

## Modelling of Energy Storage Photonic Medium by Wavelength-Based Multivariable Second-Order Differential Equation

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**Abstract:** Wavelength-dependent mathematical modelling of the differential energy change of a photon has been performed inside a proposed hypothetical optical medium. The existence of this medium demands certain mathematical constraints, which have been derived in detail. Using reverse modelling, a medium satisfying the derived conditions is proven to store energy as the photon propagates from the entry to exit point. A single photon with a given intensity is considered in the analysis and hypothesized to possess a definite non-zero probability of maintaining its energy and velocity functions analytic inside the proposed optical medium, despite scattering, absorption, fluorescence, heat generation, and other nonlinear mechanisms. The energy and velocity functions are thus singly and doubly differentiable with respect to wavelength. The solution of the resulting second-order differential equation in two variables proves that energy storage or energy flotation occurs inside a medium with a refractive index satisfying the described mathematical constraints. The minimum-value-normalized refractive index profiles of the modelled optical medium for transformed wavelengths both inside the medium and for vacuum have been derived. Mathematical proofs, design equations, and detailed numerical analyses are presented in the paper.

**Keywords:** Optical medium modelling, energy storage, multivariable second order differential equation, numerical analysis, minimum value-normalized refractive index profile.

### 1 Introduction

Optical media can affect many interaction mechanisms of a photon as it travels through them. In normal optical media, the change in light direction as it passes from one medium to another is associated with changes in velocity and wavelength, but the energy of the light remains unchanged throughout different media. For visible light, the velocity decreases to 0.66 times in linseed oil medium; for pigments such as titanium white, the corresponding decrease is 0.40 times relative to the speed in vacuum. Semiconducting

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silicon is transparent to low-energy infrared (IR) photons but opaque to photons in the visible portion of the spectrum. This IR transparency of silicon arises from its covalent bonds. Gallium arsenide and many paint pigments like titanium white, titanium oxide, red vermilion, and mercury sulfide have similar wavelength-dependent responses to light [Akinlami and Ashamu (2013); Seredin, Lenshin, Zolotukhin et al. (2018); Robinson, Cho and Gellene (2000); Høye and Stell (1982)].

Compton scattering is one of three competing processes occurring in photon-matter interactions [Mallick (2011); Nieto-Chaupis (2016)]. At energies of a few electron-volts to a few thousand electron-volts, corresponding to visible light through soft X-rays, a photon can be completely absorbed as energy that ejects an electron from the host atom, a process known as the photoelectric effect. High-energy photons of 1.022 MeV and above can bombard the nucleus, causing the formation of an electron and a positron in the pair-production process. Coherent or Rayleigh scattering may occur at low photon energies. A photon may interact with an orbital electron and scatter at a small angle with no change in energy and no other effects. The probability of photoelectric interactions, on the other hand, decreases rapidly with increasing photon energy; it is inversely proportional to the cube of the photon energy. The energy at which interactions change from predominantly photoelectric to Compton is a function of the atomic number of the material comprising the medium [Hopersky, Nadolinsky, Novikov et al. (2015); Powell (1978)].

For visible light, most transparent media have refractive indices between 1 and 2. These values are measured at the yellow doublet D-line of sodium, which has a wavelength of 589 nm, as is conventionally done [Hecht (2002); Bor, Osvay, Rác et al. (1990)]. Complex values of refractive indices possess a real component indicating the phase velocity and an imaginary part indicating attenuation; the two parts are related through the Kramers-Kronig relations. If the refractive index of a medium varies gradually with position, the material is defined as a gradient-index or GRIN medium and is described by gradient index optics [Moore (1980)].

The wavelength dependence of a material's refractive index is usually quantified by its Abbe number or its coefficients in empirical formulas such as the Cauchy or Sellmeier equations [Wei, Murray, Barnes et al. (2018); Donaldson and Caplin (1986)]. The refractive indices and absorption edges of liquid bromine and iodine have been measured over the temperature ranges 19-57°C and 114-181°C respectively, and at wavelengths from the absorption edge (700 nm in bromine and 1100 nm in iodine) to 1800 nm. At 1800 nm, the indices are 1.604 in bromine at 19°C and 1.934 in iodine at 114°C; both liquids are therefore strongly dispersive [Donaldson and Caplin (1986)].

Recent research has also demonstrated the existence of materials with negative refractive indices, which can occur if the permittivity and permeability are both negative. This can be achieved with periodically constructed metamaterials. A survey of relatively recent developments in reconfigurable and tunable metamaterial technology can be found [Chen (2011); Turpin, Bossard, Morgan et al. (2014)]. Unlike natural matter, a metamaterial's

refractive index depends on the properties of the materials composing it and how they are arranged [Padilla, Basov and Smith (2006); Hindy, Elsageer and Yasseen (2018); Soukoulis, Linden and Wegener (2007)]. The resulting negative refraction, which is a reversal of Snell's law, offers the possibility of superlenses and other exotic phenomena. Negative-index metamaterials or negative-index materials (NIMs) are metamaterials with electromagnetic (EM) refractive indices that are negative over some frequency range [Shalev (2007); Engheta and Ziolkowski (2006); Zouhdi, Sihvola and Vinogradov (2008); Shelby, Smith and Schultz (2001); Pendry (2004); Smith, Padilla, Vier et al. (2000)].

In a work [Veselago (1968)] it has been shown that the opposing directions of EM plane waves and the flow of energy arose from the individual Maxwell curl equations. In ordinary optical materials, the curl equation for the electric field shows the 'right-hand rule' for the directions of the electric field  $E$ , the magnetic induction  $B$ , and wave propagation, which has the direction of the wave vector  $k$ . However, the direction of energy flow formed by  $E \times H$  is right-handed only when the permeability is greater than zero. This means that when the permeability is less than zero, e.g. negative, the direction of wave propagation is reversed (determined by  $k$ ) and opposes the direction of energy flow. Furthermore, the relations of vectors  $E$ ,  $H$ , and  $k$  form a 'left-handed' system; Veselago coined the term 'left-handed (LH) material' for such situations. He contended that an LH material has a negative refractive index, basing his argument on the steady-state solutions of Maxwell's equations. A study of the reduction of photon group velocity in free space by measuring a change in arrival time induced by changing the beam's transverse spatial structure using time-correlated photon pairs in both a Bessel beam and a focused Gaussian beam has been mentioned in Giovannini et al. [Giovannini, Romero, Potoček et al. (2015)].

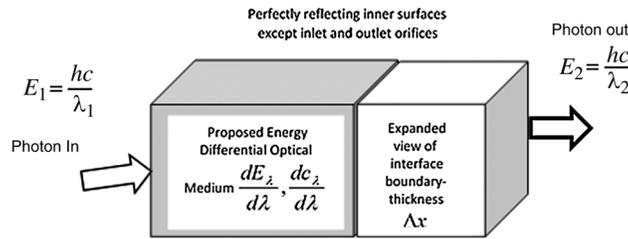
In this paper, the case of a single photon is considered; the photon conforms to all probabilities and its continuous travel is analyzed through a proposed hypothetical medium. During transit, an energy differential with respect to wavelength is assumed to be present in the medium; its existence is governed by certain mathematical conditions. This energy differential exists only with the satisfaction of these mathematical conditions, which requires special properties for parameters like the refractive index, in the optical medium. The photon, when moved out from a single outlet, possesses a lower energy than the input photon, with a change in wavelength but the same velocity. The difference in energy is considered to become stored in the medium under the forced boundary conditions. For this energy differential to exist at a given light intensity, the probability for a single photon's energy and velocity functions, excluding scattering, dispersion, absorption, or other nonlinear processes, is hypothesized to be analytic, which makes it singly and doubly differentiable with respect to wavelength.

The paper is arranged into four major sections. After the present introduction to the proposed approach, in Section 2, the proposed optical medium is physically defined with a stream of photons entering through a narrow orifice, encountering refraction only at the first interface. The energy and velocity functions are analytic for a single photon that

enters the orifice and which is assumed to exit at the other end of the medium. Inside the medium, the energy and velocity functions are singly and doubly differentiable with respect to wavelength, i.e., a gradient energy is assumed to exist. The mathematical constraints necessary for these properties are derived, permitting the development of a second-order differential equation in two variables. The wavelength shift for the single photon under consideration is due to refraction alone at the first interface of the inlet and not due to any other wavelength-shifting mechanism, whereas the proposed medium introduces a wavelength differential for energy and velocity for the photon travelling through it. In Section 3, a mathematical solution of the derived second-order differential equation in two variables is obtained by two proposed substitutions for the wavelength-dependent energy and velocity of the photon under consideration inside the medium. The boundary conditions for integration are also derived in this section. The wavelengths in the solutions are in a transformed domain and are defined as unitless. In addition, the possible numerical solutions for the proposed solution parameters of the wavelength, energy, and velocity of the photon are also estimated using computational software platforms. For the optical medium, the wavelength-energy and wavelength-velocity relations are derived using the polynomial curve fitting technique and the results are presented and plotted. In Section 4, the design of the photonic medium using the derived and estimated parameters is performed in detail. Using the derived results, the differential of the refractive index of the medium is developed; using vector analysis, the solution is derived that ultimately necessitates the mathematical condition for the refractive index of the medium to be existent. Using numerical analysis, the minimum value-normalized refractive index profile is estimated for transformed wavelengths inside the medium and also for vacuum wavelengths. The mathematical analysis proves that such a photonic medium satisfying the derived conditions can produce wavelength-differential energy accumulation, which in turn causes energy storage or energy flotation inside the medium. It is assumed in the mathematical derivation that, with a certain angle of incidence to the normal, the light beam undergoes refraction at the first boundary. The analysis is with respect to the energy-wavelength differential of a single photon; despite effects by ionic interactions, the energy and velocity functions remain continuous and differentiable. A non-zero probability assumption in the derivation for the abovementioned medium is by all means valid, conforming to the basic law that a probability of unity can never be assumed for all photons undergoing interactions. In addition, out of those undergoing ionic interactions, a nonzero probability exists for their energy and velocity functions to be continuous and differentiable, thus making those functions for the photon analytic. The hypothesis is that such a proposed medium that introduces an energy gradient with respect to wavelength in its bulk should satisfy the mathematical and boundary conditions as shown in the following derivations. Inside the medium, the energy function and velocity of the photon will be singly and doubly differentiable with respect to wavelength; therefore, gradients of energy and velocity for the photon are assumed to exist.

## 2 Energy-wavelength and velocity-wavelength derivatives for the photon in the proposed medium

The light beam comprising photons is assumed to enter the closed optical medium through a considerably narrow inlet, as illustrated in Fig. 1. With perfectly reflecting inner surfaces except for the inlet and outlet orifices, from this stream of photons, a single photon by all means is hypothesized to possess a definite nonzero probability of maintaining continuous and differentiable velocity and energy functions from the entry point to the exit, despite ionic interactions or mechanisms of scattering, absorption, fluorescence, and other nonlinear processes [Singh, Gangwar and Singh (2007); Black, Lin, Cronin et al. (2002)].



**Figure 1:** Photon medium with energy differential with respect to wavelength

Inside the proposed optical medium, for the photon under consideration, it is assumed that energy and velocity differentials exist with respect to wavelength. Then rates of change of energy and velocity exist, that can be denoted by  $\frac{dE_\lambda}{d\lambda}$  and  $\frac{dc_\lambda}{d\lambda}$  respectively in Fig. 1.

The perfectly reflecting inner surfaces eliminates any possibility of photon being absorbed or escaped through the horizontal walls. The vertical interface boundary of the optical medium and free space at the exit point of photon is illustrated in an expanded view with a very small thickness  $\Delta x$ . With a non-zero probability of the photon energy and velocity functions being continuous throughout the transit, the photon moving out of the orifice is assumed possess lesser energy compared to that at the entry point, resulting in the existence of energy differential and velocity differential inside the medium. A multivariable second order differential equation is derived using these principles for modeling the optical medium possessing these properties.

### 2.1 Multivariable second-order differential equation

With reference to Fig. 1, the medium is modelled to be having energy and velocity differentials inside it for a photon in transit. For that to exist, the quantum mechanical energy equations should be differentiable with respect to wavelength of the photon [Klein (2010)]. Additionally, velocity of the photon inside the medium will be a function of wavelength.

Let  $E_\lambda$  and  $c_\lambda$  denote the wavelength ( $\lambda$ )-dependent energy and velocity of the photon, respectively, in the medium;  $c$  is the velocity of the photon in vacuum. Let the boundary conditions be as marked in Fig. 1. For simplicity, denoting energy  $E_\lambda$  as  $E$  only and the velocity  $c_\lambda$  of the photon, a function of  $\lambda$ ,

$$c_\lambda = f(\lambda) \text{ and } E = E_\lambda = \frac{hc_\lambda}{\lambda}$$

The energy differential with respect to wavelength inside the medium will be:

$$\frac{dE}{d\lambda} = h \frac{\left( \lambda \frac{dc_\lambda}{d\lambda} - c_\lambda \right)}{\lambda^2} \quad (1)$$

$$\frac{dE}{d\lambda} = \frac{h}{\lambda} \frac{dc_\lambda}{d\lambda} - \frac{E}{\lambda} \quad (2)$$

Or

$$E = h \frac{dc_\lambda}{d\lambda} - \lambda \frac{dE}{d\lambda} \quad (3)$$

Differentiating the energy of the photon with respect to the wavelength again inside the medium and rearranging yields the multivariable second-order differential equation as follows:

$$\frac{d^2E}{d\lambda^2} + \frac{2}{\lambda} \frac{dE}{d\lambda} - \frac{h}{\lambda} \frac{d^2c_\lambda}{d\lambda^2} = 0 \quad (4)$$

Eq. (4) therefore becomes

$$\frac{d^2E}{d\lambda^2} + \frac{2}{\lambda} \frac{dE}{d\lambda} - \frac{E}{c_\lambda} \frac{d^2c_\lambda}{d\lambda^2} = 0 \quad (5)$$

The multivariable second-order differential equation in two variables, which forms the basis of the modelling of the proposed optical medium, can be made solvable by converting it to a second-order second-degree differential equation. This is allowed using mathematical substitutions for energy and velocity, as presented in Section 3.

### 3 Mathematical solution and numerical analysis

The second-order differential Eq. (5) in two variables is solved by making the following mathematical substitutions:

$$E = e^{z\lambda} \quad (6)$$

$$c_\lambda = e^{\beta E} \quad (7)$$

which in turn becomes

$$c_\lambda = e^{\beta e^{\alpha\lambda}} \tag{8}$$

The energy and velocity of the photon inside the medium is modelled in such a way that they conform to standard properties and also render the multivariable second order differential equation suitable for numerical analysis based solutions. i.e. within the differential energy medium under consideration, the energy is modelled as a function of the transformed wavelength  $\lambda$  and another wavelength-dependent parameter  $\alpha$ , where  $\alpha=f(\lambda)$ , Additionally, the velocity of the photon within the proposed medium is modelled as another function of  $\alpha$ ,  $\beta$ , and  $\lambda$ , where  $\beta$  is modelled as an independent parameter with respect to both  $\alpha$  and  $\lambda$ , which in turn are dependent variables. Eqs. (6) and (7) are proposed mathematical substitutions needed for transforming Eq. (5) so as to obtain numerical analysis based solutions. Eq. (8) follows from Eqs. (6) and (7). These mathematical substitutions are made conformed to standard energy and velocity relations as proved in Appendix A2.

For photons undergoing dispersion, the Sellmeier equation [Sellmeier (1871)] traditionally characterizes the refractive index-wavelength relation of an optical medium. The dispersion effects are modelled by considering the oscillating electric fields of the light beam interacting with the dipoles of the medium. The oscillating dipoles resonate at a specific wavelength and hence their response is modelled as Lorentz oscillators. The proposed solution considers the dipole interaction with the field of propagating light via its conformity to the Sellmeier equation of the medium with the  $m$  oscillators under consideration. The detailed derivation for the relationship of the solution to the Sellmeier coefficients of the proposed medium is presented in Appendix A1. Denoting  $\lambda_0$ , the wavelength in vacuum, and  $c$ , the photon velocity in vacuum, the relation is given by:

$$\sum_{p=0}^{\infty} \frac{\beta^p}{p!} \left( \sum_{k=0}^{\infty} \frac{(f(\lambda)\lambda)^k}{k!} \right)^p \left( 1 + \sum_{i=1}^m \frac{A_i \lambda_0^2}{\lambda_0^2 - \lambda_i^2} \right)^{\frac{1}{2}} - c = 0 \tag{9}$$

The truncated infinite summation in the first term conforms to the Sellmeier coefficients of the optical medium.

With  $\alpha=f(\lambda)$  already defined, Eq. (9) considers the field–dipole interaction occurring within the medium as the photon propagates through it. Any modification that may be introduced in the original Sellmeier equation is also reflected in the second term of Eq. (9).

Then, from Eq. (6):

$$\frac{dE}{d\lambda} = e^{\alpha x} \left( \alpha + \lambda \frac{d\alpha}{d\lambda} \right) \tag{10}$$

$$\begin{aligned} \frac{d^2 E}{d\lambda^2} &= \lambda e^{\alpha\lambda} \frac{d^2 \alpha}{d\lambda^2} + (2e^{\alpha\lambda} + \alpha\lambda e^{\alpha\lambda} + \alpha\lambda e^{2\alpha\lambda}) \frac{d\alpha}{d\lambda} + \\ &\lambda^2 e^{2\alpha\lambda} \left(\frac{d\alpha}{d\lambda}\right)^2 + \alpha^2 e^{\alpha\lambda} \end{aligned} \quad (11)$$

With reference to Eq. (5), all dependencies can be considered in the development of a complete multivariable second-order second-degree differential equation for the photon. For the velocity solution, neglecting very minute dependencies in the context of the second-order differential Eq. (5) in energy and velocity, the following is developed:

$$\frac{d^2 c_\lambda}{d\lambda^2} = \alpha\beta\lambda \frac{d\alpha}{d\lambda} e^{(\alpha\lambda + \beta e^{\alpha\lambda})} + \alpha^2 \beta e^{(\alpha\lambda + \beta e^{\alpha\lambda})} \quad (12)$$

After substitutions and simplification, Eq. (5) becomes:

$$\begin{aligned} \lambda e^{\alpha\lambda} \frac{d^2 \alpha}{d\lambda^2} + (\alpha\lambda e^{\alpha\lambda} + \alpha\lambda e^{2\alpha\lambda} + 2e^{\alpha\lambda} - \alpha\beta\lambda e^{2\alpha\lambda}) \frac{d\alpha}{d\lambda} + \\ \lambda^2 e^{2\alpha\lambda} \left(\frac{d\alpha}{d\lambda}\right)^2 + \alpha^2 e^{\alpha\lambda} + \frac{2}{\lambda} \alpha e^{\alpha\lambda} - \alpha^2 \beta e^{2\alpha\lambda} = 0 \end{aligned} \quad (13)$$

### 3.1 Boundary conditions

The boundary conditions for Eq. (13) are derived, based on the physical background presented in Fig. 1. For the photon constituting the input stream at the inlet of the optical medium, the wavelength boundaries are  $\lambda_1$  and  $\lambda_2$ . This happens because of the existence of an energy differential in the medium for the photon. Then the following boundary conditions can be derived for the parameter  $\alpha$ , making use of Eq. (6) and the quantum mechanical energy equivalence.

$$\alpha_1 = \frac{\ln\left(\frac{hc}{\lambda_1}\right)}{\lambda_1} = \frac{1}{\lambda_1} (\ln(hc) - \ln(\lambda_1)) \quad (14)$$

$$\alpha_2 = \frac{\ln\left(\frac{hc}{\lambda_2}\right)}{\lambda_2} = \frac{1}{\lambda_2} (\ln(hc) - \ln(\lambda_2)) \quad (15)$$

### 3.2 Discussions on initial conditions

The initial conditions of the photon can be assessed in the context of its transit from inlet to outlet as in Fig. 1. Since the photon is assumed to enter into the inlet from free space, the initial condition of velocity of the photon will be the velocity in free space,  $c$ . It is with this initial velocity that the photon starts its transit across the optical medium. While traversing the length across this medium, the initial velocity will undergo change, which is analysed. For the photon, the initial wavelength value is assumed to be  $\lambda_1$ , which corresponds to its initial energy. The optical medium is possessing energy differential

with respect to wavelength and consequently this initial energy will be lowered as the photon travels across the medium.

### 3.3 Function-transformed wavelength domain

The proposed solution defines and also generates a set of wavelengths in a function-transformed mode, which are hence defined as unitless in the analysis. The transform function for energy equivalence is derived in Appendix A2, as:

$$T(\lambda) = \lambda \frac{e^{(\alpha\lambda - \beta e^{\alpha\lambda})}}{h} \quad (16)$$

The system energy in a real experimental set up is correlated to quantum mechanical expressions by deriving the transform function as shown in Appendix A2. The transformed wavelengths in the analysis which are unitless can be mapped back to unit-based real wavelengths by using standard photon energy equations and applying an energy equivalence condition for the two wavelength domains. The analysis and solution of the multivariable differential equation are rendered more suitable in the transformed domain which always possesses equivalence to the real wavelength domain by the presence of the derived transform function. The numerical solutions derived in Section 3.4 belong to this transformed unitless wavelength domain and are used for further analysis.

### 3.4 Numerical solutions

The approximate solutions for  $\alpha$  and  $\lambda$  in the second-order second-degree differential Eq. (13) are estimated by numerical analysis using MATLAB<sup>®</sup> for different values of  $\beta$ . The boundary conditions are specified in Eqs. (14) and (15); however, the complex nature of Eq. (13) does not yield implicit or explicit solutions, thus requiring numerical integration. Reliability in the numerical solutions is ensured by performing the integration operation using MATLAB<sup>®</sup> toolbox which makes use of the adaptive Simpson quadrature technique. This is a robust procedure in which the interval is divided into subintervals and the quadrature rule is used in each subinterval to compute the integral [Lyness (1969); Bernsten, Espelid and Sørøvik (1991)]. The subdivision of intervals is determined recursively using an error estimate.

After estimating  $\alpha$  and  $\lambda$ , using Eqs. (6) and (7), the energy and velocity values of the photon are computed. The results are presented in Tab. 1.

The  $\lambda$ - $c_\lambda$  relation is estimated by polynomial curve fitting. The estimation yields a sixth-degree polynomial as in Eq. (17):

$$c_\lambda = -0.5015 \times 10^{12} \lambda^6 + 2.3191 \times 10^{12} \lambda^5 - 4.4088 \times 10^{12} \lambda^4 + 4.4054 \times 10^{12} \lambda^3 - 2.4376 \times 10^{12} \lambda^2 + 0.7075 \times 10^{12} \lambda - 0.0841 \quad (17)$$

Similarly, the  $\lambda$ - $E$  relation of the photon in the medium is governed by the sixth-degree polynomial as:

**Table 1:** Numerical analysis-based solutions for the second-order second-degree differential equation

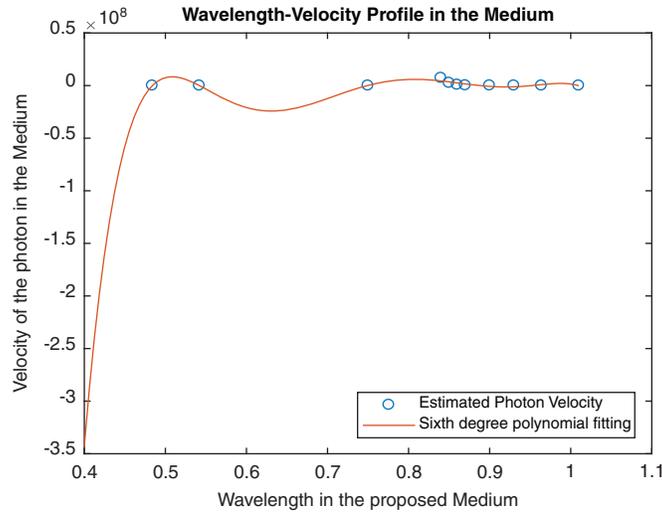
$\beta$	$\alpha$	$\lambda$	$E$	$c_\lambda$
1	-0.405	0.484	0.822	2.28
2	1.43	0.75	2.93	$3.51 \times 10^7$
3	1.03	1.01	2.83	$4.8 \times 10^3$
4	0.89	0.964	2.35	$1.2 \times 10^4$
5	0.79	0.93	2.08	$3.3 \times 10^4$
6	0.72	0.90	1.92	$1 \times 10^5$
7	0.66	0.87	1.76	$2.24 \times 10^5$
8	0.62	0.86	1.70	$8.1 \times 10^5$
9	0.58	0.85	1.64	$2.60 \times 10^6$
10	0.55	0.84	1.58	$7.3 \times 10^6$

$$E = 0.4026 \times 10^5 \lambda^6 - 1.9041 \times 10^5 \lambda^5 + 3.7040 \times 10^5 \lambda^4 - 3.7875 \lambda^3 + 2.1438 \lambda^2 - 0.6359 \lambda + 0.0771 \quad (18)$$

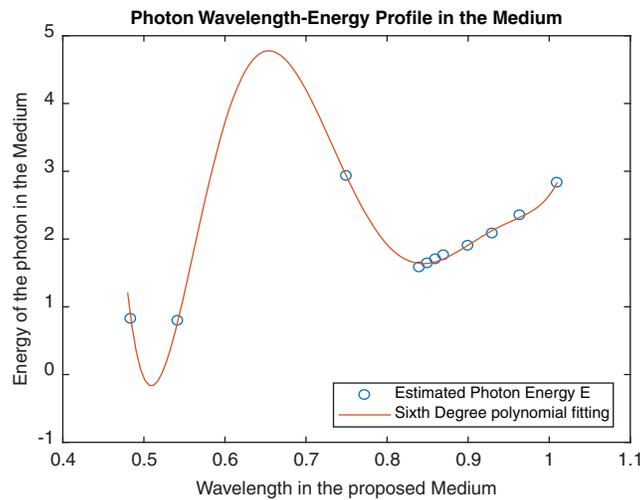
The plots of the two polynomials in Eqs. (17) and (18) for  $\lambda$  in the interval (0, 1.1) are shown in Figs. 2 and 3, respectively. The estimated photon velocities are plotted in Fig. 2 which are spread over a nonlinear curve. The lowest order nonlinear curve that fits closest these estimated values is found to be a sixth degree polynomial, which is plotted over the spread values to show the fit. Similarly, in Fig. 3, the estimated photon energies are plotted which are having a nonlinear distribution. The lowest order nonlinear curve that fits closest these estimated energy values is found again to be a sixth degree polynomial, which is plotted over the distributed values to show the fit. Thus the estimated values for both velocity and energy of the photon fit sixth-degree polynomials as derived for the two cases. Eq. (17) denotes this sixth degree polynomial for velocity and Eq. (18) represents the sixth degree polynomial fit for energy. In the velocity-wavelength and energy-wavelength profiles shown, the wavelengths, velocity, and consequently energy are in transformed unitless domains that can be mapped to real unit-based parameters in a design-based implementation setup.

#### 4 Photonic medium modelling from mathematical solution

The refractive index for a common optical medium can be considered as the factor by which the velocity and wavelength of the radiation are reduced with respect to their vacuum values and hence modified as  $v=c/n$ , where  $c$  is the speed in vacuum,  $v$  is that in the medium, and  $n$  is the refractive index. Similarly, the wavelength in that medium is  $\lambda=\lambda_0/n$ , where  $\lambda_0$  is the wavelength in vacuum. This implies that, in a common optical medium, the



**Figure 2:** The estimated  $\lambda$ - $c_\lambda$  sixth-degree polynomial. The wavelengths and velocity are in transformed unitless domains but can be mapped back to real parameters



**Figure 3:** The estimated  $\lambda$ - $E$  sixth-degree polynomial. The wavelengths and energy are in transformed unitless domains but can be mapped back to real parameters

frequency ( $f = v/\lambda$ ) of the wave is not affected by the refractive index; therefore, the energy ( $E = hf$ ) of the photon is not affected by the refraction or the refractive index of the medium. However, the proposed photonic medium that stores energy, thereby possessing energy-wavelength and velocity-wavelength differentials, can be modelled using the obtained

mathematical results. As already hypothesized in Section 2, a single photon is considered in the analysis which by all means possesses a definite nonzero probability of maintaining continuous and differentiable energy and velocity functions from the entry point to the exit. Thus, the medium can be designed as a multi-layered structure such that at any point inside it, the derivative or slope of the wavelength-velocity profile at that point matches that of the designed profile shown in Fig. 2. In analyzing the refractive index differential with respect to wavelength for the medium:

$$n = \frac{c}{c_\lambda} = \frac{\lambda_0}{\lambda} \quad (19)$$

$$\frac{dn}{d\lambda} = -\frac{c}{c_\lambda^2} \frac{dc_\lambda}{d\lambda} = -\frac{\lambda_0}{\lambda^2} \quad (20)$$

$$\frac{dn}{d\lambda} = -\frac{n}{c_\lambda} \frac{dc_\lambda}{d\lambda} = -\frac{\lambda_0}{\lambda^2} \quad (21)$$

Substituting from Eq. (17) gives the differential as:

$$\frac{dn}{d\lambda} = \left( \frac{-n}{c_\lambda} \right) (-3.01 \times 10^{12} \lambda^5 - 11.60 \times 10^{12} \lambda^4 + 17.6 \times 10^{12} \lambda^3 - 413.22 \times 10^{12} \lambda^2 + 24.88 \times 10^{12} \lambda - 0.7075 \times 10^{12}) \quad (22)$$

Again with reference to Eq. (17) and using vector notations, let the vectors be defined as:

$$p = (3.01 \quad 11.60 \quad -17.60 \quad 413.22 \quad -24.88 \quad +0.7075)$$

$$q = (\lambda^5 \quad \lambda^4 \quad \lambda^3 \quad \lambda^2 \quad \lambda \quad 1)$$

$$r = (-0.5015 \quad 2.3191 \quad -4.4088 \quad 4.4054 \quad -2.4376 \quad 0.7075 \quad -0.0841)$$

$s$  is the modified  $q$  vector of seven elements with the extra first element as the sixth power of wavelength while  $s_o$  is the  $s$ -vector with the photon vacuum wavelengths. In addition,

$$u = (n^{-6} \quad n^{-5} \quad n^{-4} \quad n^{-3} \quad n^{-2} \quad n^{-1} \quad 1)$$

Then Eq. (22) becomes

$$\frac{dn}{n} = \frac{pq^T}{rs^T} d\lambda \quad (23)$$

Integrating both sides of Eq. (23) to obtain the refractive index dependence on wavelength for the medium and considering the boundary conditions will yield neither implicit nor explicit solutions. Therefore, numerical integration is performed using MATLAB<sup>®</sup> and the adaptive quadrature technique. The approximate solutions for Eq. (23) are estimated as the dot product as follows:

$$\ln(n) = \ln\left(\frac{1}{10^{18}m \cdot s}\right) \tag{24}$$

This gives the Refractive Index profile with respect to wavelengths in the designed medium as

$$n = \frac{1}{10^{18}m \cdot s} \tag{25}$$

where the vector

$$m = (5.132 \quad -23.745 \quad 45.146 \quad -45.111 \quad 24.961 \quad -7.244 \quad 8.611 \times 10^{-13})$$

With reference to the wavelengths in vacuum  $\lambda_0$ , Eq. (25) is modified to the Hadamard product as:

$$n = \frac{1}{10^{18}m(u \circ s_0)^T} \tag{26}$$

Eq. (25) corresponds to the refractive index profile of the proposed medium for the energy–wavelength differential to exist as the photon propagates through it. The minimum value-normalized refractive index profiles for the medium with respect to  $\lambda$  and for the vacuum wavelength  $\lambda_0$  are estimated by numerical analysis and the results are presented in Tab. 2. As stated in Section 3.2, the numerical solutions of wavelength belong to the transformed unitless domain, rather than being real wavelengths, but they can be mapped back to unit-based real wavelengths in practical designs by using photon energy relations.

**Table 2:** Minimum value-normalized refractive index ( $n$ ) profiles for the proposed optical medium

Unitless transformed domain wavelengths	Normalized $n$ with respect to Min value	Possible solution set for normalized $n$ with respect to min. value for vacuum wavelength-based analysis
1	-7.4127	1.0323+j0.5548
0.1	-7.4689	1.0323-j0.5548
0.01	-7.7051	1.1178+j1.7880
0.001	-46.691	1.1178-j1.7880
0.0001	1.00	1.00

#### 4.1 Existence of solutions

The solutions of the numerical analysis indicate the mathematical constraints to be satisfied for the optical medium to possess an energy differential for a photon. Possible solutions of negative and complex values of normalized refractive indices indicate the prospect of

metamaterials in the physical design of the medium, inside which energy differential exists for a photon. The solutions that yielded using numerical analysis act as pointers towards the existence of composite materials that can constitute the proposed optical medium with differential energy storage. Design and implementation of the proposed optical medium can go a long way in the development of photon based optical energy storage structures or cells. If the challenges of exact physical design and optimization can be overcome, these non-conventional energy storage units find numerous applications in practical scenarios.

## 5 Conclusions

Mathematical modelling of an optical medium possessing the property of energy storage by the existence of differential energy and velocity for a photon of light with respect to wavelength is carried out in this paper. Inside the medium, a nonzero probability is assumed for a single photon to maintain analytic velocity and energy functions despite ion interactions, absorption, scattering, and other nonlinear mechanisms. This assumption makes the photon energy and velocity functions singly and doubly differentiable with respect to wavelength; in other words, a gradient energy with respect to wavelength is assumed to exist. Such an optical medium exists only when certain mathematical constraints are satisfied, especially for the refractive index, which are analyzed in the paper. A multivariable second-order differential equation is derived whose mathematical solutions are developed using two proposed transformations for energy and velocity. Using numerical analysis, specific solutions are estimated for different modelling parameters. The solution of the derived second-order second-degree differential equation proves that energy storage or flotation occurs inside the medium if it is designed as per the mathematical constraints. The minimum value-normalized refractive index profiles of the proposed optical medium for the transformed wavelengths both inside the medium and for vacuum wavelengths have been derived. The future prospect of the proposed model includes physical implementation by suitable design of optical materials including metamaterials, thereby resulting in the development of energy storage units or cells in a practical scenario.

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**Appendices**

***A1. Relation of the proposed solution to Sellmeier equation***

The Sellmeier equation [Sellmeier (1871)] is an empirical relationship between the refractive index and wavelength for an optical medium and can be used to analyze the dispersion of light in that medium. The oscillating dipoles of the medium resonate at a specific frequency and wavelength; thus, the dielectric response is modelled as one or more Lorentz oscillators. For  $m$  oscillators, the Sellmeier equation may be written as:

$$n^2 = 1 + \sum_{i=1}^m \frac{A_i \lambda_0^2}{\lambda_0^2 - \lambda_i^2} \tag{A1}$$

where  $n$  is the refractive index of the material,  $\lambda_0$  is the vacuum wavelength, and  $\lambda_i$  are the wavelengths associated with the corresponding relaxation frequencies.

Using the defined mathematical substitutions:  $E = e^{\alpha\lambda}$  and  $c_\lambda = e^{\beta E}$  inside the differential energy medium under consideration, the energy is modelled as a function of the wavelength  $\lambda$  and another wavelength-dependent parameter  $\alpha$ , where  $\alpha = f(\lambda)$ .

Additionally, the velocity of the photon inside the proposed medium is modelled as another function of  $\alpha$ ,  $\beta$ , and  $\lambda$ , where  $\beta$  is modelled as an independent parameter with respect to both  $\alpha$  and  $\lambda$ .

Analyzing the function as a Taylor series expansion:

$$E = e^{f(\lambda)\lambda} = \sum_{k=0}^{\infty} \frac{(f(\lambda)\lambda)^k}{k!} \tag{A2}$$

and

$$c_\lambda = e^{\beta E} = \sum_{p=0}^{\infty} \frac{\beta^p}{p!} \left( \sum_{k=0}^{\infty} \frac{(f(\lambda)\lambda)^k}{k!} \right)^p \tag{A3}$$

The photon velocity inside the medium, defined with respect to the refractive index, manifests itself from the Sellmeier equation as:

$$\frac{c^2}{c_\lambda^2} = 1 + \sum_{i=1}^m \frac{A_i \lambda_0^2}{\lambda_0^2 - \lambda_i^2} \tag{A4}$$

$$c_\lambda = \frac{c}{\sqrt{\left(1 + \sum_{i=1}^m \frac{A_i \lambda_0^2}{\lambda_0^2 - \lambda_i^2}\right)}} = \sum_{p=0}^{\infty} \frac{\beta^p}{p!} \left( \sum_{k=0}^{\infty} \frac{(f(\lambda)\lambda)^k}{k!} \right)^p \quad (\text{A5})$$

Or,

$$\sum_{p=0}^{\infty} \frac{\beta^p}{p!} \left( \sum_{k=0}^{\infty} \frac{(f(\lambda)\lambda)^k}{k!} \right)^p \left( 1 + \sum_{i=1}^m \frac{A_i \lambda_0^2}{\lambda_0^2 - \lambda_i^2} \right)^{\frac{1}{2}} - c = 0 \quad (\text{A6})$$

The infinite summation in the first term is truncated, conforming to the Sellmeier coefficients. The proposed solution parameters should satisfy the above relation, as in (A6) to conform to the Sellmeier equation or the Sellmeier coefficients of the proposed hypothetical optical medium. It considers the light-dipole interactions while the light propagates inside the medium.

### ***A2. Derivation of transform function for energy equivalence***

From Section 3, the proposed energy and wavelength dependent velocity inside the medium are  $E = e^{\alpha\lambda}$  and  $c_\lambda = e^{\beta E}$ . For obtaining energy equivalence, for the photon,

$$e^{\alpha\lambda} = \frac{he^{\beta E}}{\lambda} T(\lambda) \quad (\text{A7})$$

where  $T(\lambda)$  is the transform function applied for energy equivalence in the wavelength domain of the solution obtained. Solving,

$$T(\lambda) = \lambda \frac{e^{\alpha\lambda}}{he^{\beta e^{\alpha\lambda}}} \quad (\text{A8})$$

On simplification, the transform function is derived as:

$$T(\lambda) = \lambda \frac{e^{(\alpha\lambda - \beta e^{\alpha\lambda})}}{h} \quad (\text{A9})$$

Evaluating typically for a four-decimal-place nonzero approximation using numerical solutions from [Tab. 1](#) in Section 3.4 as follows:

For vector  $\vec{\beta} = [1 \ 2 \ 3 \ 4 \ 5]$ , the transform function value vector is estimated as:

$$\vec{T}(\lambda) = \frac{1}{h} [0.1749 \ 0.0063 \ 0.0006 \ 0.0002 \ 0.0001]$$

Estimations can be extended to higher-decimal-place nonzero approximations considering the constraint of computational complexity.

### **Section organization**

The various sections of the manuscript are organized as follows: Section 1 is an introduction to different photon interaction mechanisms in an optical medium and the studies carried out in the field. In addition, an overview of metamaterials along with an introduction to the proposed modelling is presented. In Section 2, a multivariable second degree differential equation is derived for photon propagating in the proposed optical medium. For solving, this differential equation is being transformed to a second degree second order differential equation using two proposed mathematical substitutions. After deriving the boundary conditions, numerical analysis-based solutions have been estimated. The photonic medium modeling has been carried out in Section 4. Section 5 is a conclusion of the work presented, followed by acknowledgement and declaration of conflicts of interest. In the reference section, the various citations utilized in the work are shown. Finally, in the Appendices section, the relation of the proposed solution to Sellmeier equation is derived in A1 and the derivation of transform function for energy equivalence is presented in A2.