On Caputo-Type Cable Equation: Analysis and Computation

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Abstract: In this paper, a special case of nonlinear time fractional cable equation is studied. For the equation defined on a bounded domain, the existence, uniqueness, and regularity of the solution are firstly studied. Furthermore, it is numerically studied via the weighted and shifted Grünwald difference (WSGD) methods/the local discontinuous Galerkin (LDG) finite element methods. The derived numerical scheme has been proved to be stable and convergent with order $\mathcal{O}(\Delta t^2 + h^{k+1})$, where Δt, h, k are the time stepsize, the spatial stepsize, and the degree of piecewise polynomials, respectively. Finally, a numerical experiment is presented to verify the theoretical analysis.

Keywords: Fractional cable equation, regularity, local discontinuous Galerkin method, stability, convergence.

1 Introduction

In this paper, we consider a special case of nonlinear time fractional cable equation in the following form,

$$\frac{\partial u(x,t)}{\partial t} + {}_C\mathrm{D}_{0,t}^\alpha u(x,t) - u_{xx}(x,t) + f(u) = g(x,t), \;\; x \in \Omega, \;\; t > 0, \tag{1}$$

with the initial value condition,

$$u(x,t)|_{t=0} = u_0(x), \;\; x \in \Omega, \tag{2}$$

and the boundary value condition,

$$u(x,t)|_{x \in \partial\Omega} = 0, \;\; t > 0, \tag{3}$$

where $0 < \alpha < 1$, $\Omega = (a,b)$ is a bounded domain, g, u_0 are given smooth functions, ${}_C\mathrm{D}_{0,t}^\alpha$ is the α-th order Caputo derivative operator defined by Podlubny [Podlubny (1999)]

$$_C\mathrm{D}_{0,t}^\alpha u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \frac{\partial u}{\partial \tau} \mathrm{d}\tau, \;\; 0 < \alpha < 1, \tag{4}$$

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in which $\Gamma(\cdot)$ denotes the Gamma function. We always suppose that the nonlinear source term $f(u)$ satisfies Lipschitz continuity condition with respect to u, that is, there exists a positive constant L such that for all u_1, u_2,

$$|f(u_1) - f(u_2)| \leq L|u_1 - u_2|.$$

Cable equations with fractional order temporal operators were introduced to model electrotonic properties of spiny neuronal dendrites by Henry et al. [Henry, Langlands and Wearne (2008)]. The time fractional cable equation (TFCE) is similar to the traditional cable equation except that the order of derivative with respect to the time is fractional. If there is a nonlinear source term, the equation reads as (with $0 < \alpha, \beta < 1$) [Henry, Langlands and Wearne (2008)]

$$\frac{\partial u(x,t)}{\partial t} + {}_C \mathrm{D}_{0,t}^\alpha u(x,t) - {}_C \mathrm{D}_{0,t}^\beta u_{xx}(x,t) + f(u) = g(x,t), \tag{5}$$

which has been numerically treated by a number of authors. For example, Lin et al. [Lin, Li and Xu (2011)] constructed a finite difference/Legendre spectral scheme for discretization of TFCE. Hu et al. [Hu and Zhang (2012)] proposed two implicit compact difference schemes for TFCE. A fourth-order compact finite difference scheme for 2D TFCE was studied by Yu et al. [Yu and Jiang (2016)]. Zheng et al. [Zheng and Zhao (2017)] developed and analyzed a time LDG method (LDG method is applied in time direction) for solving TFCE. Al-Maskari et al. [Al-Maskari and Karaa (2018)] discussed the lumped mass Galerkin finite element method for TFCE. Recently, a scheme combining a finite difference approach in time direction and LDG finite element method in space direction for TFCE was proposed by Li et al. [Li and Wang (2019)]. They proved that the derived scheme could reach 2-nd order in time direction, which was higher than the classical L1 method. Liu et al. [Liu, Du, Li et al. (2019)] considered some second-order θ schemes combined with Galerkin finite element method for TFCE.

It is worth noting that

- If $\alpha = 0$, $0 < \beta < 1$, Eq. (5) reduces to a time fractional subdiffusion equation, which has been theoretically and numerically discussed by many authors, see e.g., [Karaa, Mustapha and Pani (2018); McLean and Thomée (2010)].

- If $\alpha = \beta = 0$, Eq. (5) reduces to an integer order diffusion equation with nonlinear source term. Up to now, a great deal of work has also been done on this type of equation, see [Thomée (2006)].

- If $0 < \alpha < 1$, $\beta = 0$, Eq. (5) reduces to a time fractional parabolic equation (i.e., (1)). There seems no work on the mathematical analysis and LDG method for it. This motivates our interest in studying Eq. (1).

LDG method is a special class of discontinuous Galerkin method, proposed by Cockburn et al. [Cockburn and Shu (1998)]. The main technique of LDG method is to rewrite higher-order derivative equation into an equivalent system containing only the first

derivative, and then discretize it by the standard discontinuous Galerkin method. For more information about this method, see the review paper by Xu et al. [Xu and Shu (2010)]. Here we propose the LDG finite element methods to numerically study Eq. (1). The main contributions of this paper are twofold: One is to provide a complete mathematical analysis for Eq. (1), including existence, uniqueness, and regularity of the solution; The other is to numerically studied Eq. (1) using WSGD method in time domain and using the LDG finite element method in space domain. The derived numerical scheme is stable and convergent with order $\mathcal{O}(\Delta t^2 + h^{k+1})$.

The rest of this paper is organized as follows. In Section 2, we introduce some preliminaries, which will be used in the following section. In Section 3, we discuss the existence, uniqueness, and regularity for the solution to Eq. (1). In Section 4, a fully discrete LDG scheme is proposed and the stability and convergence of the presented scheme is analyzed too. A numerical experiment is given in Section 5 to illustrate the effectiveness of the proposed numerical method. Finally, the last section concludes this paper.

2 Preliminaries

2.1 Notations

We first recall some notations and preliminary facts, which are used throughout this paper. The L^2-norm and inner product on Ω are given by

$$\|v\|_\Omega^2 = (v, v), \quad (u, v) = \int_\Omega uv \mathrm{d}x.$$

Likewise, we define the L^∞-norm on Ω by $\|u\|_\infty = \sup_{x \in \Omega} |u|$.

The Sobolev space $H^\ell(\Omega)$ with $1 \le \ell \le \infty$ on Ω is defined by Cao et al. [Cao, Song, Wang et al. (2019)]

$$H^\ell(\Omega) = \left\{ v \in L^2(\Omega) : \sum_{|\Bbbk| \le \ell} \|\mathrm{D}^\Bbbk v\|_\Omega^2 < \infty \right\},$$

and endow this space with the following norm

$$\|v\|_{H^\ell(\Omega)} = \left(\sum_{|\Bbbk| \le \ell} \|\mathrm{D}^\Bbbk v\|_\Omega^2 \right)^{\frac{1}{2}},$$

where $\Omega \subset \mathbb{R}^n, \Bbbk = (k_1, \cdots, k_n), |\Bbbk| = k_1 + \cdots + k_n \le \ell$.

The Laplace transform of a given function $v(t)$ is defined as Li et al. [Li and Zeng (2015)]

$$\hat{v}(s) = \mathscr{L}\{v(t); s\} = \int_0^\infty e^{-st} v(t) \mathrm{d}t, \tag{6}$$

and the inverse Laplace transform is given by

$$v(t) = \mathscr{L}^{-1}\{\hat{v}(s); t\} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} \hat{v}(s) \mathrm{d}s, \quad c = \mathrm{Re}(s) > c_0, \tag{7}$$

where c_0 lies in the right half plane of the absolute convergence of the Laplace transform (6).

We denote by Σ_ζ the sector

$$\Sigma_\zeta = \{s \in \mathbb{C} \ : \ |\arg s| < \zeta, s \neq 0\}, \ \ 0 < \zeta < \pi.$$

To this end, let $(\mathcal{X}, \mathcal{D}) = \left(L^2(\Omega), H_0^1(\Omega) \cap H^2(\Omega)\right)$, and $\| \cdot \|_{\mathcal{X} \to \mathcal{X}}$ be the operator norm on the space $\mathcal{X}$. Then the operator $A =: \Delta$ satisfies [Atluri, Batty, Hieber et al. (2011)]

$$\|(s - A)^{-1}\|_{\mathcal{X} \to \mathcal{X}} \leq C_\zeta |s|^{-1}, \ \ \forall s \in \Sigma_\zeta, \ \ \forall \zeta \in (0, \pi), \tag{8}$$

where C_ζ is a positive constant depending on ζ.

2.2 Solution representation

Let $w = u - u_0$, then (1) can be rewritten as the following equivalent system

$$\begin{cases} \frac{\partial w(x,t)}{\partial t} + {}_C\mathrm{D}_{0,t}^\alpha w - Aw = Au_0 - f(u) + g(x,t), \ x \in \Omega, \ t > 0, \ \Omega = (a,b), \\ w(x,0) = 0, \ \ x \in \Omega. \end{cases} \tag{9}$$

Using Laplace transform, we obtain

$$(s + s^\alpha - A)\, \hat{w}(s) = s^{-1}Au_0 + \mathscr{L}\{-f(u); s\} + \hat{g}(s),$$

which further implies

$$\hat{w}(s) = (s + s^\alpha - A)^{-1}\left(s^{-1}Au_0 + \mathscr{L}\{-f(u); s\} + \hat{g}(s)\right).$$

Then by inverse Laplace transform and convolution rule, the solution of Eq. (9) can be represented by

$$w = \mathscr{E}(t)Au_0 - \int_0^t \mathscr{F}(t - \tau)f\big(u(\tau)\big)\mathrm{d}\tau + \int_0^t \mathscr{F}(t - \tau)g(\tau)\mathrm{d}\tau,$$

where the operators $\mathscr{E}(t), \mathscr{F}(t) : \mathcal{X} \to \mathcal{X}$ are defined by

$$\mathscr{E}(t) = \frac{1}{2\pi i} \int_{\Gamma_{\theta,\delta}} e^{st}s^{-1}(s + s^\alpha - A)^{-1}\mathrm{d}s,$$

$$\mathscr{F}(t) = \frac{1}{2\pi i} \int_{\Gamma_{\theta,\delta}} e^{st}(s + s^\alpha - A)^{-1}\mathrm{d}s.$$

For fixed $\delta > 0$ and $\theta \in (\frac{\pi}{2}, \pi)$, the contour of integration $\Gamma_{\theta,\delta}$ is defined by

$$\Gamma_{\theta,\delta} = \left\{s \in \mathbb{C} \ : \ |s| = \delta, |\arg s| \leq \theta\right\} \cup \left\{s \in \mathbb{C} \ : \ s = \rho e^{\pm i\theta}, \rho \geq \delta\right\},$$

and with $\mathrm{Im}s$ increasing.

Now we obtain a representation of the solution of Eq. (1),

$$u = u_0 + \mathscr{E}(t)Au_0 - \int_0^t \mathscr{F}(t - \tau)f\big(u(\tau)\big)\mathrm{d}\tau + \int_0^t \mathscr{F}(t - \tau)g(\tau)\mathrm{d}\tau. \tag{10}$$

3 Regularity of the solution

Before we present the main theorem of this section, we would like to give two useful lemmas here.

Lemma 3.1. For the operators $\mathscr{E}(t)$ and $\mathscr{F}(t)$, the following estimates hold for any $t \in (0, T]$, $m \in \mathbb{N}_0$, and $\nu = 0, 1$:

(i) $\|A^\nu \mathscr{F}^{(m)}(t)\|_{\mathcal{X} \to \mathcal{X}} \leq C t^{-m-\nu}$;

(ii) $\|A^\nu \mathscr{E}^{(m)}(t)\|_{\mathcal{X} \to \mathcal{X}} \leq C t^{-m+1-\nu}$;

(iii) $\mathscr{E}(t) : \mathcal{X} \to \mathcal{D}$ *is continuous with respect to* $t \in [0, T]$, *and* $A\mathscr{E}(0) = 0$;

with $\mathscr{E}^{(m)}(t) = \frac{\mathrm{d}^m \mathscr{E}(t)}{\mathrm{d}t^m}$ and $\mathscr{F}^{(m)}(t) = \frac{\mathrm{d}^m \mathscr{F}(t)}{\mathrm{d}t^m}$.

Proof. The proof line of (i) is similar to that of Theorem 2.1 in Al-Maskari et al. [Al-Maskari and Karaa (2018)] (we refer to Theorem 2.1 for the case $\alpha_2 = 1$). Since $\mathscr{F}(t) = \mathscr{E}'(t)$, (ii) immediately follows from (i).

Note that $A\mathscr{E} = \mathscr{E}A : \mathcal{X} \to \mathcal{X}$ is continuous with respect to $t \in [0, T]$. Then taking $t \to 0$ in (10) implies (iii). This ends the proof. ∎

Lemma 3.2 (Zeng, Cao and Li (2013), Gronwall's inequality)**.** Let $q(t)$ be continuous and nonnegative on $[0, T]$. If

$$q(t) \leq c(t) + h \int_0^t \frac{q(s)}{(t-s)^\mu} \mathrm{d}s, \quad 0 \leq t \leq T,$$

where $0 \leq \mu < 1$, $c(t)$ is nonnegative monotonic increasing continuous function on $[0, T]$, and h is a positive constant, then

$$q(t) \leq c(t) E_{1-\mu,1}\big(h\Gamma(1-\mu)t^{1-\mu}\big), \quad 0 \leq t \leq T.$$

Now we consider the existence, uniqueness, and regularity of the solution to Eq. (1).

Theorem 3.1. For a given $T > 0$, suppose that $u_0 \in \mathcal{D}$ and $g \in C\big([0, T]; H^2(\Omega)\big)$. $f: \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous. Then Eq. (1) has a unique solution u such that

$$u \in C^\alpha\big([0, T]; \mathcal{X}\big) \cap C\big([0, T]; \mathcal{D}\big). \tag{11}$$

$$_C\mathrm{D}_{0,t}^\alpha u \in C\big([0, T]; \mathcal{X}\big). \tag{12}$$

Moreover, if $g'(t) \in C\big([0, T]; \mathcal{X}\big)$, there holds

$$u'(t) \in C\big([0, T]; \mathcal{X}\big). \tag{13}$$

Proof. *Step 1: Existence and uniqueness.* Following the idea in Li et al. [Li and Wang (2019)], we first define a map $\mathscr{M}: C\big([0, T]; \mathcal{X}\big)_\lambda \to C\big([0, T]; \mathcal{X}\big)_\lambda$ by

$$\mathscr{M}v(t) = u_0 + \mathscr{E}(t)Au_0 - \int_0^t \mathscr{F}(t-\tau)f\big(v(\tau)\big)\mathrm{d}\tau + \int_0^t \mathscr{F}(t-\tau)g(\tau)\mathrm{d}\tau, \tag{14}$$

where the space $C\big([0,T];\mathcal{X}\big)_\lambda$ is defined by

$$\|v\|_\lambda = \max_{0\le t\le T} \|e^{-\lambda t}v(t)\|_\mathcal{X}, \quad \forall v \in C\big([0,T];\mathcal{X}\big).$$

Then we only have to prove that for some $\lambda > 0$, the map $\mathscr{M}$ has a unique fixed point. For any $v_1, v_2 \in C\big([0,T];\mathcal{X}\big)_\lambda$, we have

$$\begin{aligned}
&\left\| e^{-\lambda t}\big(\mathscr{M}v_1(t) - \mathscr{M}v_2(t)\big) \right\|_\Omega \\
&= \left\| e^{-\lambda t} \int_0^t \mathscr{F}(t-\tau)\big(f\big(v_1(\tau)\big) - f\big(v_2(\tau)\big)\big)\,\mathrm{d}\tau \right\|_\Omega \\
&\le C e^{-\lambda t} \int_0^t \big\| f\big(v_1(\tau)\big) - f\big(v_2(\tau)\big)\big\|_\Omega\,\mathrm{d}\tau \\
&\le \frac{C}{\lambda}\|v_1 - v_2\|_\lambda,
\end{aligned} \tag{15}$$

where we have used Lemma 3.1 with $\nu = m = 0$ in the first inequality and f is Lipschitz continuous in the second inequality.

By choosing a sufficiently large λ such that $\tilde{C} = C/\lambda < 1$ and taking maximum of the left hand side of (15) with respect to $t \in [0,T]$, there holds

$$\|\mathscr{M}v_1(t) - \mathscr{M}v_2(t)\|_\lambda \le \tilde{C}\|v_1 - v_2\|_\lambda.$$

Finally, applying the Banach fixed point theorem, we can obtain that Eq. (1) has a unique solution $u \in C\big([0,T];\mathcal{X}\big)$.

Step 2: $C^\alpha\big([0,T];\mathcal{X}\big)$ regularity. Consider the following difference quotient for $\Delta t > 0$

$$\begin{aligned}
&\frac{u(t+\Delta t) - u(t)}{\Delta t^\alpha} \\
&= \frac{\mathscr{E}(t+\Delta t) - \mathscr{E}(t)}{\Delta t^\alpha}Au_0 - \frac{1}{\Delta t^\alpha}\int_t^{t+\Delta t}\mathscr{F}(\tau)f\big(u(t+\Delta t-\tau)\big)\mathrm{d}\tau \\
&\quad -\frac{1}{\Delta t^\alpha}\int_0^t \mathscr{F}(\tau)\big(f\big(u(t+\Delta t-\tau)\big) - f\big(u(t-\tau)\big)\big)\,\mathrm{d}\tau \\
&\quad +\frac{1}{\Delta t^\alpha}\int_0^t \big(\mathscr{F}(t+\Delta t-\tau) - \mathscr{F}(t-\tau)\big)g(\tau)\mathrm{d}\tau \\
&\quad +\frac{1}{\Delta t^\alpha}\int_t^{t+\Delta t}\mathscr{F}(t+\Delta t-\tau)g(\tau)\mathrm{d}\tau \\
&=: \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_4 + \mathcal{I}_5,
\end{aligned} \tag{16}$$

which will be estimated respectively as follows.

Applying Lemma 3.1, we arrive at

$$\|\mathcal{I}_1\|_\Omega \le \frac{\int_t^{t+\Delta t}\|\mathscr{E}'(t)\|_{X\to X}\mathrm{d}\tau}{\Delta t^\alpha}\|Au_0\|_\Omega \le C_{\alpha,T},$$

where $C_{\alpha,T}$ is a positive constant depending on α and T.

By using Lemma 3.1 and the Lipschitz continuity of f again, we have

$$\|\mathcal{I}_2\|_{\Omega} \leq \frac{1}{\Delta t^{\alpha}} \int_t^{t+\Delta t} \left\|\mathscr{F}(\tau) f\big(u(t + \Delta t - \tau)\big)\right\|_{\Omega} \mathrm{d}\tau$$

$$\leq \frac{C}{\Delta t^{\alpha}} \int_t^{t+\Delta t} \|f\big(u(t + \Delta t - \tau)\big)\|_{\Omega} \mathrm{d}\tau \leq C,$$

and

$$\|\mathcal{I}_3\|_{\Omega} = \left\|\int_0^t \mathscr{F}(t - \tau) \frac{f\big(u(\tau + \Delta t)\big) - f\big(u(\tau)\big)}{\Delta t^{\alpha}} \mathrm{d}\tau\right\|_{\Omega}$$

$$\leq C \int_0^t \left\|\frac{u(\tau + \Delta t) - u(\tau)}{\Delta t^{\alpha}}\right\|_{\Omega} \mathrm{d}\tau.$$

Similarly, the two terms $\mathcal{I}_4$ and $\mathcal{I}_5$ are shown to be bounded, respectively, by

$$\|\mathcal{I}_4\|_{\Omega} \leq \frac{1}{\Delta t^{\alpha}} \int_0^t \|\mathscr{F}(t + \Delta t - \tau) - \mathscr{F}(t - \tau)\|_{\mathcal{X} \to \mathcal{X}} \|g(\tau)\|_{\Omega} \mathrm{d}\tau$$

$$\leq \frac{C}{\Delta t^{\alpha}} \int_0^t \int_{t-\tau}^{t+\Delta t-\tau} \|\mathscr{F}'(\mu)\|_{\mathcal{X} \to \mathcal{X}} \mathrm{d}\mu \mathrm{d}\tau$$

$$\leq \frac{C_{\alpha,T}}{\Delta t^{\alpha}} \int_0^t \int_{t-\tau}^{t+\Delta t-\tau} \mu^{\alpha-2} \mathrm{d}\mu \mathrm{d}\tau \leq C_{\alpha,T},$$

and

$$\|\mathcal{I}_5\|_{\Omega} \leq \frac{1}{\Delta t^{\alpha}} \int_t^{t+\Delta t} \|\mathscr{F}(t + \Delta t - \tau) g(\tau)\|_{\Omega} \mathrm{d}\tau \leq C_{\alpha,T}.$$

Denoting $W(t) = \Delta t^{-\alpha} \|u(t + \Delta t) - u(t)\|_{\Omega}$ and substituting the estimates of $\mathcal{I}_i$ ($i = 1, 2, \ldots, 5$) into (16), we obtain

$$W(t) \leq C_{\alpha,T} + C \int_0^t W(\tau) \mathrm{d}\tau, \tag{17}$$

which together with Lemma 3.2 yields $u \in C^{\alpha}\big([0,T]; \mathcal{X}\big)$. The assertion $_C\mathrm{D}_{0,t}^{\alpha} u \in C\big([0,T]; \mathcal{X}\big)$ is a direct result of the $C^{\alpha}\big([0,T]; \mathcal{X}\big)$ regularity and the mapping property of Caputo derivative.

Step 3: $C\big([0,T]; \mathcal{D}\big)$ regularity. By applying the operator A to both sides of (10), we arrive at

$$Au(t) - Au_0 = A\mathscr{E}(t) Au_0 - \int_0^t A\mathscr{F}(t - \tau) f\big(u(\tau)\big) \mathrm{d}\tau + \int_0^t A\mathscr{F}(t - \tau) g(\tau) \mathrm{d}\tau$$

$$= A\mathscr{E}(t)\big(Au_0 - f\big(u(t)\big)\big) - \int_0^t A\mathscr{F}(t - \tau)\big(f\big(u(\tau)\big) - f\big(u(t)\big)\big) \mathrm{d}\tau$$

$$+ \int_0^t A\mathscr{F}(t - \tau) g(\tau) \mathrm{d}\tau.$$

Then by Lemma 3.1 and the regularity assumption of g, we have

$$
\begin{aligned}
\|Au(t) - Au_0\|_\Omega \leq & \|A\mathscr{E}(t)\|_{\mathcal{X}\to\mathcal{X}} \|Au_0 - f(u(t))\|_\Omega \\
& + \int_0^t \|A\mathscr{F}(t-\tau)\|_{\mathcal{X}\to\mathcal{X}} \big\|f\big(u(\tau)\big) - f\big(u(t)\big)\big\|_\Omega \, \mathrm{d}\tau \\
& + \int_0^t \|\mathscr{F}(t-\tau)Ag(\tau)\|_\Omega \mathrm{d}\tau \\
\leq & C_{\alpha,T},
\end{aligned}
$$

which further implies $u \in C\big([0,T]; H^2(\Omega)\big)$.

Step 4: Estimate of $u'(t)$. By differentiating (10) with respect to t, we have

$$
\begin{aligned}
u'(t) = & \mathscr{E}'(t)Au_0 - \mathscr{F}(t)f(u_0) - \int_0^t \mathscr{F}(\tau)f'\big(u(t-\tau)\big)u'(t-\tau)\mathrm{d}\tau \\
& + \mathscr{F}(t)g(x,0) + \int_0^t \mathscr{F}(\tau)g'(t-\tau)\mathrm{d}\tau \\
= & \mathscr{F}(t)\big(Au_0 - f(u_0)\big) - \int_0^t \mathscr{F}(t-\tau)f'\big(u(\tau)\big)u'(\tau)\mathrm{d}\tau \\
& + \mathscr{F}(t)g(x,0) + \int_0^t \mathscr{F}(t-\tau)g'(\tau)\mathrm{d}\tau.
\end{aligned}
$$

It follows from Lemma 3.1 that

$$
\begin{aligned}
\|u'(t)\|_\Omega \leq & \; \|\mathscr{F}(t)\big(Au_0 - f(u_0)\big)\|_\Omega + \int_0^t \|\mathscr{F}(t-\tau)g'(\tau)\|_\Omega \mathrm{d}\tau \\
& + \int_0^t \|\mathscr{F}(t-\tau)f'\big(u(\tau)\big)u'(\tau)\|_\Omega \mathrm{d}\tau + \|\mathscr{F}(t)g(x,0)\|_\Omega \\
\leq & \; C_{\alpha,T} + C\int_0^t \|u'(\tau)\|_\Omega \mathrm{d}\tau.
\end{aligned} \tag{18}
$$

Using Lemma 3.2 again yields the assertion of (13). The proof is thus complete. ∎

4 The LDG method and its convergence

In this section, we first present the semidiscrete scheme and fully discrete scheme for problem (1), where the time fractional derivative is discretized by WSGD method and the spatial derivative by the LDG method. Then we prove that the fully discrete LDG scheme is stable and convergent.

The usual notations of LDG method are introduced here. Assume that the mesh consisting of cells $I_j = (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$, for $1 \leq j \leq N$, where $a = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \cdots < x_{N+\frac{1}{2}} = b$, covers $\overline{\Omega} = [a,b]$. The cell center and cell length are denoted by $x_j = (x_{j-\frac{1}{2}} + x_{j+\frac{1}{2}})/2$ and $h_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}$, respectively. Denote by $h = \max_j h_j$ the maximum cell length.

Denote by $u_{j+\frac{1}{2}}^-$ and $u_{j+\frac{1}{2}}^+$, the values of u at the discontinuity point $x_{j+\frac{1}{2}}$ from the left and right cell, respectively. In what follows, we use $[\![u]\!] = u^+ - u^-$ and $\{\!\{u\}\!\} = \frac{u^+ + u^-}{2}$ to represent the jump value of u and the mean value of u at each element boundary point. The discontinuous finite element space is defined as

$$V_h = \left\{ v \in L^2(\Omega) : v|_{I_j} \in \mathcal{P}^k(I_j), j = 1, \ldots, N \right\},$$

where $\mathcal{P}^k(I_j)$ denotes the space of polynomials in I_j of degree at most $k \geq 0$.

As the usual treatment, we would like to introduce the Gauss-Radau projections $\mathscr{P}_h^\pm$ [Castillo, Kanschat, Schotzau et al. (2002)]: for any scalar function $q \in H^1(\Omega)$, the projection is the unique element in V_h, satisfying

$$\int_{I_j} \left(\mathscr{P}_h^+ q(x) - q(x) \right) v_h \mathrm{d}x = 0, \quad \forall v_h \in \mathcal{P}^{k-1}(I_j) \quad (\mathscr{P}_h^+ q)_{j-\frac{1}{2}}^+ = q(x_{j-\frac{1}{2}}^+), \tag{19}$$

$$\int_{I_j} \left(\mathscr{P}_h^- q(x) - q(x) \right) v_h \mathrm{d}x = 0, \quad \forall v_h \in \mathcal{P}^{k-1}(I_j), \quad (\mathscr{P}_h^- q)_{j+\frac{1}{2}}^- = q(x_{j+\frac{1}{2}}^-), \tag{20}$$

for any $j = 1, 2, \ldots, N$.

Suppose $q \in H^{k+1}(\Omega)$, then by a standard scaling argument [Ciarlet (1978)], there holds

$$\|\mathscr{P}_h^\pm q - q\|_\Omega \leq C\|q\|_{H^{k+1}(\Omega)} h^{k+1}, \tag{21}$$

where C is a positive constant independent of h.

4.1 Semidiscrete scheme

On the space V_h, the $L^2(\Omega)$-orthogonal projection $\mathscr{P}_h : L^2(\Omega) \to V_h$ and the discrete Laplacian $\Delta_h : V_h \to V_h$ are defined by

$$(\mathscr{P}_h \varphi, v_h) = (\varphi, v_h), \quad \forall v_h \in V_h, \tag{22}$$

and

$$-(\Delta_h w_h, v_h) = (\nabla w_h, \nabla v_h), \quad \forall w_h, v_h \in V_h, \tag{23}$$

respectively.

Replacing the exact solutions by the numerical solutions, then we can define the semidiscrete LDG scheme as follows: find $u_h(\cdot, t) \in V_h$ such that

$$\begin{cases} \frac{\partial u_h}{\partial t} + {}_C\mathrm{D}_{0,t}^\alpha u_h - \Delta_h u_h + f(u_h) = g_h(x,t), \ (x,t) \in \Omega \times (0,T], \\ u_h(0) = u_0(x), \ x \in \Omega, \end{cases} \tag{24}$$

where $g_h = \mathscr{P}_h g$. By a similar argument as (10), the solution to (24) can be represented by

$$u_h(t) = u_0 + \mathscr{E}(t)Au_0 - \int_0^t \mathscr{F}(t-\tau)f\big(u_h(\tau)\big)\mathrm{d}\tau + \int_0^t \mathscr{F}(t-\tau)g_h(\tau)\mathrm{d}\tau.$$

As proved in Theorem 3.1, we have the following similar results.

Theorem 4.1. Suppose that f, g and u_0 satisfy the conditions in Theorem 3.1. Then Eq. (24) has a unique solution u_h such that

$u_h \in C^\alpha\big([0,T];\mathcal{X}\big) \cap C\big([0,T];\mathcal{D}\big).$

$_C\mathrm{D}_{0,t}^\alpha u_h \in C\big([0,T];\mathcal{X}\big).$

Moreover, if $g'(t) \in C\big([0,T];\mathcal{X}\big)$, then there holds

$u_h'(t) \in C\big([0,T];\mathcal{X}\big).$

4.2 Fully discrete LDG scheme

Let $\Delta t = T/M$ be the time mesh size, $t_n = n\Delta t$, $n = 0, 1, \ldots, M$ be the mesh point, $M \in \mathbb{Z}^+$. For simplicity of notations, we denote $u^{n+1} = u(x, t_{n+1})$ and $\delta_t^{n+1} u^{n+1} = \frac{u^{n+1}-u^n}{\Delta t}$. Suppose $u(t) \in C^2[0, t_{n+1}]$, then the time fractional derivative (4) at time t_{n+1} can be approximated as Wang et al. [Wang and Vong (2014)]

$$_C\mathrm{D}_{0,t}^\alpha u\Big|_{t=t_{n+1}} = \sum_{i=0}^{n+1} \frac{b^\alpha(i)}{\Delta t^\alpha} u^{n+1-i} + \mathcal{O}(\Delta t^2), \tag{25}$$

where $b^\alpha(i) = \begin{cases} \frac{\alpha+2}{2} g_0^{(\alpha)}, & i = 0, \\ \frac{\alpha+2}{2} g_i^{(\alpha)} - \frac{\alpha}{2} g_{i-1}^{(\alpha)}, & i > 0, \end{cases}$ $\quad g_i^{(\alpha)} = \Gamma(i-\alpha)/\big(\Gamma(-\alpha)\Gamma(i+1)\big).$

In what follows, we would like to introduce several lemmas which are very important in obtaining the error estimate.

Lemma 4.1 [Liu, Du, Li et al. (2016)]. For series $\{b^\alpha(i)\}_{i=1}^\infty$ given as above, the following inequality holds for any integer n

$$\sum_{i=0}^{n+1} |b^\alpha(i)| \leq 2\alpha + 2.$$

Lemma 4.2 [Wang and Vong (2014)]. Let $\{b^\alpha(i)\}_{i=1}^\infty$ be defined as above. Then for any positive integer k and real vector $(v_1, v_2, \ldots, v_k) \in \mathbb{R}^k$, it holds that

$$\sum_{n=0}^{k-1} \left(\sum_{i=0}^n b^\alpha(i) v_{n+1-i} \right) v_{n+1} \geq 0.$$

Lemma 4.3 [Li and Zeng (2015), Discrete Gronwall's inequality]. Let x_n be real positive numbers. Assume that H, C and Δt are positive and also $x_0 \leq H$. Suppose the inequality

$$x_n \leq C\Delta t \sum_{k=0}^{n-1} x_k + H$$

holds. Then one has

$x_n \leq H\mathrm{e}^{Cn\Delta t}.$

In order to get the LDG formulation, we firstly rewrite Eq. (1) into the following lower-order system of two equations by introducing an auxiliary variable $p = \partial u / \partial x$

$$\begin{cases} p = u_x, \\ \frac{\partial u}{\partial t} + {}_C D_{0,t}^\alpha u - p_x + f(u) = g(x, t). \end{cases} \tag{26}$$

Then we can get the weak form of Eq. (26) at t_{n+1} as

$$\begin{cases} (p^{n+1}, w) + (u^{n+1}, w_x) - \sum_{j=1}^{N} \left(u^{n+1} w^- |_{j+\frac{1}{2}} - u^{n+1} w^+ |_{j-\frac{1}{2}} \right) \\ \qquad = 0, \\ (\frac{3}{2} \delta_t^{n+1} u^{n+1} - \frac{1}{2} \delta_t^n u^n, v) + \sum_{i=0}^{n+1} \frac{b^\alpha(i)}{\Delta t^\alpha} (u^{n+1-i}, v) - (p_x^{n+1}, v) \\ \qquad = \left(-2f(u^n) + f(u^{n-1}), v \right) + (g^{n+1}, v) + (E_0 + E_1 + E_2, v), \end{cases} \tag{27}$$

where

$$E_0 = \sum_{i=0}^{n+1} \frac{b^\alpha(i)}{\Delta t^\alpha} u^{n+1-i} - {}_C D_{0,t}^\alpha u^{n+1} = \mathcal{O}(\Delta t^2),$$

$$E_1 = \frac{3}{2} \delta_t^{n+1} u^{n+1} - \frac{1}{2} \delta_t^n u^n - u_t^{n+1} = \mathcal{O}(\Delta t^2),$$

$$E_2 = 2f(u^n) - f(u^{n-1}) - f(u^{n+1}) = \mathcal{O}(\Delta t^2).$$

When $n = 0$, we take $u^{-1} = 2u^0 - u^1 + \mathcal{O}(\Delta t^2)$ by Taylor expansion.

Let $u_h^{n+1}, p_h^{n+1} \in V_h$ be the approximation of u^{n+1} and p^{n+1}, respectively. We get the fully discrete LDG scheme as follows: find $u_h^{n+1}, p_h^{n+1} \in V_h$ such that for all test functions $v_h, w_h \in V_h$,

$$\begin{cases} (p_h^{n+1}, w_h) + (u_h^{n+1}, (w_h)_x) - \sum_{j=1}^{N} \left(\widetilde{u}_h^{n+1} w_h^- |_{j+\frac{1}{2}} - \widetilde{u}_h^{n+1} w_h^+ |_{j-\frac{1}{2}} \right) \\ \qquad = 0, \\ (\frac{3}{2} \delta_t^{n+1} u_h^{n+1} - \frac{1}{2} \delta_t^n u_h^n, v_h) + \sum_{i=0}^{n+1} \frac{b^\alpha(i)}{\Delta t^\alpha} (u_h^{n+1-i}, v_h) + (p_h^{n+1}, (v_h)_x) \\ \qquad - \sum_{j=1}^{N} \left(\widetilde{p}_h^{n+1} v_h^- |_{j+\frac{1}{2}} - \widetilde{p}_h^{n+1} v_h^+ |_{j-\frac{1}{2}} \right) \\ \qquad = \left(-2f(u_h^n) + f(u_h^{n-1}), v_h \right) + (g_h^{n+1}, v_h), \end{cases} \tag{28}$$

where $u_h^{-1} = 2u_h^0 - u_h^1$. The "tilde" terms are the so-called "numerical fluxes", which are taken as the "alternating" numerical flux

$$\widetilde{u}_h^n = (u_h^n)^-, \quad \widetilde{p}_h^n = (p_h^n)^+. \tag{29}$$

Remark 4.1. The choice for the fluxes (29) is not unique. We can also take the numerical fluxes as $\widetilde{u}_h^n = (u_h^n)^+$, $\widehat{p}_h^n = (p_h^n)^-$ on each cell interface.

4.2.1 Stability analysis

In this subsection, we consider the stability of the LDG scheme (28). Let (U_h^n, P_h^n) be the perturbed solution of (u_h^n, p_h^n), i.e., (U_h^n, P_h^n) and (u_h^n, p_h^n) satisfy (27) with different initial conditions.

Theorem 4.2. Suppose that f and u_0 satisfy the conditions in Theorem 3.1, then the fully discrete LDG scheme (28) with flux (29) is stable.

Proof. Denoting $e_{u_h}^{n+1} = u_h^{n+1} - U_h^{n+1}$ and $e_{p_h}^{n+1} = p_h^{n+1} - P_h^{n+1}$, we obtain the following perturbation equation

$$
\begin{cases}
(e_{p_h}^{n+1}, w_h) + (e_{u_h}^{n+1}, (w_h)_x) \\
\quad = \sum_{j=1}^{N} \left((e_{u_h}^{n+1})^- w_h^-|_{j+\frac{1}{2}} - (e_{u_h}^{n+1})^- w_h^+|_{j-\frac{1}{2}} \right), \\
(\frac{3}{2}\delta_t^{n+1} e_{u_h}^{n+1} - \frac{1}{2}\delta_t^n e_{u_h}^n, v_h) + \sum_{i=0}^{n+1} \frac{b^\alpha(i)}{\Delta t^\alpha} (e_{u_h}^{n+1-i}, v_h) + (e_{p_h}^{n+1}, (v_h)_x) \\
\quad - \sum_{j=1}^{N} \left((e_{p_h}^{n+1})^+ v_h^-|_{j+\frac{1}{2}} - (e_{p_h}^{n+1})^+ v_h^+|_{j-\frac{1}{2}} \right) \\
\quad = \left(-2f(u_h^n) + f(u_h^{n-1}) + 2f(U_h^n) - f(U_h^{n-1}), v_h \right).
\end{cases}
\tag{30}
$$

Let $v_h = e_{u_h}^{n+1}$ and $w_h = e_{p_h}^{n+1}$. Then (30) can be written as

$$
\begin{cases}
(e_{p_h}^{n+1}, e_{p_h}^{n+1}) + (e_{u_h}^{n+1}, (e_{p_h}^{n+1})_x) \\
\quad = \sum_{j=1}^{N} \left((e_{u_h}^{n+1})^- (e_{p_h}^{n+1})^-|_{j+\frac{1}{2}} - (e_{u_h}^{n+1})^- (e_{p_h}^{n+1})^+|_{j-\frac{1}{2}} \right), \\
(\frac{3}{2}\delta_t^{n+1} e_{u_h}^{n+1} - \frac{1}{2}\delta_t^n e_{u_h}^n, e_{u_h}^{n+1}) + \sum_{i=0}^{n+1} \frac{b^\alpha(i)}{\Delta t^\alpha} (e_{u_h}^{n+1-i}, e_{u_h}^{n+1}) + (e_{p_h}^{n+1}, (e_{u_h}^{n+1})_x) \\
\quad - \sum_{j=1}^{N} \left((e_{p_h}^{n+1})^+ (e_{u_h}^{n+1})^-|_{j+\frac{1}{2}} - (e_{p_h}^{n+1})^+ (e_{u_h}^{n+1})^+|_{j-\frac{1}{2}} \right) \\
\quad = \left(-2f(u_h^n) + f(u_h^{n-1}) + 2f(U_h^n) - f(U_h^{n-1}), e_{u_h}^{n+1} \right).
\end{cases}
\tag{31}
$$

Firstly, we prove that the theorem holds true for $n = 0$. Taking $n = 0$ in (31) and adding the two equations together lead to

$$
\begin{aligned}
& \left(\tfrac{3}{2}\delta_t^1 e_{u_h}^1 - \tfrac{1}{2}\delta_t^0 e_{u_h}^0, e_{u_h}^1 \right) + \|e_{p_h}^1\|_\Omega^2 + \sum_{i=0}^{1} \frac{b^\alpha(i)}{\Delta t^\alpha} (e_{u_h}^{1-i}, e_{u_h}^1) \\
& \quad = \left(-2f(u_h^0) + f(u_h^{-1}) + 2f(U_h^0) - f(U_h^{-1}), e_{u_h}^1 \right).
\end{aligned}
\tag{32}
$$

Multiplying (32) by $2\Delta t$ and using Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
3\|e_{u_h}^1\|_\omega^2 + 2\Delta t\|e_{p_h}^1\|_\Omega^2 &+ 2\Delta t^{1-\alpha}\sum_{i=0}^{1} b^\alpha(i)(e_{u_h}^{1-i}, e_{u_h}^1) \\
&= 2\Delta t\big(-2f(u_h^0) + f(u_h^{-1}) + 2f(U_h^0) - f(U_h^{-1}), e_{u_h}^1\big) \\
&\quad + 4(e_{u_h}^0, e_{u_h}^1) - (e_{u_h}^{-1}, e_{u_h}^1) \\
&\leq (4\Delta tL + 4)\|e_{u_h}^0\|_\Omega\|e_{u_h}^1\|_\Omega + (2\Delta tL + 1)\|e_{u_h}^{-1}\|_\Omega\|e_{u_h}^1\|_\Omega.
\end{aligned}
$$

Noticing that $b^\alpha(0) > 0$ and $b^\alpha(1) < 0$, we get

$$
\begin{aligned}
3\|e_{u_h}^1\|_\Omega &\leq (4\Delta tL + 4)\|e_{u_h}^0\|_\Omega + (2\Delta tL + 1)\|2e_{u_h}^0 - e_{u_h}^1\|_\Omega \\
&\quad - 2\Delta t^{1-\alpha}b^\alpha(1)\|e_{u_h}^0\|_\Omega \\
&\leq \big(8\Delta tL + 6 - 2\Delta t^{1-\alpha}b^\alpha(1)\big)\|e_{u_h}^0\|_\Omega + (2\Delta tL + 1)\|e_{u_h}^1\|_\Omega.
\end{aligned}
$$

Thus, if $\Delta t < \frac{1}{2L}$, we have

$$
\|e_{u_h}^1\|_\Omega \leq C\|e_{u_h}^0\|_\Omega. \tag{33}
$$

Now we are going to prove the case of $n \geq 1$. Adding the two equations of (31) together results in

$$
\begin{aligned}
\|e_{p_h}^{n+1}\|_\Omega^2 &+ \sum_{i=0}^{n+1} \frac{b^\alpha(i)}{\Delta t^\alpha}(e_{u_h}^{n+1-i}, e_{u_h}^{n+1}) + \frac{1}{4\Delta t}\Big(\|e_{u_h}^{n+1}\|_\Omega^2 + \|2e_{u_h}^{n+1} - e_{u_h}^n\|_\Omega^2 \\
&- \big(\|e_{u_h}^n\|_\Omega^2 + \|2e_{u_h}^n - e_{u_h}^{n-1}\|_\Omega^2\big) + \|e_{u_h}^{n+1} - 2e_{u_h}^n + e_{u_h}^{n-1}\|_\Omega^2\Big) \\
&= \big(-2f(u_h^n) + f(u_h^{n-1}) + 2f(U_h^n) - f(U_h^{n-1}), e_{u_h}^{n+1}\big). \tag{34}
\end{aligned}
$$

Multiplying (34) by $4\Delta t$ and using Cauchy-Schwarz inequality and Young's inequality, we arrive at

$$
\begin{aligned}
&4\Delta t\|e_{p_h}^{n+1}\|_\Omega^2 + \sum_{i=0}^{n+1} 4\Delta t^{1-\alpha}b^\alpha(i)(e_{u_h}^{n+1-i}, e_{u_h}^{n+1}) + \|e_{u_h}^{n+1}\|_\Omega^2 + \|2e_{u_h}^{n+1} - e_{u_h}^n\|_\Omega^2 \\
&\quad + \|e_{u_h}^{n+1} - 2e_{u_h}^n + e_{u_h}^{n-1}\|_\Omega^2 \\
&= \|e_{u_h}^n\|_\Omega^2 + \|2e_{u_h}^n - e_{u_h}^{n-1}\|_\Omega^2 \\
&\quad + 4\Delta t\big(-2f(u_h^n) + f(u_h^{n-1}) + 2f(U_h^n) - f(U_h^{n-1}), e_{u_h}^{n+1}\big) \\
&\leq \|e_{u_h}^n\|_\Omega^2 + \|2e_{u_h}^n - e_{u_h}^{n-1}\|_\Omega^2 \\
&\quad + C\Delta t(\|e_{u_h}^n\|_\Omega^2 + \|e_{u_h}^{n-1}\|_\Omega^2) + \|e_{u_h}^{n+1} - 2e_{u_h}^n + e_{u_h}^{n-1}\|_\Omega^2,
\end{aligned}
$$

which further implies

$$
\sum_{i=0}^{n+1} 4\Delta t^{1-\alpha} b^{\alpha}(i)(e_{u_h}^{n+1-i}, e_{u_h}^{n+1}) + \|e_{u_h}^{n+1}\|_{\Omega}^2 + \|2e_{u_h}^{n+1} - e_{u_h}^n\|_{\Omega}^2
$$
$$
\leq \|e_{u_h}^n\|_{\Omega}^2 + \|2e_{u_h}^n - e_{u_h}^{n-1}\|_{\Omega}^2 + C\Delta t(\|e_{u_h}^n\|_{\Omega}^2 + \|e_{u_h}^{n-1}\|_{\Omega}^2).
$$

Summing n from 1 to k and using Lemma 4.2, we can get

$$
\|e_{u_h}^{k+1}\|_{\Omega}^2 + \|2e_{u_h}^{k+1} - e_{u_h}^k\|_{\Omega}^2
$$
$$
\leq \|e_{u_h}^1\|_{\Omega}^2 + \|2e_{u_h}^1 - e_{u_h}^0\|_{\Omega}^2 + C\Delta t \sum_{n=1}^{k} (\|e_{u_h}^n\|_{\Omega}^2 + \|2e_{u_h}^n - e_{u_h}^{n-1}\|_{\Omega}^2).
$$

Then from discrete Gronwall's lemma (i.e., Lemma 4.3) and (33), it yields that

$$
\|e_{u_h}^{k+1}\|_{\Omega}^2 + \|2e_{u_h}^{k+1} - e_{u_h}^k\|_{\Omega}^2 \leq C(\|e_{u_h}^1\|_{\Omega}^2 + \|2e_{u_h}^1 - e_{u_h}^0\|_{\Omega}^2). \tag{35}
$$

Combining (33) with (35), we complete the proof of this theorem.

∎

4.2.2 Error estimate

In this subsection, we will give the error estimate for the fully discrete LDG scheme (28).

Theorem 4.3. Assume that f, g and u_0 satisfy the conditions in Theorem 3.1. Let u^{n+1} be the exact solution of (1) and u_h^{n+1} be the numerical solution of the fully discrete LDG scheme (28) with fluxes (29). If we assume that $u(x,t) \in C^2\left([0,T]; H^{k+1}(\Omega)\right)$, then there holds

$$
\|u^{n+1} - u_h^{n+1}\|_{\Omega} \leq C(\Delta t^2 + h^{k+1}), \tag{36}
$$

where C is a positive constant independent of Δt and h.

Proof. Denote

$$
\begin{aligned}
e_u^n &= u^n - u_h^n = \mathscr{P}_h^- u^n - u_h^n + u^n - \mathscr{P}_h^- u^n = \xi_u^n + \eta_u^n, \\
e_p^n &= p^n - p_h^n = \mathscr{P}_h^+ p^n - p_h^n + p^n - \mathscr{P}_h^+ p^n = \xi_p^n + \eta_p^n.
\end{aligned} \tag{37}
$$

According to (21), we have

$$
\|\eta_u^n\|_{\Omega} \leq Ch^{k+1}. \tag{38}
$$

Thus in what follows, we will focus on the estimate for ξ_u^n.

In order to estimate ξ_u^n, we would like to set up the corresponding error equation first. Subtracting (28) with (27) and using the flux (29), we get

$$
\left\{
\begin{aligned}
&(e_p^{n+1}, w_h) + (e_u^{n+1}, (w_h)_x) \\
&\quad = \sum_{j=1}^{N} \left((e_u^{n+1})^- (w_h)^- |_{j+\frac{1}{2}} - (e_u^{n+1})^- (w_h)^+ |_{j-\frac{1}{2}} \right), \\
&(\tfrac{3}{2}\delta_t^{n+1} e_u^{n+1} - \tfrac{1}{2}\delta_t^n e_u^n, v_h) + \sum_{i=0}^{n+1} \frac{b^\alpha(i)}{\Delta t^\alpha}(e_u^{n+1-i}, v_h) + \left(e_p^{n+1}, (v_h)_x \right) \\
&\quad - \sum_{j=1}^{N} \left((e_p^{n+1})^+ (v_h)^- |_{j+\frac{1}{2}} - (e_p^{n+1})^+ (v_h)^+ |_{j-\frac{1}{2}} \right) \\
&\quad + \left(2f(u^n) - 2f(u_h^n) - f(u^{n-1}) + f(u_h^{n-1}), v_h \right) \\
&\quad = (E_0 + E_1 + E_2, v_h).
\end{aligned}
\right.
\tag{39}
$$

Substituting (37) into (39), we can get the following equations.

Case I: $n = 0$

$$
\begin{aligned}
&(\tfrac{3}{2}\delta_t^1 e_u^1 - \tfrac{1}{2}\delta_t^0 e_u^0, v_h) + \sum_{i=0}^{1} \frac{b^\alpha(i)}{\Delta t^\alpha}(\xi_u^{1-i}, v_h) + \left(\xi_p^1, (v_h)_x \right) \\
&\quad - \sum_{j=1}^{N} \left((\xi_p^1)^+ v_h^- |_{j+\frac{1}{2}} - (\xi_p^1)^+ v_h^+ |_{j-\frac{1}{2}} \right) + (\xi_p^1, w_h) + \left(\xi_u^1, (w_h)_x \right) \\
&\quad - \sum_{j=1}^{N} \left((\xi_u^1)^- w_h^- |_{j+\frac{1}{2}} - (\xi_u^1)^- w_h^+ |_{j-\frac{1}{2}} \right) \\
&= \left(-2f(u^0) + 2f(u_h^0) + f(u^{-1}) - f(u_h^{-1}), v_h \right) \\
&\quad - \sum_{i=0}^{1} \frac{b^\alpha(i)}{\Delta t^\alpha}(\eta_u^{1-i}, v_h) - (\eta_p^1, w_h) - \left(\eta_u^1, (w_h)_x \right) \\
&\quad + \sum_{j=1}^{N} \left((\eta_u^1)^- w_h^- |_{j+\frac{1}{2}} - (\eta_u^1)^- w_h^+ |_{j-\frac{1}{2}} \right) + (E_0 + E_1 + E_2, v_h) \\
&\quad - (\eta_p^1, (v_h)_x) + \sum_{j=1}^{N} \left((\eta_p^1)^+ v_h^- |_{j+\frac{1}{2}} - (\eta_p^1)^+ v_h^+ |_{j-\frac{1}{2}} \right).
\end{aligned}
\tag{40}
$$

Case II: $n \geq 1$

$$(\frac{3}{2}\delta_t^{n+1}\xi_u^{n+1} - \frac{1}{2}\delta_t^n\xi_u^n, v_h) + (\xi_p^{n+1}, (v_h)_x) + \sum_{i=0}^{n+1}\frac{b^\alpha(i)}{\Delta t^\alpha}(\xi_u^{n+1-i}, v_h)$$

$$+(\xi_p^{n+1}, w_h) - \sum_{j=1}^N \left((\xi_p^{n+1-i})^+ v_h^-|_{j+\frac{1}{2}} - (\xi_p^{n+1-i})^+ v_h^+|_{j-\frac{1}{2}}\right)$$

$$+(\xi_u^{n+1}, (w_h)_x) - \sum_{j=1}^N \left((\xi_u^{n+1})^- w_h^-|_{j+\frac{1}{2}} - (\xi_u^{n+1})^- w_h^+|_{j-\frac{1}{2}}\right)$$

$$= \left(-2f(u^n) + 2f(u_h^n) + f(u^{n-1}) - f(u_h^{n-1}), v_h\right) - (\frac{3}{2}\delta_t^{n+1}\eta_u^{n+1} - \frac{1}{2}\delta_t^n\eta_u^n, v_h)$$

$$-\sum_{i=0}^{n+1}\frac{b^\alpha(i)}{\Delta t^\alpha}(\eta_u^{n+1-i}, v_h) - (\eta_p^{n+1}, w_h) - (\eta_u^{n+1}, (w_h)_x)$$

$$+\sum_{j=1}^N \left((\eta_u^{n+1})^- w_h^-|_{j+\frac{1}{2}} - (\eta_u^{n+1})^- w_h^+|_{j-\frac{1}{2}}\right) + (E_0 + E_1 + E_2, v_h)$$

$$-(\eta_p^{n+1}, (v_h)_x) + \sum_{j=1}^N \left((\eta_p^{n+1})^+ v_h^-|_{j+\frac{1}{2}} - (\eta_p^{n+1})^+ v_h^+|_{j-\frac{1}{2}}\right). \tag{41}$$

We first prove Case I. Taking the test functions $v_h = \xi_u^1$ and $w_h = \xi_p^1$ in (40), and a simple use of (19), (20), and (22), we get

$$(\frac{3}{2}\delta_t^1 e_u^1 - \frac{1}{2}\delta_t^0 e_u^0, \xi_u^1) + \sum_{i=0}^1 \frac{b^\alpha(i)}{\Delta t^\alpha}(\xi_u^{1-i}, \xi_u^1) + \|\xi_p^1\|_\Omega^2$$

$$= \left(-2f(u^0) + 2f(u_h^0) + f(u^{-1}) - f(u_h^{-1}), \xi_u^1\right) - \sum_{i=0}^1 \frac{b^\alpha(i)}{\Delta t^\alpha}(\eta_u^{1-i}, \xi_u^1)$$

$$-(\eta_p^1, \xi_p^1) + (E_0 + E_1 + E_2, \xi_u^1). \tag{42}$$

Since f is Lipschitz continuous, we have

$$\| - 2f(u^n) + 2f(u_h^n) + f(u^{n-1}) - f(u_h^{n-1})\|_\Omega$$
$$\leq L(2\|e_u^n\|_\Omega + \|e_u^{n-1}\|_\Omega), \quad n = 0, 1, \ldots, M. \tag{43}$$

Owing to the property (21), we get

$$|(e_u^{-1}, \xi_u^1)| \leq \frac{5}{4}\|\xi_u^1\|_\Omega^2 + C(h^{2k+2} + \Delta t^4). \tag{44}$$

Multiplying (42) by $2\Delta t$, it is easy to see that

$$3\|\xi_u^1\|_\Omega^2 + 2\Delta t^{1-\alpha}\sum_{i=0}^{1} b^\alpha(i)(\xi_u^{1-i}, \xi_u^1) + 2\Delta t\|\xi_p^1\|_\Omega^2$$
$$= -3(\eta_u^1, \xi_u^1) + 4(\eta_u^0, \xi_u^1) - (e_u^{-1}, \xi_u^1)$$
$$+2\Delta t\big(-2f(u^0) + 2f(u_h^0) + f(u^{-1}) - f(u_h^{-1}), \xi_u^1\big)$$
$$+2\Delta t(E_0 + E_1 + E_2, \xi_u^1) - 2\Delta t\sum_{i=0}^{1}\frac{b^\alpha(i)}{\Delta t^\alpha}(\eta_u^{1-i}, \xi_u^1) - 2\Delta t(\eta_p^1, \xi_p^1).$$

Then applying the Cauchy-Schwarz inequality, Young's inequality, (21), (43), as well as (44), we have

$$3\|\xi_u^1\|_\Omega^2 + 2\Delta t^{1-\alpha}\|\xi_u^1\|_\Omega^2 + 2\Delta t\|\xi_p^1\|_\Omega^2$$
$$\leq 18\|\eta_u^1\|_\Omega^2 + (\frac{5}{2}\Delta t L + \frac{7}{4})\|\xi_u^1\|_\Omega^2 + 2\Delta t^{1-\alpha}\|\xi_u^1\|_\Omega^2 + 2\Delta t^{1+\alpha}L^2\|\eta_u^0\|_\Omega^2$$
$$+C(\Delta t^4 + h^{2k+2}) + \Delta t\|\eta_p^1\|_\Omega^2 + \Delta t\|\xi_p^1\|_\Omega^2$$
$$\leq (\frac{5}{2}\Delta t L + \frac{7}{4})\|\xi_u^1\|_\Omega^2 + 2\Delta t^{1-\alpha}\|\xi_u^1\|_\Omega^2 + \Delta t\|\xi_p^1\|_\Omega^2 + C(\Delta t^4 + h^{2k+2}).$$

As a consequence, if we let $\Delta t < \frac{1}{2L}$, we can get

$$\|\xi_u^1\|_\Omega^2 \leq C(\Delta t^4 + h^{2k+2}). \tag{45}$$

Next we are going to prove Case II. By taking $(v_h, w_h) = (\xi_u^{n+1}, \xi_p^{n+1})$ in (41), we can derive

$$(\frac{3}{2}\delta_t^{n+1}\xi_u^{n+1} - \frac{1}{2}\delta_t^n\xi_u^n, \xi_u^{n+1}) + \sum_{i=0}^{n+1}\frac{b^\alpha(i)}{\Delta t^\alpha}(\xi_u^{n+1-i}, \xi_u^{n+1}) + \|\xi_p^{n+1}\|_\Omega^2$$
$$= \big(-2f(u^n) + 2f(u_h^n) + f(u^{n-1}) - f(u_h^{n-1}), \xi_u^{n+1}\big)$$
$$+(E_0 + E_1 + E_2, \xi_u^{n+1}) - (\frac{3}{2}\delta_t^{n+1}\eta_u^{n+1} - \frac{1}{2}\delta_t^n\eta_u^n, \xi_u^{n+1})$$
$$-\sum_{i=0}^{n+1}\frac{b^\alpha(i)}{\Delta t^\alpha}(\eta_u^{n+1-i}, \xi_u^{n+1}) - (\eta_p^{n+1}, \xi_p^{n+1}). \tag{46}$$

Multiplying (46) by $4\Delta t$ and employing the definitions of δ_t^{n+1} for ξ_u^{n+1} and δ_t^n for ξ_u^n, we

have

$$\|\xi_u^{n+1}\|_\Omega^2 + \|2\xi_u^{n+1} - \xi_u^n\|_\Omega^2 - \left(\|\xi_u^n\|_\Omega^2 + \|2\xi_u^n - \xi_u^{n-1}\|_\Omega^2\right)$$

$$+\|\xi_u^{n+1} - 2\xi_u^n + \xi_u^{n-1}\|_\Omega^2 + 4\Delta t^{1-\alpha}\sum_{i=0}^{n+1} b^\alpha(i)(\xi_u^{n+1-i}, \xi_u^{n+1}) + 4\Delta t\|\xi_p^{n+1}\|_\Omega^2$$

$$= 4\Delta t\Bigg[\left(-2f(u^n) + 2f(u_h^n) + f(u^{n-1}) - f(u_h^{n-1}), \xi_u^{n+1} - 2\xi_u^n + \xi_u^{n-1}\right)$$

$$+(E_0 + E_1 + E_2, \xi_u^{n+1} - 2\xi_u^n + \xi_u^{n-1})$$

$$-\left(\frac{3}{2}\delta_t^{n+1}\eta_u^{n+1} - \frac{1}{2}\delta_t^n\eta_u^n, \xi_u^{n+1} - 2\xi_u^n + \xi_u^{n-1}\right)$$

$$-\sum_{i=0}^{n+1}\frac{b^\alpha(i)}{\Delta t^\alpha}(\eta_u^{n+1-i}, \xi_u^{n+1} - 2\xi_u^n + \xi_u^{n-1}) - (\eta_p^{n+1}, \xi_p^{n+1})\Bigg]$$

$$-4\Delta t\Bigg[\left(-2f(u^n) + 2f(u_h^n) + f(u^{n-1}) - f(u_h^{n-1}), -2\xi_u^n + \xi_u^{n-1}\right)$$

$$+(E_0 + E_1 + E_2, -2\xi_u^n + \xi_u^{n-1}) - \left(\frac{3}{2}\delta_t^{n+1}\eta_u^{n+1} - \frac{1}{2}\delta_t^n\eta_u^n, -2\xi_u^n + \xi_u^{n-1}\right)$$

$$-\sum_{i=0}^{n+1}\frac{b^\alpha(i)}{\Delta t^\alpha}(\eta_u^{n+1-i}, -2\xi_u^n + \xi_u^{n-1})\Bigg]$$

$$\leq C\Delta t\Big(\|\xi_u^n\|_\Omega^2 + \|\eta_u^n\|_\Omega^2 + \|\xi_u^{n-1}\|_\Omega^2 + \|\eta_u^{n-1}\|_\Omega^2 + \Delta t^4$$

$$+\|\frac{3}{2}\delta_t^{n+1}\eta_u^{n+1} - \frac{1}{2}\delta_t^n\eta_u^n\|_\Omega^2 + \|_C\mathrm{D}_{0,t}^\alpha\eta_u^{n+1} + \mathcal{O}(\Delta t^2)\|_\Omega^2\Big)$$

$$+C\Delta t\| - 2\xi_u^n + \xi_u^{n-1}\|_\Omega^2 + 4\Delta t\|\xi_p^{n+1}\|_\Omega^2 + \Delta t\|\eta_p^{n+1}\|_\Omega^2$$

$$+\|\xi_u^{n+1} - 2\xi_u^n + \xi_u^{n-1}\|_\Omega^2,$$

where Cauchy-Schwarz inequality, Young's inequality, and (43) are used in the last step. With the help of (21), we can further obtain

$$\|\xi_u^{n+1}\|_\Omega^2 + \|2\xi_u^{n+1} - \xi_u^n\|_\Omega^2 + 4\Delta t^{1-\alpha}\sum_{i=0}^{n+1} b^\alpha(i)(\xi_u^{n+1-i}, \xi_u^{n+1})$$

$$\leq \|\xi_u^n\|_\Omega^2 + \|2\xi_u^n - \xi_u^{n-1}\|_\Omega^2 + C\Delta t(\|\xi_u^n\|_\Omega^2 + \|\xi_u^{n-1}\|_\Omega^2)$$

$$+C\Delta t\|\frac{3}{2}\delta_t^{n+1}\eta_u^{n+1} - \frac{1}{2}\delta_t^n\eta_u^n\|_\Omega^2 + C(\Delta t^4 + h^{2k+2}). \tag{47}$$

By virtue of Lemma 4.2, we have

$$\sum_{n=1}^{K-1}\sum_{i=0}^{n+1} b^\alpha(i)(\xi_u^{n+1-i}, \xi_u^{n+1}) \geq 0.$$

Then summing (47) for n from 1 to $K - 1$ leads to

$$\|\xi_u^K\|_\Omega^2 \leq C\|\xi_u^1\|_\Omega^2 + C\Delta t \sum_{n=0}^{K-1} \|\xi_u^n\|_\Omega^2 + C \int_{t_0}^{t_K} \|\frac{\partial \eta_u}{\partial s}\|_\Omega^2 ds + C(\Delta t^4 + h^{2k+2})$$

$$\leq C\Delta t \sum_{n=0}^{K-1} \|\xi_u^n\|_\Omega^2 + C(\Delta t^4 + h^{2k+2}),$$

where the property (21) and the result of (45) are used for the second inequality. Finally, it follows straightforwardly by using Lemma 4.3 that

$$\|\xi_u^K\|_\Omega \leq C(\Delta t^2 + h^{k+1}), \tag{48}$$

which combine the triangle inequality to complete the proof of this theorem. ∎

Remark 4.2. (1) We must remark here that the above error estimate is optimal both in time and space.

(2) Compared with the classical L1 method with time convergence rate of $(2 - \alpha)$, our scheme can arrive at second order in time.

(3) Our discussions focus on Caputo-type partial differential equation, it may be interesting to extend the analysis to Riesz-type fractional differential equation [Cai and Li (2019)].

5 Numerical example

In this section, we present a numerical example to verify the theoretical results.

Example 5.1. Consider the following Caputo-type cable equation with compactly supported boundary condition,

$$u_t + _C D_{0,t}^\alpha u - u_{xx} + u^2 = g(x, t), \tag{49}$$

on $\Omega = (0, 2\pi)$, $t \in (0, 1]$. The initial value condition is

$$u(x, 0) = 0, \quad x \in (0, 1),$$

and the source term is

$$g(x, t) = \left(2t + \frac{2t^{2-\alpha}}{\Gamma(3 - \alpha)} + t^2\right) \sin(x) + t^4 \sin^2(x).$$

The exact solution of (49) is given by $u(x, t) = t^2 \sin(x)$.

In Tab. 1, by taking $\alpha = 0.01, 0.5, 0.99$ and fixed temporal step length $\Delta t = 1/100$, we show the L^2-norm errors and convergence orders of space at $t = 1$ for Example 5.1. The L^2-norm errors and convergence orders of time with different α are listed in Tab. 2. Clearly, the first-order convergence in space and second-order convergence in time are observed, which is in agreement with the theoretical analysis.

Fig. 1 demonstrates the comparison of the exact solution and the numerical solution at $t = 1$, when $\alpha = 0.5$, $h = 2\pi/40$, $\Delta t = 1/100$. We can see that the numerical solutions
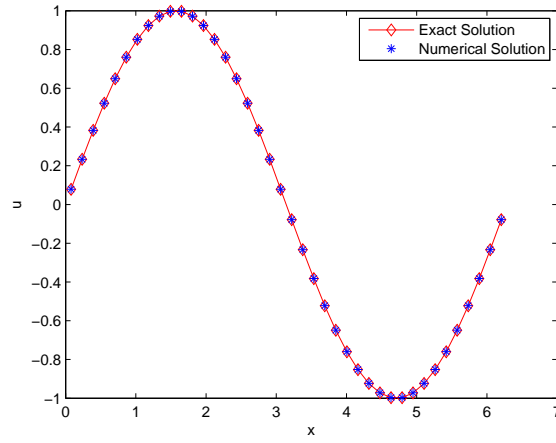
fit well with the exact solutions. The L^2-norm errors versus h (Δt) between the numerical solution and the exact solution at $t = 1$ for Example 5.1 with different α are displayed in Figs. 2 and 3. From Figs. 2 and 3, we can observe that the errors decay rapidly as h and Δt decrease.

**Table 1:** The L^2 errors at $t = 1$ and convergence orders in space for Example 5.1, $\Delta t = 1/100$, $k = 0$

$2\pi/h$	$\alpha = 0.01$		$\alpha = 0.5$		$\alpha = 0.99$	
	Error	Order	Error	Order	Error	Order
5	6.3111e-1	-	6.3054e-1	-	6.2995e-1	-
10	3.2002e-1	0.9797	3.1994e-1	0.9788	3.1987e-1	0.9778
20	1.6056e-1	0.9950	1.6055e-1	0.9948	1.6054e-1	0.9946
40	8.0350e-2	0.9987	8.0349e-2	0.9987	8.0348e-2	0.9986
80	4.0185e-2	0.9996	4.0185e-2	0.9996	4.0184e-2	0.9996
160	2.0097e-2	0.9997	2.0095e-2	0.9998	2.0094e-2	0.9999

**Table 2:** The L^2 errors at $t = 1$ and convergence orders in time for Example 5.1, $h = 2\pi\Delta t^2$, $k = 0$

$1/\Delta t$	$\alpha = 0.01$		$\alpha = 0.5$		$\alpha = 0.99$	
	Error	Order	Error	Order	Error	Order
5	1.5542e-1	-	1.4927e-1	-	1.4475e-1	-
10	4.3922e-2	1.8232	4.0466e-2	1.8831	3.7797e-2	1.9372
20	1.1966e-2	1.8760	1.0758e-2	1.9113	9.8140e-3	1.9454
40	3.1359e-3	1.9320	2.7868e-3	1.9487	2.5132e-3	1.9653

Figure 1: Comparison of the exact solution and the numerical solution at $t = 1$ when $\alpha = 0.5$, $h = 2\pi/40$, $\Delta t = 1/100$
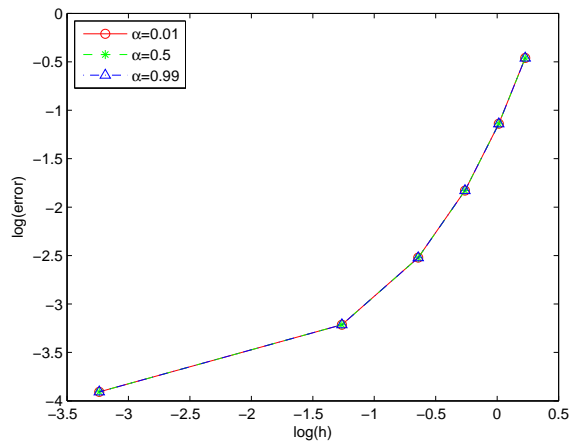
Figure 2: L^2-norm errors versus $\log(h)$ at $t = 1$ for Example 5.1 with different α
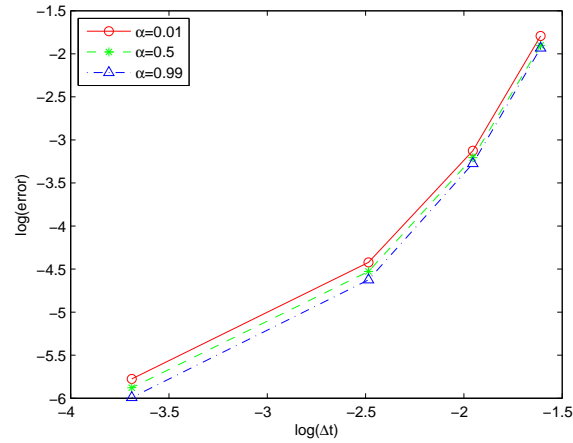
Figure 3: L^2-norm errors versus $\log(\Delta t)$ at $t = 1$ for Example 5.1 with different α

6 Concluding remarks

In this paper, we have studied the existence, uniqueness, and regularity of the solutions of a special case of Caputo-type cable equation. Based on these theoretical results, we apply the WSGD method to approximating the temporal fractional derivative, and apply the LDG method to approximating the space derivative. The resulting fully discrete scheme is proved to be stable and convergent. Finally, a numerical example is presented to verify the theoretical analysis.

The results presented in this paper indicate that the proposed LDG scheme enjoys the same accuracy as the spectral schemes in Liu et al. [Liu and Lü (2019); Yang, Jiang and Zhang (2018)]. However, if the geometry and boundary conditions are complicated, LDG method may be more suitable and can achieve the uniformly high-order accuracy, which is what I will do next [Xu and Shu (2010)]. Besides, it is of much interest to investigate the blow-up phenomenon of the solution, see for example [Cao, Song, Wang et al. (2019)]. In the future work, I will consider using LDG method to deal with variable order fractional differential equations.

Conflicts of Interest: The authors declare that they have no conflicts of interest to report regarding the present study.

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