Computer Modeling in Engineering & Sciences CMES, vol.122, no.1, pp.49-59, 2020

Dynamic Analysis of Stochastic Friction Systems Using the Generalized Cell Mapping Method

Shichao Ma[1, 2, *], Xin Ning[1, 2, *] and Liang Wang[3]

Abstract: Friction systems are a kind of typical non-linear dynamical systems in the actual engineering and often generate abundant dynamics phenomena. Because of non-smooth characteristics, it is difficult to handle these systems by conventional analysis methods directly. At the same time, random perturbation often affects friction systems and makes these systems more complicated. In this context, we investigate the steady-state stochastic responses and stochastic P-bifurcation of friction systems under random excitations in this paper. And in order to retain the non-smooth of friction system, the generalized cell mapping (GCM) method is first used to the original stochastic friction systems without any approximate transformation. To verify the accuracy and validate the applicability of the suggested approach, we present two classical nonlinear friction systems, i.e., Coulomb force model and Dahl force model as examples. Meanwhile, this method is in good agreement with the Monte Carlo simulation method and the computation time is greatly reduced. In addition, further discussion finds that the adjustable parameters can induce the stochastic P-bifurcation in the two examples, respectively. The stochastic P-bifurcation phenomena indicate that the stability of the friction system changes very sensitively with the parameters. Research of responses analysis and stochastic P-bifurcation has certain significances for the reliability and stability analysis of practical engineering.

Keywords: Friction systems, non-smooth, stochastic responses, cell mapping.

1 Introduction

Spacecraft are complex mechanical systems which contain a wealth of nonlinear dynamic behavior with each component. Among them, the instability of systems that may result from the non-smooth features requires designers to develop the most stringent solution to ensure the overall system security. As a typical non-smooth factor, friction is a very complicated phenomenon arising at contact surfaces in the space manipulator [Awrejcewicz and Olenjnik (2005)]. And many researchers have paid attention to the

[1] Northwestern Polytechnical University School of Astronautics, Xi'an, 710072, China.

[2] National Key Laboratory of Aerospace Flight Dynamics, Xi'an, 710072, China.

[3] Northwestern Polytechnical University School of Science, Xi'an, 710129, China.

[*] Corresponding Authors: Shichao Ma. Email: shichaoma@mail.nwpu.edu.cn;
Xin Ning. Email: ningxin@nwpu.edu.cn.

Received: 10 April 2019; Accepted: 03 August 2019.

problems of friction systems in the actual production. Some interesting results have been proposed [Xu, Wang, Feng et al. (2018)]. And dynamic friction models, such as Dahl, Bliman-Sorine, LuGre, Coulomb as well as atomic scale and fractal models, have been developed to reasonably describe some special phenomena [Bowden and Tabor (1973); Berger (2002); Astrom and Wit (2008)]. On the other hand, the uncertainty caused by random forces increases the complexity of studying such systems. Stochastic response analysis is an important research content of nonlinear stochastic dynamics. And stochastic bifurcation refer to the sudden qualitative change in the dynamic behavior of stochastic system when some parameters change slightly continuously, especially the stochastic P-bifurcation [Arnold (1995); Xu, Gu, Zhang et al. (2011); Yang, Sanjuan, Liu et al. (2016)]. In response analysis of nonlinear stochastic dynamical systems, the study of random bifurcation is of great significance.

Some researchers used the various analysis methods to obtain the stochastic responses of friction systems [Baule, Touchette and Cohen (2010); Kumar, Narayanan and Gupta (2016); Guerine, Hami, Walha et al. (2016); Fang, Liang and Zuo (2018); Sun, Xu and Lin (2018); Jin, Wang and Huang (2018)]. Among these methods, the generalized cell mapping (GCM) method has been demonstrated to be a very efficient tool due to its ability of global analysis of the strongly nonlinear systems, especially for the stochastic systems. Recently, researchers do many research based on the cell mapping method which was first proposed by Hsu based on the Markov theory [Hsu (1987)]. Various improvements were applied to analyze the dynamical phenomena. For example, Tongue et al. [Tongue and Gu (1988)] had proposed the interpolated cell mapping (ICM) method, in which an orbital is generated by numerical integration. The first-passage problem of a non-linear system with dry friction damping was studied by Sun [Sun (1995)]. And based on the concepts of set theory and graph theory, the generalized cell mapping digraph (GCMD) method was proposed by Hong et al. to study the crises and chaotic transients [Hong and Xu (1999)]. In addition, the various improved cell mapping method can also obtain the stochastic response and global analysis of strongly non-linear systems [Yue, Xu, Jia et al. (2013); Wang, Xue, Xu et al. (2017); Wang, Ma, Sun et al. (2018); Yue, Xu, Xu et al. (2019)]. And we can see that the cell mapping method is a fast and effective numerical method.

However, for friction systems, existing methods would be hard to approximate discontinuities to smooth functions. In this paper, we research the nonlinear dynamics of friction systems by using stochastic generalized cell mapping (SGCM) method. The purpose is to preserve the non-smooth properties of the friction systems. And this paper is arranged as follows. In Section 2, we introduce the friction system model and the SGCM method. In Section 3, we present the responses analysis of two different friction force models by this method. And Monte Carlo (MC) simulation method is used to verify the effectiveness. Finally, conclusions are drawn in Section 4.

2 Friction system model and the SGCM method

Consider a mechanical model of a single-degree-of-freedom Duffing friction oscillator subjected to Gaussian white noise excitation (Fig. 1)

$$\ddot{x} + \alpha\dot{x} + \kappa x + \mu x^3 + F(x, \dot{x}) = \xi(t) \tag{1}$$

where $x, \dot{x}, \ddot{x}$ mean displacement, velocity and acceleration, respectively. The dot $\cdot$ represents the differentiation with respect to the time t. α is the damping coefficient, κ is the linear stiffness coefficient and μ is the nonlinear stiffness coefficient representing the intensity of stiffness nonlinearity, and $F(x, \dot{x})$ is the discontinuous friction model. $\xi(t)$ is the Gaussian white noise excitation which satisfies the following condition [Zhu and Cai (2017)]:

$$E[\xi(t)] = 0, \quad E[\xi(t)\xi(t+\tau)] = \sigma^2\delta(\tau) \tag{2}$$

And the 2σ represents the intensity of Gaussian white noise, $\delta(\cdot)$ is the Dirac delta function.

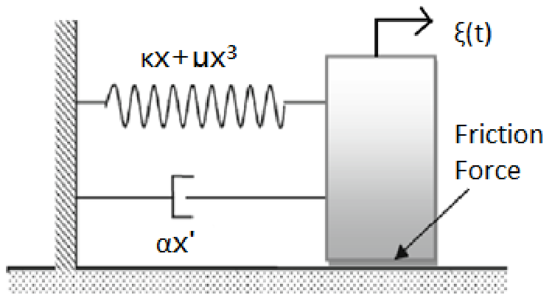


Figure 1: Schematic of friction system with random excitation

For the friction system described above, we will introduce the stochastic generalized cell mapping method. Suppose that Ω is the selected domain of the system and then divide Ω evenly into N_c intervals with $N_l \times N_w$, each interval is named a cell. And these cells are labeled with integers $N = \{1, 2, ..., N_c\}$. Then, we establish a one-step transition probability matrix. Each cell generates $\overline{s}$ random sample trajectories for these N_c cells. And if the cell i $(1 \le i \le N_c)$ has $\overline{s}_i (1 \le \overline{s}_i \le \overline{s})$ sample points falling from the cell j $(1 \le j \le N_c)$, then the one step transition probability from cell j to i is assigned to be $p_{ij} = \overline{s}_i / \overline{s}$ with $\sum_i \overline{s}_i = \overline{s}$ and $\sum_i p_{ij} = 1$. Here, P represents the one-step transition probability matrix with the element p_{ij}. Finally, according to the C-K equation [Yue, Xu, Jia et al. (2013)].

$$p(\mathbf{x}, m\Delta t) = \int p(\mathbf{x}, t \mid \mathbf{x}_0, 0) p(\mathbf{x}_0, (m-1)\Delta t) d\mathbf{x}_0 \tag{3}$$

We can obtain the probability distribution vector $\mathbf{p}(m)$ after m cycles with the initial vector $\mathbf{p}(0)$. Among,

$$\mathbf{p}(m+1) = P \cdot \mathbf{p}(m) \text{ or } \mathbf{p}(m) = P^m \cdot \mathbf{p}(0) \tag{4}$$

3 Dynamic responses analysis of the friction systems

In this section, we directly utilize the SGCM method to obtain the stochastic responses by the marginal and joint probability densities. Two classical nonlinear Duffing systems with different non-smooth friction force models are presented as examples. One is the

Coulomb friction force, and the other one is the Dahl friction force. And we compare the results obtained by SGCM methods with those from MC simulation method and evaluate the effectiveness and the applicability of the proposed procedure for different friction force models.

In addition, we also find the stochastic P-bifurcation with variable stiffness. The stochastic P-bifurcation, which is a complex nonlinear dynamic phenomenon, mainly illustrates the change of topological shape of the steady-state PDFs induced by parameters. The stochastic P-bifurcation usually refers to a change of shapes or a change in the number of peaks in the PDFs.

3.1 Example 1: coulomb friction force

Now, we consider this Duffing system with Coulomb friction excited by Gaussian white noise in which

$$\ddot{x} + \alpha\dot{x} + \kappa x + \mu x^3 + f_c\,\mathbf{sgn}(\dot{x}) = \xi(t) \tag{5}$$

Or can write it as

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\alpha y - \kappa x - \mu x^3 - f_c\,\mathbf{sgn}(y) + \xi(t) \end{cases} \tag{6}$$

which f_c is the amplitude of friction and sgn (.) represents the signum function. The parameter values of the system are initially chosen as $\alpha = 1, \kappa = 0.01, \mu = 0.01$. And in this case, the intensity of Gaussian white noise σ^2 is chosen as 0.01.

In order to compute the one-step transition probability matrix, the interesting domain is chosen as $\Omega = \{-3 \le x \le 3, -3 \le \dot{x} \le 3\}$. And the selected domain is divided into 50×50 cells, 2000 random sample trajectories are generated from each cell. Therefore, 5000000 random sample trajectories in total are used to construct the one-step transition probability matrix for the selected domain. MC simulation method takes the same number of samples. Hence, using the method illustrated in Section 2, the steady-state probability density functions (PDFs) of displacement x and velocity $y(y = \dot{x})$ are shown in Figs. 2 and 3 with the adjustable friction force coefficient f_c.

In Fig. 2(a), we can see that the marginal PDF of displacement x presents a bimodal state (red solid line) with $f_c = -0.02$. As f_c increases, the topological structure of marginal PDFs of x changes from bimodal state to unimodal state. And when $f_c = 0.01$, the topological structure of marginal PDF is a unimodal state (brown solid line) obviously. The marginal PDFs of velocity $\dot{x}$ have similar trends with the friction coefficient f_c increasing, which are shown in Fig. 2(b). In Fig. 3, the joint PDFs of displacement x and velocity $\dot{x}$ are displayed with the adjustable coefficient f_c. Among them, the results of the SGCM method is in the left side of Figs. 3(a)-3(d) when $f_c = -0.02, -0.01, 0, 0.01$, respectively. The corresponding results of MC simulation are on the right side. With the increasing of f_c, the steady-state joint PDFs vary from a "crater" into one "peak". By combining Figs. 2 and 3, we can find that the value of the friction coefficient f_c will induce changes in the topological structure of the stochastic system, thus affecting the stability of the system. This phenomenon is called stochastic P-bifurcation [Yang, Sanjuan, Liu et al. (2016)]. In addition, it can be seen that the results of the SGCM method are in good agreement with the results of the MC simulation method. And Tab. 1 is the time comparison of the SGCM method and MC method, which shows the high efficiency of SGCM obviously. As we can see, this method is an efficient approach to analyze the responses of a friction system with noise fluctuation.

Table 1: The calculation time comparison of two methods for the system (6)

Values (f_c) / Methods	-0.02	-0.01	0	0.01
SGCM	249.7 s	252.4 s	248.2 s	260.9 s
MC	54208.2 s	54036.7 s	54912.7 s	55610.5 s

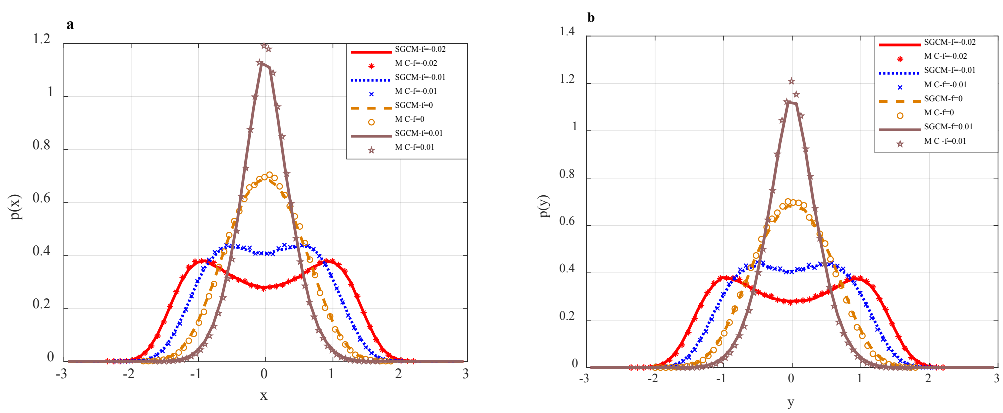
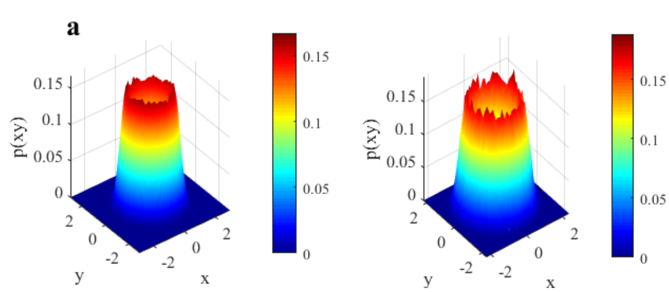


Figure 2: The steady-state marginal PDFs of system (6) with $f_c = -0.02, -0.01, 0, 0.01$. (a) The steady-state marginal PDFs of x; (b) The steady-state marginal PDFs of y. Solid lines: the SGCM method results; symbols: MC simulation method results



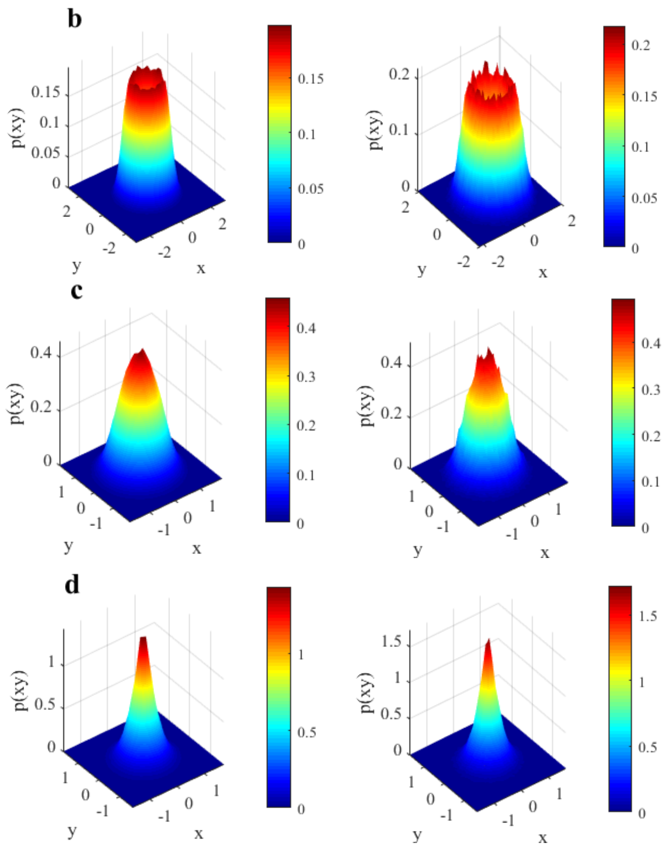


Figure 3: The steady-state joint PDFs of system (6) for x and y with $f_c = -0.02, -0.01, 0, 0.01$. Left sides: the SGCM method results; right sides: MC simulation method results. (a) $f_c = -0.02$; (b) $f_c = -0.01$; (c) $f_c = 0$; (d) $f_c = 0.01$

3.2 Example 2: Dahl friction force

The second example considers the following Duffing system with Dahl friction subjected to Gaussian white noise. The equations of motion of the system are written as:

$$\ddot{x} + \kappa\dot{x} + \alpha x + \mu x^3 + \lambda f_D = \xi(t) \tag{7}$$

$$\dot{f}_D = \dot{x} - \frac{\lambda|\dot{x}|}{f_C} x \tag{8}$$

where λf_D is the Dahl friction force, suppose that $z = f_D$. Rewrite the system as

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\kappa y - \alpha x - \mu x^3 - \lambda z + \xi(t) \\ \dot{z} = y - \dfrac{\lambda|y|}{f_c} x. \end{cases} \tag{9}$$

For Eq. (9), the effects of different values of α are considered under the fixed parameters $\kappa = 0.02, \mu = 1, f_c = 0.05$ and the Dahl friction force parameter $\lambda = 0.06$. The intensity of Gaussian white noise σ^2 is chosen as 0.02. The interested domain of the SGCM method is chosen as $\Omega = \{-2 \le x \le 2, -2 \le y \le 2\}$ with a cell structure of 50×50. And 2000 random sample trajectories are also generalized from each cell. Hence, out of cells, there are a total of 5000000 sample trajectories to construct the one-step transition probability matrix. For comparison, the MC simulation method is used. In Figs. 4 and 5, the steady-state PDFs of displacement x and velocity y of system (9) are respectively plotted with the adjustable damping coefficient α.

In Fig. 4, blue line, green line, pink line and orange line represent the steady-state responses with $\alpha = 0.5, 0, -0.5, -1$, respectively. As shown in Fig. 4(a), the topological structure of marginal PDF of displacement x is unimodal shape when $\alpha = 0.5$. But as α decreases, the peaks of the marginal PDFs start to go down and it can be seen that the shape of the marginal PDFs of x changes from one peak to two peaks. However, there is no significant change in the topological structure velocity y with the α decreasing in Fig. 4(b). The marginal PDFs of velocity always maintain the shape of single peak no matter how α changes. Fig. 5 shows the joint PDFs of displacement and velocity when α changes. Among them, the results of the SGCM method is in the left side of Figs. 5 (a)-(d) when $\alpha = 0.5, 0, -0.5, -1$, respectively. The corresponding results of the MC simulation method are on the right side. The joint PDFs of x and y display the change from one peak to two peaks. These figures demonstrate the occurrence of the stochastic P-bifurcation when the damping coefficient α decreases from 0.5 to -1. Of course, the results of the SGCM method are well coincident with the MC simulations, which verify the effectiveness of the method. The calculate time comparison of the SGCM method and MC simulation method are shown in Tab. 2, it can be seen that the SGCM method has the advantage of time obviously.

Table 2: The calculation time comparison of two methods for the system (9)

Methods \ Values (α)	0.5	0	-0.5	-1
SGCM	248.4 s	246.0 s	240.5 s	243.2 s
MC	80670.1 s	78120.2 s	77960.4 s	77420.8 s

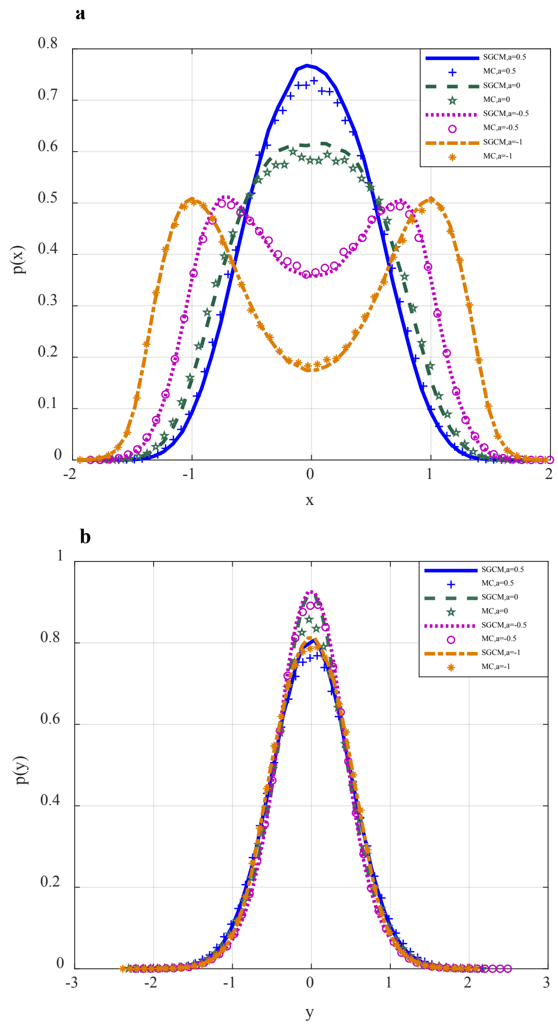
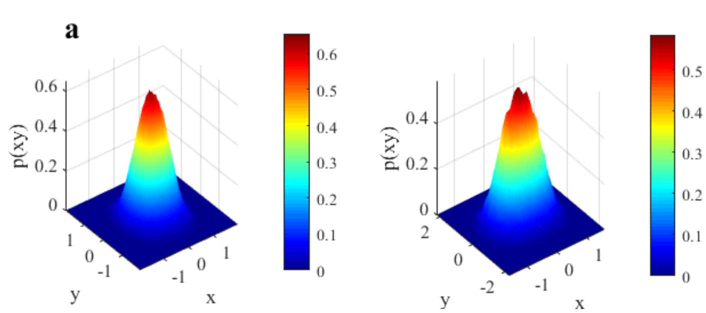


Figure 4: The steady-state marginal PDFs of system (9) with $\alpha = 0.5, 0, -0.5, -1$. (a) The steady-state marginal PDFs of x; (b) The steady-state marginal PDFs of y. Solid lines: the SGCM method results; symbols: MC simulation method results



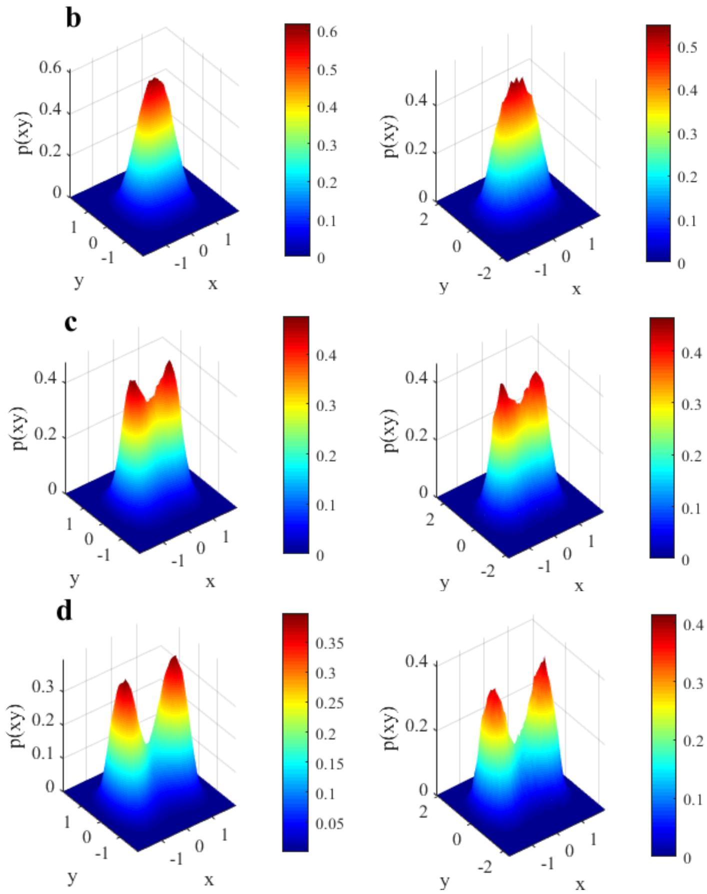


Figure 5: The steady-state joint PDFs of system (9) with $\alpha = 0.5, 0, -0.5, -1$. Left sides: the SGCM method results; right sides: MC simulation method results. (a) $\alpha = 0.5$; (b) $\alpha = 0$; (c) $\alpha = -0.5$; (d) $\alpha = -1$

4 Conclusions

In this paper, we investigate the stochastic responses of the nonlinear friction systems with the adjustable coefficient property under Gaussian white noise excitation. Using the SGCM method, we can obtain the steady-state probability density functions of systems displacement and velocity with different friction force model.

To verify the accuracy and validate the applicability of the suggested approach, we present two classical nonlinear friction systems with the adjustable coefficient, i.e., Coulomb force model and Dahl force model as examples. By comparing with the Monte Carlo simulation method, it is proved that the high efficiency of the SGCM method under two examples. In addition, adjustable coefficients f_c and α can induce stochastic P-bifurcation in the two examples, respectively. It is showed that this phenomena can affect the stability of the systems with the adjustable coefficient.

Without any approximation transformation to the original systems, the non-smooth of the friction systems still retain, we conclude that the proposed method is more advanced than the common analytic method. We are currently working in addressing for engineering interests, the dynamic analysis of stochastic response and bifurcation is of great significance to practical engineering, we will report results of more complicated problems in future publications.

Acknowledgment: This work was supported by the National Science Foundation of China through the Grants (11872306, 11772256), the Central University Fundamental Research Fund (3102018zy043).

Conflicts of Interest: The authors declare that they have no conflicts of interest to report regarding the present study.

References

Arnold, L. (1995): *Random Dynamical Systems*. Springer-Verlag Press.

Astrom, K. J.; de Wit, C. C. (2008): Revisiting the LuGre friction model, stick-slip motion and friction dependence. *IEEE Control Systems Magazine*, vol. 28, pp. 101-114.

Awrejcewicz, J.; Olejnik, P. (2005): Analysis of dynamic systems with various friction laws. *Applied Mechanics Reviews*, vol. 58, pp. 389-411.

Baule, A.; Touchette, H.; Cohen, E. G. D. (2010): Stick-slip motion of solids with dry friction subject to random vibrations and an external field. *Nonlinearity*, vol. 24, pp. 351-372.

Berger, B. J. (2002): Friction model for dynamic system simulation. *Applied Mechanics Reviews*, vol. 55, pp. 535-777.

Bowden, F. P.; Tabor, D. (1982): *Friction-An Introduction to Tribology*. China Machine Press, Beijing.

Fang, Y. N.; Liang, X. H.; Zuo, M. J. (2018): Effects of friction and stochastic load on transient characteristics of a spur gear pair. *Nonlinear Dynamics*, vol. 93, pp. 599-609.

Guerine, A.; Hami, E. A.; Walha, L.; Fakhfakh, T.; Haddar, M. (2018): Dynamic response of a Spur gear system with uncertain friction coefficient. *Advances in Engineering Software*, vol. 120, pp. 45-54.

Hong, L.; Xu, J. X. (1999): Crises and chaotic transients studied by the generalized cell mapping digraph method. *Physics Letters A*, vol. 262, pp. 1206-1214.

Hsu, C. S. (1987): *Cell-to-Cell Mapping: A Method Global Analysis for Nonlinear System.* Springer.

Jin, X. L.; Wang, Y.; Huang, Z. L. (2018): Approximately analytical technique for random response of LuGre friction system. *International Journal of Non-Linear Mechanics*, vol. 104, pp. 1-7.

Kumar, P.; Narayanan, S.; Gupta, S. (2016): Stochastic bifurcation analysis of a Duffing oscillator with Coulomb friction excited by Poisson white noise. *Procedia Engineering*, vol. 144, pp. 998-1006.

Sun, J. J.; Xu, W.; Lin, Z. F. (2018): Research on the reliability of the friction system under combined additive and multiplicative random excitations. *Communications in*

Nonlinear Science and Numerical Simulation, vol. 54, pp. 1-12.

Sun, J. Q. (1995): Random vibration analysis of a non-linear system with dry friction damping by the short-time Gaussian cell mapping method. *Journal of Sound and Vibration*, vol. 180, pp. 785-795.

Tongue, B. H.; Gu, K. Q. (1988): A theory basis for interpolated cell mapping. *SIAM Journal on Applied Mathematics*, vol. 48, no. 5, pp. 1206-1214.

Wang, L.; Xue, L. L.; Xu, W.; Yue, X. L. (2017): Stochastic P-bifurcation analysis of a fractional smooth and discontinuous oscillator via the generalized cell mapping method. *International Journal of Non-Linear Mechanics*, vol. 96, pp. 56-63.

Wang, L.; Ma, S. C.; Sun, C. Y.; Jia, W. T.; Xu, W. (2018): Stochastic response of a class of impact systems calculated by a new strategy based on generalized cell mapping method. *Journal of Applied Mechanics-Transition of the ASME*, vol. 85, 054502.

Xu, W.; Wang, L.; Feng, J. Q.; Qiao, Y.; Han, P. (2018): Some new advance on the research of stochastic non-smooth systems. *Chinese Physical B*, vol. 27, pp. 110501-110506.

Xu, Y.; Gu, R. C.; Zhang, H. Q.; Xu, W.; Duan, J. Q. (2011): Stochastic bifurcations in a bistable Duffing-Van del Pol oscillator with colored noise, *Physical Review E*, vol. 83, 056215.

Yang, J. H.; Sanjuan, M. A. F.; Liu, H. G.; Litak, G.; Li, X. (2016): Stochastic P-bifurcation and stochastic resonance in a noisy bistable frictional-order system, *Communications in Nonlinear Science and Numerical Simulation*, vol. 41, pp. 104-117.

Yue, X. L.; Xu, W.; Jia, W. T.; Wang, L. (2013): Stochastic response of a φ^6 oscillator subjected to combined harmonic and Poisson white noise excitations. *Physica A*, vol. 392, pp. 2988-2998.

Yue, X. L.; Xu, Y.; Xu, W.; Sun, J. Q. (2019): Probabilistic response of dynamical systems based on the global attractor with the compatible cell mapping method. *Physica A*, vol. 516, pp. 509-519.

Zhu, W. Q.; Cai, G. Q. (2017): *Introduction to Stochastic Dynamics.* Science and Technology Press, Beijing.