to perform the system frequency analysis and therefore the correct reconstruction of the stress state, while the second is related to the methods used to evaluate the load spectra and/or the fatigue damage.

The two aspects, non linearity and non Gaussianity, are intrinsically connected [Benasciutti and Tovo (2003, 2005); Braccesi et al. (2005c, 2009)]. In fact one can safely assume as a frequent origin of non Gaussianity any non-linear transformation between input (ex. road unevenness) and output (ex. body input loads generated by the suspension). A typical example would be the transfer of road loads, understood as the unevenness of the ground, which can be considered Gaussian and which, “passing” through the suspension (example of a non-linear system) assume, as input to the chassis, non Gaussian characteristics. It has been amply demonstrated [Braccesi et al. (2005c, 2009)] that, at least in proximity of the interface zones between the flexible components and the system, the stress state shows a greater non Gaussian behaviour proportional to the non-linear behaviour of the system, that is, the inputs to the component are more non Gaussian.

The research activity conducted to date by the authors has intentionally separated the two aspects, always seeking solutions to the two problems separately in a scenario of simulation and analysis that calls for their integration in the final phase of the evaluation process of fatigue behaviour and/or damage.

However, the authors wish to stress in this document that the contribution of these "errors" in the evaluation of fatigue behaviour is not yet clear nor has it yet been systematically analysed and, therefore, how great an effect they can have on a design process in which such techniques are involved in the initial phases of product development.

In this work in particular a methodological solution is proposed for the correct evaluation of the stress state (psd matrix of the stress tensor) of a component integrated into a system with non-linear behaviour. The solution for non-gaussianity, formulated in previous works [Braccesi et al. (2005c, 2009)], will also be illustrated using a flow chart representation.

2 Non Linearity and Current Simulation Techniques

To date, faced with a clearly non-liner system behaviour, one is lead to adopt dynamic transient analysis in the evaluation of dynamic or fatigue behaviour.

Obviously, starting from results obtained in the time domain, it is possible to then conduct an analysis of fatigue behaviour not necessarily directly in time domain (ex. Rainflow counting, multiaxial fatigue methods, ...) but reconstructing a representation nonetheless in frequency domain of the stress state, through signal analysis, and adopting the consolidated techniques for the evaluation of damage and/or for the recovery of the load spectra [Braccesi et al. (2005b)]. This calculation method is a compromise solution which accepts the dependence of the refinement level of the spectra (PSD) on the parameters of signal integration and analysis; this determines a considerable irregularity of the spectra (unless one conducts analyses that are extremely onerous from a computational viewpoint) while maintaining constant and "unfillable" the gap, in terms of computation time and quality of results, existing between the evaluation in time domain (when the fatigue damage per loading cycle reaches a stabilized value [Lori (2005); Benasciutti and Tovo (2003, 2005); Braccesi et al. (2005c, 2009, 2004)]) of the load spectrum and of the damage (ex. "Rainflow" counting and the law of Palmgren-Miner) and that conducted in frequency domain (ex. formula of Dirlik [Braccesi et al. (2005b)] and law of Palmgren-Miner).

According to the authors, the reason for the existence and development of the frequency domain approach to the evaluation of fatigue damage lies in the possibility of obtaining an estimation of fatigue behaviour by adopting as input the representation in frequency domain of the stress state evaluated in an extremely short computational time through the dynamic frequency analysis of the system. Having to abandon this possibility or being able to obtain such a representation in a way that cannot be compared with the continuity of the results obtainable through the classic frequency approach was the main motivation that lead the authors to seek a methodological solution that preserved the representation of the non-linear behaviour of the system, obtaining a frequency domain representation of the stress state in that is not dependent on the typical parameters of time domain dynamic analysis.

2.1 Analysis in the time domain

Transient analysis can be used both to generate the sample of the time process associated with the stress state and necessary and sufficient to lead the entire fatigue behaviour evaluation process into the time domain, and, as previously mentioned, to obtain instead its representation in frequency domain.

It should nevertheless be remembered [Braccesi et al. (2004)] that any statistical parameter (ex. mean value, standard deviation, kurtosis and skewness) evaluated starting from time histories belonging to a random stationary Gaussian process is characterised by a convergence with the expected value that increases with proportionality k like the square root of the dimension T of the time histories $k/T^{0.5}$. This property is a direct consequence of the theory of sample survey [Ross (2000)], whose standard deviation σ_c is, as is well known, tied to that of the universe σ by the following relation:

$$\sigma_C = \frac{\sigma}{\sqrt{n}} \tag{1}$$

in which n indicate the numerousness of the sample.

This attests that transient analysis, although it is the simulation which best guarantees the representation of the non-linear behaviour of the system, must nevertheless offer in terms of simulation time and sampling characteristics, as well as integration, a sufficient level of statistical representativeness to allow itself to fully be an image of the non-linear process.

The evaluation of fatigue behaviour totally conducted in the time domain (the former of the two mentioned approaches) has a greater potential to correspond with the real behaviour but at the same time a higher level of onerousness as far as computations are concerned. It is well known in literature and the subject of previous works by the authors that a correct evaluation of damage, or rather the stabilization of its value as a function of time, can be reached by adopting a time sample of the process with a numerousness of the sample equal to $4 \cdot 10^5$ cycles [Braccesi et al. (2004)]. This result is both fruit of the relationship illustrated earlier, connected with the stabilization of the statistical parameters of the process, and of the "filter" constituted by the strength curve of the material which amplifies its needs in terms of simulation time. This proposed value demonstrates the onerousness of dynamic transient analyses suitable for obtaining such a database; this appears even more evident if we think of a typical example of the application of damage evaluation techniques for random processes such as the evaluation of the fatigue behaviour of a vehicle chassis, an application that calls for the adoption of complex multibody non-linear models and chassis modal models with a high number of Lagrangian coordinates obtained from finite element models of as notable dimensions [Braccesi and Cianetti (2005); Bel Knani et al. (2007)].

In the second case the time analysis aims at obtaining stress time histories that allow a sufficiently detailed representation of the PSD matrix of the stress state from which to reconstruct with frequency methods the load spectrum or damage, in the hypothesis of the Gaussianity of the process, or as an alternative, to adopt damage evaluation methods which allow one to consider the non-Gaussian nature of the time processes [Benasciutti and Tovo (2003, 2005); Braccesi et al. (2005c, 2009)].

This approach considerably reduces the onerousness of the dynamic analysis computation compared to the previous one. In fact the stabilization of the PSD of a time process requires significantly inferior acquisition times than those required by damage evaluation, which can be establish in the order of 100 s [Lori (2005); Braccesi et al. (2004)], considering usual input conditions and typical sampling frequencies. Obviously the characteristics of the PSD function that can be obtained by the acquisition and mean process is a function of the sampling time step (dt), of the window size (*time* and *number of samples*) and of the type of windowing (ex. *hanning*) used for the buffering. These parameters determine the size of the frequency range that can be obtained by the analysis of the signal as well as its sampling (df). In turn these values allow one to obtain, or not, a detailed reconstruction of the "real" power spectral density function and the correct representation of any peaks [Bendat and Piersol (2000); Ewins (1984)]. The advantage of this second approach in terms of computational time is certainly considerable but it must be emphasised that the analyses must in any event guarantee a simulation time that will allow a sufficient number of means and, therefore, of acquisitions to properly represent the frequency content of the signal with the most detailed frequency resolution (the ability to identify peaks and their correct representation).

The purpose of the methodology developed is, by adopting this second approach, to minimise the computational burden guaranteeing a more refined response, independent of the typical parameters of signal analysis.

2.2 Analysis in the frequency domain

The classical procedure for the frequency evaluation of the PSD matrix of the stress state calls for the linearization of the non-linear system in an arbitrary equilibrium configuration determining a representation of the behaviour of the system characterised by constant parameters (ex. damping, stiffness, ...).

The technique developed to date by the authors [Braccesi and Cianetti (2003); Braccesi et al. (2005a)] in particular calls for the representation of the system by means of a state space approach using as input those to the system and as output the Lagrangian coordinates of the flexible component modelled into the multibody simulation environment with modal approach [Braccesi and Cianetti (2005)]. Therefore from here on reference will be made to this method of component modelling, assuming with the expression "modal model" the modal representation developed by Craig and Bampton [Craig and Bampton (1968)].

The state-space representation is the following, expressed in terms of state variable **x** and defined by the classic matrix **A**(*state to state*), **B** (*input to state*), **C** (*state to output*), **D** (*input to output*) in the hypothesis of a system with multiple inputs **u** and multiple outputs **y**:

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\cdot\mathbf{x}(t)+\mathbf{B}\cdot\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}\cdot\mathbf{x}(t)+\mathbf{D}\cdot\mathbf{u}(t) \end{aligned} \tag{2}$$

A substantial peculiarity of the method is the choice of the generalised coordinates **q** of the generic flexible body as the output variables of the system. In this way it is possible to define the matrix $\mathbf{H}_q(\omega)$ $(m \times n)$ of the frequency response functions obtained starting with n generic inputs and with outputs being the m Lagrangian coordinates; $\mathbf{H}_q(\omega)$ represents the frequency contribution of the individual Lagrangian coordinates to the deformation of the flexible component.

$$\mathbf{H}_q(\omega) = \mathbf{C}\cdot(j\omega\cdot\mathbf{I}-\mathbf{A})^{-1}\mathbf{B} \tag{3}$$

Therefore by defining the generic Power Spectral Density matrix of n inputs in general partially correlated $\mathbf{G}_x(\omega)$ it is possible to obtain the Power Spectral Density matrix $\mathbf{S}(\omega)$ of the stress tensor (generally of dimensions 6×6) of the i-th element or node of the model through the following relationship [Braccesi and Cianetti (2003); Braccesi et al. (2005a)]:

$$\mathbf{S} = \mathbf{\Phi}^{\sigma} \cdot \left[\mathbf{H}_q \cdot \mathbf{G}_x \cdot \mathbf{H}'^{t}_q\right] \cdot \mathbf{\Phi}^{\sigma t} \tag{4}$$

in which $\mathbf{\Phi}^{\sigma}$ is the modal matrix of the i-th element or node of the model, expressed in terms of stress ($6 \times m$), the apex $'$ means the complex conjugate, the apex ^{t} is used to indicate the transposed matrix and the term between parentheses represents the PSD matrix $\mathbf{G}_q(\omega)$ ($m \times m$) of the Lagrangian coordinates (matrix defined *Hermitian*).

$$\mathbf{G}_q = \mathbf{H}_q \cdot \mathbf{G}_x \cdot \mathbf{H}'^{t}_q \tag{5}$$

Unlike the time approach, the accuracy of the output PSD is extreme and exclusively a function of the resolution of the frequency vector used to obtain the frequency response function matrix of the Lagrangian coordinates and from which, through simple matrix operations, the PSD matrix of the stress state of the generic element of the flexible component can be evaluated.

Despite this positive aspect of the frequency reconstruction procedure the arbitrariness should be noted of the choice to consider one configuration rather than another to represent the behaviour of the system. Effectively, according to the authors, the linearization is the most delicate step of the entire damage valuation procedure especially if the system, as almost always happens, assumes a non-linear behaviour. This can happen because of the intrinsic characteristics of certain components (ex. springs, shock absorbers) or of the overall characteristics of the behaviour of the entire system (ex. large displacements, contacts). What is "serious" is that the error that inevitably is committed in linearising a non-linear system translates into a wrong evaluation of the stress state (that is, of the Lagrangian coordinates) and consequently in the wrong evaluation of the load spectrum and a wrong evaluation of the damage. In this way the ulterior steps to correct the results, that are intended to take into consideration the non-Gaussianity of the process and which are based on the premise of having a "correct" evaluation of the stress PSD function [Benasciutti and Tovo (2003, 2005); Braccesi et al. (2005c, 2009)], are vanified .

3 Proposed Procedure

The objective of this paper has been to develop and verify a technique for obtaining the PSD matrix of any physical parameters of a non linear system and the PSD matrix of the Lagrangian coordinates of each flexible components and, therefore, of the stress state of each i-th element of the j-th flexible component. The aim of the procedure is to obtain a PSD matrix that is representative of the nonlinear behaviour of the system/component and furthermore that is time invariant, maintaining intact the characteristics of continuity and a detail of the power spectral density function that can be obtained using the classical frequency approach. Another aim is to obtain it by minimizing the time domain analysis computational onerousness.

The proposed procedure especially turns itself on multibody simulation environment in which the flexible behaviour of components is modelled by using modal approach that is by using Lagrangian coordinates.

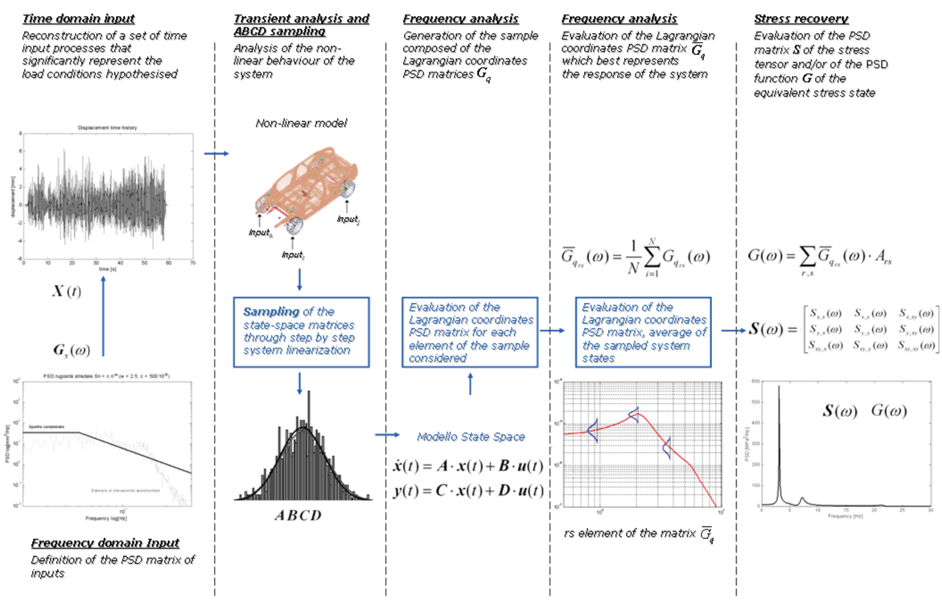

Figure 1: Flow chart of the proposed procedure

This procedure (fig. 1) is based on the statistical consideration that the linearization of a system in an arbitrary instant is like the extraction of a single sample from a population of a stochastic variable and that, therefore, the attempt to represent such a population with a sample composed of a single sample is the most mistaken thing it is possible to do. A consequence of this consideration is the desire to extract, from the population, a significant sample through which to give a better representation of the process. As a random variable we have chosen the state-space representation of the system (that is, the matrices **A**, **B**, **C**, **D**) and as a statistically significant base a subset of states extracted from a transient simulation of a time length suitable for stabilizing the PSD matrix of the physical parameters and/or of the Lagrangian coordinates and the statistical parameters of the signal.

definition of the input PSD matrix $\mathbf{G}_x(\omega)$ that represents the load conditions the designer would apply to the system.

generation of a set of time processes $\mathbf{X}(t)$ starting from the input PSD matrix $\mathbf{G}_x(\omega)$; the generation performs a fast and exact simulation of stationary zero mean Gaussian process from spectrum through circulant embedding of the covariance matrix or by summation of sinus functions with random amplitudes and random phase angle [Brodtkorb et al. (2000)];

dynamic transient analysis and contextual extraction, through step by step linearization, of the state matrices **A**, **B**, **C**, **D** characteristic of the state space representation of the system.

The dynamic transient analysis could be performed by using a symbolic, numeric or multibody code. The choice depends on type of model used: analytical, numerical or multibody model.

Extraction can be made for each instant of the analysis steps or for an arbitrary subset of these having as objective that of gathering the sample (with numerousness N) representative of the non-linear behaviour of the system. The state space system is characterised by having the Lagrangian coordinates of the component, or other physical parameters of the system, as output and therefore by being able to supply a square matrix of the PSD functions of output characterised by self-correlation or cross-correlation terms. Faced with a single degree of freedom model, characterised by a non-linear stiffness k and by a non-linear damping c and by a simple analytical state space representation (see table 2), in collecting a sample of these representations one performs an operation that can be compared to that of collecting, for each simulation step, couples of stiffness and damping values. The random variable in this case coincides in fact with the couple of stiffness and damping values from which the state space matrices are easily obtainable.

The extraction step is easy if the state space matrices are previously defined (see table 2), that is for analytical models. As concerns multibody models, all commercial MB codes can linearize the system on fly and export the state space matrices in various forms (i.e. ASCII, MATLAB, etc.).

for each element of the sample, composed of a set of matrices **A**, **B**, **C**, **D**, a frequency domain calculation is made following the formulas presented previously which can be carried out simultaneously with or after the transient analysis starting from matrix $\mathbf{G}_x(\omega)$ of the input in order to generate another sample of a random variable composed of N PSD matrices of the Lagrangian coordinates $\mathbf{G}_q(\omega)$ or of the others physical parameters;

the final and fundamental step of the procedure is that of a simple average of the PSD in perfect analogy to what is commonly done in experimental acquisitions when time signals are stored under the form of power spectral density function. The average extended to each element of the matrix, whether this be a term of self-correlation or cross-correlation, supplies the PSD matrix of output (i.e. of the Lagrangian coordinates $\bar{\mathbf{G}}_q$) which best represents the response of the component/system, a response to which to associate a probability of approximately 50 % , given the definition of mean value. This representation is time invariant, that is, unaffected by the time discretisation of the input signal or by its output sampling frequency.

For the generic element of the PSD matrix of the Lagrangian coordinates $\bar{\mathbf{G}}_q$ we have

$$\bar{G}_{q_{rs}}(\omega) = \frac{1}{N}\sum_{i=1}^{N} G_{q_{rs}}(\omega) \tag{6}$$

And as a consequence the stress PSD matrix will be:

$$\bar{\mathbf{S}} = \mathbf{\Phi}^{\sigma} \cdot \bar{\mathbf{G}}_q \cdot \mathbf{\Phi}^{\sigma t} \tag{7}$$

It must also be pointed out that this operation can be done using as a random variable not only the instantaneous matrix $\mathbf{G}_q$ but also the matrix of the frequency response function $\mathbf{H}_{q.}$ This leads to obtaining not only the non-linear response of the system (matrix of PSD functions) but also the input/output frequency response functions representing non-linear behaviour, that is, the mean of the states collected when sampling the state matrices. For the generic element of the matrix $\bar{\mathbf{H}}_q$ we have:

$$\bar{H}_{q_{rs}}(\omega) = \frac{1}{N}\sum_{i=1}^{N} H_{q_{rs}}(\omega) \tag{8}$$

So with this proposal we have joined the time domain and the frequency domain approach, in particular joining the potential of both and we have obtained: fully ability to simulate the non-linear behaviour of whatever mechanical system, time histories length and computational time extremely short, high accuracy of the result on all the frequency range with whatever detail (free choose of the user), characteristics of continuity and detail of the power spectral density function typical of the classical frequency approach, extremely short post processing time for the evaluation in frequency domain of the load spectrum (i.e. by using Dirlik formula) or directly of the fatigue damage. The only need is a Transient analyses time to have a sufficient length suitable for storing of a significant sample of states.

4 Validation of the Developed Methodology

The validity of the proposed procedure was verified by means of a statistically significant sample of transient analyses (classified by length and records numerousness) which by definition ought to be representative of real behaviour. Results obtained by time domain analysis will be referred as "real" in the next sections.

Starting directly from results obtained in time domain by transient analyses and subsequent signal analysis, the output “real” PSD matrices were obtained. These were compared with those obtained by the proposed procedure and using the same analyses. The transformation step from time domain to frequency domain, that is from the time history to its power spectral density function is based on fast Fourier transform (FFT) signal analysis.

In order to avoid the complexity of typical multi-body models and therefore to develop a basic research we have chosen to develop and verify the reconstruction methodology using simple lumped parameters non-linear models, developed analytically and then numerically in a reference mathematical code such as MATLAB; subsequently these models were translated into a multibody simulation code environment (ADAMS) in order to verify the transferability of the procedure into an industrial application environment. In this simulation scenario simple lumped parameters models were enriched with equally simple modal models coming from the finite element modelling environment in order to fully reproduce the problems of the simulation of a generic multibody model with flexible components up to the recovery of the stress state.

4.1 Description of the models used

Having as application reference the behaviour of an automobile suspension, analytical and numerical models were created (figures 2 and 3) which could reproduce on various levels of complexity the behaviour of the dynamic behaviour of a typical spring damper system.

To be brief only two of the various analytical models developed will be described: a model with one degree of freedom (*1 dof*) and one with two degrees of freedom (*2 dof*). They are lumped parameters models. In figures 2, 3, 4 and in table 1 the functional diagrams are indicated as well as the values assumed by the parameters themselves and the non-linear characteristics considered for the various analyses.

Although developed to be able to consider input expressed also in terms of force, only the results obtained with base motion conditions will be analysed.

The analytical models were developed in a numerical code (MATLAB) adopting a state-space representation in analogy with what is done in multibody dynamic simulation when the non-linear model is linearized. The classic equations necessary

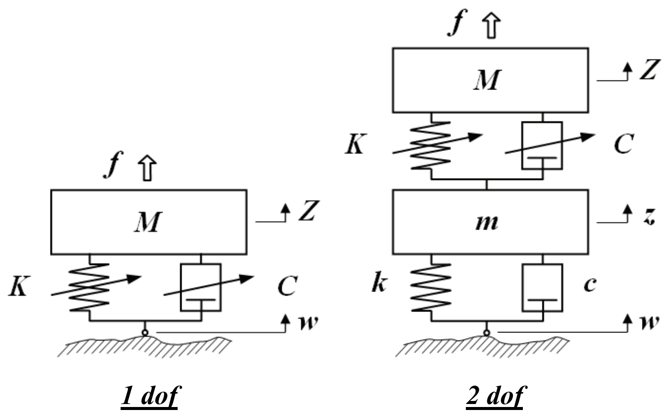

Figure 2: Representation of the developed two analytical models (*1 dof* and *2 dof*)

for the definition of the two analytical models are indicated in table 2.

The non-linearity connected with the spring and damper elements indicated with K and C are considered thanks to the development of special routines that work together with the numerical code solver used for the solution of ordinary differential equations (ODE45) and which step by step up-date the parameters in function of reference variables (in this case respectively r and $\dot{r}$ indicating with r the relative displacement and with $\dot{r}$ the relative velocity).

The values of the mass, stiffness and damping parameters respond to the characteristics requested by a typical "quarter-car" model. The non-linear characteristics (fig. 4) were considered in the first place to exalt the problem analysed, that is, the difference in behaviour in terms of the PSD function of the output which can be found between the non-linear model and its generic linearized representation ($K_1(r)$, $C_1(\dot{r})$), and in the second place to simulate as faithfully as possible the real behaviour and the non-linearity of an automobile suspension ($K_2(r)$, $C_2(\dot{r})$) [Braccesi and Cianetti (2005)].

As far as the multibody model is concerned, this has been developed using the commercial multibody code ADAMS/View. The model simulates the behaviour of a semi-car, modelling the couple of the left or right anterior and posterior suspensions with two systems with two degrees of freedom whose parameters are the same as those used for the quarter-car models and introducing the chassis through a modal model synthesised in the finite elements code ANSYS represented by a shell structure modelled with shell elements characterised by a negligible mass and by stiffness characteristics defining a dynamic behaviour similar to that of a real

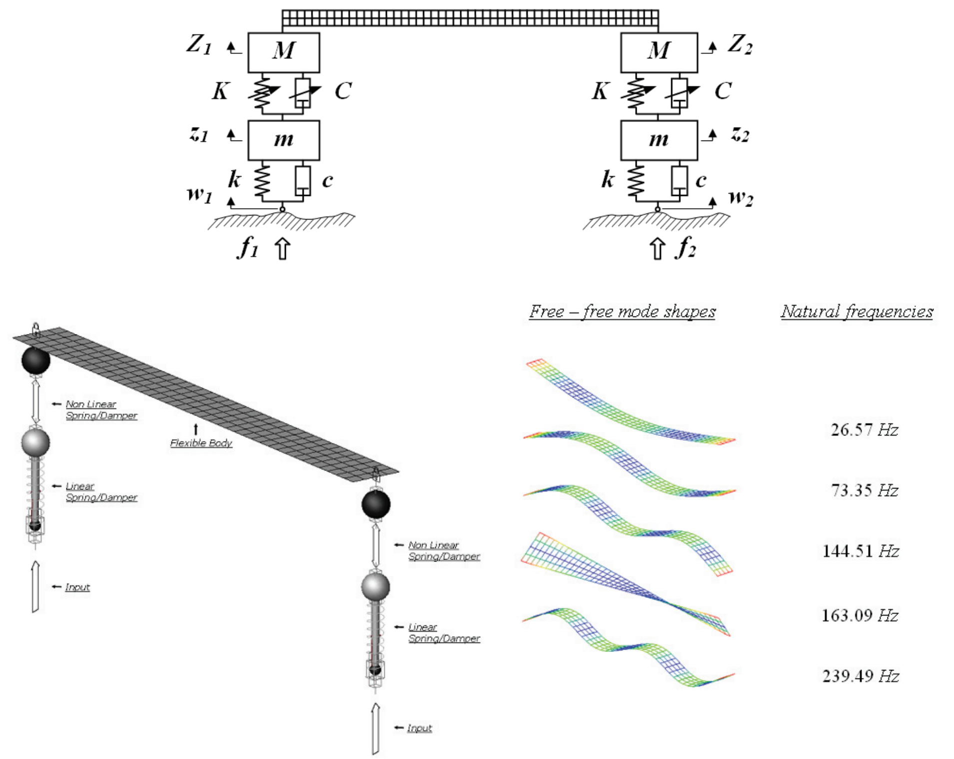

Figure 3: Schematic representation (upper) of the developed multibody model (*MBS*) and image of the *MBS* model realised in ADAMS/View (lower left). Brief description of the modal model of the flexible component (lower right)

chassis. The frequencies of the first free-free normal modes are indicated in figure 3 with a representation of the associated mode shapes. The modal model (characterised by 17 modal degrees of freedom) was realised using the modal synthesis of Craig and Bampton [Craig and Bampton (1968)] considering 12 static correction modes at the extremis and 5 constrained normal modes (the boundary areas were previously stiffened through constraint equations that define a rigid region and two reference masters). The model scheme and its representation within the code are shown in the figure. To allow the recovery of the stress state for all the elements the normal modes $\mathbf{\Phi}^{\sigma}$ expressed in terms of stress have been exported.

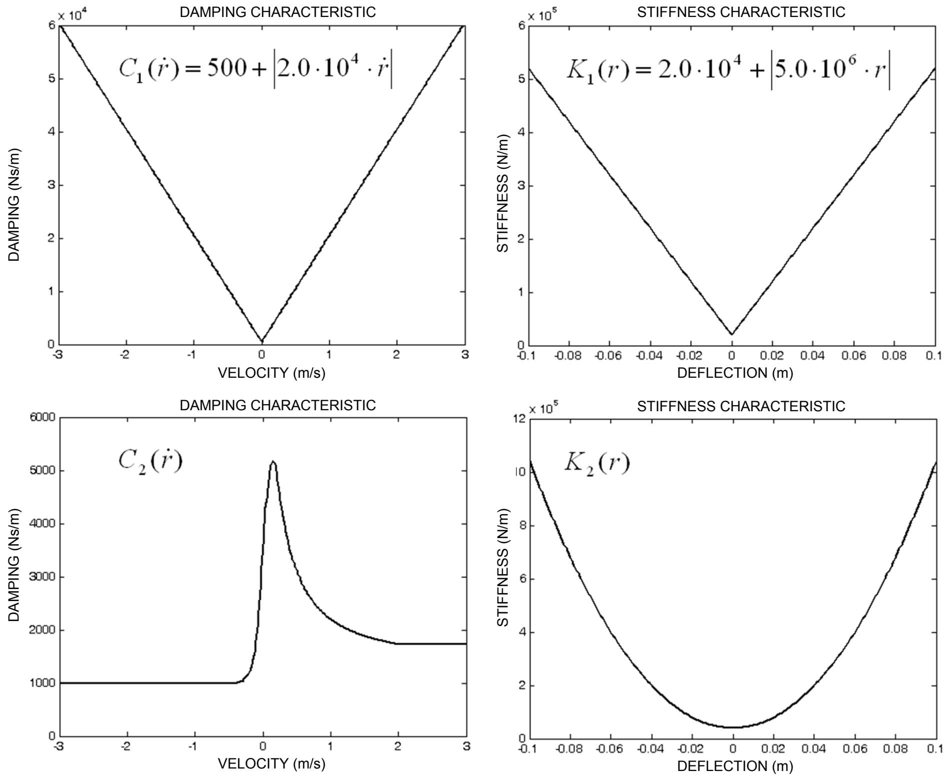

Figure 4: Representation of the non-linear characteristics of damping (left column) and stiffness (right column) $K_1(r)$, $C_1(\dot{r})$, $K_2(r)$, $C_2(\dot{r})$ used in the realisation of the models represented in table 1

4.2 Description of the analyses conducted and of the results obtained

The analyses that are reported in this paper are only a part of those carried out in the research activity and will be used to point out the positive and negative aspects of the classic procedure and of that proposed by the authors.

Only the analyses with inputs in the form of base motion (displacements) w and, in the case of the multibody model, perfectly correlated have been considered. The matrix $\mathbf{G}_x$ degenerates therefore into a single PSD function G_x.

The analysed results are the time histories and the PSD functions of some significant outputs (i.e. displacement, velocity or acceleration of the mass of model *1dof*, the stress state of one of the most stressed element of the flexible body belonging to the *MBS* model). The root mean square (*rms*) value of the signal and of the PSD function was chosen as quantitative parameter of the result.

Transient analyses characterised by various lengths (a significant number of analyses for each length) were conducted with the aim to highlight the statistical dispersion of the result reachable both by using the classical time domain approach and the proposed procedure.

This choice allows and it has allowed to show how much the result of the classical procedure is dispersed for a homogeneous analyses sample (i.e. results obtained from same length time histories) and, moreover, how much the numerousness of the records has an impact on the result dispersion (i.e. results obtained from time histories with different length). This result allows to define an absolute reference with which to compare the result reachable by the proposed procedure and its dispersion. As concern the proposed approach the result dispersion, depending on the numerousness of the states extracted from the single analysis (sampling), was analysed.

In table 3 the summary of the analyses and of their characteristics is shown. Analyses ID 1−6 referred to $1\ dof_d$ model and analysis ID 7 to *MBS* model.

The power spectral density function considered for the definition of the load conditions (characterised by a df equal to 0.1 Hz for analyses 1-6 and 0.5 Hz for analysis 7) is that described in figure 5a. To emphasise the behaviour at medium-high frequencies, not sufficiently amplified by the typical road inputs [Robson (1979)], the PSD considered extends the constant amplitude excitation region to 70 Hz assuming the appearance of a flat excitation spectrum. The *rms* value of this input is 9.55 10^{-4} m (see “Reference Input” column of table 4).

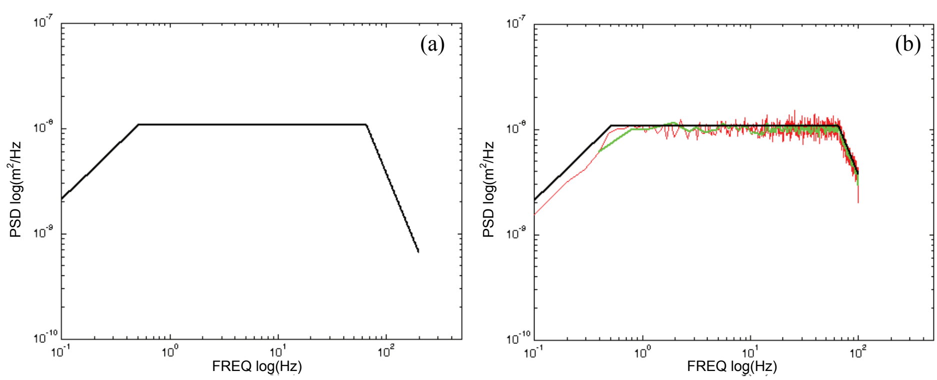

Figure 5: PSD function of design input (a) (black curve) compared with that obtained using the reconstructed input time history (b) (green curve obtained by 230 averages and red curve obtained by 60 averages)

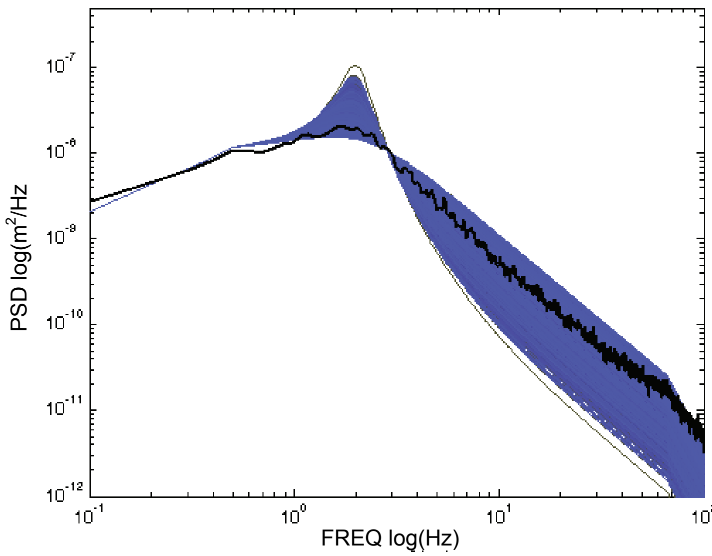

Figure 6: PSD function of output Z of the model *1 dof*$_d$ obtained by signal analysis of the Z time history (black curve), compared with some of those obtained by frequency domain analysis from instantaneous linearization of the system (blue curves)

4.2.1 Description of the steps followed for one of the analysis type conducted (analysis ID n.1)

Taking the example of analysis ID n.1 of table 3, in this paragraph the steps followed performing the analysis and obtaining the results are described.

The first phase of the analysis and of the proposed procedure calls for the reconstruction, from the PSD function $G_x(\omega)$, of the time processes $X(t)$ with which to conduct the transient dynamic analysis, in this case one hundred time histories. The sampling time step used for the present analysis type (analysis ID n.1) was $5.0 \cdot 10^{-3}$ s. The length of the time history considered as a reference was 600 s. This length, associated with the sampling time step adopted, exceeds the values indicated as the sample length and numerousness sufficient to guarantee the stabilisation of the PSD and of the statistical parameters of the output signal (120832 records). By adopting a window size of 2048 points, the discretisation step df of the frequency vector is established at approximately 0.1 Hz (0.0977 Hz) and the minimum number of averages equal to 60.

Despite this, the PSD function of the single time process (fig. 5b) shows that the exiguous input dynamics do not allow us to obtain and therefore to use, even with such a high numerousness of the sample, an input that responds to the intentions of the analyst (red curve of figure 5b). Moreover each time histories shows a variation of the statistical parameters of the signal and on its frequency content.

In table 4 as in table 5 the variation of the statistical characteristics of the process reconstructed starting from the reference PSD input is evident both as synthetic parameter *rms* and as frequency content. In table 5 a statistical representation of input signals in frequency domain is reported. In this table, for three value of the frequency range, the PSD value of the input time histories is statically analysed evaluating mean value, standard deviation (*std*) and coefficient of variation (*cov*) at 0.5, 1 and 2 Hz.

Only great numerousnesses, so great that they would not permit us to obtain results in a reasonable amount of time, would allow a better representation of input. This is certainly one of the positive aspects of the proposed procedure which by definition adopts as input that designed in the first step of the procedure without dispersion (see table 5 "Proposed approach input signal"). One can obtain a better representation of the PSD if, decreasing the size of the sampling window (512 points) and increasing the averages (230 averages), one accepts a greater df (in this case 0.4 Hz) with a poor output resolution as a result (green curve of figure 5b).

As it is possible to see in table 4 all the generated one hundred time inputs of analysis ID n.1 show however a little statistical dispersion of *rms* value. The coefficient of variation of *rms* is 2.60 10^{-3}.

On analysing the results obtained in one of the one hundred analyses conducted on model $1\ dof_d$ the problem which lead us to develop this activity emerges immediately. Figure 6 shows the instantaneous PSD functions (frequency domain analysis) of displacement Z of the mass $M$and the "real" one obtained using the output obtained by time domain analysis (black curve). The sample of matrices **A**, **B**, **C**, ***D*** used and whose results are shown in the figure has a numerousness N equal to 1209 (with a frequency sampling of 0.5 s) in comparison to the numerousness of the entire history of 120832 points (see table 3).

As can be seen, the PSD function obtained by analysing the displacement time history does not correspond with any of those sampled in the various instants of the simulation. The sensation is that there is a modulation of the system behaviour in function of the variability of the parameters, in this case K and C.

If we now (figures 7a and 7b) represent the "real" PSD function, comparing it with the averaged one obtained by the proposed method (red curve) and with some of the instantaneous ones representative of extreme conditions of linearization, that we could find if we arbitrarily linearized the system, it can be seen that the proposed methodology responds well to the objectives set in this work.

From an analysis of the representation in linear scale of the comparison (fig. 7b) the error committed by arbitrarily linearizing and therefore the validity of the method proposed is evident.

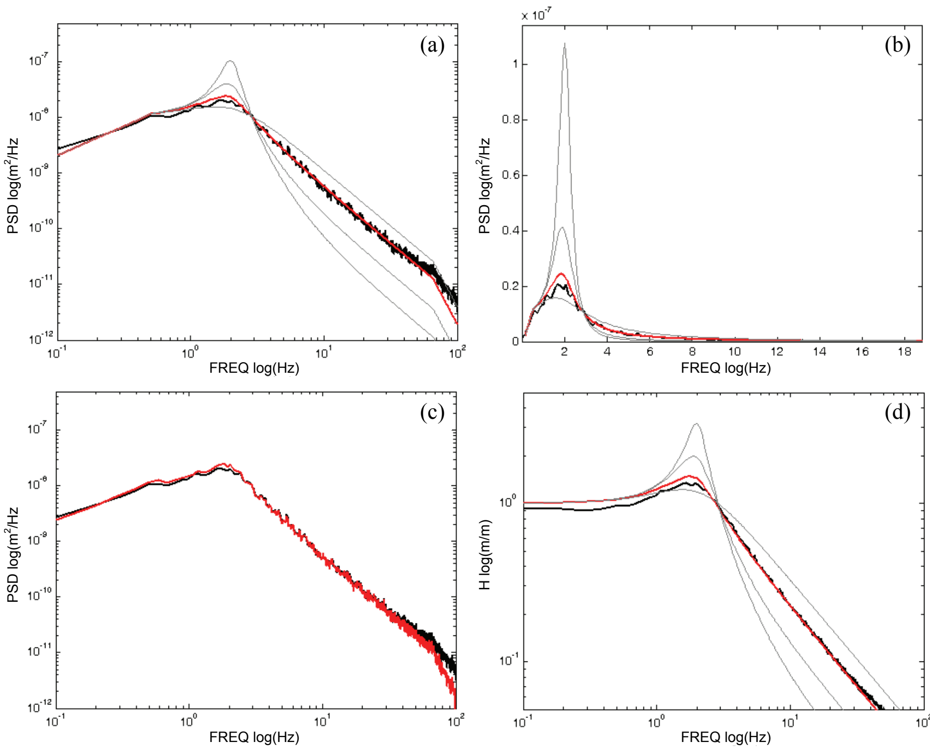

Figure 7: In figures (a) and (b) the “real” PSD function (black curve) of the output Z is compared with the averaged one obtained by the proposed method (red curve) and with some instantaneous ones (grey curves). In figure (c) the “real” PSD function (black curve) is compared with the averaged one obtained by the proposed method starting with the input PSD obtained from the time process (red curve). In figure (d) the ”real” frequency response function (black curve) is compared with the averaged one obtained by the proposed method (red curve) and with some instantaneous ones (grey curves).

To further demonstrate how the result obtained in time domain is sensitive to the input frequency content, which we have previously shown to be not perfectly superimposable on the designed one, we could obtain the $\bar{\mathbf{G}}_q$ by adopting as input the

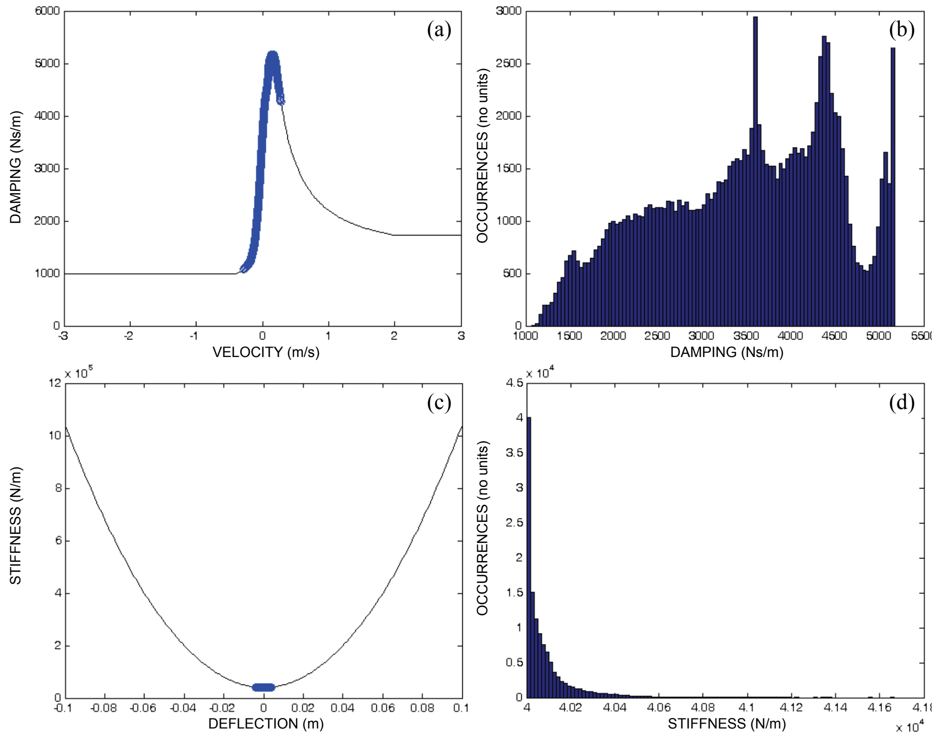

Figure 8: The representation of the range of variability that the non-linear parameters damping (C) and stiffness (K) have assumed is shown in figures (a) and (c). The representation of the respective distributions is shown in figures (b) and (d).

PSD obtained from the excitation time history (fig. 7c). The result demonstrates that the small differences that can be found in the two trends obtained earlier have been reduced even further. This confirms that the adoption of the proposed method is independent of time regarding both the definition of the input and of the output.

The good results obtained in the PSD analysis can be verified even in terms of the frequency response function $\bar{\mathbf{H}}_q$ (fig. 7d) which demonstrates the possibility of also obtaining a characterisation in frequency of the non-linear behaviour of the system with the same method, which would be useful in generating results without necessarily solving the model but simply using the well-know relations that connect input and output through the operator *FRF*.

Figure 8 shows an entire series of accessory results which illustrate the range that the non-linear parameters of damping and stiffness have assumed and the characteristics of their distributions.

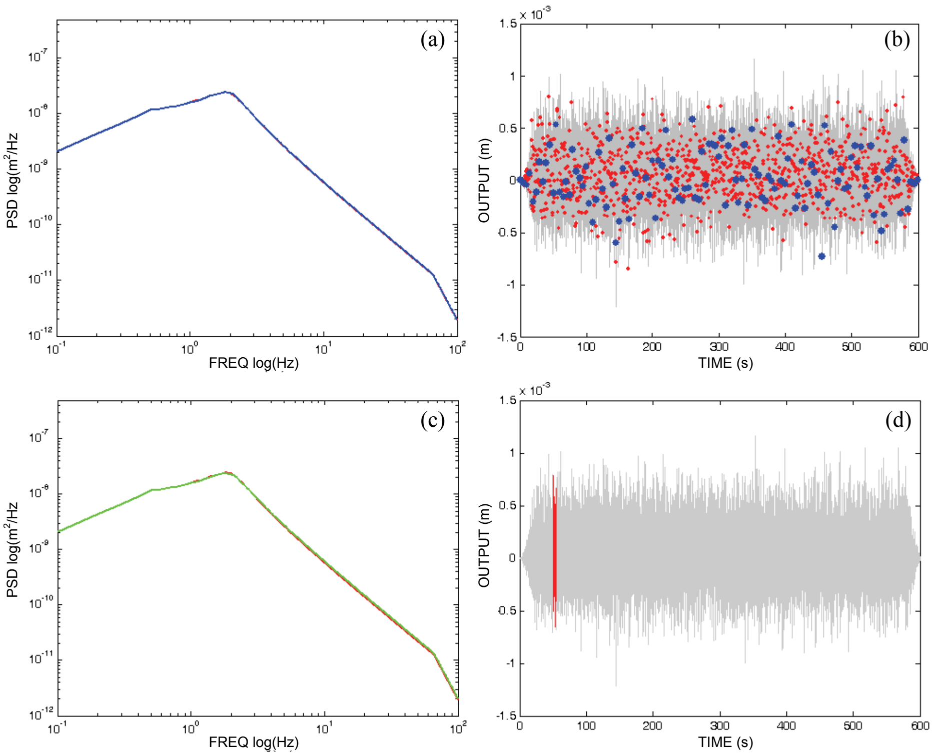

Figure 9: In figure (a) a comparison of the PSD functions of output Z obtained with a sample of 1200 system states (linearising the system every 0.5 s) (red curve) and one of 120 states (linearising the system every 5 s) (blue curve) is shown. In figure (b) a representation of the sample of 1200 system states (red points) and of that of 120 system states (blue points) of the previous PSD functions is shown. In figure (c) a comparison of the PSD functions obtained using a sample of 1200 system states (red curve) and one of 120 states (green curve) extracted from the time window of 5 s (d) by linearising the system every 0.05 s is shown. In figure (d) a time window of the output Z of 5 s (in red) adopted to reconstruct the PSD function of the output (green curve in (c)) is shown.

If now all the one hundred output time histories (classical time domain approach) and all the one hundred output PSD functions (proposed approach) are analysed we could evaluate (in table n. 4 and n.6) how much is the dispersion of the results, that could be obtained with the two approaches, both from a global point of view (*rms*) and from a frequency content point of view. The mean values of *rms* are very close (as figure 7 confirms) but the dispersion of the *rms* (*rms cov*, coefficient of variation) obtained using the proposed approach is one or two order of magnitude less than that obtained using the classical one.

As observed in figure 6, table 7 illustrates how much the linearization of the system in an arbitrary configuration of equilibrium with subsequent dynamic frequency analysis is source of extreme variability in the results both in terms of the maximum output values and in terms of *rms*. In this table the dispersion on the result (in term of PSD *rms*) obtainable using the classical frequency domain approach is compared with the result obtainable using the proposed one, that is with the "real" result.

The standard deviation (*rms std*), the coefficient of variation (*rms cov*), the maximum (*max rms*) and the minimum (*min rms*) value of *rms* demonstrate, for example for analysis ID n.1, how much great is the error we could accept if we arbitrarily linearise the system in all the 1209 instants of the 100 transient analysis.

But the potential of the method is demonstrated even better by the fact that if, instead of adopting a sample of 1209 system states (one every 0.5 s) (fig. 9b, red points), we were to consider a sample of 120 states (linearizing the system every 5 s) (fig. 9b, blue points); this would not make any appreciable difference in the results as demonstrated in figure 9a, where the two PSDs obtained with the two statistical bases considered are not distinguishable.

This leads to the conclusion that if to obtain a stabilized PSD of a generic time process a transient analysis of, say, 600 s must be conducted and therefore a set of 120832 sampling points handled, to obtain the same results with the proposed method, moreover not affected by irregularities in the generation of input and much more detailed in terms of frequency resolution, one needs a sample of several orders of magnitude smaller (in the case examined reduced to 120).

If therefore one considers a window of 5 seconds (fig. 9d) arbitrarily extracted from a reference window of 600 seconds (in this case from 50 to 55 s) and proceeds to identify 100 system states (one every 0.05 s) it emerges that even in this case the PSD perfectly superimposes that obtained from the sub-sampling of the reference simulation using 1209 samples (figure 9c).

This is the definitive demonstration that through the proposed method the objective of obtaining a detailed representation of the frequency response of a non-linear system can be reached thanks to the adoption of a sufficiently limited sample of states with the need to conduct a transient analysis that is extremely "light" from a computational viewpoint (i.e. 5 s with an output time step equal to $5{\cdot}10^{-3}$ s); the analysis is "light" even on complex multibody models (i.e. vehicle with flexible chassis and subject to pavè type load condition). A similar analysis could never be used in the classic approach which calls for obtaining a stabilized PSD starting from a time process which, to permit all of this, would have to have a length on the

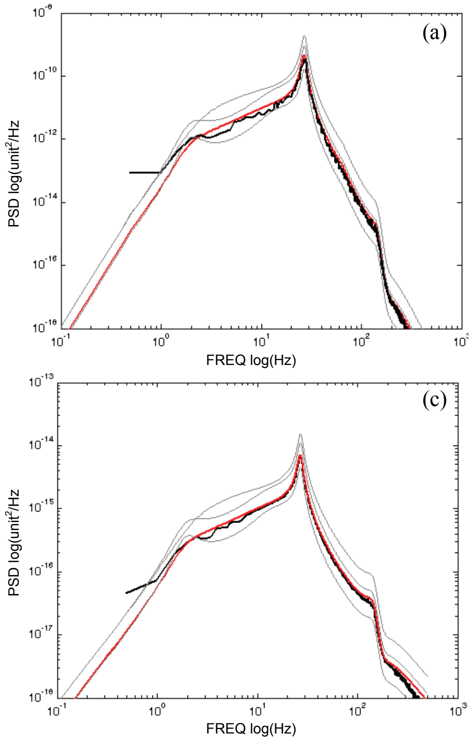

Figure 10: In figure (a) the term 11 of the "real" PSD matrix of lagrangian coordinates (black curve) is shown with the same term of the averaged matrix obtained by the proposed method (red curve) and with the term (1,1) of some of the instantaneous matrices (gray curves), representative of extreme conditions of linearization that we could find if we arbitrarily linearise the system. The right figure (b) is a zoom of the previous one (a). In figure (c) the term 11 of the "real" FRF matrix (black curve) is shown with the same term of the averaged one obtained by the proposed method (red curve) and with term (1,1) of some of the instantaneous matrices (gray curves). The lower right figure (d) is a zoom of the previous one (c).

order of 100 s.

All the results and the conclusions reached for the analysis ID n.1 are comparable with those reachable and reached for the other analysis types as we can read in tables 3-7.

In conclusion the time domain approach needs a very long transient analysis or a lot of transient analyses to obtain a significant sample from which evaluate a PSD function representative of the non linear process. The proposed approach needs a

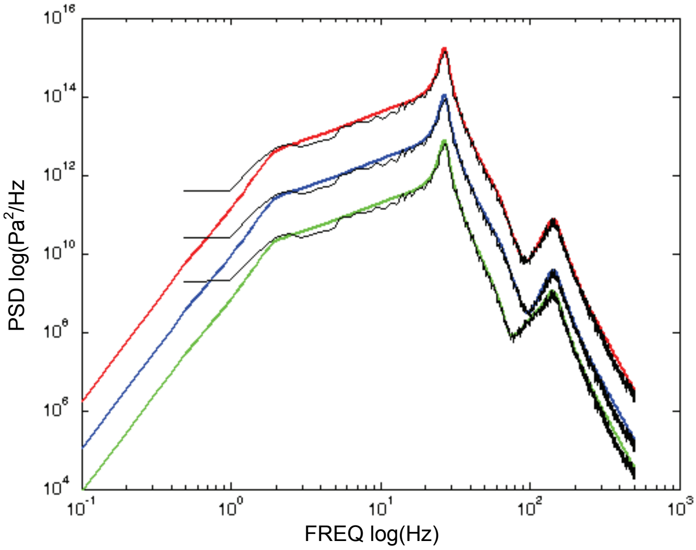

Figure 11: Comparison between the self-correlation terms of matrix $\bar{\mathbf{S}}$ (stresses σ_x, σ_y and τ_{xy}) (respectively the red, blue and green curves) of element no. 40 obtained with a sample of 120 states and those of the matrix that can be obtained directly from the time recovery of the stress state starting from the reference transient analysis (black curves)

single short transient analysis that is extremely "light" from a computational viewpoint to reach the same result.

4.2.2 Description of the results obtained using the MBS model (analysis ID n.7)

To verify the reached conclusions and to demonstrate the possibility to translate this approach into multibody simulation environment, a multibody commercial code as solver (ADAMS/Solver) and a previously shown multibody model (developed in ADAMS/View and similar to that of a vehicle but surely less complex) were used (see paragraph 4.1). A procedure was implemented for the automatic extraction of the state-space matrices during the execution of a transient analysis and the simultaneous calculation (in co-simulation) of the PSD matrix $\bar{\mathbf{G}}_q$, of the FRF matrix $\bar{\mathbf{H}}_q$ and of the stress matrix $\bar{\mathbf{S}}$of each element, utilising equations (2-8).

As a reference a transient analysis of 60 s was conducted (analsys ID n.7), with sampling frequency of 1000 Hz (60417 records), integration time step of $1.0{\cdot}10^{-4}$ s, buffering window of 2048 points, establishing in approximately 0.5 Hz (0.488 Hz) the step resolution df of the frequency vector and a minimum number of av-

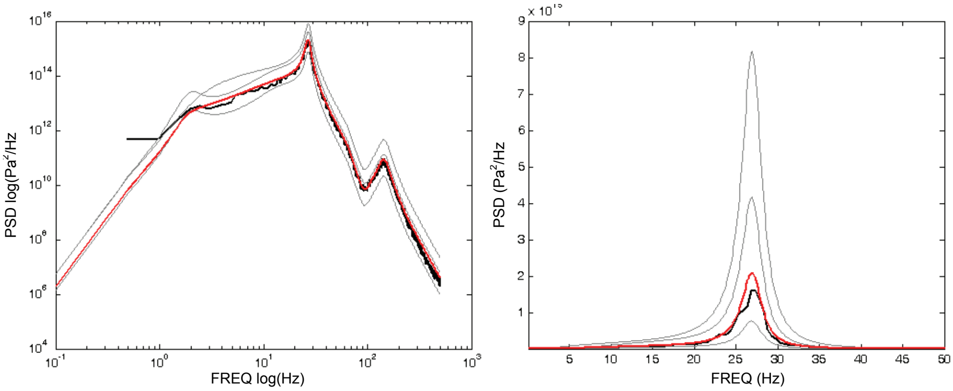

Figure 12: Comparison of the term (1,1) (σ_x) of the “real” PSD matrix of the stress tensor of element no. 40 (black curve), the same term of the averaged matrix $\bar{\mathbf{S}}$(red curve), obtained using the proposed procedure and method with a sample of 120 states, and the term (1,1) of some of the instantaneous matrices (grey curves). The representations are in logarithmic (left) and in linear scale (right).

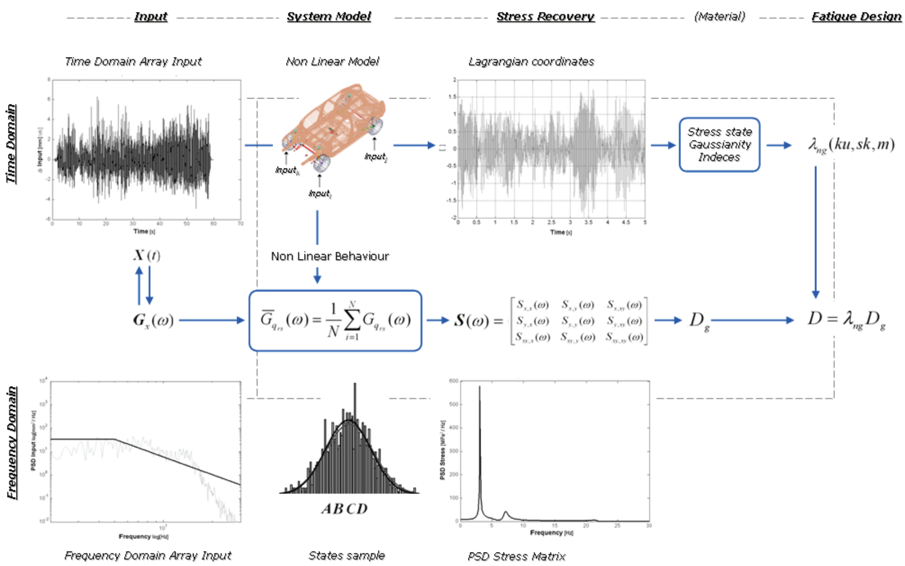

Figure 13: Flow chart of the evaluation scenario that considers non-Gaussianity of the stress state

erages equal to approximately 300. The adoption of an smaller sampling time step has allowed us to extend the observable frequency range and therefore any amplifications associated with the flexible behaviour of the "chassis".

The sample of matrices **A**, **B**, **C**, **D** initially used has a numerousness N equal to 120 (with a frequency sampling of 0.5 s) in comparison to the numerousness of the entire history of 60417 points (see table 3). In table 3, 4 and 7 the analogous results and conclusions evaluated and reached for the analysis ID n.1 are shown starting from a single transient analysis and from the previous sample.

Table 1: Parameters of the models developed

Model	*M* (kg)	*m* (kg)	*K* (KN/m)	*k* (KN/m)	*C* (Ns/m)	*c* (Ns/m)
$1\ dof_a$	250	-	$K_1(r)$	-	1000	-
$1\ dof_b$	250	-	25	-	$C_1(\dot{r})$	-
$1\ dof_c$	250	-	$K_1(r)$	-	$C_1(\dot{r})$	-
$1\ dof_d$	250	-	$K_2(r)$	-	$C_2(\dot{r})$	-
$2\ dof_a$	250	35	$K_1(r)$	250	1000	1000
$2\ dof_b$	250	35	25	250	$C_1(\dot{r})$	1000
$2\ dof_c$	250	35	$K_1(r)$	250	$C_1(\dot{r})$	1000
$2\ dof_d$	250	35	$K_2(r)$	250	$C_2(\dot{r})$	1000
MBS_a	250	35	$K_1(r)$	250	1000	1000
MBS_b	250	35	25	250	$C_1(\dot{r})$	1000
MBS_c	250	35	$K_1(r)$	250	$C_1(\dot{r})$	1000
MBS_d	250	35	$K_2(r)$	250	$C_2(\dot{r})$	1000

Indicating with r the relative displacement and $\dot{r}$ the relative velocity.

Considering then a time window of the same length as that adopted previously (5 s) and extracting 100 system states (one every 0.05 s) equally positive results were obtained as those illustrated earlier and reachable in table 4 and totally analogous to them.

In figure 10 the "real" PSD function, term 11 of PSD matrix of lagrangian coordinates, (black curve) is shown together with the averaged one obtained by the proposed method (red curve) and with some of the instantaneous ones (gray curves), representative of extreme conditions of linearization that we could find if we arbitrarily linearised the system. In the same the term 11 of the "real" FRF matrix

Table 2: Characteristic relationships of the state space models (*1 dof* and *2 dof* models)

1 dof model	***2 dof model***
Change of variable	
$\dot{Z} = p + (C/M)\cdot w$ $\ddot{Z} = \dot{p} + (C/M)\cdot \dot{w}$	$\dot{z} = p + (c/m)\cdot w$ $\ddot{z} = \dot{p} + (c/m)\cdot \dot{w}$
States, input and output arrays	
$\boldsymbol{x} = \begin{Bmatrix} Z \\ p \end{Bmatrix},\ \boldsymbol{u} = \begin{Bmatrix} w \\ f \end{Bmatrix},\ \boldsymbol{y} = \begin{Bmatrix} Z \\ \dot{Z} \end{Bmatrix}$	$\boldsymbol{x} = \begin{Bmatrix} Z \\ z \\ \dot{Z} \\ p \end{Bmatrix},\ \boldsymbol{u} = \begin{Bmatrix} w \\ f \end{Bmatrix},\ \boldsymbol{y} = \begin{Bmatrix} Z \\ z \\ \dot{Z} \\ \dot{z} \end{Bmatrix}$
State to state matrix	
$\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ -K/M & -C/M \end{bmatrix}$	$\boldsymbol{A} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -K/M & K/M & -C/M & -C/M \\ K/m & -(K+k)/m & -(C+c)/m & -(C+c)/m \end{bmatrix}$
Input to state matrix	
$\boldsymbol{B} = \begin{bmatrix} C/M & 0 \\ K/M - (C/M)^2 & 1/M \end{bmatrix}$	$\boldsymbol{B} = \begin{bmatrix} 0 & 0 \\ c/m & 0 \\ c\cdot C/(m\cdot M) & 1/M \\ k/m - (C+c)\cdot c/m^2 & 0 \end{bmatrix}$
State to output matrix	
$\boldsymbol{C} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\boldsymbol{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
Input to output matrix	
$\boldsymbol{D} = \begin{bmatrix} 0 & 0 \\ C/M & 0 \end{bmatrix}$	$\boldsymbol{D} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ c/m & 0 \end{bmatrix}$

Table 3: Analyses characteristics and dynamic simulation parameters

				Parameters of the classical time domain approach				***Parameters of the proposed approach***			
Analysis ID	*Type*	*Time histories length*	*Histories number*	*Output time step*	*Records number*	*Window points number*	*Frequency discretisation step*	*Frequency discretisation step*	*Frequency range*	*Numerousness of the sample extracted from the single time history*	*Sampling time step*
		(s)		*(s)*			*(Hz)*	*(Hz)*	*(Hz)*		*(s)*
0	Linear	600	100	$5 \cdot 10^{-3}$	120832	2048	0.0977	0.1	0÷100	1209	0.5
1	Non linear	600	100	$5 \cdot 10^{-3}$	120832	2048	0.0977	0.1	0÷100	1209	0.5
2								0.1	0÷100	120	5
3	Non linear	6000	10	$5 \cdot 10^{-3}$	1200128	2048	0.0977	0.1	0÷100	1200	0.5
4								0.1	0÷100	120	5
5	Non linear	60	100	$5 \cdot 10^{-3}$	12288	2048	0.0977	0.1	0÷100	1200	0.05
6								0.1	0÷100	120	0.5
7	MBS non linear	60	1	$1 \cdot 10^{-3}$	60417	2048	0.488	0.5	0÷500	120	0.5

Table 4: Statistical representation of input and output signals. Comparison between results obtained by classical approach and the proposed one

Analysis ID	***Reference Input***	***Reconstructed input signals***				***Classical approach output***			***Proposed approach output***		
	Rms	*Histories number*	*Mean of rms*	*Std of rms*	*Cov of rms*	*Mean of rms*	*Std of rms*	*Cov of rms*	*Mean of rms*	*Std of rms*	*Cov of rms*
	(m)		*(m)*	*(m)*		*(m)*	*(m)*		*(m)*	*(m)*	
0	$9.55\ 10^{-4}$	100	$9.54\ 10^{-4}$	$2.46\ 10^{-6}$	$2.57\ 10^{-3}$	$2.42\ 10^{-4}$	$2.56\ 10^{-6}$	$1.06\ 10^{-2}$	$2.35\ 10^{-4}$	0.00	0.00
1	$9.55\ 10^{-4}$	100	$9.54\ 10^{-4}$	$2.48\ 10^{-6}$	$2.60\ 10^{-3}$	$2.60\ 10^{-4}$	$2.43\ 10^{-6}$	$9.35\ 10^{-3}$	$2.60\ 10^{-4}$	$2.97\ 10^{-7}$	$1.14\ 10^{-3}$
2	$9.55\ 10^{-4}$	100	$9.54\ 10^{-4}$	$2.49\ 10^{-6}$	$2.61\ 10^{-3}$	$2.60\ 10^{-4}$	$2.87\ 10^{-6}$	$1.10\ 10^{-2}$	$2.60\ 10^{-4}$	$9.28\ 10^{-7}$	$3.57\ 10^{-3}$
3	$9.55\ 10^{-4}$	10	$9.55\ 10^{-4}$	$9.01\ 10^{-7}$	$9.44\ 10^{-4}$	$2.61\ 10^{-4}$	$7.60\ 10^{-7}$	$2.91\ 10^{-3}$	$2.60\ 10^{-4}$	$2.24\ 10^{-7}$	$8.60\ 10^{-4}$
4	$9.55\ 10^{-4}$	10	$9.54\ 10^{-4}$	$9.24\ 10^{-7}$	$9.68\ 10^{-4}$	$2.61\ 10^{-4}$	$6.72\ 10^{-7}$	$2.57\ 10^{-3}$	$2.60\ 10^{-4}$	$9.62\ 10^{-7}$	$3.70\ 10^{-3}$
5	$9.55\ 10^{-4}$	100	$9.53\ 10^{-4}$	$8.39\ 10^{-6}$	$8.81\ 10^{-3}$	$2.52\ 10^{-4}$	$8.35\ 10^{-6}$	$3.32\ 10^{-2}$	$2.60\ 10^{-4}$	$2.68\ 10^{-7}$	$1.03\ 10^{-3}$
6	$9.55\ 10^{-4}$	100	$9.53\ 10^{-4}$	$8.39\ 10^{-6}$	$8.81\ 10^{-3}$	$2.52\ 10^{-4}$	$8.35\ 10^{-6}$	$3.32\ 10^{-2}$	$2.60\ 10^{-4}$	$7.82\ 10^{-7}$	$3.01\ 10^{-3}$
	Rms	*Histories number*	*Mean of rms*	*Std of rms*	*Cov of rms*	*Mean of rms*	*Std of rms*	*Cov of rms*	*Mean of rms*	*Std of rms*	*Cov of rms*
	(m)		*(m)*	*(m)*		*(Pa)*	*(Pa)*		*(Pa)*	*(Pa)*	
7	$9.55\ 10^{-4}$	1	$9.52\ 10^{-4}$	0	0	$8.75\ 10^{+7}$	0	0	$9.04\ 10^{+7}$	0	0

(black curve) is shown with the averaged one obtained by the proposed method (red curve) and with some of the instantaneous ones (gray curves). The comparison fully demonstrates the validity of the method developed.

The comparison between the self-correlation terms of the matrix$\bar{\mathbf{S}}$ related to the stresses σ_x, σ_y e τ_{xy} of element no. 40 chosen in correspondence to the fixed joint of the shell structure (verified as the most damageable zone) and those of the matrix

Table 5: Statistical representation of input signals in frequency domain. Comparison between classical and proposed approach

		Classical approach input signals								
		0.5 Hz			*1 Hz*			*2 Hz*		
Analysis ID	*Histories number*	*Mean*	*Std*	*Cov*	*Mean*	*Std*	*Cov*	*Mean*	*Std*	*Cov*
		(m^2/Hz)	*(m^2/Hz)*		*(m^2/Hz)*	*(m^2/Hz)*		*(m^2/Hz)*	*(m^2/Hz)*	
0	100	1.01 10^{-8}	1.21 10^{-9}	1.20 10^{-1}	1.06 10^{-8}	1.22 10^{-9}	1.16 10^{-1}	1.07 10^{-8}	1.07 10^{-9}	9.98 10^{-2}
1	100	1.01 10^{-8}	1.20 10^{-9}	1.19 10^{-1}	1.06 10^{-8}	1.22 10^{-9}	1.16 10^{-1}	1.07 10^{-8}	1.07 10^{-9}	1.01 10^{-1}
2	100	9.64 10^{-9}	1.05 10^{-9}	1.09 10^{-1}	1.08 10^{-8}	1.42 10^{-9}	1.32 10^{-1}	1.07 10^{-8}	1.21 10^{-9}	1.13 10^{-1}
3	10	9.91 10^{-9}	2.13 10^{-10}	2.15 10^{-2}	1.07 10^{-8}	3.98 10^{-10}	3.71 10^{-2}	1.06 10^{-8}	3.49 10^{-10}	3.29 10^{-2}
4	10	9.95 10^{-9}	2.65 10^{-10}	2.67 10^{-2}	1.04 10^{-8}	3.56 10^{-10}	3.43 10^{-2}	1.06 10^{-8}	3.92 10^{-10}	3.69 10^{-2}
5	100	1.02 10^{-8}	3.90 10^{-9}	3.84 10^{-1}	1.10 10^{-8}	4.34 10^{-9}	3.94 10^{-1}	1.03 10^{-8}	3.81 10^{-9}	3.71 10^{-1}
6	100	1.02 10^{-8}	3.90 10^{-9}	3.84 10^{-1}	1.10 10^{-8}	4.34 10^{-9}	3.94 10^{-1}	1.03 10^{-8}	3.81 10^{-9}	3.71 10^{-1}
		Proposed approach input signal								
		0.5 Hz			*1 Hz*			*2 Hz*		
Analysis ID	*Histories number*	*Mean*	*Std*	*Cov*	*Mean*	*Std*	*Cov*	*Mean*	*Std*	*Cov*
		(m^2/Hz)	*(m^2/Hz)*		*(m^2/Hz)*	*(m^2/Hz)*		*(m^2/Hz)*	*(m^2/Hz)*	
0	100	1.06 10^{-8}	0.00	0.00	1.06 10^{-8}	0.00	0.00	1.06 10^{-8}	0.00	0.00
1	100	1.06 10^{-8}	0.00	0.00	1.06 10^{-8}	0.00	0.00	1.06 10^{-8}	0.00	0.00
2	100	1.06 10^{-8}	0.00	0.00	1.06 10^{-8}	0.00	0.00	1.06 10^{-8}	0.00	0.00
3	10	1.06 10^{-8}	0.00	0.00	1.06 10^{-8}	0.00	0.00	1.06 10^{-8}	0.00	0.00
4	10	1.06 10^{-8}	0.00	0.00	1.06 10^{-8}	0.00	0.00	1.06 10^{-8}	0.00	0.00
5	100	1.06 10^{-8}	0.00	0.00	1.06 10^{-8}	0.00	0.00	1.06 10^{-8}	0.00	0.00
6	100	1.06 10^{-8}	0.00	0.00	1.06 10^{-8}	0.00	0.00	1.06 10^{-8}	0.00	0.00

that can be obtained directly by the time reconstruction of the stress starting with the reference transient analysis is illustrated in figure 11. The stress recovery takes place by modal superimposition according to what has been described in [Braccesi and Cianetti (2005)]. Even in this case, the comparison fully demonstrates the validity of the method developed. However, in order to evaluate the error that could be made in the evaluation of fatigue behaviour by linearising the system exclusively in an arbitrary instant, in figure 12 the trend of the term (1,1) (σ_x) of the "real" PSD of the stress tensor of element no. 40 is indicated, comparing it with the same

Table 6: Statistical representation of output signals in frequency domain. Comparison between classical and proposed approach

		Classical approach output								
		0.5 Hz			1 Hz			2 Hz		
Analysis ID	Histories number	Mean	Std	Cov	Mean	Std	Cov	Mean	Std	Cov
		(m^2/Hz)	(m^2/Hz)		(m^2/Hz)	(m^2/Hz)		(m^2/Hz)	(m^2/Hz)	
0	100	$1.24\ 10^{-8}$	$1.41\ 10^{-9}$	$1.14\ 10^{-1}$	$1.81\ 10^{-8}$	$1.99\ 10^{-8}$	$1.10\ 10^{-1}$	$1.66\ 10^{-8}$	$1.57\ 10^{-9}$	$9.44\ 10^{-2}$
1	100	$1.08\ 10^{-8}$	$1.22\ 10^{-9}$	$1.13\ 10^{-1}$	$1.53\ 10^{-8}$	$1.62\ 10^{-9}$	$1.06\ 10^{-1}$	$2.00\ 10^{-8}$	$1.90\ 10^{-9}$	$9.49\ 10^{-2}$
2	100	$1.03\ 10^{-8}$	$1.02\ 10^{-9}$	$9.90\ 10^{-2}$	$1.55\ 10^{-8}$	$1.88\ 10^{-9}$	$1.21\ 10^{-1}$	$2.01\ 10^{-8}$	$2.16\ 10^{-9}$	$1.07\ 10^{-1}$
3	10	$1.07\ 10^{-8}$	$2.53\ 10^{-10}$	$2.36\ 10^{-2}$	$1.55\ 10^{-8}$	$5.56\ 10^{-10}$	$3.60\ 10^{-2}$	$1.99\ 10^{-8}$	$6.89\ 10^{-10}$	$3.47\ 10^{-2}$
4	10	$1.05\ 10^{-8}$	$2.44\ 10^{-10}$	$2.31\ 10^{-2}$	$1.52\ 10^{-8}$	$5.45\ 10^{-10}$	$3.58\ 10^{-2}$	$1.98\ 10^{-8}$	$6.42\ 10^{-10}$	$3.24\ 10^{-2}$
5	100	$1.03\ 10^{-8}$	$3.96\ 10^{-9}$	$3.85\ 10^{-1}$	$1.51\ 10^{-8}$	$5.98\ 10^{-9}$	$3.96\ 10^{-1}$	$1.83\ 10^{-8}$	$6.87\ 10^{-9}$	$3.76\ 10^{-1}$
6	100	$1.03\ 10^{-8}$	$3.96\ 10^{-9}$	$3.85\ 10^{-1}$	$1.51\ 10^{-8}$	$5.98\ 10^{-9}$	$3.96\ 10^{-1}$	$1.83\ 10^{-8}$	$6.87\ 10^{-9}$	$3.76\ 10^{-1}$
		Proposed approach output								
		0.5 Hz			1 Hz			2 Hz		
Analysis ID	Histories number	Mean	Std	Cov	Mean	Std	Cov	Mean	Std	Cov
		(m^2/Hz)	(m^2/Hz)		(m^2/Hz)	(m^2/Hz)		(m^2/Hz)	(m^2/Hz)	
0	100	$1.24\ 10^{-8}$	0.00	0.00	$1.75\ 10^{-8}$	0.00	0.00	$1.63\ 10^{-8}$	0.00	0.00
1	100	$1.19\ 10^{-8}$	$1.72\ 10^{-12}$	$1.44\ 10^{-4}$	$1.60\ 10^{-8}$	$3.05\ 10^{-11}$	$1.91\ 10^{-3}$	$2.33\ 10^{-8}$	$3.45\ 10^{-10}$	$1.48\ 10^{-2}$
2	100	$1.19\ 10^{-8}$	$5.18\ 10^{-12}$	$4.33\ 10^{-4}$	$1.60\ 10^{-8}$	$9.75\ 10^{-11}$	$6.11\ 10^{-3}$	$2.33\ 10^{-8}$	$1.17\ 10^{-9}$	$5.02\ 10^{-2}$
3	10	$1.19\ 10^{-8}$	$2.14\ 10^{-12}$	$1.79\ 10^{-4}$	$1.60\ 10^{-8}$	$3.87\ 10^{-11}$	$2.42\ 10^{-3}$	$2.35\ 10^{-8}$	$4.05\ 10^{-10}$	$1.73\ 10^{-2}$
4	10	$1.19\ 10^{-8}$	$3.95\ 10^{-12}$	$3.31\ 10^{-4}$	$1.60\ 10^{-8}$	$8.15\ 10^{-11}$	$5.10\ 10^{-3}$	$2.35\ 10^{-8}$	$1.06\ 10^{-9}$	$4.51\ 10^{-2}$
5	100	$1.19\ 10^{-8}$	$1.48\ 10^{-12}$	$1.24\ 10^{-4}$	$1.60\ 10^{-8}$	$2.62\ 10^{-11}$	$1.64\ 10^{-3}$	$2.33\ 10^{-8}$	$2.97\ 10^{-10}$	$1.27\ 10^{-2}$
6	100	$1.19\ 10^{-8}$	$5.55\ 10^{-12}$	$4.65\ 10^{-4}$	$1.59\ 10^{-8}$	$9.59\ 10^{-11}$	$6.01\ 10^{-3}$	$2.32\ 10^{-8}$	$9.60\ 10^{-10}$	$4.15\ 10^{-2}$

term of the averaged $\bar{\mathbf{S}}$ obtained with the proposed procedure and method and with some of the instantaneous ones, which are representative of the extreme conditions of linearization. An analysis of the curves represented in logarithmic and linear scale highlight the considerable and "serious" error that could be committed in considering the "wrong" instants for linearising the system. Table 7 illustrates how much the linearization of the system in an arbitrary configuration of equilibrium with subsequent dynamic frequency analysis is source of extreme variability in the results both in terms of the maximum stress values and in terms of *rms*. In particular

Table 7: Result summary. Proposed approach vs classical frequency domain one

		Frequency domain outputs vs proposed approach output					
Analysis ID	*Histories number*	*Sample numerousness*	*Rms (Prop. Appr.)*	*Rms std (Freq. Appr.)*	*Rms cov (Freq. Appr.)*	*Max rms (Freq. Appr.)*	*Min rms (Freq. Appr.)*
			(m)	*(m)*		*(m)*	*(m)*
0	100	1209	$2.35\ 10^{-4}$	0.00	0.00	$2.35\ 10^{-4}$	$2.35\ 10^{-4}$
1	100	1209	$2.60\ 10^{-4}$	$9.44\ 10^{-6}$	$3.63\ 10^{-2}$	$3.15\ 10^{-4}$	$2.53\ 10^{-4}$
2	100	120	$2.60\ 10^{-4}$	$1.07\ 10^{-5}$	$4.11\ 10^{-2}$	$3.19\ 10^{-4}$	$2.53\ 10^{-4}$
3	10	1200	$2.60\ 10^{-4}$	$9.78\ 10^{-6}$	$3.76\ 10^{-2}$	$3.26\ 10^{-4}$	$2.53\ 10^{-4}$
4	10	120	$2.60\ 10^{-4}$	$9.43\ 10^{-6}$	$3.62\ 10^{-2}$	$3.15\ 10^{-4}$	$2.53\ 10^{-4}$
5	100	1200	$2.60\ 10^{-4}$	$8.55\ 10^{-6}$	$3.30\ 10^{-2}$	$3.20\ 10^{-4}$	$2.53\ 10^{-4}$
6	100	120	$2.60\ 10^{-4}$	$7.09\ 10^{-6}$	$2.74\ 10^{-2}$	$2.88\ 10^{-4}$	$2.53\ 10^{-4}$
Analysis ID	*Histories number*	*Sample numerousness*	*Rms (Prop. Appr.)*	*Max rms (Freq. Appr.)*	*Min rms (Freq. Appr.)*		
			(Pa)	*(Pa)*	*(Pa)*		
7	1	120	$9.04\ 10^{+7}$	$1.948\ 10^{+8}$	$5.82\ 10^{+7}$		

we could observe a ratio of 2 or 0.5 between the "real" value of stress *rms* and its minimum or the maximum value.

With the present proposed approach the scenario for the analyst is that of a symbiosis between transient analysis and frequency analysis of the system that minimises the burden of computational time of time domain simulation and guarantees a high level of refinement of the stress state evaluation exclusively as a function of the discretisation of the frequency vector and of the numerousness of the sample of states considered for the averaging operation.

Therefore, after one has obtained a correct representation of the non-linear frequency response, one must face the problem of the hypothesis of the Gaussianity of the process made by the majority of the methods of fatigue damage evaluation developed in the frequency domain (i.e. Dirlik). With reference to what has been obtained by the authors in previous works ([Braccesi et al. (2005c, 2009)]), and that is to an approach through corrective factors, one can foreshadow a scenario which adopts transient simulation, necessary for the correct representation of the non-linear behaviour of the system, as a tool to obtain the statistical properties of the Lagrangian coordinates, which, associated with the modal shapes, permit us to predict the characteristics of non-Gaussianity of the stress state (*kurtosis* and *skewness*) without reconstructing its tensor in the time domain. Thanks to the evaluation of the two parameters it is then possible to arrive at a corrective factor (λ_{ng}) of the damage, a function of the parameters mentioned earlier and of the slope of the fatigue strength curve, with which to correct the damage obtained in the hypothesis of Gaussianity (D_g). The flow chart of this scenario is shown in figure 13. It should also be pointed out that the transient analysis necessary to contextually reach the two objectives must, however, respond to the need for stabilization of the statistical parameters as already discussed in paragraph 2.1 of this paper.

5 Conclusions

The results reached in this research, of which those illustrated in this paper are only an exiguous testimony, have demonstrated that it is indispensable to obtain a correct and accurate evaluation in frequency of the stress state to be able to correctly estimate fatigue damage using the classic frequency approach. The linearization of the system in an arbitrary configuration of equilibrium with subsequent dynamic frequency analysis has proven to be the source of extreme variability in the results both in terms of the maximum stress values and in terms of rms. At the same time, transient analysis does not appear to be sufficiently accurate and adequately rapid in obtaining the matrix of the PSD functions of the stress tensor.

The methodology developed and object of this paper has proven to be a valid response to the need for accuracy and speed in analysis. The correct representation in frequency of the behaviour of the system and of the component is guaranteed on a statistical basis by a limited sample of system states obtained from a transient analysis nevertheless necessary when faced with a non-linear behaviour of the system; the exiguousness of the sample requested reduces the need to conduct dynamic analyses of significant length and therefore of great computational time.

The proposed method also fits well into the scenario of the evaluation of fatigue behaviour prefigured by the authors which calls for, after the transient dynamic analysis, the evaluation of the parameters of non Gaussianity of the component stress state (obtained exclusively through the opportune combination of the stress modal shapes and of the spectral moments of the Lagrangian coordinates) and the evaluation of a corrective coefficient which, element by element, allows one to correct the damage obtained in the case of the Gaussianity of the process with the classic frequency methods starting with the PSD matrix evaluated using the method proposed in this work.

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