*Computer Modeling in Engineering & Sciences*
DOI:10.32604/cmes.2020.08656
***Article***

# Addition Formulas of Leaf Functions and Hyperbolic Leaf Functions

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Received: 21 September 2019; Accepted: 06 February 2020

**Abstract:** Addition formulas exist in trigonometric functions. Double-angle and half-angle formulas can be derived from these formulas. Moreover, the relation equation between the trigonometric function and the hyperbolic function can be derived using an imaginary number. The inverse hyperbolic function $\mathrm{arsinh}(r) = \int_0^r \frac{1}{\sqrt{1+t^2}} \mathrm{d}t$ is similar to the inverse trigonometric function $\arcsin(r) = \int_0^r \frac{1}{\sqrt{1-t^2}} \mathrm{d}t$, such as the second degree of a polynomial and the constant term 1, except for the sign − and +. Such an analogy holds not only when the degree of the polynomial is 2, but also for higher degrees. As such, a function exists with respect to the leaf function through the imaginary number *i*, such that the hyperbolic function exists with respect to the trigonometric function through this imaginary number. In this study, we refer to this function as the hyperbolic leaf function. By making such a definition, the relation equation between the leaf function and the hyperbolic leaf function makes it possible to easily derive various formulas, such as addition formulas of hyperbolic leaf functions based on the addition formulas of leaf functions. Using the addition formulas, we can also derive the double-angle and half-angle formulas. We then verify the consistency of these formulas by constructing graphs and numerical data.

**Keywords:** Leaf functions; hyperbolic leaf functions; lemniscate functions; Jacobi elliptic functions; ordinary differential equations; nonlinear equations

## 1 Introduction

### *1.1 Leaf Functions and Hyperbolic Leaf Functions*

An ordinary differential equation consists of both a function raised to the $2n - 1$ power and the second derivative of the function.

$$\frac{\mathrm{d}^2 r(l)}{\mathrm{d}l^2} = -n r(l)^{2n-1} \tag{1}$$

The preceding equation is the ODE that motivated this study. Although the Eq. (1) is a simple ordinary differential equation, it has a very important meaning because it generates characteristic waves. By numerically analyzing the solution that satisfies this equation, we can obtain regular and periodic waves [1,2]. The form of these waves differs from the form of the waves based on trigonometric functions. The function that satisfies this ordinary differential equation is called a leaf function, and it describes the features of these functions. Eq. (1) is transformed as follows:

$$l = \int_0^r \frac{\mathrm{d}t}{\sqrt{1 - t^{2n}}} (= \mathrm{arcsleaf}_n(r)) \tag{2}$$

The preceding integral is defined as the inverse function $\mathrm{arcsleaf}_n$ ($l$) of the leaf function. Another function can be defined as follows:

$$l = \int_r^1 \frac{\mathrm{d}t}{\sqrt{1 - t^{2n}}} (= \mathrm{arccleaf}_n(r)) \tag{3}$$

The preceding integral is also defined as the inverse function $\mathrm{arccleaf}_n(r)$ of the leaf function with a different integral domain compared to Eq. (2). The variable $n$ represents a natural number, and it is referred to as the basis. Moreover, the ordinary differential equation that is satisfied by the hyperbolic functions $r(l) = \sinh(l)$ and $r(l) = \cosh(l)$ is described as follows.

$$\frac{\mathrm{d}^2 r(l)}{\mathrm{d}l^2} = r(l) \tag{4}$$

Compared to Eq. (1), the difference in Eq. (4) is the positive sign on the right hand side of the equation. The inverse hyperbolic functions arsinh($r$) and arcosh($r$) are well known as:

$$l = \int_0^r \frac{\mathrm{d}t}{\sqrt{1 + t^2}} (= \mathrm{asinh}(r)) \tag{5}$$

$$l = \int_1^r \frac{\mathrm{d}t}{\sqrt{t^2 - 1}} (= \mathrm{acosh}(r)) \tag{6}$$

The contents of the root in the integrand constitute a polynomial. The polynomial of the inverse hyperbolic function and that of the inverse trigonometric function both have a degree of 2. The magnitude 1 of the constant term in the root is also the same. The difference between the inverse functions of the trigonometric function and the hyperbolic function is the sign ("+" and "–") of the polynomial in the root. Using Eqs. (5) and (6), it is seen that trigonometric functions and hyperbolic functions have relational equation through imaginary numbers. Based on this relationship, similar functions also could be paired with leaf functions though analogy relation (see Appendix D in detail). These functions are called hyperbolic leaf functions and consist of two functions. One function is defined as follows.

$$r(l) = \mathrm{sleafh}_n(l)(n = 1, 2, 3 \cdots) \tag{7}$$

The limit exists for the function $\mathrm{sleafh}_n(l)$ (see Appendix F). The domain of the variable $l$ is defined as follows:

$$-\zeta_n < l < \zeta_n \tag{8}$$

The initial conditions of the preceding equation are defined as follows.

$$r(0) = \mathrm{sleafh}_n(0) = 0(n = 1, 2, 3 \cdots) \tag{9}$$

$$\frac{\mathrm{d}r(0)}{\mathrm{d}l} = \frac{\mathrm{d}}{\mathrm{d}l}\mathrm{sleafh}_n(0) = 1(n = 1, 2, 3 \cdots) \tag{10}$$

Next, the another function is defined as follows:

$$r(l) = \mathrm{cleafh}_n(l)(n = 1, 2, 3 \cdots) \tag{11}$$

The limit exists for the function $\mathrm{cleafh}_n(l)$ (see Appendix G). The domain of the variable $l$ is as follows:

$$-\eta_n < l < \eta_n \tag{12}$$

The initial conditions of the preceding equation are defined as follows.

$$r(0) = \mathrm{cleafh}_n(0) = 1(n = 1, 2, 3 \cdots) \tag{13}$$

$$\frac{\mathrm{d}r(0)}{\mathrm{d}l} = \frac{\mathrm{d}}{\mathrm{d}l}\mathrm{cleafh}_n(0) = 0(n = 1, 2, 3 \cdots) \tag{14}$$

The ordinary differential equations that are satisfied by the hyperbolic leaf functions that correspond to both Eqs. (7) and (11) are as follows.

$$\frac{\mathrm{d}^2 r(l)}{\mathrm{d}l^2} = nr(l)^{2n-1} \tag{15}$$

The inverse function of the hyperbolic leaf function is derived as follows:

$$l = \int_0^r \frac{\mathrm{d}t}{\sqrt{1 + t^{2n}}} (= \mathrm{asleafh}_n(r))(n = 1, 2, 3 \cdots) \tag{16}$$

$$l = \int_1^r \frac{\mathrm{d}t}{\sqrt{t^{2n} - 1}} (= \mathrm{acleafh}_n(r))(n = 1, 2, 3 \cdots) \tag{17}$$

Here, the prefix $a$ of both hyperbolic leaf functions $\mathrm{sleafh}_n$ $(l)$ and $\mathrm{cleafh}_n(l)$ are defined as the inverse functions.

### *1.2 Comparison of Legacy Functions*

The leaf functions and the hyperbolic leaf functions based on the basis $n$ = 1 are as follows:

$$\mathrm{sleaf}_1(l) = \sin(l) \tag{18}$$

$$\mathrm{cleaf}_1(l) = \cos(l) \tag{19}$$

$$\mathrm{sleafh}_1(l) = \sinh(l) \tag{20}$$

$$\mathrm{cleafh}_1(l) = \cosh(l) \tag{21}$$

Lemniscate functions were proposed by Johann Carl Friedrich Gauss [3]. The relation equations between these functions and leaf functions are as follows:

$$\mathrm{sleaf}_2(l) = \mathrm{sl}(l) \tag{22}$$

$$\mathrm{cleaf}_2(l) = \mathrm{cl}(l) \tag{23}$$

$$\mathrm{sleafh}_2(l) = \mathrm{slh}(l) \tag{24}$$

The definition of the function slh($l$) in Eq. (24) can be confirmed based on references [4,5]. A function corresponding to the hyperbolic leaf function $\mathrm{cleafh}_2(l)$ is not described in the literature [6]. In the case where the basis $n \geqq 3$, the leaf function or the hyperbolic leaf function cannot be represented by a legacy function such as the lemniscate function.

### *1.3 Originality and Purpose*

The purpose of this report is to propose addition formulas for the hyperbolic leaf functions with basis $n = 2$ and $n = 3$, in addition to establishing both double-angle and the half-angle formulas using addition formulas. A similar analogy exists in the relation between the leaf function and the hyperbolic leaf function such that the relation between the trigonometric function and the hyperbolic function can be derived using imaginary numbers. Using this analogy, the addition formulas of hyperbolic leaf functions based on $n = 3$ can be derived from the addition formulas of leaf functions based on $n = 3$. Using addition formulas, we present numerical data and curves derived from the hyperbolic leaf function and show that these addition formulas in the Section 2 are consistent.

### *1.4 Contribution*

The leaf functions are closely related to the Jacobi elliptic functions. The Jacobi elliptic function originated from the lemniscate function. In 1691, Jacob Bernoulli noticed that the arc length OP of the lemniscate curve was the same as the integral of the equation [7]. As shown in Fig. 1, $l$ represents the length of arc OP. The arc OP is represented as:

$$arcOP = \int_0^r \frac{dt}{\sqrt{1-t^4}} (= l) \tag{25}$$

The curve in Fig. 1 can be expressed using variables $x$ and $y$ as:

$$(x^2 + y^2)^2 = x^2 - y^2 \tag{26}$$

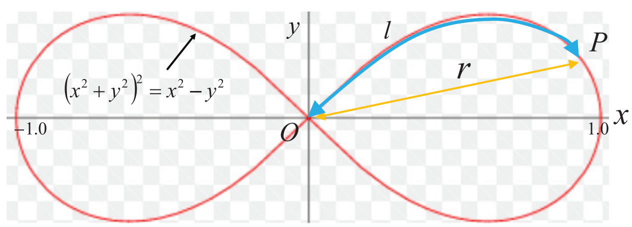


**Figure 1:** Lemniscate of Bernoulli

When the basis of the leaf function, n = 2, the curve for the leaf function is the same as the lemniscate curve. In 1718, Giulio Carlo de' Toschi di Fagnano published a paper explaining how arc OP could be divided into two equal parts using only a straightedge and a compass [8]. He discovered that the length $r$ was twice the length $u$, as shown in Fig. 2. This led to the derivation of the addition theorem for the lemniscate function:

$$r = \frac{2u\sqrt{1-u^4}}{1+u^4} \text{ under the condition } arcOP = 2arcOA \tag{27}$$

$$arcOA = \int_0^u \frac{dt}{\sqrt{1-t^4}} \tag{28}$$

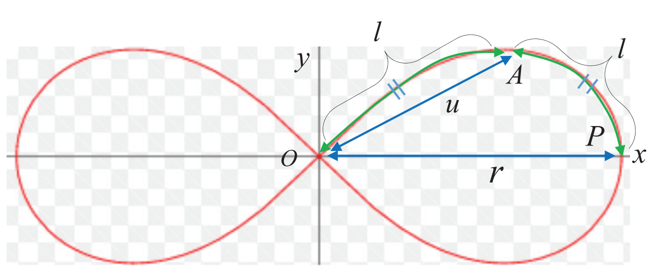


**Figure 2:** Fagnano and the lemniscate

After reading Fagnano's paper, Leonhard Euler found the addition formula for the lemniscate function in 1752 [9]. In the formula, the sum of the integral forms of arbitrary variables $u$ and $v$ equals the integral form $r$:

$$r = \frac{u\sqrt{1-v^4} + v\sqrt{1-u^4}}{1+u^2v^2} \tag{29}$$

The above-mentioned relation satisfies the integral equation as follows:

$$\int_0^r \frac{\mathrm{d}t}{\sqrt{1-t^4}} = \int_0^u \frac{\mathrm{d}t}{\sqrt{1-t^4}} + \int_0^v \frac{\mathrm{d}t}{\sqrt{1-t^4}} \tag{30}$$

In 1796, Carl Friedrich Gauss derived the addition formula using the lemniscate function [3]. The inverse lemniscate function is expressed as:

$$\mathrm{arcsl}(r) = \int_0^r \frac{\mathrm{d}t}{\sqrt{1-t^4}} (= l_1 \text{ or } l_2) \tag{31}$$

Using this definition of inverse arc sl, the addition formula of lemniscate function sl($l_1 + l_2$) is derived as:

$$\mathrm{sl}(l_1 + l_2) = \frac{sl(l_1)\sqrt{1 - sl(l_2)^4} + sl(l_2)\sqrt{1 - sl(l_1)^4}}{1 + sl(l_1)^2 sl(l_2)^2} \tag{32}$$

Phases $l_1$ or $l_2$ of lemniscate function sl can be extended to complex variables $i \cdot l_1$ or $i \cdot l_2$, respectively. In 1827, Carl Gustav Jacob Jacobi derived the inverses of the Jacobi elliptic functions [10]. To derive the formula, the term $t^2$ is added to the root of the integrand denominator in Eq. (31):

$$\mathrm{arcsn}(r, k) = \int_0^r \frac{\mathrm{d}t}{\sqrt{1-(1+k^2)t^2 + k^2t^4}} \tag{33}$$

Eq. (33) represents the inverse Jacobi elliptic function sn, where $k$ is a constant. There exist 12 Jacobi elliptic functions, including cn and dn. In the Eq. (33), the variable $t$ is to the fourth power in the denominator. Jacobi did not discuss the variable $t$ to higher powers, such as follows:

$$\int_0^r \frac{\mathrm{d}t}{\sqrt{1-t^6}}, \int_0^r \frac{\mathrm{d}t}{\sqrt{1-t^8}}, \int_0^r \frac{\mathrm{d}t}{\sqrt{1-t^{10}}} \cdots \tag{34}$$

In other words, there had been no discussion for $n$ = 3 in Eqs. (2), (3), (16), and (17). Therefore, the addition formulas for the leaf function were investigated for n = 3 [11]. In case of $n \geqq 3$, no clear description for the addition formulas of hyperbolic leaf functions exists. On the contrary, $n$ = 1 represents hyperbolic functions sinh($l$) and cosh($l$). Therefore, the addition formulas of the hyperbolic leaf function and hyperbolic function are the same. The hyperbolic leaf function with $n$ = 2 represents hyperbolic lemniscate function slh($l$). No clear description of the addition formulas of function slh($l$) exists.

### *1.5 Advantage and Disadvantage*

In physics, the nonlinear duffing equation represents a model for the spring pendulum whose spring stiffness does not obey Hooke's law. This undamped duffing equation is represented as:

$$\text{Cubic} - \text{Quintic Duffing Equation}: \frac{\mathrm{d}^2 r}{\mathrm{d}l^2} + \alpha r + \beta r^3 + \mu r^5 = 0 \tag{35}$$

To solve the above equation, numerical analysis or analytical approximate solutions have been applied [12–24]. Additionally, literatures describe the application of the cubic duffing equation, using Jacobi elliptic functions [25–27]. As in the leaf function represented by Eq. (2), the term $t^2$ is added to the root of inverse Jacobian elliptic function sn in Eq. (33). Variables $r$ and $k$ control Jacobian function sn. The scope of applying the duffing equation to the Jacobian elliptic function is wider compared with the leaf function that has only one parameter $l$. Over time, the nonlinear duffing equation has witnessed improvements and further numerical analysis [28–31].

$$\text{Cubic} - \text{Quintic} - \text{Septic Duffing Equation} : \frac{\mathrm{d}^2 r}{\mathrm{d}l^2} + \alpha r + \beta r^3 + \mu r^5 + \delta r^7 = 0 \tag{36}$$

A high-order exact solution using the Jacobi elliptic function has not been found yet. Furthermore, the high-order addition theorem necessary to derive the exact solution is not defined in the Jacobi elliptic function. To find the exact solution, the addition theorem is important to apply superposition principle. By using the addition theorem, one term in the exact solution can be divided into several terms, or the several terms can be integrated into one term. In this paper, we derive the addition theorem to further derive an exact solution for a high-order duffing equation, followed by applying the superposition principle.

## 2 Addition Formulas

### *2.1 Addition Formulas of Leaf Functions*

Let there be two variables, $l_1$ and $l_2$. The addition formulas of the function $\mathrm{sleaf}_2(l)$ can be stated as follows:

$$\mathrm{sleaf}_2(l_1 + l_2) = \frac{\mathrm{sleaf}_2(l_1)\dfrac{\partial \mathrm{sleaf}_2(l_2)}{\partial l_2} + \mathrm{sleaf}_2(l_2)\dfrac{\partial \mathrm{sleaf}_2(l_1)}{\partial l_1}}{1 + (\mathrm{sleaf}_2(l_1))^2(\mathrm{sleaf}_2(l_2))^2} \tag{37}$$

Depending on the domain of the variable $l$ of the leaf function, the signs of both $\partial\mathrm{sleaf}_2(l_2)/\partial l_2$ and $\partial\mathrm{sleaf}_2(l_1)/\partial l_1$ change. Eq. (37) can be summarized according a number of cases based on the domains of variables $l_1$ and $l_2$ (See Fig. 3). The parameters $m$ and $k$ represent integers.

(i) In the case where $\frac{\pi_2}{2}(4m-1) \leqq l_1 \leqq \frac{\pi_2}{2}(4m+1)$, $\frac{\pi_2}{2}(4k-1) \leqq l_2 \leqq \frac{\pi_2}{2}(4k+1)$(see Appendix E for the constant $\pi_2$), Eq. (37) is transformed into:

$$\mathrm{sleaf}_2(l_1 + l_2) = \frac{\mathrm{sleaf}_2(l_1)\sqrt{1 - (\mathrm{sleaf}_2(l_2))^4} + \mathrm{sleaf}_2(l_2)\sqrt{1 - (\mathrm{sleaf}_2(l_1))^4}}{1 + (\mathrm{sleaf}_2(l_1))^2(\mathrm{sleaf}_2(l_2))^2} \tag{38}$$

(ii) In the case where $\frac{\pi_2}{2}(4m-1) \leqq l_1 \leqq \frac{\pi_2}{2}(4m+1)$, $\frac{\pi_2}{2}(4k+1) \leqq l_2 \leqq \frac{\pi_2}{2}(4k+3)$, Eq. (37) is transformed into:

$$\mathrm{sleaf}_2(l_1 + l_2) = \frac{-\mathrm{sleaf}_2(l_1)\sqrt{1 - (\mathrm{sleaf}_2(l_2))^4} + \mathrm{sleaf}_2(l_2)\sqrt{1 - (\mathrm{sleaf}_2(l_1))^4}}{1 + (\mathrm{sleaf}_2(l_1))^2(\mathrm{sleaf}_2(l_2))^2} \tag{39}$$

(iii) In the case where $\frac{\pi_2}{2}(4m+1) \leqq l_1 \leqq \frac{\pi_2}{2}(4m+3)$, $\frac{\pi_2}{2}(4k-1) \leqq l_2 \leqq \frac{\pi_2}{2}(4k+1)$, Eq. (37) is tansformed into:

$$\mathrm{sleaf}_2(l_1 + l_2) = \frac{\mathrm{sleaf}_2(l_1)\sqrt{1 - (\mathrm{sleaf}_2(l_2))^4} - \mathrm{sleaf}_2(l_2)\sqrt{1 - (\mathrm{sleaf}_2(l_1))^4}}{1 + (\mathrm{sleaf}_2(l_1))^2(\mathrm{sleaf}_2(l_2))^2} \tag{40}$$

(iv) In the case where $\frac{\pi_2}{2}(4m+1) \leqq l_1 \leqq \frac{\pi_2}{2}(4m+3)$, $\frac{\pi_2}{2}(4k+1) \leqq l_2 \leqq \frac{\pi_2}{2}(4k+3)$, Eq. (37) is tansformed into:

$$\mathrm{sleaf}_2(l_1+l_2) = \frac{-\mathrm{sleaf}_2(l_1)\sqrt{1-(\mathrm{sleaf}_2(l_2))^4} - \mathrm{sleaf}_2(l_2)\sqrt{1-(\mathrm{sleaf}_2(l_1))^4}}{1+(\mathrm{sleaf}_2(l_1))^2(\mathrm{sleaf}_2(l_2))^2} \tag{41}$$

Next, the addition formula of $\mathrm{cleaf}_2(l)$ can be stated as follows:

$$\mathrm{cleaf}_2(l_1+l_2) = \frac{\mathrm{cleaf}_2(l_1)\dfrac{\partial \mathrm{sleaf}_2(l_2)}{\partial l_2} + \mathrm{sleaf}_2(l_2)\dfrac{\partial \mathrm{cleaf}_2(l_1)}{\partial l_1}}{1+(\mathrm{cleaf}_2(l_1))^2(\mathrm{sleaf}_2(l_2))^2} \tag{42}$$

Depending on the domain of the variable $l$ of the leaf function, the signs of both $\partial \mathrm{sleaf}_2(l_2)/\partial l_2$ and $\partial \mathrm{cleaf}_2(l_1)/\partial l_1$ change. Eq. (42) can be summarized according a number of cases based on the domains of variables $l_1$ and $l_2$.

(i) In the case where $2m\,\pi_2 \leqq l_1 \leqq (2m+1)\,\pi_2$, $\frac{\pi_2}{2}(4k-1) \leqq l_2 \leqq \frac{\pi_2}{2}(4k+1)$, Eq. (42) is transformed into:

$$\mathrm{cleaf}_2(l_1+l_2) = \frac{\mathrm{cleaf}_2(l_1)\sqrt{1-(\mathrm{sleaf}_2(l_2))^4} - \mathrm{sleaf}_2(l_2)\sqrt{1-(\mathrm{cleaf}_2(l_1))^4}}{1+(\mathrm{cleaf}_2(l_1))^2(\mathrm{sleaf}_2(l_2))^2} \tag{43}$$

(ii) In the case where $(2m-1)\,\pi_2 \leqq l_1 \leqq 2m\,\pi_2$, $\frac{\pi_2}{2}(4k-1) \leqq l_2 \leqq \frac{\pi_2}{2}(4k+1)$, Eq. (42) is transformed into:

$$\mathrm{cleaf}_2(l_1+l_2) = \frac{\mathrm{cleaf}_2(l_1)\sqrt{1-(\mathrm{sleaf}_2(l_2))^4} + \mathrm{sleaf}_2(l_2)\sqrt{1-(\mathrm{cleaf}_2(l_1))^4}}{1+(\mathrm{cleaf}_2(l_1))^2(\mathrm{sleaf}_2(l_2))^2} \tag{44}$$

(iii) In the case where $(2m-1)\,\pi_2 \leqq l_1 \leqq 2m\,\pi_2$, $\frac{\pi_2}{2}(4k+1) \leqq l_2 \leqq \frac{\pi_2}{2}(4k+3)$, Eq. (42) is transformed into:

$$\mathrm{cleaf}_2(l_1+l_2) = \frac{-\mathrm{cleaf}_2(l_1)\sqrt{1-(\mathrm{sleaf}_2(l_2))^4} + \mathrm{sleaf}_2(l_2)\sqrt{1-(\mathrm{cleaf}_2(l_1))^4}}{1+(\mathrm{cleaf}_2(l_1))^2(\mathrm{sleaf}_2(l_2))^2} \tag{45}$$

(iv) In the case where $2m\ \pi_2 \leqq l_1 \leqq (2m+1)\ \pi_2$, $\frac{\pi_2}{2}(4k+1) \leqq l_2 \leqq \frac{\pi_2}{2}(4k+3)$, Eq. (42) is transformed into:

$$\mathrm{cleaf}_2(l_1+l_2) = \frac{-\mathrm{cleaf}_2(l_1)\sqrt{1-(\mathrm{sleaf}_2(l_2))^4} - \mathrm{sleaf}_2(l_2)\sqrt{1-(\mathrm{cleaf}_2(l_1))^4}}{1+(\mathrm{cleaf}_2(l_1))^2(\mathrm{sleaf}_2(l_2))^2} \tag{46}$$

Next, the addition formulas of $\mathrm{sleaf}_3(l)$ can be described as follows:

$$(\mathrm{sleaf}_3(l_1+l_2))^2 = \frac{\left\{\mathrm{sleaf}_3(l_1)\dfrac{\partial \mathrm{sleaf}_3(l_2)}{\partial l_2} + \mathrm{sleaf}_3(l_2)\dfrac{\partial \mathrm{sleaf}_3(l_1)}{\partial l_1}\right\}^2}{1+4(\mathrm{sleaf}_3(l_1))^4(\mathrm{sleaf}_3(l_2))^2+4(\mathrm{sleaf}_3(l_1))^2(\mathrm{sleaf}_3(l_2))^4} + \frac{\left\{(\mathrm{sleaf}_3(l_1))^3\mathrm{sleaf}_3(l_2) - \mathrm{sleaf}_3(l_1)(\mathrm{sleaf}_3(l_2))^3\right\}^2}{1+4(\mathrm{sleaf}_3(l_1))^4(\mathrm{sleaf}_3(l_2))^2+4(\mathrm{sleaf}_3(l_1))^2(\mathrm{sleaf}_3(l_2))^4} \tag{47}$$

The preceding equation can be summarized as follows according to a number of cases based on the domains of the variables $l_1$ and $l_2$.

(i) In the case where both $(4m-1)\frac{\pi_3}{2} \leqq l_1 \leqq (4m+1)\frac{\pi_3}{2}$ and $(4k-1)\frac{\pi_3}{2} \leqq l_2 \leqq (4k+1)\frac{\pi_3}{2}$ or both $(4m+1)\frac{\pi_3}{2} \leqq l_1 \leqq (4m+3)\frac{\pi_3}{2}$ and $(4k+1)\frac{\pi_3}{2} \leqq l_2 \leqq (4k+3)\frac{\pi_3}{2}$, Eq. (47) is tansformed into:

$$(\mathrm{sleaf}_3(l_1+l_2))^2 = \frac{\left\{\mathrm{sleaf}_3(l_1)\sqrt{1-(\mathrm{sleaf}_3(l_2))^6} + \mathrm{sleaf}_3(l_2)\sqrt{1-(\mathrm{sleaf}_3(l_1))^6}\right\}^2}{1+4(\mathrm{sleaf}_3(l_1))^4(\mathrm{sleaf}_3(l_2))^2+4(\mathrm{sleaf}_3(l_1))^2(\mathrm{sleaf}_3(l_2))^4} + \frac{\left\{(\mathrm{sleaf}_3(l_1))^3\mathrm{sleaf}_3(l_2) - \mathrm{sleaf}_3(l_1)(\mathrm{sleaf}_3(l_2))^3\right\}^2}{1+4(\mathrm{sleaf}_3(l_1))^4(\mathrm{sleaf}_3(l_2))^2+4(\mathrm{sleaf}_3(l_1))^2(\mathrm{sleaf}_3(l_2))^4} \tag{48}$$

The symbol $\pi_3$ represents a constant (see Appendix E).

(ii) In the case where both $(4m+1)\frac{\pi_3}{2} \leqq l_1 \leqq (4m+3)\frac{\pi_3}{2}$ and $(4k-1)\frac{\pi_3}{2} \leqq l_2 \leqq (4k+1)\frac{\pi_3}{2}$ or both $(4m-1)\frac{\pi_3}{2} \leqq l_1 \leqq (4m+1)\frac{\pi_3}{2}$ and $(4k+1)\frac{\pi_3}{2} \leqq l_2 \leqq (4k+3)\frac{\pi_3}{2}$, Eq. (47) is tansformed into:

$$(\mathrm{sleaf}_3(l_1+l_2))^2 = \frac{\left\{\mathrm{sleaf}_3(l_1)\sqrt{1-(\mathrm{sleaf}_3(l_2))^6} - \mathrm{sleaf}_3(l_2)\sqrt{1-(\mathrm{sleaf}_3(l_1))^6}\right\}^2}{1+4(\mathrm{sleaf}_3(l_1))^4(\mathrm{sleaf}_3(l_2))^2+4(\mathrm{sleaf}_3(l_1))^2(\mathrm{sleaf}_3(l_2))^4} + \frac{\left\{(\mathrm{sleaf}_3(l_1))^3\mathrm{sleaf}_3(l_2) - \mathrm{sleaf}_3(l_1)(\mathrm{sleaf}_3(l_2))^3\right\}^2}{1+4(\mathrm{sleaf}_3(l_1))^4(\mathrm{sleaf}_3(l_2))^2+4(\mathrm{sleaf}_3(l_1))^2(\mathrm{sleaf}_3(l_2))^4} \tag{49}$$

Next, the addition formulas of $\mathrm{cleaf}_3(l)$ can be defined as follows:

$$(\mathrm{cleaf}_3(l_1+l_2))^2 = \frac{\left\{\mathrm{cleaf}_3(l_1)\frac{\partial \mathrm{sleaf}_3(l_2)}{\partial l_2} + \mathrm{sleaf}_3(l_2)\frac{\partial \mathrm{cleaf}_3(l_1)}{\partial l_1}\right\}^2}{1+4(\mathrm{sleaf}_3(l_2))^4(\mathrm{cleaf}_3(l_1))^2+4(\mathrm{sleaf}_3(l_2))^2(\mathrm{cleaf}_3(l_1))^4} + \frac{\left\{(\mathrm{sleaf}_3(l_2))^3\mathrm{cleaf}_3(l_1) - \mathrm{sleaf}_3(l_2)(\mathrm{cleaf}_3(l_1))^3\right\}^2}{1+4(\mathrm{sleaf}_3(l_2))^4(\mathrm{cleaf}_3(l_1))^2+4(\mathrm{sleaf}_3(l_2))^2(\mathrm{cleaf}_3(l_1))^4} \tag{50}$$

The preceding equation can be summarized as follows according to a number of cases based on the domains of the variables $l_1$ and $l_2$.

(i) In the case where both $2k\,\pi_3 \leqq l_1 \leqq (2k+1)\,\pi_3$ and $(4m-1)\frac{\pi_3}{2} \leqq l_2 \leqq (4m+1)\frac{\pi_3}{2}$ or both $(2k+1)\,\pi_3 \leq l_1 \leq (2k+2)\,\pi_3$ and $(4m+1)\frac{\pi_3}{2} \leqq l_2 \leqq (4m+3)\frac{\pi_3}{2}$, Eq. (50) is transformed into:

$$(\mathrm{cleaf}_3(l_1+l_2))^2 = \frac{\left\{\mathrm{cleaf}_3(l_1)\sqrt{1-(\mathrm{sleaf}_3(l_2))^6} - \mathrm{sleaf}_3(l_2)\sqrt{1-(\mathrm{cleaf}_3(l_1))^6}\right\}^2}{1+4(\mathrm{sleaf}_3(l_2))^4(\mathrm{cleaf}_3(l_1))^2+4(sleaf_3(l_2))^2(\mathrm{cleaf}_3(l_1))^4} + \frac{\left\{(\mathrm{sleaf}_3(l_2))^3\mathrm{cleaf}_3(l_1) - \mathrm{sleaf}_3(l_2)(\mathrm{cleaf}_3(l_1))^3\right\}^2}{1+4(\mathrm{sleaf}_3(l_2))^4(\mathrm{cleaf}_3(l_1))^2+4(\mathrm{sleaf}_3(l_2))^2(\mathrm{cleaf}_3(l_1))^4} \tag{51}$$

(ii) In the case where both $(2k+1)\,\pi_3 \leqq l_1 \leqq (2k+2)\,\pi_3$ and $(4m-1)\frac{\pi_3}{2} \leqq l_2 \leqq (4m+1)\frac{\pi_3}{2}$ or both $2k\,\pi_3 \leq l_1 \leq (2k+1)\,\pi_3$ and $(4m+1)\frac{\pi_3}{2} \leqq l_2 \leqq (4m+3)\frac{\pi_3}{2}$, Eq. (50) is transformed into:

$$(\mathrm{cleaf}_3(l_1+l_2))^2 = \frac{\left\{\mathrm{cleaf}_3(l_1)\sqrt{1-(\mathrm{sleaf}_3(l_2))^6}+\mathrm{sleaf}_3(l_2)\sqrt{1-(\mathrm{cleaf}_3(l_1))^6}\right\}^2}{1+4(\mathrm{sleaf}_3(l_2))^4(\mathrm{cleaf}_3(l_1))^2+4(\mathrm{sleaf}_3(l_2))^2(\mathrm{cleaf}_3(l_1))^4} + \frac{\left\{(\mathrm{sleaf}_3(l_2))^3\mathrm{cleaf}_3(l_1)-\mathrm{sleaf}_3(l_2)(\mathrm{cleaf}_3(l_1))^3\right\}^2}{1+4(\mathrm{sleaf}_3(l_2))^4(\mathrm{cleaf}_3(l_1))^2+4(\mathrm{sleaf}_3(l_2))^2((\mathrm{cleaf}_3(l_1))^4} \tag{52}$$

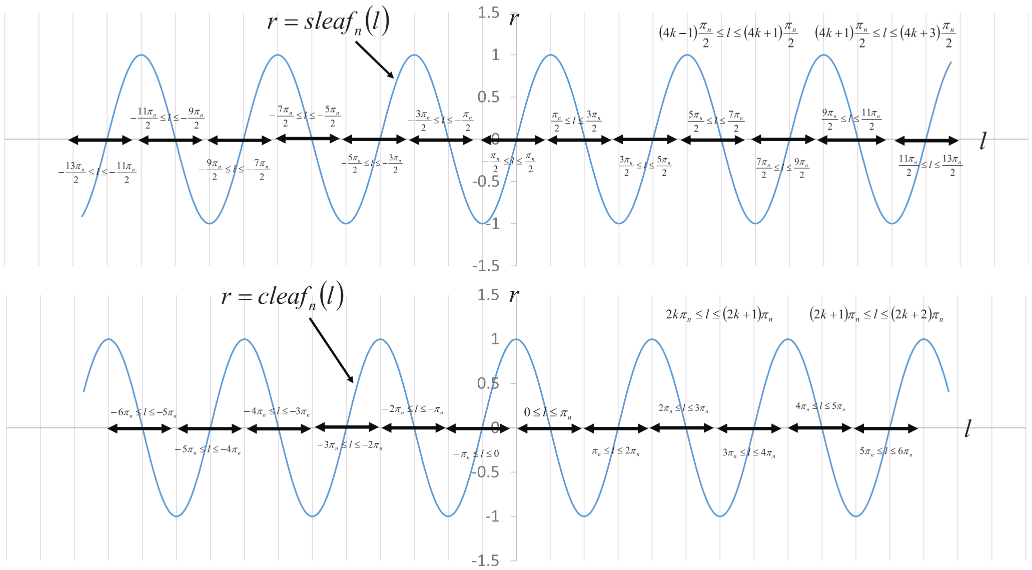


**Figure 3:** Curves of the functions sleaf$_n$($l$) and cleaf$_n$($l$)

### *2.2 Addition Formulas of Hyperbolic Leaf Function*

Let there be two variables, $l_1$ and $l_2$. Considering the imaginary number $i$, the relation between sleaf$_2$($l$) and sleafh$_2$($l$), and the relation between cleaf$_2$($l$) and cleafh$_2$($l$) can be obtained as follows (see Appendix D in detail):

$$\mathrm{sleaf}_2(i\cdot l) = i\cdot \mathrm{sleaf}_2(l) \tag{53}$$

$$\mathrm{sleafh}_2(i\cdot l) = i\cdot \mathrm{sleafh}_2(l) \tag{54}$$

$$\mathrm{cleaf}_2(i\cdot l) = \mathrm{cleafh}_2(l) \tag{55}$$

As shown in Eqs. (53) and (54), in the case where $n$ = 2, the functions sleaf$_2$($i \cdot l$) and sleafh$_2$($i \cdot l$) are equal to the functions $i \cdot$ sleaf$_2$($l$) and $i \cdot$ sleafh$_2$($l$), respectively. Therefore, we cannot derive the addition formulas of sleafh$_2$($l$) by replacing $i \cdot l$ with $l$ in Eqs. (38)–(41). Using the relation between the function sleaf$_2$($l$) and the function sleafh$_2$($l$) (see Appendix B), the addition formulas of sleafh$_2$($l$) can be obtained. By substituting Eq. (96) into Eqs. (38)–(41), the following equation is obtained:

$$\text{sleafh}_2(l_1+l_2)=\frac{\text{sleafh}_2(l_1)\sqrt{1+(\text{sleafh}_2(l_2))^4}+\text{sleafh}_2(l_2)\sqrt{1+(\text{sleafh}_2(l_1))^4}}{1-(\text{sleafh}_2(l_1))^2(\text{sleafh}_2(l_2))^2} \tag{56}$$

In the work [32], the addition formulas of $\text{cleafh}_2(l)$ are obtained using Eq. (95):

$$\text{cleafh}_2(l_1+l_2)=\frac{2\text{cleafh}_2(l_1)\text{cleafh}_2(l_2)+\dfrac{\partial\text{cleaf}_2(l_1)}{\partial l_1}\dfrac{\partial\text{cleaf}_2(l_2)}{\partial l_2}}{1+(\text{cleafh}_2(l_1))^2+(\text{cleafh}_2(l_2))^2-(\text{cleafh}_2(l_1))^2(\text{cleafh}_2(l_2))^2} \tag{57}$$

The preceding equation can be summarized as follows according to a number of cases based on the domains of variables $l_1$ and $l_2$.

(i) In the case where both $0 \leqq l_1 \leqq \eta_2$ and $0 \leqq l_2 \leqq \eta_2$ or both $-\eta_2 \leqq l_1 \leqq 0$ and $-\eta_2 \leqq l_2 \leqq 0$ (see Appendix G for the constant $\eta_2$.), Eq. (57) is transformed into:

$$\text{cleafh}_2(l_1+l_2)=\frac{2\text{cleafh}_2(l_1)\text{cleafh}_2(l_2)+\sqrt{(\text{cleafh}_2(l_1))^4-1}\sqrt{(\text{cleafh}_2(l_2))^4-1}}{1+(\text{cleafh}_2(l_1))^2+(\text{cleafh}_2(l_2))^2-(\text{cleafh}_2(l_1))^2(\text{cleafh}_2(l_2))^2} \tag{58}$$

(ii) In the case where both $0 \leqq l_1 \leqq \eta_2$ and $-\eta_2 \leqq l_2 \leqq 0$ or both $-\eta_2 \leqq l_1 \leqq 0$ and $0 \leqq l_2 \leqq \eta_2$, Eq. (57) is transformed into:

$$\text{cleafh}_2(l_1+l_2)=\frac{2\text{cleafh}_2(l_1)\text{cleafh}_2(l_2)-\sqrt{(\text{cleafh}_2(l_1))^4-1}\sqrt{(\text{cleafh}_2(l_2))^4-1}}{1+(\text{cleafh}_2(l_1))^2+(\text{cleafh}_2(l_2))^2-(\text{cleafh}_2(l_1))^2(\text{cleafh}_2(l_2))^2} \tag{59}$$

Next, let us consider the case of $n = 3$. The relation between $\text{sleaf}_3(l)$ and $\text{sleafh}_3(l)$ and the relation between $\text{cleaf}_3(l)$ and $\text{cleafh}_3(l)$ are as follows (see Appendix D):

$$\text{sleaf}_3(l)=-i\cdot\text{sleafh}_3(i\cdot l) \tag{60}$$

$$\text{cleaf}_3(l)=\text{cleafh}_3(i\cdot l) \tag{61}$$

In Eqs. (47)–(52), the variables $l_1$ and $l_2$ are replaced with the expressions $i\cdot l_1$ and $i\cdot l_2$, respectively. The addition formulas of $\text{sleafh}_3(l)$ are defined as follows:

$$(\text{sleafh}_3(l_1+l_2))^2=\frac{\left\{\text{sleafh}_3(l_1)\sqrt{1+(\text{sleafh}_3(l_2))^6}+\text{sleafh}_3(l_2)\sqrt{1+(\text{sleafh}_3(l_1))^6}\right\}^2}{1-4(\text{sleafh}_3(l_1))^4(\text{sleafh}_3(l_2))^2-4(\text{sleafh}_3(l_1))^2(\text{sleafh}_3(l_2))^4} - \frac{\left\{(\text{sleafh}_3(l_1))^3\text{sleafh}_3(l_2)-\text{sleafh}_3(l_1)(\text{sleafh}_3(l_2))^3\right\}^2}{1-4(\text{sleafh}_3(l_1))^4(\text{sleafh}_3(l_2))^2-4(\text{sleafh}_3(l_1))^2(\text{sleafh}_3(l_2))^4} \tag{62}$$

The addition formulas of $\text{cleafh}_3(l)$ are defined as follows:

$$(\mathrm{cleafh}_3(l_1+l_2))^2 = \frac{\left\{\mathrm{cleafh}_3(l_1)\dfrac{\partial \mathrm{sleafh}_3(l_2)}{\partial l_2} + \mathrm{sleafh}_3(l_2)\dfrac{\partial \mathrm{cleafh}_3(l_1)}{\partial l_1}\right\}^2}{1+4(\mathrm{sleafh}_3(l_2))^4(\mathrm{cleafh}_3(l_1))^2 - 4(\mathrm{sleafh}_3(l_2))^2(\mathrm{cleafh}_3(l_1))^4} - \frac{\left\{(\mathrm{sleafh}_3(l_1))^3\mathrm{cleafh}_3(l_2) + \mathrm{sleafh}_3(l_2)(\mathrm{cleafh}_3(l_1))^3\right\}^2}{1+4(\mathrm{sleafh}_3(l_2))^4(\mathrm{cleafh}_3(l_1))^2 - 4(\mathrm{sleafh}_3(l_2))^2(\mathrm{cleafh}_3(l_1))^4} \tag{63}$$

The preceding equation can be summarized as follows according to a number of cases based on the domains of the variables $l_1$ and $l_2$.

(i) In the case where both $-\eta_3 \leqq l_1 \leqq 0$ (see Appendix G for the constant $\eta_3$), Eq. (63) is transformed into:

$$(\mathrm{cleafh}_3(l_1+l_2))^2 = \frac{\left\{\mathrm{cleafh}_3(l_1)\sqrt{1+(\mathrm{sleafh}_3(l_2))^6} - \mathrm{sleafh}_3(l_2)\sqrt{(\mathrm{cleafh}_3(l_1))^6-1}\right\}^2}{1+4(\mathrm{sleafh}_3(l_2))^4(\mathrm{cleafh}_3(l_1))^2 - 4(\mathrm{sleafh}_3(l_2))^2(\mathrm{cleafh}_3(l_1))^4} - \frac{\left\{(\mathrm{sleafh}_3(l_2))^3\mathrm{cleafh}_3(l_1) + \mathrm{sleafh}_3(l_2)(\mathrm{cleafh}_3(l_1))^3\right\}^2}{1+4(\mathrm{sleafh}_3(l_2))^4(\mathrm{cleafh}_3(l_1))^2 - 4(\mathrm{sleafh}_3(l_2))^2(\mathrm{cleafh}_3(l_1))^4} \tag{64}$$

(ii) In the case where $0 \leqq l_1 \leqq \eta_3$, Eq. (63) is transformed into:

$$(\mathrm{cleafh}_3(l_1+l_2))^2 = \frac{\left\{\mathrm{cleafh}_3(l_1)\sqrt{1+(\mathrm{sleafh}_3(l_2))^6} + \mathrm{sleafh}_3(l_2)\sqrt{(\mathrm{cleafh}_3(l_1))^6-1}\right\}^2}{1+4(\mathrm{sleafh}_3(l_2))^4(\mathrm{cleafh}_3(l_1))^2 - 4(\mathrm{sleafh}_3(l_2))^2(\mathrm{cleafh}_3(l_1))^4} - \frac{\left\{(\mathrm{sleafh}_3(l_2))^3\mathrm{cleafh}_3(l_1) + \mathrm{sleafh}_3(l_2)(\mathrm{cleafh}_3(l_1))^3\right\}^2}{1+4(\mathrm{sleafh}_3(l_2))^4(\mathrm{cleafh}_3(l_1))^2 - 4(\mathrm{sleafh}_3(l_2))^2(\mathrm{cleafh}_3(l_1))^4} \tag{65}$$

## 3 Double Angle Formulas and Half Angle Formulas

### *3.1 Double Angle Formulas of Leaf Functions*

In the case where the basis $n = 2$, the variables $l_1$ and $l_2$ in Eq. (37) are replaced with the variable $l$, and the double-angle formula can be expressed as follows:

$$\mathrm{sleaf}_2(2l) = \frac{2\mathrm{sleaf}_2(l)\dfrac{\partial \mathrm{sleaf}_2(l)}{\partial l}}{1+(\mathrm{sleaf}_2(l))^4} \tag{66}$$

The preceding equation can be summarized as follows according to a number of cases based on the domain of the variable $l$.

(i) In the case where $\frac{\pi_2}{2}(4m-1) \leqq l \leqq \frac{\pi_2}{2}(4m+1)$, Eq. (66) is transformed into:

$$\mathrm{sleaf}_2(2l) = \frac{2\mathrm{sleaf}_2(l)\sqrt{1-(\mathrm{sleaf}_2(l))^4}}{1+(\mathrm{sleaf}_2(l))^4} \tag{67}$$

(ii) In the case where $\frac{\pi_2}{2}(4m+1) \leqq l \leqq \frac{\pi_2}{2}(4m+3)$, Eq. (66) is transformed into:

$$\mathrm{sleaf}_2(2l) = -\frac{2\mathrm{sleaf}_2(l)\sqrt{1-(\mathrm{sleaf}_2(l))^4}}{1+(\mathrm{sleaf}_2(l))^4} \tag{68}$$

The variables $l_1$ and $l_2$ in Eqs. (43)–(46) are replaced with the variable $l$. The double-angle formula can be defined as follows:

$$\mathrm{cleaf}_2(2l) = \frac{1-2(\mathrm{cleaf}_2(l))^2-(\mathrm{cleaf}_2(l))^4}{-1-2(\mathrm{cleaf}_2(l))^2+(\mathrm{cleaf}_2(l))^4} \tag{69}$$

In the case where the basis $n = 3$, the variables $l_1$ and $l_2$ of Eq. (47) are replaced with the variable $l$, and the double-angle formula of the function $\mathrm{sleaf}_3(2l)$ can be expressed as follows:

$$\mathrm{sleaf}_3(2l) = \frac{2\mathrm{sleaf}_3(l)\dfrac{\partial \mathrm{sleaf}_3(l)}{\partial l}}{\sqrt{1+8(\mathrm{sleaf}_3(l))^6}} \tag{70}$$

(i) In the case where $\frac{\pi_3}{2}(4m-1) \leqq l \leqq \frac{\pi_3}{2}(4m+1)$ (see Appendix E for the constant $\pi_3$), Eq. (70) is transformed into:

$$\mathrm{sleaf}_3(2l) = \frac{2\mathrm{sleaf}_3(l)\sqrt{1-(\mathrm{sleaf}_3(l))^6}}{\sqrt{1+8(\mathrm{sleaf}_3(l))^6}} \tag{71}$$

(ii) In the case where $\frac{\pi_3}{2}(4m+1) \leqq l \leqq \frac{\pi_3}{2}(4m+3)$, Eq. (70) is transformed into:

$$\mathrm{sleaf}_3(2l) = -\frac{2\mathrm{sleaf}_3(l)\sqrt{1-(\mathrm{sleaf}_3(l))^6}}{\sqrt{1+8(\mathrm{sleaf}_3(l))^6}} \tag{72}$$

In the case where the basis $n = 3$, the variable $l_1$ and the variable $l_2$ of Eqs. (51)–(52) are replaced with the variable $l$. The double-angle formula of the function $\mathrm{cleaf}_3(2l)$ is then expressed as follows:

$$\mathrm{cleaf}_3(2l) = \frac{2(\mathrm{cleaf}_3(l))^2+2(\mathrm{cleaf}_3(l))^4-1}{\sqrt{1+8(\mathrm{cleaf}_3(l))^2+8(\mathrm{cleaf}_3(l))^6-8(\mathrm{cleaf}_3(l))^8}} \tag{73}$$

### *3.2 Half Angle Formulas of Leaf Functions*

In the case where the basis $n = 2$, the variables $l_1$ and $l_2$ in Eqs. (38)–(41) are replaced with the expression $l/2$, and the half-angle formula is defined as follows:

(i) In the case where $\frac{\pi_2}{2}(4m+1) \leqq l \leqq \frac{\pi_2}{2}(4m+3)$ (see Appendix E for the constant $\pi_2$), the half-angle formula is expressed as follows:

$$\left(\mathrm{sleaf}_2\left(\frac{l}{2}\right)\right)^2 = \frac{-1-\sqrt{1-(\mathrm{sleaf}_2(l))^2}}{(\mathrm{sleaf}_2(l))^2} + \frac{\sqrt{1+(\mathrm{sleaf}_2(l))^2}}{(\mathrm{sleaf}_2(l))^2}\sqrt{2-(\mathrm{sleaf}_2(l))^2+2\sqrt{1-(\mathrm{sleaf}_2(l))^2}} \tag{74}$$

(ii) In the case where $\frac{\pi_2}{2}(4m-1) \leqq l \leqq \frac{\pi_2}{2}(4m+1)$, the half-angle formula is defined as follows:

$$\left(\mathrm{sleaf}_2\left(\frac{l}{2}\right)\right)^2 = \frac{-1+\sqrt{1-(\mathrm{sleaf}_2(l))^2}}{(\mathrm{sleaf}_2(l))^2} + \frac{\sqrt{1+(\mathrm{sleaf}_2(l))^2}}{(\mathrm{sleaf}_2(l))^2}\sqrt{2-(\mathrm{sleaf}_2(l))^2-2\sqrt{1-(\mathrm{sleaf}_2(l))^2}} \tag{75}$$

In the case where the basis $n = 2$, the variables $l_1$ and $l_2$ in Eqs. (43)–(46) are replaced with the expression $l/2$ and the half-angle formula is expressed as follows:

$$\left(\mathrm{cleaf}_2\left(\frac{l}{2}\right)\right)^2 = \frac{-1+\mathrm{cleaf}_2(l)+\sqrt{2}\sqrt{1+(\mathrm{cleaf}_2(l))^2}}{1+\mathrm{cleaf}_2(l)} \tag{76}$$

In the case where the basis $n = 3$, the variables $l_1$ and $l_2$ in Eqs. (47)–(49) are replaced with the expression $l/2$ and the half-angle formula of the function $\mathrm{sleaf}_3(l)$ is defined as follows:

(i) In the case where $\frac{\pi_3}{2}(4m-1) \leqq l \leqq \frac{\pi_3}{2}(4m+1)$ (see Appendix E for the constant $\pi_3$), the half-angle formula is defined as follows:

$$\left(\mathrm{sleaf}_3\left(\frac{l}{2}\right)\right)^2 = -\frac{1}{2}(\mathrm{sleaf}_3(l))^2 + \frac{1}{2}\sqrt{1+(\mathrm{sleaf}_3(l))^2+(\mathrm{sleaf}_3(l))^4} - \frac{1}{2}\sqrt{-1-(\mathrm{sleaf}_3(l))^2+2(\mathrm{sleaf}_3(l))^4+\frac{2-2(\mathrm{sleaf}_3(l))^6}{\sqrt{1+(\mathrm{sleaf}_3(l))^2+(\mathrm{sleaf}_3(l))^4}}} \tag{77}$$

(ii) In the case where $\frac{\pi_3}{2}(4m+1) \leqq l \leqq \frac{\pi_3}{2}(4m+3)$, the half-angle formula is expressed as follows:

$$\left(\mathrm{sleaf}_3\left(\frac{l}{2}\right)\right)^2 = -\frac{1}{2}(\mathrm{sleaf}_3(l))^2 + \frac{1}{2}\sqrt{1+(\mathrm{sleaf}_3(l))^2+(\mathrm{sleaf}_3(l))^4} + \frac{1}{2}\sqrt{-1-(\mathrm{sleaf}_3(l))^2+2(\mathrm{sleaf}_3(l))^4+\frac{2-2(\mathrm{sleaf}_3(l))^6}{\sqrt{1+(\mathrm{sleaf}_3(l))^2+(\mathrm{sleaf}_3(l))^4}}} \tag{78}$$

In the case where the basis $n = 3$, the variables $l_1$ and $l_2$ in Eqs. (51)–(52) are replaced with the expression $l/2$ and the half-angle formula of the function $\mathrm{cleaf}_3\left(\frac{l}{2}\right)$ is defined as follows:

$$\left(\mathrm{cleaf}_3\left(\frac{l}{2}\right)\right)^2=\frac{(\mathrm{cleaf}_3(l))^2-1}{4(\mathrm{cleaf}_3(l))^2+2}+\frac{\sqrt{3}\sqrt{1+(\mathrm{cleaf}_3(l))^2+(\mathrm{cleaf}_3(l))^4}}{2\sqrt{1+4(\mathrm{cleaf}_3(l))^2+4(\mathrm{cleaf}_3(l))^4}}$$
$$+\frac{\sqrt{3}\mathrm{cleaf}_3(l)\sqrt{-3-6(\mathrm{cleaf}_3(l))^2+2\sqrt{3}\{1+2(\mathrm{cleaf}_3(l))^2\}\sqrt{1+(\mathrm{cleaf}_3(l))^2+(\mathrm{cleaf}_3(l))^4}}}{2\{1+2(\mathrm{cleaf}_3(l))^2\}^{\frac{3}{2}}} \tag{79}$$

### *3.3 Double Angle Formulas of Hyperbolic Leaf Functions*

In the case where the basis $n = 2$, the variables $l_1$ and $l_2$ in Eq. (56) are replaced with the variable $l$, and the double-angle formula is defined as follows:

$$\mathrm{sleafh}_2(2l)=\frac{2\mathrm{sleafh}_2(l)\sqrt{1+(\mathrm{sleafh}_2(l))^4}}{1-(\mathrm{sleafh}_2(l))^4} \tag{80}$$

The variables $l_1$ and $l_2$ in Eqs. (58) and (59) are replaced with the variable $l$. The double-angle formula is then defined as follows:

$$\mathrm{cleafh}_2(2l)=\frac{(\mathrm{cleafh}_2(l))^4+2(\mathrm{cleafh}_2(l))^2-1}{-(\mathrm{cleafh}_2(l))^4+2(\mathrm{cleafh}_2(l))^2+1} \tag{81}$$

In the case where the basis $n = 3$, the variables $l_1$ and $l_2$ of Eq. (62) are replaced with the variable $l$, and the double-angle formula of the function $\mathrm{sleafh}_3(2l)$ is defined as follows:

$$\mathrm{sleafh}_3(2l)=\frac{2\mathrm{sleafh}_3(l)\sqrt{1+(\mathrm{sleafh}_3(l))^6}}{\sqrt{1-8(\mathrm{sleafh}_3(l))^6}} \tag{82}$$

In the case where the basis $n = 3$, the variables $l_1$ and $l_2$ of Eqs. (64) and (65) are replaced with the variable $l$, and the double-angle formula of the function $\mathrm{cleafh}_3(2l)$ is defined as follows:

$$\mathrm{cleafh}_3(2l)=\frac{2(\mathrm{cleafh}_3(l))^2+2(\mathrm{cleafh}_3(l))^4-1}{\sqrt{1+8(\mathrm{cleafh}_3(l))^2+8(\mathrm{cleafh}_3(l))^6-8(\mathrm{cleafh}_3(l))^8}} \tag{83}$$

### *3.4 Half Angle Formulas of Hyperbolic Leaf Functions*

In the case where the basis $n = 2$, the variables $l_1$ and $l_2$ in Eq. (56) are replaced with the expression $l/2$, and the half-angle formula is defined as follows:

(i) In the case where $|l| \leqq \zeta_2$ (see Appendix F for the constant $\zeta_2$ and Appendix H for the periodicity $n = 2$), the half-angle formula is expressed as follows:

$$\left(\mathrm{sleafh}_2\left(\frac{l}{2}\right)\right)^2=\frac{1+\sqrt{1+(\mathrm{sleafh}_2(l))^4}}{(\mathrm{sleafh}_2(l))^2}-\frac{\sqrt{2}}{\sqrt{-1+\sqrt{1+(\mathrm{sleafh}_2(l))^4}}} \tag{84}$$

(ii) In the case where $\zeta_2 \leq |l|$, the half-angle formula is defined as follows:

$$\left(\mathrm{sleaf}_2\left(\frac{l}{2}\right)\right)^2 = \frac{1+\sqrt{1+(\mathrm{sleafh}_2(l))^4}}{(\mathrm{sleafh}_2(l))^2} + \frac{\sqrt{2}}{\sqrt{-1+\sqrt{1+(\mathrm{sleafh}_2(l))^4}}} \tag{85}$$

In the case where the basis $n = 2$, the variables $l_1$ and $l_2$ in Eqs. (58) and (59) are replaced with the expression $l/2$, and the half-angle formula can be expressed as follows (see Appendix G for the constant $\eta_2$ and Appendix H for the periodicity $n = 2$):

(i) In the case where $|l| \leqq \eta_2$, the half-angle formula is defined as follows:

$$\left(\mathrm{cleafh}_2\left(\frac{l}{2}\right)\right)^2 = \frac{-1+\mathrm{cleafh}_2(l)+\sqrt{2}\sqrt{1+(\mathrm{cleafh}_2(l))^2}}{1+\mathrm{cleafh}_2(l)} \tag{86}$$

(ii) In the case where $\eta_2 \leqq |l|$, the half-angle formula is defined as follows:

$$\left(\mathrm{cleafh}_2\left(\frac{l}{2}\right)\right)^2 = \frac{-1+\mathrm{cleafh}_2(l)-\sqrt{2}\sqrt{1+(\mathrm{cleafh}_2(l))^2}}{1+\mathrm{cleafh}_2(l)} \tag{87}$$

In the case where the basis $n = 3$, the variables $l_1$ and $l_2$ in Eq. (62) are replaced with the expression $l/2$, and the half-angle formula of the function $\mathrm{sleafh}_3(l)$ is defined as follows:

$$\begin{aligned}\left(\mathrm{sleafh}_3\left(\frac{l}{2}\right)\right)^2 = &-\frac{1}{2}(\mathrm{sleafh}_3(l))^2 - \frac{1}{2}\sqrt{1-(\mathrm{sleafh}_3(l))^2+(\mathrm{sleafh}_3(l))^4} \\ &+\frac{1}{2}\sqrt{-1+(\mathrm{sleafh}_3(l))^2+2(\mathrm{sleafh}_3(l))^4+\frac{2+2(\mathrm{sleafh}_3(l))^6}{\sqrt{1-(\mathrm{sleafh}_3(l))^2+(\mathrm{sleafh}_3(l))^4}}}\end{aligned} \tag{88}$$

In the case where the basis $n = 3$, the variables $l_1$ and $l_2$ in Eqs. (64) and (65) are replaced with the expression $l/2$ and the half-angle formula of the function $\mathrm{cleafh}_3(l)$ is defined as follows:

$$\begin{aligned}\left(\mathrm{cleafh}_3\left(\frac{l}{2}\right)\right)^2 = &\frac{-1+\mathrm{cleafh}_3(l))^2+\sqrt{3}\sqrt{1+\mathrm{cleafh}_3(l))^2+(\mathrm{cleafh}_3(l))^4}}{4(\mathrm{cleafh}_3(l))^2+2} \\ &+\frac{\sqrt{3}\mathrm{cleafh}_3(l)\sqrt{-3-6(\mathrm{cleafh}_3(l))^2+2\sqrt{3}\{1+2(\mathrm{cleafh}_3(l))^2\}\sqrt{1+(\mathrm{cleafh}_3(l))^2+(\mathrm{cleafh}_3(l))^4}}}{2\{1+2\mathrm{cleafh}_3(l))^2\}^{3/2}}\end{aligned} \tag{89}$$

## 4 Numerical Analysis

### *4.1 Numerical Analysis of Leaf Functions*

The curves of the leaf functions $\mathrm{sleaf}_2(l)$ and $\mathrm{cleaf}_2\,(l)$ are shown in Figs. 4 and 5. Numerical data for these two leaf functions are summarized in Tab. 1. These curves are the same curves as those of the lemniscate elliptic functions $r = \mathrm{sl}(l)$ and $r = \mathrm{cl}(l)$. Using the addition formulas of Eqs. (38)–(46), the curves of the leaf functions $\mathrm{sleaf}_2(l)$ and $\mathrm{cleaf}_2(l)$ are translated in the direction of the axis $l$. Fig. 6 shows graphs of the double-angle $\mathrm{sleaf}_2(2l)$ and the half-angle $\mathrm{sleaf}_2(l/2)$ obtained using Eqs. (67)–(68) and Eqs. (74)–(75). Fig. 7 shows

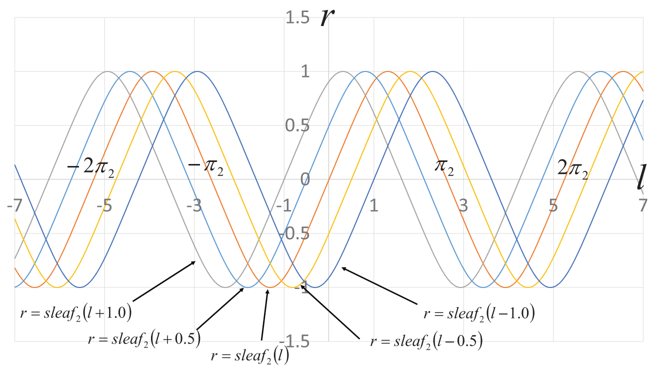


**Figure 4:** Translation of the curves of the function $\mathrm{sleaf}_2(l)$ obtained using the addition formulas with the basis $n = 2$

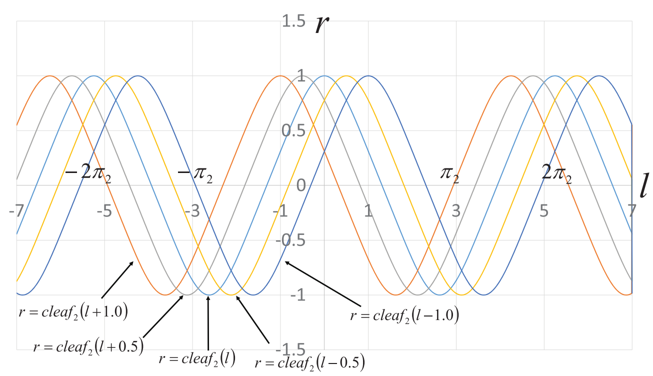


**Figure 5:** Translation of the curves of the function $\mathrm{cleaf}_2(l)$ obtained using the addition formulas with the basis $n = 2$

graphs of the double-angle $\mathrm{cleaf}_2(2l)$ and the half-angle $\mathrm{cleaf}_2(l/2)$ obtained using Eqs. (69) and (76). The amplitude of the wave is 1 and one period of the function $\mathrm{cleaf}_2(l)$ is 2 $\pi_2$ ( = 2 × 2.622 ...).

As shown in Fig. 4, curves $\mathrm{sleaf}_2(l)$ are translated using only the addition theorem, so that the period remains constant at $2\pi_2$. On the contrary, as shown in Fig. 6, the period changes to $\pi_2$ and $4\pi_2$, when the phase becomes $2l$ and $l/2$, respectively. Similarly, as shown in Fig. 5, curves $\mathrm{cleaf}_2(l)$ are translated using only the addition theorem, so that the period remains constant at $2\pi_2$. On the contrary, as shown in Fig. 7, the period changes to $\pi_2$ and $4\pi_2$, when the phase becomes $2l$ and $l/2$, respectively. Additionally, the leaf function can be expressed as the following trigonometric function:

$$\mathrm{sleaf}_n\left(l + \frac{\pi_n}{2}\right) = \mathrm{cleaf}_n(l)\ (n = 1, 2, 3, \cdots) \tag{90}$$

Using the Eq. (90) and with constant $\frac{\pi_2}{2}$, the waves are translated in the direction $l$. The curve shown in the Fig. 6 represents the wave translated in the positive direction $l$, as shown in the Fig. 7. Similarly, the curve shown in the Fig. 4 represents the wave translated in the positive direction $l$, as shown in the Fig. 5.

Next, the graph of the leaf function with the basis $n = 3$ is shown. The curves of the leaf functions $\mathrm{sleaf}_3(l)$ and $\mathrm{cleaf}_3(l)$ are shown in Figs. 8 and 9. The horizontal and vertical axes represent the variables $l$ and $r$, respectively. The numerical data of the leaf functions $\mathrm{sleaf}_3(l)$ and $\mathrm{cleaf}_3(l)$ are summarized in Tab. 1. The curves of the leaf functions $\mathrm{sleaf}_3(l)$ and $\mathrm{cleaf}_3(l)$ are translated in the direction of the axis $l$.

**Table 1:** Numerical data of the leaf functions

| $l$ | $\mathrm{sleaf}_1(l)$ | $\mathrm{cleaf}_1(l)$ | $\mathrm{sleaf}_2(l)$ | $\mathrm{cleaf}_2(l)$ | $\mathrm{sleaf}_3(l)$ | $\mathrm{cleaf}_3(l)$ |
|---|---|---|---|---|---|---|
| 0.0 | 0.000000000 | 1.000000000 | 0.000000000 | 1.000000000 | 0.000000000 | 1.000000000 |
| 0.1 | 0.099833417 | 0.995004165 | 0.099998987 | 0.990049602 | 0.099999991 | 0.98518434 |
| 0.2 | 0.198669331 | 0.980066578 | 0.199967976 | 0.960781145 | 0.199999064 | 0.942809514 |
| 0.3 | 0.295520207 | 0.955336489 | 0.299757126 | 0.913842132 | 0.299984331 | 0.878183695 |
| 0.4 | 0.389418342 | 0.921060994 | 0.398978135 | 0.851676083 | 0.39988294 | 0.797825011 |
| 0.5 | 0.479425539 | 0.877582562 | 0.496891146 | 0.777159391 | 0.499442694 | 0.70763201 |
| 0.6 | 0.564642473 | 0.825335615 | 0.592307034 | 0.693234267 | 0.598009242 | 0.611978813 |
| 0.7 | 0.644217687 | 0.764842187 | 0.683522566 | 0.602609146 | 0.694183101 | 0.513646507 |
| 0.8 | 0.717356091 | 0.696706709 | 0.768312999 | 0.507563306 | 0.785387303 | 0.414175714 |
| 0.9 | 0.78332691 | 0.621609968 | 0.844009686 | 0.409858439 | 0.867486256 | 0.314303714 |
| 1.0 | 0.841470985 | 0.540302306 | 0.90768321 | 0.310738001 | 0.934767593 | 0.214323891 |
| 1.1 | 0.89120736 | 0.453596121 | 0.956432623 | 0.210987025 | 0.980707849 | 0.114325366 |
| 1.2 | 0.932039086 | 0.362357754 | 0.987748032 | 0.111027204 | 0.999692203 | 0.014325392 |
| 1.3 | 0.963558185 | 0.267498829 | 0.999878378 | 0.011028912 | 0.989089542 | –0.085674597 |
| 1.4 | 0.98544973 | 0.169967143 | 0.99211532 | –0.088970511 | 0.950392842 | –0.185674048 |
| 1.5 | 0.997494987 | 0.070737202 | 0.96491412 | –0.188946955 | 0.888559535 | –0.285663493 |
| 1.6 | 0.999573603 | –0.029199522 | 0.919815574 | –0.288769649 | 0.810063642 | –0.385583945 |
| 1.7 | 0.99166481 | –0.128844494 | 0.859192306 | –0.388082304 | 0.720971617 | –0.485219858 |
| 1.8 | 0.973847631 | –0.227202095 | 0.785891649 | –0.486189025 | 0.6258955 | –0.583992736 |
| 1.9 | 0.946300088 | –0.323289567 | 0.702864932 | –0.581954203 | 0.527828311 | –0.680635105 |
| 2.0 | 0.909297427 | –0.416146837 | 0.612857981 | –0.673732946 | 0.428461029 | –0.772765772 |
| 2.1 | 0.863209367 | –0.504846105 | 0.518203565 | –0.759356014 | 0.328621294 | –0.856486525 |
| 2.2 | 0.808496404 | –0.588501117 | 0.420721859 | –0.836196738 | 0.228648563 | –0.92628646 |
| 2.3 | 0.745705212 | –0.666276021 | 0.3217114 | –0.90134206 | 0.128650882 | –0.975673073 |
| 2.4 | 0.675463181 | –0.737393716 | 0.222003575 | –0.951870972 | 0.028650956 | –0.998769949 |
| 2.5 | 0.598472144 | –0.801143616 | 0.122054841 | –0.985211764 | –0.071349009 | –0.992412076 |
| 2.6 | 0.515501372 | –0.856888753 | 0.022057545 | –0.999513456 | –0.171348665 | –0.95749878 |
| 2.7 | 0.42737988 | –0.904072142 | –0.077942171 | –0.993943297 | –0.2713412 | –0.898594215 |
| 2.8 | 0.33498815 | –0.942222341 | –0.177924624 | –0.968828424 | –0.371279371 | –0.822087294 |
| 2.9 | 0.239249329 | –0.970958165 | –0.277776677 | –0.925599649 | –0.470980082 | –0.734191026 |
| 3.0 | 0.141120008 | –0.989992497 | –0.37717265 | –0.866554268 | –0.569933963 | –0.639752776 |

These curves of the leaf functions were obtained using the addition formulas of Eqs. (48)–(49) and Eqs. (51)–(52). Fig. 10 shows graphs of the double-angle $\mathrm{sleaf}_3(2l)$ and the half-angle $\mathrm{sleaf}_3(l/2)$ obtained using Eqs. (71)–(72) and Eqs. (77)–(78). Fig. 11 shows graphs of the double-angle $\mathrm{cleaf}_3(2l)$ and the half-angle $\mathrm{cleaf}_3(l/2)$ obtained using Eqs. (73) and (79). The amplitude of the wave is 1 and one period of the function $\mathrm{cleaf}_3(l)$ is $2\pi_3$ ( $= 2 \times 2.429 \ \ldots$ ).

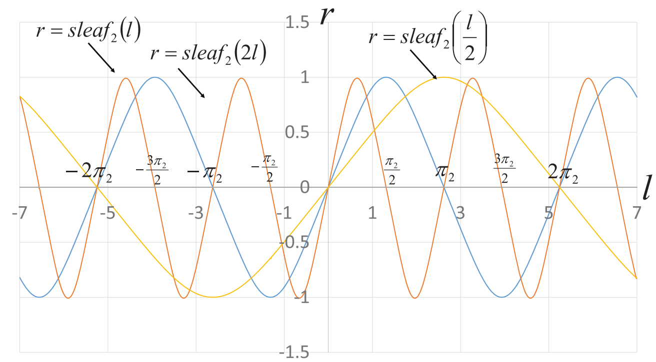


**Figure 6:** Translation of the curves of the functions $\mathrm{sleaf}_2(l)$, $\mathrm{sleaf}_2(2l)$, and $\mathrm{sleaf}_2(l/2)$ obtained using the addition formulas based on the basis $n = 2$

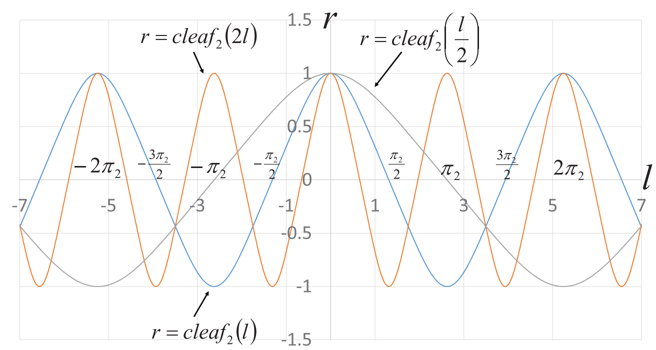


**Figure 7:** Translation of the curves of the functions $\mathrm{cleaf}_2(l)$, $\mathrm{cleaf}_2(2l)$, and $\mathrm{cleaf}_2(l/2)$ obtained using the addition formulas based on the basis $n = 2$

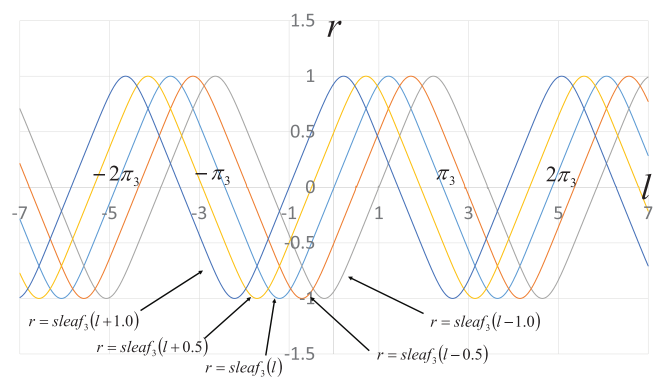


**Figure 8:** Translation of the curves of the function $\mathrm{sleaf}_3(l)$ obtained using the addition formulas with the basis $n = 3$

When the phase is doubled, the period is halved, and vice-versa. Even with a change in the phase, the amplitude remains constant at 1, and initial condition $\mathrm{cleaf}_3(0) = 1$ is maintained at $l = 0$, as confirmed from the graph.

### *4.2 Numerical Analysis of Hyperbolic Leaf Functions*

The curves of the leaf functions $\mathrm{sleafh}_2(l)$ and $\mathrm{cleafh}_2(l)$ are shown in Figs. 12 and 13. The horizontal and vertical axes represent the variables $l$ and $r$. The numerical data for the leaf functions $\mathrm{sleafh}_2(l)$ and $\mathrm{cleafh}_2(l)$

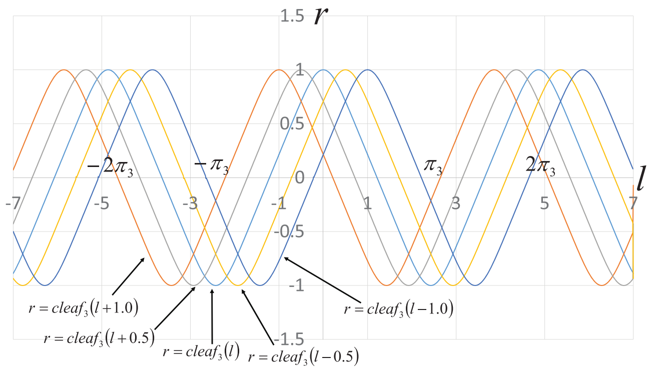


**Figure 9:** Translation of the curves of the function cleaf$_3$($l$) obtained using the addition formulas with the basis $n$ = 3

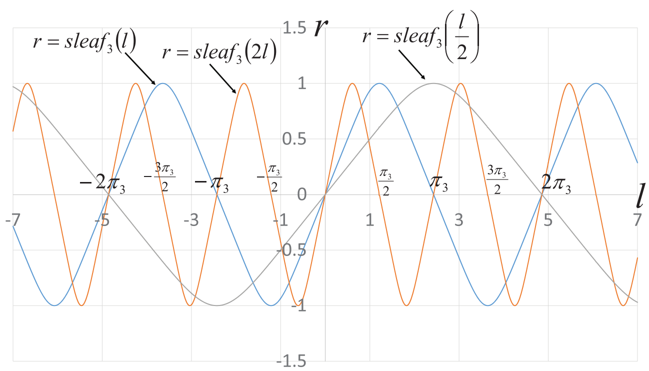


**Figure 10:** Translation of the curves of the functions sleaf$_3$($l$), sleaf$_3$(2$l$), and sleaf$_3$($l$/2) obtained using the addition formulas based on the basis $n$ = 3

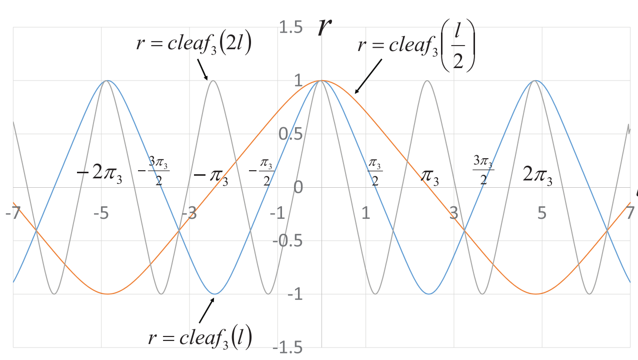


**Figure 11:** Translation of the curves of the functions cleaf$_3$($l$), cleaf$_3$(2$l$), and cleaf$_3$($l$/2) obtained using the addition formulas with the basis $n$ = 3

are summarized in Tab. 2. Using the addition formulas of Eq. (56) and the Eqs. (58)–(59), the curves of the leaf functions sleafh$_2$($l$) and cleafh$_2$($l$) are translated in the direction $l$. Fig. 14 shows graphs of the double-angle sleafh$_2$(2$l$) and the half-angle sleafh$_2$($l$/2) obtained using Eq. (80) and Eqs. (84)–(85). Fig. 15 shows graphs of the double-angle cleafh$_2$(2$l$) and the half-angle cleafh$_2$($l$/2) obtained using Eq. (81) and Eqs. (86)–(87). Limits exist for the functions sleafh$_2$($l$) and cleafh$_2$($l$), respectively (see Appendix F and Appendix G). Next, curves of the leaf functions sleafh$_3$ ($l$) and cleafh$_3$($l$) are shown in Figs. 16 and 17.

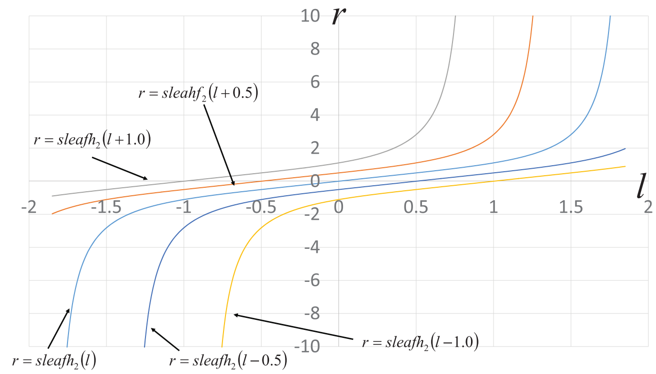


**Figure 12:** Translation of the curves of the function sleafh$_2$($l$) obtained using the addition formulas with the basis $n$ = 2

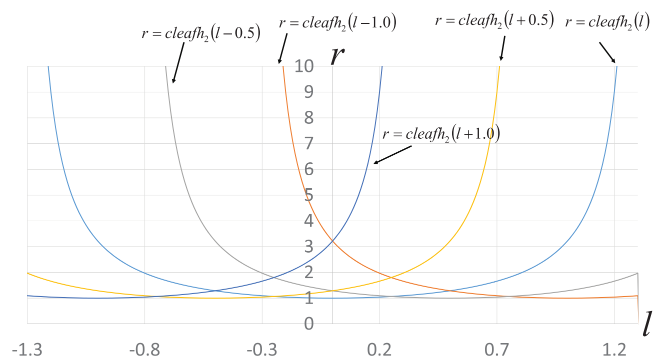


**Figure 13:** Translation of the curves of the function cleafh$_2$($l$) obtained using the addition formulas with the basis $n$ = 2

**Table 2:** Numerical data of the hyperbolic leaf functions

| $l$ | sleafh$_1$($l$) | cleafh$_1$($l$) | sleafh$_2$($l$) | cleafh$_2$($l$) | sleafh$_3$($l$) | cleafh$_3$($l$) |
|---|---|---|---|---|---|---|
| 0.0 | 0.000000000 | 1.000000000 | 0.000000000 | 1.000000000 | 0.000000000 | 1.000000000 |
| 0.1 | 0.10016675 | 1.005004168 | 0.100001013 | 1.010050409 | 0.100000009 | 1.015190873 |
| 0.2 | 0.201336003 | 1.020066756 | 0.200032033 | 1.040819784 | 0.200000936 | 1.063219846 |
| 0.3 | 0.304520293 | 1.045338514 | 0.300243205 | 1.094280966 | 0.300015671 | 1.152957367 |
| 0.4 | 0.410752326 | 1.081072372 | 0.401026247 | 1.174155432 | 0.400117152 | 1.306327433 |
| 0.5 | 0.521095305 | 1.127625965 | 0.503141445 | 1.286737533 | 0.500558986 | 1.583264962 |
| 0.6 | 0.636653582 | 1.185465218 | 0.607861028 | 1.442514133 | 0.6020087 | 2.225120045 |
| 0.7 | 0.758583702 | 1.255169006 | 0.717150413 | 1.659450947 | 0.705950043 | 21.4096535 |
| 0.8 | 0.888105982 | 1.337434946 | 0.833926854 | 1.97019847 | 0.815368602 | – |
| 0.9 | 1.026516726 | 1.433086385 | 0.962467567 | 2.439868366 | 0.936017909 | – |
| 1.0 | 1.175201194 | 1.543080635 | 1.10910404 | 3.218148246 | 1.079143503 | – |
| 1.1 | 1.33564747 | 1.668518554 | 1.283479658 | 4.739635312 | 1.26866512 | – |
| 1.2 | 1.509461355 | 1.810655567 | 1.500980956 | 9.006830737 | 1.566095647 | – |
| 1.3 | 1.698382437 | 1.97091423 | 1.787828613 | 90.67397241 | 2.210887381 | – |
| 1.4 | 1.904301501 | 2.150898465 | 2.192926988 | – | 15.13849028 | – |

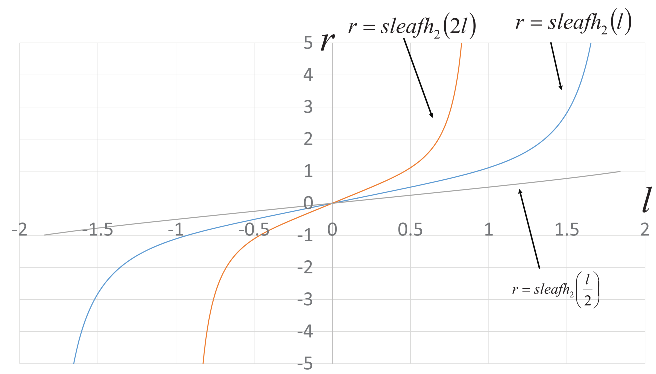


**Figure 14:** Translation of the curves of the functions sleafh$_2$($l$), sleafh$_2$(2$l$) and sleafh$_2$($l$/2) obtained using the addition formulas with the basis $n$ = 2

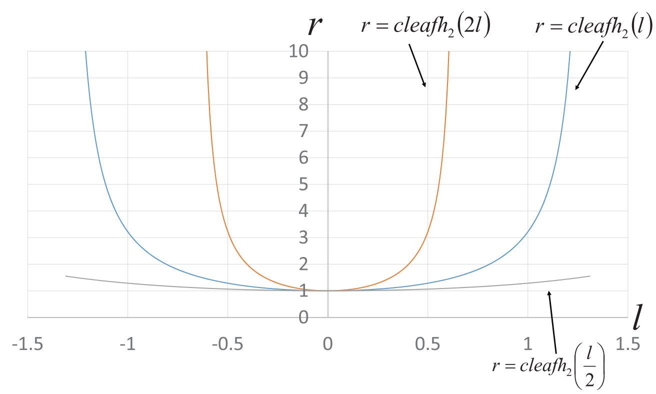


**Figure 15:** Translation of the curves of the functions cleafh$_2$($l$), cleafh$_2$(2$l$), and cleafh$_2$($l$/2) obtained using the addition formulas with the basis $n$ = 2

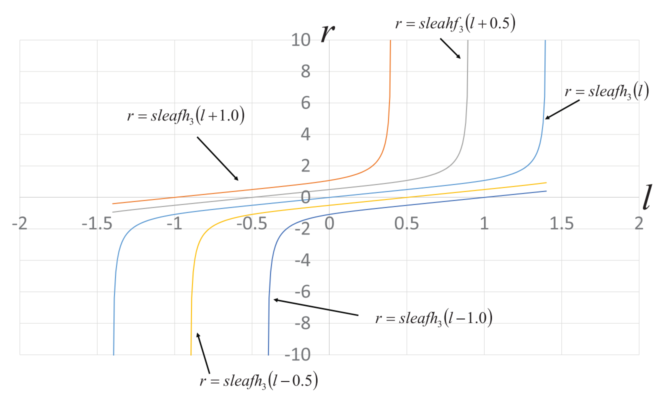


**Figure 16:** Translation of the curves of the function sleafh$_3$($l$) obtained using the addition formulas with the basis $n$ = 3

The horizontal and vertical axes represent the variables $l$ and $r$, respectively. The numerical data of the leaf functions sleafh$_3$($l$) and cleafh$_3$($l$) are summarized in Tab. 2. Using the addition formulas of Eq. (62) and Eqs. (64)–(65), the curves of the leaf functions sleafh$_3$($l$) and cleafh$_3$($l$) are translated in the

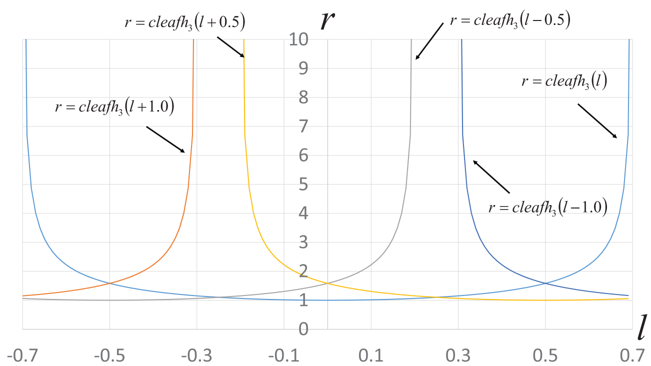


**Figure 17:** Translation of the curves of the functions cleafh$_3$($l$) obtained using the addition formulas with the basis $n$ = 3

direction $l$. Fig. 18 shows graphs of the double-angle sleafh$_3$(2$l$) and the half-angle sleafh$_3$($l$/2) obtained using Eqs. (82) and (88).

Fig. 19 shows graphs of the double-angle cleafh$_3$(2$l$) and the half-angle cleafh$_3$($l$/2) obtained using Eqs (83) and (89). Limits exist in the functions sleafh$_3$($l$) and cleafh$_3$($l$), respectively. For the function sleafh$_3$($l$), the limit exists at $\pm\zeta_3$ (see Appendix F for the constant $\zeta_3$). The curve of the function sleafh$_3$($l$) monotonically increases in the domain $-\zeta_3 < l < \zeta_3$. In the case of the function cleafh$_3$($l$), the limit exists at $\pm\,\eta_3$ (see Appendix G for the constant $\eta_3$). The domain of the function cleafh$_3$($l$) is $-\eta_3 < l < \eta_3$.

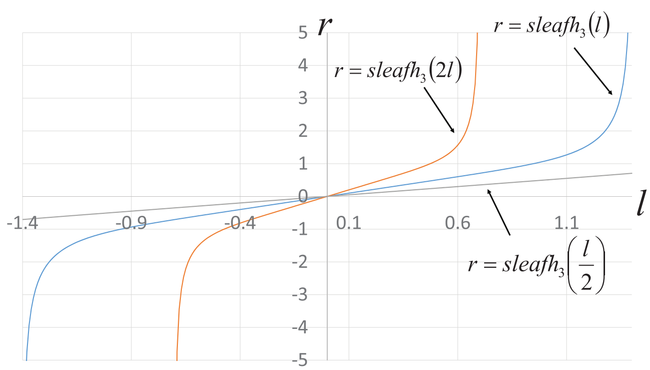


**Figure 18:** Translation of the curves of the functions sleafh$_3$($l$), sleafh$_3$(2$l$), and sleafh$_3$($l$/2) obtained using the addition formulas with the basis $n$ = 3

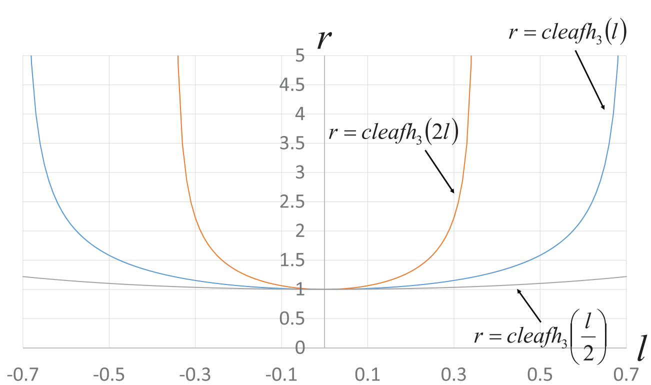


**Figure 19:** Translation of the curves of the functions cleafh$_3$($l$), cleafh$_3$(2$l$), and cleafh$_3$($l$/2) obtained using the addition formulas with the basis $n$ = 3

## 5 Conclusion

Based on the analogy between the trigonometric and hyperbolic function, the hyperbolic leaf function paired with the leaf function was defined. The main conclusions can be summarized as follows:

- The relation equations between the leaf function and the hyperbolic leaf function were derived using imaginary numbers.
- The addition formulas of the hyperbolic leaf function were derived by using addition formulas of the leaf function with the basis $n$ = 1, 2, 3.
- For both the leaf function and hyperbolic leaf function for the basis $n$ = 1, 2, 3, half-angle and double-angle formulas were derived using addition formulas

As a future research topic, we will investigate whether the periodicity of the hyperbolic leaf function exists. In the case where the basis $n = 2$, a limit exists in the hyperbolic function. By appropriately setting the initial conditions, the addition formulas for $n = 2$ can be applied in all domains over the limit. Although the periodicity of the hyperbolic leaf function $n = 2$ is evident, questions remain concerning the periodicity of the hyperbolic leaf function with $n = 3$. In the case where the basis is $n = 3$, a limit also exists for the hyperbolic leaf function. However, the addition formulas of the hyperbolic leaf function cannot be applied outside of its domain. At basis $n = 3$, the periodicity of the hyperbolic leaf function is not observed. Another unaddressed issue is that the addition formulas of the leaf function with the basis $n = 4$ or more are not known.

**Acknowledgement:** The author gratefully acknowledges the helpful comments and suggestions of the reviewers, which have improved the presentation of this paper.

**Funding Statement:** This research is supported by Daido University research grants 2019.

**Conflicts of Interest:** The authors declare that they have no conflicts of interest to report regarding the present study.

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## Appendix A. Relationships of Leaf Functions and Hyperbolic Leaf Functions (n = 1)

The relation equations with the basis $n = 1$ are described. The relation equation between the leaf function $\mathrm{sleaf}_1(l)$ and the leaf function $\mathrm{cleaf}_1(l)$ is as follows:

$$(\mathrm{sleaf}_1(l))^2 + (\mathrm{cleaf}_1(l))^2 = 1 \tag{91}$$

The relation equation between the hyperbolic leaf function $\mathrm{sleafh}_1(l)$ and the hyperbolic leaf function $\mathrm{cleafh}_1(l)$ is as follows:

$$(\mathrm{cleafh}_1(l))^2 - (\mathrm{sleafh}_1(l))^2 = 1 \tag{92}$$

## Appendix B. Relationships of Leaf Functions and Hyperbolic Leaf Functions (n = 2)

The relation equations with the basis $n = 2$ are described. The relation equation between the leaf function $\mathrm{sleaf}_2(l)$ and the leaf function $\mathrm{cleaf}_2(l)$ is as follows [1]:

$$(\mathrm{sleaf}_2(l))^2 + (\mathrm{cleaf}_2(l))^2 + (\mathrm{sleaf}_2(l))^2 \cdot (\mathrm{cleaf}_2(l))^2 = 1 \tag{93}$$

The relation equation between the hyperbolic leaf function $\mathrm{sleafh}_2(l)$ and the hyperbolic leaf function $\mathrm{cleafh}_2(l)$ is as follows [32,33]:

$$\mathrm{cleafh}_2(\sqrt{2}l) = \frac{1 + (\mathrm{sleafh}_2(l))^2}{1 - (\mathrm{sleafh}_2(l))^2} \tag{94}$$

The relation equation between the hyperbolic leaf function $\mathrm{cleaf}_2(l)$ and the hyperbolic leaf function $\mathrm{cleafh}_2(l)$ is as follows:

$$\mathrm{cleaf}_2(l) \cdot \mathrm{cleafh}_2(l) = 1 \tag{95}$$

The relation equation between the hyperbolic leaf function $\mathrm{sleaf}_2(l)$ and the hyperbolic leaf function $\mathrm{sleafh}_2(l)$ is as follows:

$$(\mathrm{sleaf}_2(\sqrt{2}l))^2 = \frac{2(\mathrm{sleafh}_2(l))^2}{1 + (\mathrm{sleafh}_2(l))^4} \tag{96}$$

## Appendix C. Relationships of Leaf Functions and Hyperbolic Leaf Functions (n = 3)

The relation equations with the basis $n = 3$ are described. The relation equation between the leaf functions $\mathrm{sleaf}_3(l)$ and $\mathrm{cleaf}_3(l)$ is as follows [1]:

$$(\mathrm{sleaf}_3(l))^2 + (\mathrm{cleaf}_3(l))^2 + 2(\mathrm{sleaf}_3(l))^2 \cdot (\mathrm{cleaf}_3(l))^2 = 1 \tag{97}$$

The relation equation between the hyperbolic leaf functions $\text{sleafh}_3(l)$ and $\text{cleafh}_3(l)$ is as follows [32,33]:

$$(\text{cleafh}_3(l))^2 - (\text{sleafh}_3(l))^2 - 2(\text{sleafh}_3(l))^2 \cdot (\text{cleafh}_3(l))^2 = 1 \tag{98}$$

**Appendix D. Relationships between Leaf Functions and Hyperbolic Functions**

Using the imaginary number, the relations between the leaf function and hyperbolic leaf function are described in the works [32,33]. To derive the relation between these two functions, the following equation is defined:

$$r = i \cdot u \tag{99}$$

The symbol $i$ represents the imaginary number. Substituting the preceding equation yields the following:

$$l = \int_0^{i \cdot u} \frac{\mathrm{d}t}{\sqrt{1 - t^{2n}}} (= \text{arcsleaf}_n(i \cdot u)) \tag{100}$$

Here, the parameter $t$ is replaced with $i \cdot \xi$ $(t = i \cdot \xi)$. In the case where $t = 0$, $\xi$ is zero. In the case where $t = i \cdot u$, $\xi$ is $u$. Thus, the following equation is obtained:

$$l = \int_0^{u} \frac{i \cdot \mathrm{d}\xi}{\sqrt{1 - (i \cdot \xi)^{2n}}} = i \cdot \int_0^{u} \frac{\mathrm{d}\xi}{\sqrt{1 - i^{2n} \cdot \xi^{2n}}} \tag{101}$$

Let $n$ be an odd number, that is, $n = 2m - 1$ $(m = 1, 2, 3, \ldots)$. The following equation is then obtained:

$$l = i \cdot \int_0^{u} \frac{\mathrm{d}\xi}{\sqrt{1 - i^{2n} \cdot \xi^{2n}}} = i \cdot \int_0^{u} \frac{\mathrm{d}\xi}{\sqrt{1 + \xi^{2n}}} = i \cdot \text{asleafh}_n(u) \tag{102}$$

The following equation is obtained based on the preceding equation as follows:

$$\text{sleafh}_n\left(\frac{l}{i}\right) = u \tag{103}$$

$$\text{sleafh}_n(-i \cdot l) = u \tag{104}$$

Here, the leaf function $\text{sleafh}_n(l)$ has the following relation [33]:

$$\text{sleafh}_n(-l) = -\text{sleafh}_n(l) \tag{105}$$

Eq. (103) can be transformed as follows:

$$-\text{sleafh}_n(i \cdot l) = u \tag{106}$$

The following equation is obtained using Eqs. (100) and (106):

$$\text{sleaf}_n(l) = -i \cdot \text{sleafh}_n(i \cdot l) \tag{107}$$

Next, let us consider the case where $n$ is an even number. In the case where $n = 2m$ $(m = 1, 2, 3 \ldots)$, the following equation is obtained:

$$l = i \cdot \int_0^{u} \frac{\mathrm{d}\xi}{\sqrt{1 - i^{2n} \cdot \xi^{2n}}} = i \cdot \int_0^{u} \frac{\mathrm{d}\xi}{\sqrt{1 - \xi^{2n}}} = i \cdot \text{arcsleaf}_n(u) \tag{108}$$

The following equation is obtained:

$$\mathrm{sleaf}_n\left(\frac{l}{i}\right) = u \tag{109}$$

$$\mathrm{sleaf}_n(-i \cdot l) = u \tag{110}$$

Here, the leaf function $\mathrm{sleaf}_n(l)$ has the following relation [2]:

$$\mathrm{sleaf}_n(-l) = -\mathrm{sleaf}_n(l) \tag{111}$$

Eq. (110) can be expressed as follows:

$$-\mathrm{sleaf}_n(i \cdot l) = u \tag{112}$$

The following equation is obtained using Eqs. (100) and (112):

$$\mathrm{sleaf}_n(l) = -i \cdot \mathrm{sleaf}_n(i \cdot l) \tag{113}$$

In the case where $n$ is an even number, the following equation is also derived:

$$\mathrm{sleafh}_n(l) = -i \cdot \mathrm{sleafh}_n(i \cdot l) \tag{114}$$

Next, the equation can be transformed as follows:

$$l = \int_1^r \frac{\mathrm{d}t}{\sqrt{t^{2n} - 1}} = \int_1^r \frac{\mathrm{d}t}{i\sqrt{1 - t^{2n}}} = \frac{1}{i} \cdot \int_1^r \frac{\mathrm{d}t}{\sqrt{1 - t^{2n}}} = \frac{1}{i} \mathrm{arccleaf}_n(r) \tag{115}$$

The following equation is obtained by the Eq. (115):

$$r = \mathrm{cleaf}_n(i \cdot l) \tag{116}$$

The following equation is also obtained by the Eq. (115):

$$r = \mathrm{cleafh}_n(l) \tag{117}$$

The following equation is obtained using Eqs. (116) and (117):

$$\mathrm{cleaf}_n(i \cdot l) = \mathrm{cleafh}_n(l) \tag{118}$$

Alternatively, the following equation is obtained by substituting $i \cdot l$ into $l$:

$$\mathrm{cleaf}_n(l) = \mathrm{cleafh}_n(i \cdot l) \tag{119}$$

In the preceding equation, the following equation is applied:

$$\mathrm{cleaf}_n(l) = \mathrm{cleaf}_n(-l) \tag{120}$$

**Appendix E. Periods of Leaf Functions**

The constants $\pi_n$ are defined as follows [1,2]:

$$\pi_n = 2\int_0^1 \frac{1}{\sqrt{1 - t^{2n}}} \mathrm{d}t (n = 1, 2, 3 \cdots) \tag{121}$$

In the case where $n = 1$, the constant $\pi_1$ represents the circular constant $\pi$. The constants $\pi_n$ $(n = 1, 2, 3\ \ldots)$ are summarized in Tab. 3.

**Table 3:** Values of constants $\pi_n$

| $n$ | $\pi_n$ |
|---|---|
| 1 | 3.1415 ... |
| 2 | 2.6220 ... |
| 3 | 2.4286 ... |
| ... | ... |

### Appendix F. Limits of Hyperbolic Leaf Functions *sleafh$_n$(l)*

Except for the basis $n = 1$, the limit of the variable $l$ exists in the hyperbolic leaf function sleafh$_n$($l$) [33]. The limit with the basis $n$ is defined as $\zeta_n$. The limit $\zeta_n$ is obtained by the following equation:

$$\zeta_n = \int_0^{\infty} \frac{1}{\sqrt{1+t^{2n}}} \mathrm{d}t (n = 2, 3 \cdots) \tag{122}$$

The constants $\zeta_n$ ($n$ = 2, 3 ...) are summarized in Tab. 4.

**Table 4:** Values of constants $\pi_n$

| $n$ | $\pi_n$ |
|---|---|
| 1 | Not applicable ... |
| 2 | 1.8540 ... |
| 3 | 1.4021 ... |
| ... | ... |

### Appendix G. Limits of Hyperbolic Leaf Functions *cleafh$_n$(l)*

Except for the basis $n = 1$, the limit of the variable $l$ exists in the hyperbolic leaf function cleafh$_n$($l$) [32]. The limit with the basis $n$ is defined as $\eta_n$. The limit $\eta_n$ is obtained using the following equation:

$$\eta_n = \int_1^{\infty} \frac{1}{\sqrt{t^{2n}-1}} \mathrm{d}t (n = 2, 3 \cdots) \tag{123}$$

The constants $\eta_n$ ($n$ = 2, 3 ...) are summarized in Tab. 5.

**Table 5:** Limits $\eta_n$ of the hyperbolic leaf function cleafh$_n$($l$)

| $n$ | $\eta_n$ |
|---|---|
| 1 | Not applicable ... |
| 2 | 1.31102 ... |
| 3 | 0.70109 ... |
| ... | ... |

### Appendix H. Periods of Hyperbolic Leaf Functions

The function sleafh$_n$($l$) and cleafh$_n$($l$) have limits. The domains of the variable $l$ are defined as Eqs. (8) and (12), respectively. Therefore, the values of the hyperbolic leaf function cannot be defined under the domain $|l| > \zeta_n$ in the function $\mathrm{sleafh}_n(l)$ or $|l| > \eta_n$ in the function $\mathrm{cleafh}_n(l)$. In the case where $n = 1$, the limits do not exist in the hyperbolic leaf function as $\mathrm{sleafh}_1(l)$ and $\mathrm{cleafh}_1(l)$ represent $\sinh(l)$ and $\cosh(l)$, respectively. In the case where $n = 2$ ($\mathrm{sleafh}_2(l)$ and $\mathrm{cleafh}_2(l)$), the initial values of the variables $r(0)$ and $dr(0)/dt$ are defined by Eqs. (9) and (10), or Eqs. (13) and (14). The initial values in the function $\mathrm{sleafh}_2(l)$ are redefined as follows:

$$r(2m\zeta_2) = \mathrm{sleafh}_2(2m\zeta_2) = 0 \tag{124}$$

$$\frac{\mathrm{d}r(2m\zeta_2)}{\mathrm{d}l} = \frac{\mathrm{d}}{\mathrm{d}l}\mathrm{sleafh}_2(2m\zeta_2) = 1 \tag{125}$$

The initial values of the function $\mathrm{cleafh}_2(l)$ are redefined as follows:

$$r(4m\eta_2) = \mathrm{cleafh}_2(4m\eta_2) = 1 \tag{126}$$

$$r((4m-2)\eta_2) = \mathrm{cleafh}_2((4m-2)\eta_2) = -1 \tag{127}$$

$$\frac{\mathrm{d}r(2m\eta_2)}{\mathrm{d}l} = \frac{\mathrm{d}}{\mathrm{d}l}\mathrm{cleafh}_2(2m\eta_2) = 0 \tag{128}$$

The variable $m$ represents an integer. The graph based on these definitions is shown in Fig. 20 ($\mathrm{sleafh}_2(l)$) and Fig. 21 ($\mathrm{cleafh}_2(l)$ ), respectively. Such definitions are consistent for all the formulas such as the addition, double-angle, and half-angle formulas. These formulas work under all domains. In the case $n = 2$, the hyperbolic leaf functions can be extended for all domains. In the case where $n = 3$ in the hyperbolic leaf function, the addition, double-angle, and half-angle formulas do not work in the domain $|l| > \zeta_3$ of $|l| >$

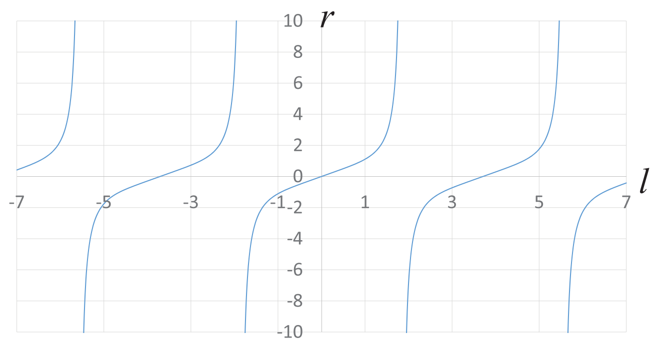


**Figure 20:** Curves of the extended hyperbolic leaf function $\mathrm{sleafh}_2(l)$ for the initial conditions: Eqs. (124) and (125)

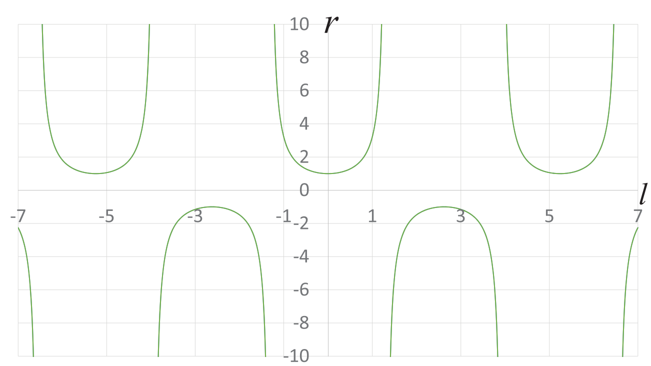


**Figure 21:** Curves of the extended hyperbolic leaf function $\mathrm{cleafh}_2(l)$ for the initial conditions: Eqs. (126)–(128)

$\eta_3$, even if the initial conditions are defined by equations such as $r(2m\,\zeta_3)$ = $\mathrm{sleafh}_3(2m\,\zeta_3)$ = 0. In the case where $n \geqq 3$, the values of $\mathrm{sleafh}_n(l)$ and $\mathrm{cleafh}_n$ are unknown for the domain $|l| > \zeta_n$ of $|l| > \eta_n$.

**Appendix I.**

Eq. (47) is set as follows:

$$\{g(l_1, l_2)\}^2 = \frac{\{p_1(l_1, l_2)\}^2}{p_3(l_1, l_2)} + \frac{\{p_2(l_1, l_2)\}^2}{p_3(l_1, l_2)} \tag{129}$$

$$g(l_1, l_2) = \mathrm{sleaf}_3(l_1 + l_2) \tag{130}$$

$$p_1(l_1, l_2) = \mathrm{sleaf}_3(l_1)\frac{\partial \mathrm{sleaf}_3(l_2)}{\partial l_2} + \mathrm{sleaf}_3(l_2)\frac{\partial \mathrm{sleaf}_3(l_1)}{\partial l_1} \tag{131}$$

$$p_2(l_1, l_2) = (\mathrm{sleaf}_3(l_1))^3\mathrm{sleaf}_3(l_2) - \mathrm{sleaf}_3(l_1)(\mathrm{sleaf}_3(l_2))^3 \tag{132}$$

$$p_3(l_1, l_2) = 1 + 4(\mathrm{sleaf}_3(l_1))^4(\mathrm{sleaf}_3(l_2))^2 + 4(\mathrm{sleaf}_3(l_1))^2(\mathrm{sleaf}_3(l_2))^4 \tag{133}$$

The following equations are obtained by differentiating with respect to variable $l_1$:

$$\frac{\partial p_1(l_1, l_2)}{\partial l_1} = \frac{\partial \mathrm{sleaf}_3(l_1)}{\partial l_1}\frac{\partial \mathrm{sleaf}_3(l_2)}{\partial l_2} - 3\mathrm{sleaf}_3(l_2)(\mathrm{sleaf}_3(l_1))^5 \tag{134}$$

$$\frac{\partial p_2(l_1, l_2)}{\partial l_1} = 3(\mathrm{sleaf}_3(l_1))^2\mathrm{sleaf}_3(l_2)\frac{\partial \mathrm{sleaf}_3(l_1)}{\partial l_1} - (\mathrm{sleaf}_3(l_2))^3\frac{\partial \mathrm{sleaf}_3(l_1)}{\partial l_1} \tag{135}$$

$$\frac{\partial p_3(l_1, l_2)}{\partial l_1} = 16(\mathrm{sleaf}_3(l_1))^3(\mathrm{sleaf}_3(l_2))^2\frac{\partial \mathrm{sleaf}_3(l_1)}{\partial l_1} + 8\mathrm{sleaf}_3(l_1)(\mathrm{sleaf}_3(l_2))^4\frac{\partial \mathrm{sleaf}_3(l_1)}{\partial l_1} \tag{136}$$

The following equations are obtained by differentiating with respect to variable $l_2$:

$$\frac{\partial p_1(l_1, l_2)}{\partial l_2} = \frac{\partial \mathrm{sleaf}_3(l_1)}{\partial l_1}\frac{\partial \mathrm{sleaf}_3(l_2)}{\partial l_2} - 3\mathrm{sleaf}_3(l_1)(\mathrm{sleaf}_3(l_2))^5 \tag{137}$$

$$\frac{\partial p_2(l_1, l_2)}{\partial l_2} = -3(\mathrm{sleaf}_3(l_2))^2\mathrm{sleaf}_3(l_1)\frac{\partial \mathrm{sleaf}_3(l_2)}{\partial l_2} + (\mathrm{sleaf}_3(l_1))^3\frac{\partial \mathrm{sleaf}_3(l_2)}{\partial l_2} \tag{138}$$

$$\frac{\partial p_3(l_1, l_2)}{\partial l_2} = 16(\mathrm{sleaf}_3(l_2))^3(\mathrm{sleaf}_3(l_1))^2\frac{\partial \mathrm{sleaf}_3(l_2)}{\partial l_2} + 8\mathrm{sleaf}_3(l_2)(\mathrm{sleaf}_3(l_1))^4\frac{\partial \mathrm{sleaf}_3(l_2)}{\partial l_2} \tag{139}$$

Using Eq. (129), the following equations are obtained by differentiating with respect to variable $l_1$:

$$\frac{\partial g(l_1, l_2)}{\partial l_1} = \frac{\left(2p_1(l_1, l_2)\frac{\partial p_1(l_1, l_2)}{\partial l_1} + 2p_2(l_1, l_2)\frac{\partial p_2(l_1, l_2)}{\partial l_1}\right)p_3(l_1, l_2)}{2g(l_1, l_2)p_3(l_1, l_2)^2} - \frac{(p_1(l_1, l_2)^2 + p_2(l_1, l_2)^2)\frac{\partial p_3(l_1, l_2)}{\partial l_1}}{2g(l_1, l_2)p_3(l_1, l_2)^2} \tag{140}$$

Using Eqs. (134)–(136), the numerator in the Eq. (140) is expanded as:

$$
\begin{aligned}
&(2p_1(l_1,l_2)\frac{\partial p_1(l_1,l_2)}{\partial l_1}+2p_2(l_1,l_2)\frac{\partial p_2(l_1,l_2)}{\partial l_1})p_3(l_1,l_2)\\
&\quad-(p_1(l_1,l_2)^2+p_2(l_1,l_2)^2)\frac{\partial p_3(l_1,l_2)}{\partial l_1}\\
&\quad=(2\mathrm{sleaf}_3(l_1)-8(\mathrm{sleaf}_3(l_1))^5(\mathrm{sleaf}_3(l_2))^2-24(\mathrm{sleaf}_3(l_1))^3(\mathrm{sleaf}_3(l_2))^4\\
&\quad-8\mathrm{sleaf}_3(l_1)(\mathrm{sleaf}_3(l_2))^6-16(\mathrm{sleaf}_3(l_1))^5(\mathrm{sleaf}_3(l_2))^8)\frac{\partial \mathrm{sleaf}_3(l_1)}{\partial l_1}\\
&\quad+(2\mathrm{sleaf}_3(l_2)-8(\mathrm{sleaf}_3(l_1))^5(\mathrm{sleaf}_3(l_2))^2-24(\mathrm{sleaf}_3(l_1))^4(\mathrm{sleaf}_3(l_2))^3\\
&\quad-8(\mathrm{sleaf}_3(l_1))^6\mathrm{sleaf}_3(l_2)-16(\mathrm{sleaf}_3(l_1))^8(\mathrm{sleaf}_3(l_2))^5)\frac{\partial \mathrm{sleaf}_3(l_2)}{\partial l_2}
\end{aligned}
\tag{141}
$$

Using the Eq. (129), the following equation is obtained by differentiating with respect to the variable $l_2$:

$$
\begin{aligned}
\frac{\partial g(l_1,l_2)}{\partial l_2}=&\frac{\left(2p_1(l_1,l_2)\frac{\partial p_1(l_1,l_2)}{\partial l_2}+2p_2(l_1,l_2)\frac{\partial p_2(l_1,l_2)}{\partial l_2}\right)p_3(l_1,l_2)}{2g(l_1,l_2)p_3(l_1,l_2)^2}\\
&-\frac{(p_1(l_1,l_2)^2+p_2(l_1,l_2)^2)\frac{\partial p_3(l_1,l_2)}{\partial l_2}}{2g(l_1,l_2)p_3(l_1,l_2)^2}
\end{aligned}
\tag{142}
$$

Using Eqs. (137)–(139), the numerator in the Eq. (142) is expanded to obtain the following relation:

$$
\frac{\partial g(l_1,l_2)}{\partial l_1}=\frac{\partial g(l_1,l_2)}{\partial l_2}
\tag{143}
$$

The following equation is derived from the Eq. (143) (see Appendix J).

$$
g(l_1,l_2)=g(l_1+l_2,0)
\tag{144}
$$

Using the initial condition $\mathrm{sleaf}_3(0)=0$ and $\frac{\partial \mathrm{sleaf}_3(0)}{\partial l}=1$, the function $g(l_1+l_2,0)$ is obtained as follows:

$$
\{g(l_1+l_2,0)\}^2=\frac{\{p_1(l_1+l_2,0)\}^2}{p_3(l_1+l_2,0)}+\frac{\{p_2(l_1+l_2,0)\}^2}{p_3(l_1+l_2,0)}=(\mathrm{sleaf}_3(l_1+l_2))^2
\tag{145}
$$

Using Eqs. (129), (144) and (145), the following equation is obtained.

$$
\begin{aligned}
&(\mathrm{sleaf}_3(l_1+l_2))^2=\{g(l_1+l_2,0)\}^2=\{g(l_1,l_2)\}^2\\
&=\frac{\left\{\mathrm{sleaf}_3(l_1)\frac{\partial \mathrm{sleaf}_3(l_2)}{\partial l_2}+\mathrm{sleaf}_3(l_2)\frac{\partial \mathrm{sleaf}_3(l_1)}{\partial l_1}\right\}^2}{1+4(\mathrm{sleaf}_3(l_1))^4(\mathrm{sleaf}_3(l_2))^2+4(\mathrm{sleaf}_3(l_1))^2(\mathrm{sleaf}_3(l_2))^4}\\
&+\frac{\left\{(\mathrm{sleaf}_3(l_1))^3\mathrm{sleaf}_3(l_2)-\mathrm{sleaf}_3(l_1)(\mathrm{sleaf}_3(l_2))^3\right\}^2}{1+4(\mathrm{sleaf}_3(l_1))^4(\mathrm{sleaf}_3(l_2))^2+4(\mathrm{sleaf}_3(l_1))^2(\mathrm{sleaf}_3(l_2))^4}
\end{aligned}
\tag{146}
$$

**Appendix J.**

The necessary and sufficient condition to satisfy $g(l_1,l_2) = g(l_1 + l_2,0)$ is that $\frac{\partial g(l_1,l_2)}{\partial l_1} = \frac{\partial g(l_1,l_2)}{\partial l_2}$ holds. Function $h(x,y)$ is defined as follows.

$$h(x,y) = g(x+y, x-y) \tag{147}$$

$$l_1 = x+y \tag{148}$$

$$l_2 = x-y \tag{149}$$

By differentiating the Eq. (147) equation with respect to $y$, the following equation is obtained.

$$\begin{aligned}\frac{\partial h(x,y)}{\partial y} &= \frac{\partial g(x+y,x-y)}{\partial l_1}\frac{\partial l_1}{\partial y} + \frac{\partial g(x+y,x-y)}{\partial l_2}\frac{\partial l_2}{\partial y} \\ &= \frac{\partial g(x+y,x-y)}{\partial l_1} - \frac{\partial g(x+y,x-y)}{\partial l_2}\end{aligned} \tag{150}$$

Therefore, if the equation $\frac{\partial g}{\partial l_1} = \frac{\partial g}{\partial l_2}$ holds, the following equation holds.

$$\frac{\partial h(x,y)}{\partial y} = 0 \tag{151}$$

Using the Eq. (151), we find that $h(x,y)$ is a function of $x$ and not of $y$. Therefore, the following equation holds for any constant $a$ and $b$:

$$h(x,a) = h(x,b) \tag{152}$$

Here, we set the following equation:

$$x = b = \frac{l_1 + l_2}{2} = 0 \tag{153}$$

$$a = \frac{l_1 - l_2}{2} = 0 \tag{154}$$

The following equation is obtained by using the Eqs. (150), (153), and (154):

$$h(x,a) = h\left(\frac{l_1+l_2}{2}, \frac{l_1-l_2}{2}\right) = g\left(\frac{l_1+l_2}{2} + \frac{l_1-l_2}{2}, \frac{l_1+l_2}{2} - \frac{l_1-l_2}{2}\right) = g(l_1, l_2) \tag{155}$$

$$h(x,b) = h\left(\frac{l_1+l_2}{2}, \frac{l_1+l_2}{2}\right) = g\left(\frac{l_1+l_2}{2} + \frac{l_1+l_2}{2}, \frac{l_1+l_2}{2} - \frac{l_1+l_2}{2}\right) = g(l_1+l_2, 0) \tag{156}$$

The following equation is obtained by using the Eqs. (152), (155) and (156).

$$g(l_1, l_2) = g(l_1 + l_2, 0) \tag{157}$$

Conversely, if the Eq. (157) holds, the following relational expression can be obtained by using Eqs. (147) and (157).

$$h(x,y) = g(x+y, x-y) = g(2x, 0) \tag{158}$$

Eq. (158) is differentiated with respect to variable $y$ to obtain the following equation:

$$\frac{\partial h(x,y)}{\partial y}=\frac{\partial g(2x,0)}{\partial y}=0 \tag{159}$$

Further, the following equation is obtained by using the Eq. (150).

$$\frac{\partial g(x+y,x-y)}{\partial l_1}=\frac{\partial g(x+y,x-y)}{\partial l_2} \tag{160}$$

Using the Eqs. (148) and (149), the following equation is obtained.

$$\frac{\partial g(l_1,l_2)}{\partial l_1}=\frac{\partial g(l_1,l_2)}{\partial l_2} \tag{161}$$