Residual Correction Procedure with Bernstein Polynomials for Solving Important Systems of Ordinary Differential Equations

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Abstract: One of the most attractive subjects in applied sciences is to obtain exact or approximate solutions for different types of linear and nonlinear systems. Systems of ordinary differential equations like systems of second-order boundary value problems (BVPs), Brusselator system and stiff system are significant in science and engineering. One of the most challenge problems in applied science is to construct methods to approximate solutions of such systems of differential equations which pose great challenges for numerical simulations. Bernstein polynomials method with residual correction procedure is used to treat those challenges. The aim of this paper is to present a technique to approximate solutions of such differential equations in optimal way. In it, we introduce a method called residual correction procedure, to correct some previous approximate solutions for such systems. We study the error analysis of our given method. We first introduce a new result to approximate the absolute solution by using the residual correction procedure. Second, we introduce a new result to get appropriate bound for the absolute error. The collocation method is used and the collocation points can be found by applying Chebyshev roots. Both techniques are explained briefly with illustrative examples to demonstrate the applicability, efficiency and accuracy of the techniques. By using a small number of Bernstein polynomials and correction procedure we achieve some significant results. We present some examples to show the efficiency of our method by comparing the solution of such problems obtained by our method with the solution obtained by Runge-Kutta method, continuous genetic algorithm, rational homotopy perturbation method and adomian decomposition method.

Keywords: Bernstein polynomials, residual correction, Runge-Kutta method, stiff system.

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1 Introduction

Systems of ordinary differential equations have been studied in many areas of science such as applied mathematics, engineering and physics. For example, systems of second-order boundary value problems (BVPs) have been employed to describe a variety of systems in applied sciences, engineering, theoretical physics and biology. Several techniques are used to solve such systems. Momani et al. [Momani, Abu Arqub and Abu Hammour (2014)] used the continuous genetic algorithm with convergence analysis to solve some nonlinear systems of second-order boundary value problems. The nonexistence of spurious solutions is applied to discrete two-point BVPs, see in Thompson et al. [Thompson and Tisdell (2003)]. Some authors applied the B-spline method to solve some linear systems of second-order BVPs, for instance see in Caglar et al. [Caglar (2009); Manni, Reali and Speleers (2015)]. New method based on homotopy perturbation method, called reproducing kernel method (RKM), is used to find the solutions of some nonlinear systems of second order boundary value problems (BVPs), see in Geng et al. [Geng and Cui (2011)]. Artificial neural network methods are used to find the solution of second order boundary value problems, see in Anitescu et al. [Anitescu, Atroshchenko, Alajlan et al. (2019)].

The Brusselator system is a fundamental model that displays Biological and Chemical oscillations. Adomian decomposition method (ADM) is used to solve the Brusselator system [Ayati and Biazar (2007)].

The stiff systems are considered to be the most important systems of equations that have been studied in many areas of science, chemistry and physics. In 1980, Cash [Cash (1980)] used extended backward differentiation formula with its modifications to solve stiff systems of ODEs. While, Hosseini et al. [Hosseini and Hojjati (1999)] developed EBDF and BDF methods to new one, called an adaptive method (A-EBDF) and they employed their new method to solve stiff system. Also, in Biazar et al. [Biazar, Ali and Salehi (2015)] introduced a new modification of the homotopy perturbation method, called rational homotopy perturbation method (RHPM), they used their new method to solve some stiff systems of ordinary differential equations. Recentely, in Alshbool et al. [Alshbool and Hashim (2016)] considered a modification of Bernstein polynomials method, called Multistage Bernstein polynomials, to solve some fractional-order stiff system.

Bernstein polynomials method (B-polynomials) is one of the most important methods that can be used to solve linear and non-linear differential equations, for some works in this topics, refer to Bhatti et al. [Bhatti and Bracken (2007); Alshbool, Bataineh, Hashim et al. (2015); Isik and Sezer (2013); Khataybeh, Hashim and Alshbool (2015)]. B-polynomials and the residual correction procedure are used to correct the solution of system differential equations. Both techniques hve attracted the attention of many researchers. B-polynomials with the residual correction procedure are used to solve a class of Lane-Emden type equations, see in Isik et al. [Isik and Sezer (2013)]. Alshbool et al. [Alshbool, Bataineh, Hashim et al. (2017)] used Caputo fractional derivative to obtain fractional Bernstein polynomials and converted $x \to x^{\alpha}$ in the operational matrices of Bernstein polynomials to solve some fractional-order differential equations. Isik et al. [Isik, Sezer and Guney (2012)] applied Bernstein to solve some linear second-order partial differential equations and error analysis. Normalized B-polynomials are presented to find the solution of the space-time fractional diffusion equation, see Baseri et al. [Baseri, Babolian and Abbasbandy (2017)], the authors have used normalized Bernstein polynomials. For the space domain, which is a semi-infinite domain. Recently, multidimensional B-polynomials and their applications are investigated by generating functions [Simsek (2018)].

The motivation underlying this work is to provide an efficient method to reduce the error in solving the systems of differential equations. The Bernstein polynomials method with residual correction procedure is applied to achieve our purpose. The problems will be solved for different values of m and n. In the examples, we investigate several systems of ordinary differential equations as systems of second-order BVPs, the Brusselator system and stiff system. The results will demonstrate the efficiency and accuracy of the techniques. We have been able to reduce the error in solving the systems to the point of reaching the exact solution in some results.

This paper is structured as follows: In Section 2, we describe our proposed method in details to give approximate solutions of such ordinary differential equations. In Section 3, we introduce two corollaries to give approximate solutions of such ordinary differential equations by minimizing the absolute error as much as we can. In Section 4, we obtain approximate solution of some known boundary value problems by using our technique and we compare our solution with some old methods to show the efficiency of our technique. In Section 5, we give a conclusion on our paper.

2 Description of the proposed method

The $(m+1)$ B-polynomials of degree m are defined by

$$B_{i,m}(x) = \binom{m}{i} x^i (1-x)^{m-i}, \quad i = 0,1,\dots,m$$

where the binomial coefficient is

$$\binom{m}{i} = \frac{m!}{i!\,(m-i)!}.$$

For mathematical convenience, the equation $B_{i,m} = 0$ if $i < 0$ or $i > m$.

In general, any function $y(x)$ with the first $(m+1)$ B-polynomials are approximated as the following:

$$y(x) \approx \sum_{i=0}^{m} c_i\, B_{i,m}(x) = \mathrm{C}^{\mathrm{T}}\phi(x),$$

where

$\mathrm{C}^{\mathrm{T}} = [\mathrm{c}_0, \mathrm{c}_1, \dots, \mathrm{c}_m]$, and

$\phi(x) = [B_{0,m}(x), B_{1,m}(x), \dots, B_{m,m}(x)]^{\mathrm{T}}$.

The vector $\phi(x)$ can be expressed as the following:

$$\phi(x) = AX, \tag{1}$$

where

$$A = \begin{pmatrix} d_{00} & d_{01} & \dots & d_{0m} \\ d_{10} & d_{11} & \dots & d_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ d_{m0} & d_{m1} & \dots & d_{mm} \end{pmatrix}, \qquad X = \begin{pmatrix} 1 \\ x \\ x^2 \\ \vdots \\ x^m \end{pmatrix} \text{ and}$$

$$d_{ij} = \begin{cases} \frac{(-1)^{j-i}}{R^j} \binom{m}{i} \binom{m-i}{j-i}, & i \le j \\ 0 & , \quad i > j \end{cases}.$$

The derivatives of the vector $\phi(x)$ can be expressed as the following:

$$\mathrm{D}^1\phi(x) = A\frac{d}{dx}X. \tag{2}$$

Eq. (2) can be written as

$$\mathrm{D}^1\phi(x) = A.\Omega.X, \tag{3}$$

where

$$\Omega = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ 0 & 2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots (m-1) & 0 \end{pmatrix}. \tag{4}$$

For example, if $m = 4$, then

$$\Omega = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{pmatrix}$$

From Eq. (1),

$$\phi(x) = AX \Rightarrow A^{-1}\phi(x) = X. \tag{5}$$

Since A is a square matrix with rows and columns are not zero, not equals, and independent vectors in $\mathbb{R}^n$, then the determinant of A is exist and not equal zero. So A is invertible.

From Eqs. (3) and (5), we have $\phi(x) = A.\Omega.A^{-1}\phi(x)$. So the derivative ($\mathrm{D}^1$) of the vector $\phi(x)$ can be expressed as

$$\mathrm{D}^1 = A.\Omega.A^{-1}, \tag{6}$$

where D^1 is the $(m+1) \times (m+1)$ operational matrix of derivative. Therefore, the derivatives of the vector $\phi(x)$ can be generalized as the following:

$$\frac{d\phi(x)}{dx} = \mathrm{D}^1\phi(x), \frac{d^2\phi(x)}{dx^2} = (\mathrm{D}^1)^2\phi(x), \dots, \frac{d^m\phi(x)}{dx^m} = (\mathrm{D}^1)^m\phi(x). \tag{7}$$

By means of the operational matrix of derivative, we approximate $y(x)$ by B-polynomials as the following:

$$y(x) \simeq \mathrm{C}^{\mathrm{T}}\phi(x). \tag{8}$$

Then

$$y'(x) \simeq \mathrm{C}^{\mathrm{T}}\mathrm{D}^1\phi(x), \tag{9}$$

$y''(x) \simeq \mathrm{C}^{\mathrm{T}}(\mathrm{D}^1)^2\phi(x),$

and

$$g(x) \simeq \mathrm{G}^{\mathrm{T}}\phi(x), \tag{10}$$

where the vector $\mathrm{G}^{\mathrm{T}} = [g_0(x), \dots, g_m(x)]^{\mathrm{T}}$.

Let us consider the system of ordinary differential equations as the following:

$$\frac{dy_j}{dx} + f_j(x, y) = g_j(x), \tag{11}$$

subject to the initial conditions:

$$y_j(x_0) = \alpha_j, \tag{12}$$

where α_j are constants, for $j = 1,2, \dots k$.

By applying (7)-(10) on system (11), the residual $\mathfrak{R}(x)$ is obtained as the following:

$$\begin{gathered}\mathrm{C}_1^{\mathrm{T}}\mathrm{D}^1\phi(x) + f_1[x, \mathrm{C}_1^{\mathrm{T}}\phi(x)] - \mathrm{G}_1^{\mathrm{T}}\phi(x) = 0,\\ \mathrm{C}_2^{\mathrm{T}}\mathrm{D}^1\phi(x) + f_2[x, \mathrm{C}_2^{\mathrm{T}}\phi(x)] - \mathrm{G}_2^{\mathrm{T}}\phi(x) = 0,\\ \vdots\\ \mathrm{C}_k^{\mathrm{T}}\mathrm{D}^1\phi(x) + f_k[x, \mathrm{C}_k^{\mathrm{T}}\phi(x)] - \mathrm{G}_k^{\mathrm{T}}\phi(x) = 0,\end{gathered} \tag{13}$$

with the initial conditions

$$\mathrm{C}_1^{\mathrm{T}}\phi(x_0) = \alpha_1, \quad \mathrm{C}_2^{\mathrm{T}}\phi(x_0) = \alpha_2, \quad \dots, \quad \mathrm{C}_k^{\mathrm{T}}\phi(x_0) = \alpha_k. \tag{14}$$

The collocation points $x_0, x_1, \dots, x_m$ in system (13) can be found by applying Chebyshev roots:

$$x_i = \frac{1}{2} + \frac{1}{2\cos\left((2i+1)\frac{\pi}{2n}\right)}, \quad i = 0,1, \dots, m-1. \tag{15}$$

3 Error analysis and residual correction procedure

In this section, we introduce two corollaries to give approximate solutions of such ordinary differential equations by minimizing the absolute error as much as we can.

Let $y_{j_m}(x)$ and $y_j(x)$ be the approximate solution and the exact solution of (11) respectively. In the following procedure, the residual correction can be given for the estimation of the absolute.

First, adding and subtracting the term

$$R_j := {y_j}'_m + f_j(x, y) \tag{16}$$

to (11) yields the following differential equation

$${e_j}'_m(x) + f_j(x, e_{j_m}(x)) = g_j(x) - R_j, \tag{17}$$

with the initial conditions:

$y_j(x_0) = 0.$

Solve it by B-polynomials of degree n, $n > m$, where $e_{j_m}(x) = y_j(x) - y_{j_m}(x)$.

3.1 Corollary 1

Let $y_{j_m}(x)$ be an approximate solution of (11) and $E_{j_n}(x)$ be an approximate solution of (17). Then $y_{j_m}(x) + E_{j_n}(x)$ is also an approximate solution of (11) and its error function is $e_{j_m}(x) - E_{j_n}(x)$. We will call the approximate solution $y_{j_m}(x) + E_{j_n}(x)$ the corrected approximate solution. Note that if $\|e_{j_m}(x) - E_{j_n}(x)\|_\infty < \epsilon$, then the absolute error can be estimated by $E_{j_n}(x)$. Moreover, if $\|e_{j_m}(x) - E_{j_n}(x)\|_\infty < \|y_j(x) - y_{j_m}(x)\|$, then $y_{j_m}(x) + E_{j_n}(x)$ is more accurate solution than $y_{j_m}(x)$ in any given norm. Optimal m, which gives the minimal absolute error, might be found by measuring $E_{j_n}(x)$ in any given norm.

3.2 Corollary 2

Let us find the approximate solutions for different values of m. The triangle inequality implies

$$||y(x) - y_{m_1}(x)| - |y(x) - y_{m_2}(x)|| \leq |y_{m_1}(x) - y_{m_2}(x)|.$$

If the previous errors are not too close to each other, we can get a rough upper bound for the resulting error. We can test the upper bound as follows:

If the error sequence is decreasing (or increasing), then

$$\big||y(x) - y_{m+1}(x)| - |y(x) - y_m(x)|\big| = (1 - \mathrm{C})|y(x) - y_m(x)| \leq |y_{m+1}(x) - y_m(x)|$$

or

$$|y(x) - y_{m+1}(x)| < |y(x) - y_m(x)| \leq \frac{1}{(1 - C)}|y_{m+1}(x) - y_m(x)|, \tag{18}$$

where

$$|y(x) - y_{m+1}(x)| = \mathrm{C}|y(x) - y_m(x)|, \quad 0 \leq x < 1.$$

If $\mathrm{C} < 1$, then the bounds of both absolute errors are very well.

4 Numerical results and discussion

In this section,we introduce some examples to show the efficiency of our new technique.

Example 1

Consider the following system of second-order BVPs, see Momani et al. [Momani, Abu Arqub and Abu Hammour (2014)]:

$$\begin{cases} y_1''(x) + xy_1(x) + xy_2(x) = g_1(x), \\ y_2''(x) + 2xy_1(x) + 2xy_2(x) = g_2(x), \end{cases} \tag{19}$$

with the boundary conditions:

$$y_1(0) = 0, \quad y_1(1) = 0$$
$$y_2(0) = 0, \quad y_2(1) = 0,$$

where $g_1(x) = 2, \quad g_2(x) = -2, \quad 0 \leq x \leq 1.$

The exact solution is

$$\begin{cases} y_1(x) = x^2 - x, \\ y_2(x) = x - x^2. \end{cases}$$

By applying the technique of Section 2, for the case $m = 4$, we have

$$y_1(x) = \mathrm{C}_1^T \phi(x) = (c_{1,0} \quad c_{1,1} \quad c_{1,2} \quad c_{1,3} \quad c_{1,4}) \begin{pmatrix} x^4 - 4x^3 + 6x^2 - 4x + 1 \\ -4x^4 + 12x^3 - 12x^2 + 4x \\ 6x^4 - 12x^3 + 6x^2 \\ -4x^4 + 4x^3 \\ x^4 \end{pmatrix}, \quad (20)$$

and

$$y_2(x) = \mathrm{C}_2^T \phi(x)$$
$$= (c_{2,0} \quad c_{2,1} \quad c_{2,2} \quad c_{2,3} \quad c_{2,4}) \begin{pmatrix} x^4 - 4x^3 + 6x^2 - 4x + 1 \\ -4x^4 + 12x^3 - 12x^2 + 4x \\ 6x^4 - 12x^3 + 6x^2 \\ -4x^4 + 4x^3 \\ x^4 \end{pmatrix}. \quad (21)$$

The residual $\Re(x)$ for system (19) is obtained as:

$$\mathrm{C}_1^{\mathrm{T}}(\mathrm{D}^1)^2\phi(x) + x\mathrm{C}_1^{\mathrm{T}}\phi(x) + x\mathrm{C}_2^{\mathrm{T}}\phi(x) - 2 = 0, \text{ and}$$
$$\mathrm{C}_2^{\mathrm{T}}(\mathrm{D}^1)^2\phi(x) + 2x\mathrm{C}_1^{\mathrm{T}}\phi(x) + 2x\mathrm{C}_2^{\mathrm{T}}\phi(x) + 2 = 0, \quad (22)$$

with the boundary conditions

$$\mathrm{C}_1^{\mathrm{T}}\phi(0) = 0, \quad \mathrm{C}_1^{\mathrm{T}}\phi(1) = 0$$
$$\mathrm{C}_2^{\mathrm{T}}\phi(0) = 0, \quad \mathrm{C}_2^{\mathrm{T}}\phi(1) = 0.$$

By substituting collocation points in Eq. (22) and applying maple program the unknowns c_i will be found, then the approximate solutions for $y_1(x)$ and $y_2(x)$ can be calculated.

The unknowns c_i found as

$$c_{1,0} = 0, c_{1,1} = 0.25, c_{1,2} = -0.\bar{3}, c_{1,3} = -0.25, c_{1,4} = 0,$$
$$c_{2,0} = 0, c_{2,1} = 0.25, c_{2,2} = 0.\bar{3}, c_{2,3} = 0.25, c_{2,4} = 0.$$

Substitutethe values of c_i Eqs. (20) and (21), the exact solution of system (19) is

$$\begin{cases} y_1(x) = x^2 - x \\ y_2(x) = x - x^2. \end{cases}$$

Note that our technique generated the exact solution of system (19), while the technique of Momani et al. [Momani, Abu Arqub and Abu Hammour (2014)] generated approximate solution. Thus our technique is efficient.

Example 2

Consider the following system of second-order BVPs, see Momani et al. [Momani, Abu Arqub and Abu Hammour (2014)]:

$$\begin{cases} y_1''(x) + xy_1(x) + 2xy_2(x) + xy_1^2(x) = g_1(x), \\ y_2''(x) + y_2(x) + x^2y_1(x) + \sin(x)y_2^2(x) = g_2(x), \end{cases} \quad (23)$$

with the boundary conditions:

$$y_1(0) = 0, y_1(1) = 0$$

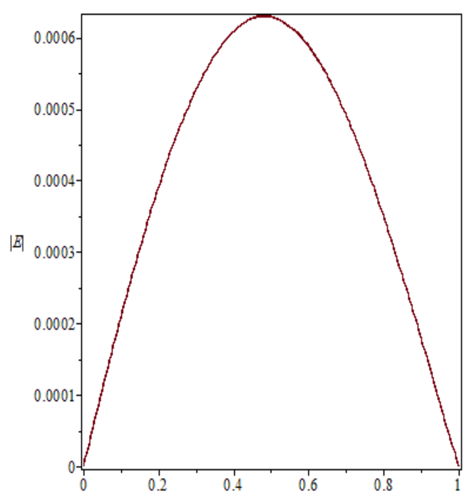
$y_2(0) = 0, y_2(1) = 0,$
where

$$\begin{cases} g_1(x) = 2x\sin(\pi x) + x^2 - 2x^4 + x^5 - 2, \\ g_2(x) = \sin(\pi x)(1 + \sin(x)\sin(\pi x)) + \pi\cos(\pi x) + x^3 - x^4, \quad 0 \le x \le 1. \end{cases}$$

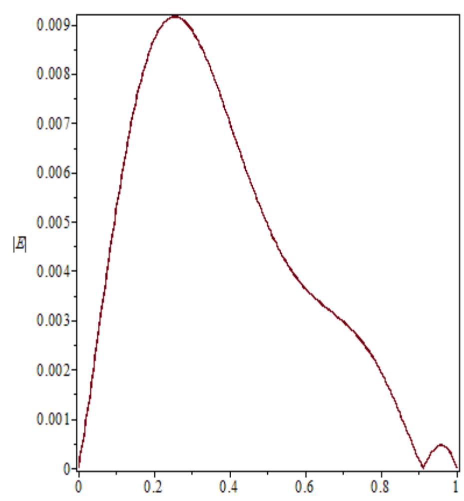
The exact solution of system (23) is

$$\begin{cases} y_1(x) = x - x^2, \\ y_2(x) = \sin(\pi x). \end{cases}$$

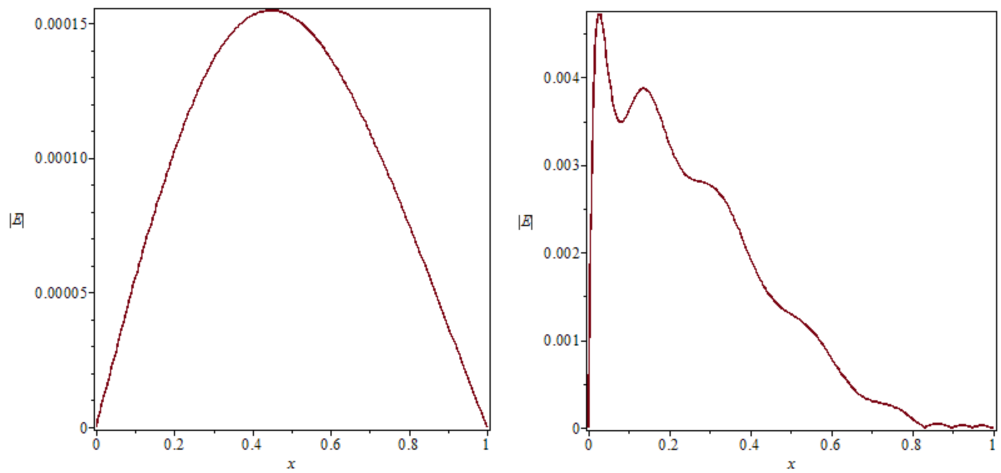
The problem is solved for the case $m = 4$ and $m = 8$. Some errors are found in the solution, so we need to improve the solution to obtain a more accurate solution, and the residual correction procedure for the case $n = 14$ is applied to correct the solutions. Fig. 1 shows the absolute error and the corrected absolute error, for the case $m = 4, n = 14$. We can see that the solutions are corrected by applying the residual correction procedure, which was described in Section 3. Fig. 2 shows the absolute error and the absolute corrected error for the case $m = 8, n = 14$, where the solutions are also corrected by applying the residual correction procedure. We were able to reduce the error in solving the system. In Figs. 3 and 4 comparison between $n = 14$ and $n = 29$, for the case $m = 5, m = 8$ consequently is showed, as seen in the Figs. 3 and 4 the absolute error is reduced in the case $m = 5$ but very close in the case $m = 8$. In Fig. 5 the upper bound of error is found for consecutive numbers $m = 8, and\ m = 9$. As seen in the figure, the absolute of e_m and e_{m+1} are b ounded by $|y_m - y_{m+1}|$ approximately.



Absolute error for y1, for case m=4



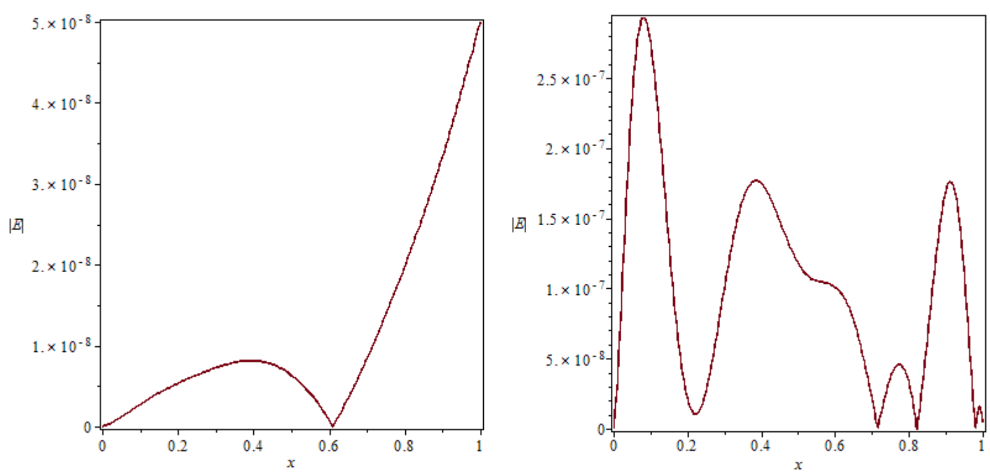
Absolute error for y2, for case m=4



Corrected absolute error for y1, for case m=4 and n=14

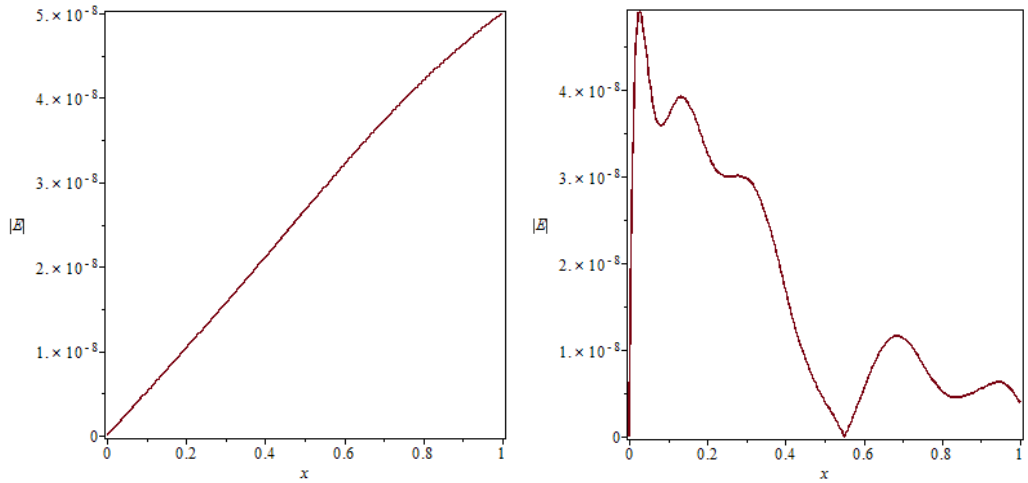
Corrected absolute error for y2, for case m=4 and n=14.

Figure 1: Absolute error and corrected absolute error to Example 2, for the case $\boldsymbol{m = 4, n = 14}$



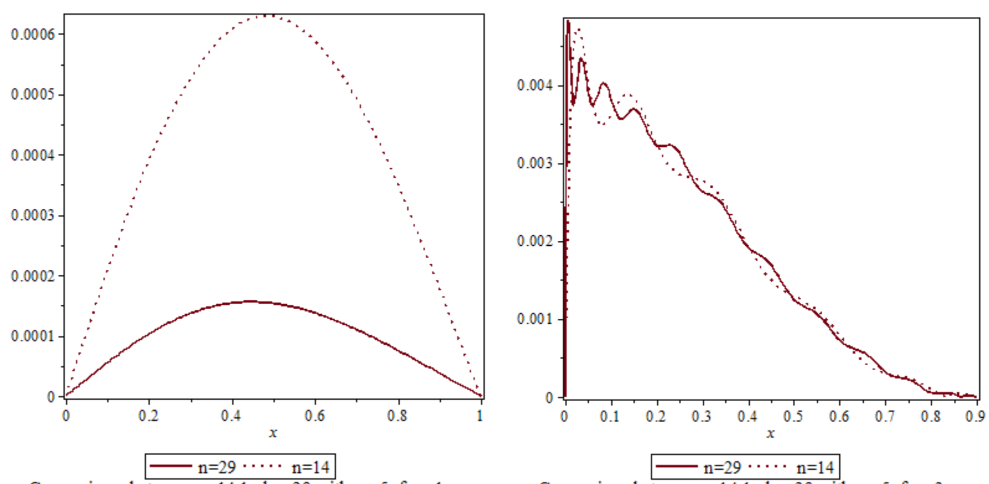
Absolute error for y1, for case m=8

Absolute error for y2, for case m=8



Corrected absolute error for y1, for case m=8 and n=14. Corrected error for y2, for case m=8 and n=14.

Figure 2: The absolute error and corrected absolute error to Example 2, for the case $\boldsymbol{m=8, n=14}$



Comparison between n=14 1nd n=29 with m=5, for y1 Comparison between n=14 1nd n=29 with m=5, for y2

Figure 3: Comparison between n=14 and n=29 with m=5 for Example 2

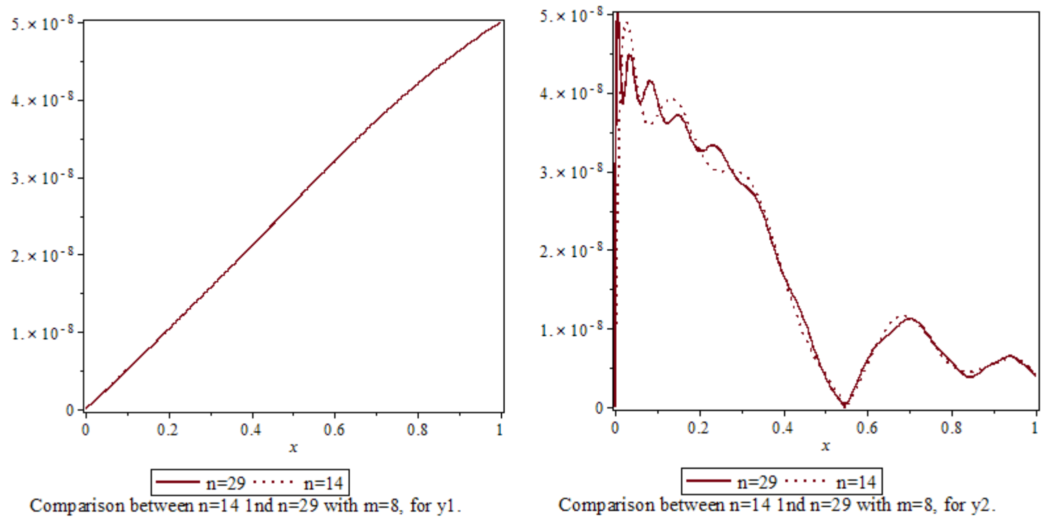


Figure 4: Comparison between n=14 and n=29 with m=8 for Example 2

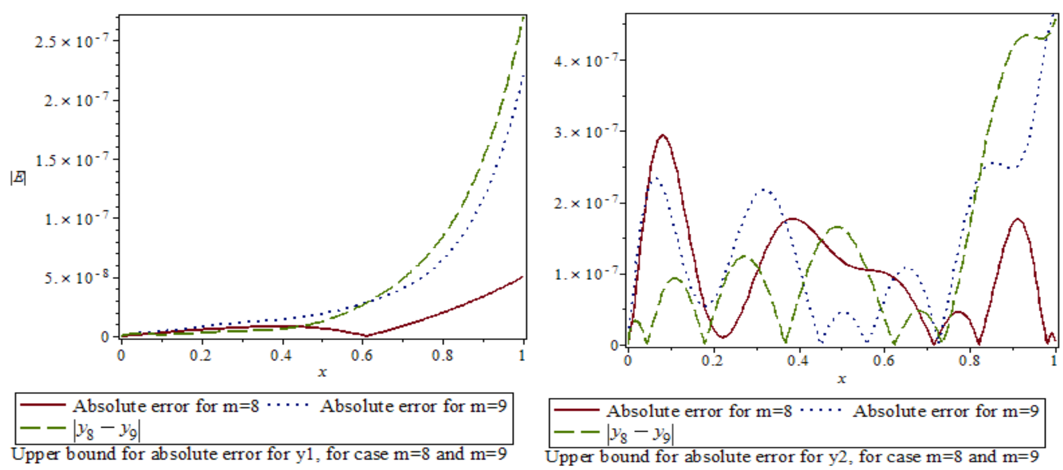


Figure 5: Upper bound for the error to Example 2, for the cases $\boldsymbol{m = 8, m = 9}$

Example 3

Consider the Brusselator system, see Ayati et al. [Ayati and Biazar (2007)]:

$$\begin{cases} y_1{}'(x) = -2y_1(x) + y_1^2(x)y_2(x), \\ y_2{}'(x) = y_1(x) - y_1^2(x)y_2(x), \end{cases} \tag{24}$$

with the boundary conditions:

$y_1(0) = 1, \quad y_2(0) = 1$

By using B-polynomials and the residual corrected procedure as in Section 3, with $m = 5$ and $n = 14$, the solutions were corrected. In Tab. 1, we listed the computed result and compared it with that given by the adomian decomposition method and the Runge-

Kutta method. Fig. 6 shows the error function and corrected absolute error function for the case $m = 5, n = 14$. The results that were obtained by using theour newprocedure are more accurate.

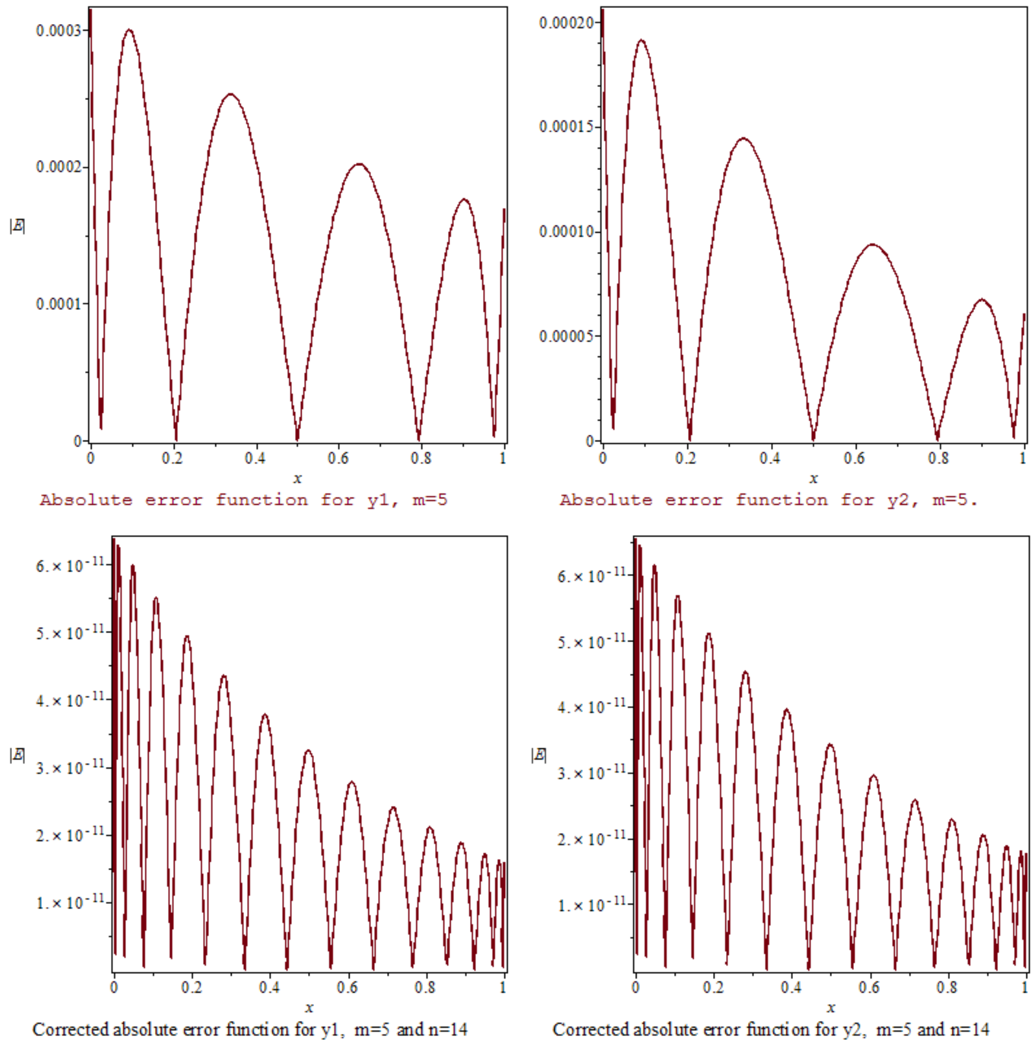


Figure 6: Absolute of error function and corrected absolute error function of Example 3, for the case $\boldsymbol{m = 5}$ and $\boldsymbol{n = 14}$

Table 1: A comparison between the solutions by B-polynomials method, residual correction procedure and RungeKutta method for Example 3

x	y_i	[Ayati and Biazar (2007)]	B-polynomials	Correction procedure	Runge-Kutta
0.1	y_1	0.900464015	0.900476626	0.9004640153	0.9004640153
	y_2	1.004524209	1.004516040	1.0045242093	1.0045242093
0.2	y_1	0.803448287	0.803477416	0.8034482894	0.8034482894
	y_2	1.016374098	1.016353898	1.0163740976	1.0163740976
0.3	y_1	0.710823962	0.710838632	0.7108240883	0.7108240883
	y_2	1.033327532	1.033314369	1.0333274656	1.0333274656
0.4	y_1	0.623890064	0.623882726	0.6238925325	0.6238925325
	y_2	1.053576241	1.053575456	1.0535747860	1.0535747860
0.5	y_1	0.543479975	0.543485270	0.5435045368	0.5435045368
	y_2	1.075665396	1.075656741	1.0756499935	1.0756499935
0.6	y_1	0.469991106	0.470141879	0.4701498027	0.4701498027
	y_2	1.098485574	1.098383767	1.0983818413	1.0983818413
0.7	y_1	0.403262683	0.404035137	0.4040240246	0.4040240246
	y_2	1.121371970	1.120852415	1.1208593146	1.1208593146

Example 4

Consider the nonlinear stiff system of the ordinary differential equation, see Biazar et al. [Biazar, Ali and Salehi (2015)]:

$$\begin{cases} y_1'(x) = -1002y_1(x) + 1000y_2^2(x), \\ y_2'(x) = y_1(x) - y_2(x) - y_2^2(x), \end{cases} \tag{25}$$

with the boundary conditions:

$y_1(0) = 1, \quad y_2(0) = 1.$

The exact solution of system (25) is

$$\begin{cases} y_1(x) = \mathrm{e}^{-2x} \\ y_2(x) = \mathrm{e}^{-x} \end{cases} \tag{26}$$

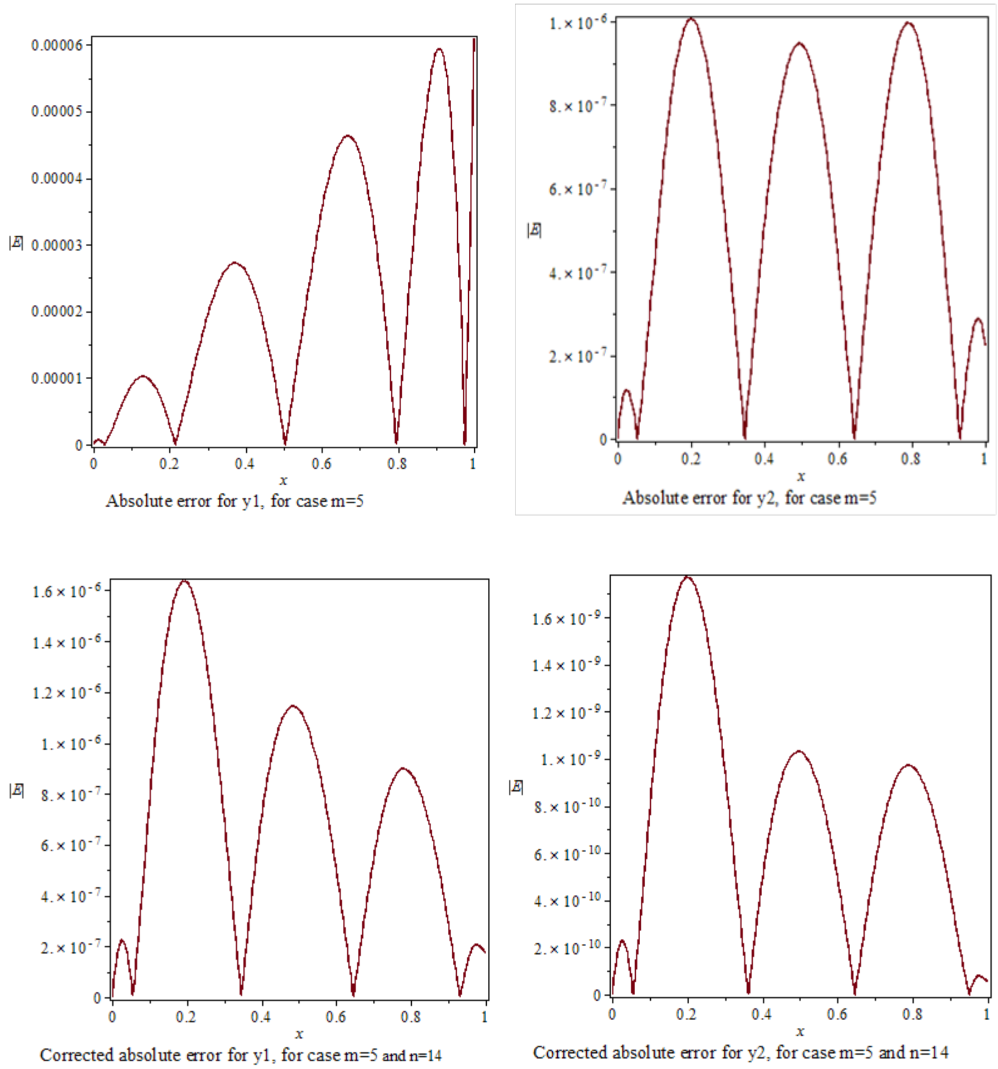


Figure 7: The absolute error and corrected absolute error to Example 4, for the case $\boldsymbol{m = 5, n = 14}$

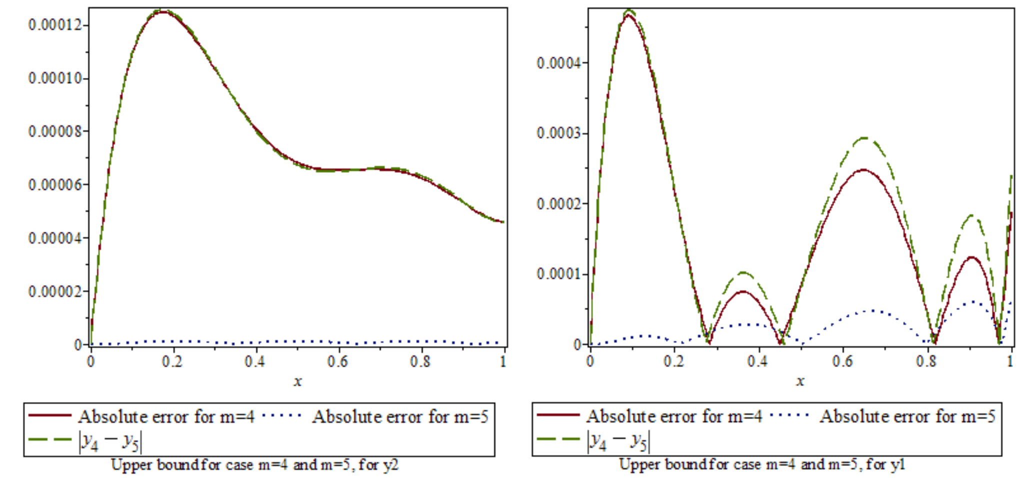


Figure 8: Upper bound for the error to Example 4, for the case $\boldsymbol{m = 4, m = 5}$

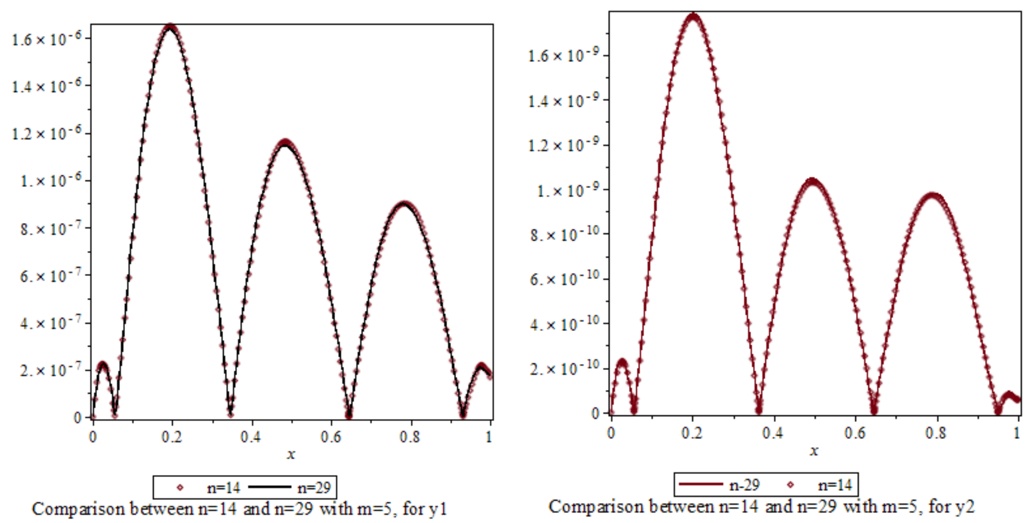


Figure 9: Comparison between n=14 and n=29 with m=5 for Example 4

Table 2: A comparison of absolute error between the solutions by B-polynomials method, residual correction procedure and other method for Example 4

x	y_i	[Biazar, Ali and Salehi (2015)]	HPM $U_{15}(x)$	B-poly. $m = 5$	Correction procedure $n = 14$
0	y_1	0	0	0	0
	y_2	0	0	0	0
0.5	y_1	$3.0e^{-10}$	0	$1.5e^{-6}$	$1.1e^{-7}$
	y_2	$1.2e^{-8}$	0	$9.4e^{-7}$	$1.04e^{-9}$
1.0	y_1	$1.5e^{-8}$	0	$6.0e^{-6}$	$1.6e^{-8}$
	y_2	$1.4e^{-7}$	$2.8e^{-9}$	$2.2e^{-7}$	$4.7e^{-11}$

The problem is solved for the case $m = 5$. The residual correction procedure for the case $n = 14$ is applied to correct the problem. Fig. 7 shows the absolute error and the corrected absolute error for the case $m = 5$ and $n = 14$. In Fig. 8, the upper bound of error is found for consecutive numbers $m = 4$ and $m = 5$. As seen in Fig. 8, the absolute of e_m and e_{m+1} are bounded by $|y_m - y_{m+1}|$ approximately. In Fig. 9 comparison between $n = 14$ and $n = 29$, for the case $m = 5, m = 8$ consequently is showed. As seen in Fig. 9, the absolute error is very close between $n = 14$ and $n = 29$ for residual correction procedure. A comparison of absolute error between the solutions by B-polynomials method, residual correction procedure, homotopy perturbation method, and rational homotopy perturbation method are displayed in Tab. 2. The above comparisons show that residual correction procedure can solve stiff problem more accurately with less number of iterations.

5 Conclusions

In this work, the Bernstein polynomials method and residual correcting procedure are applied to solve a system of second-order BVPs, Brusselator system and nonlinear stiff system. The main goal has been achieved by correcting the solutions, which are solved by the classical B-polynomials method. Some numerical examples are given to show the efficiency of our new technique by comparing the solutions obtained by our technique with some old methods such as the technique of Momani et al. [Momani, Abu Arqub and Abu Hammour (2014); Ayati and Biazar (2007)] and the Runge-Kutta method. For future work, we will apply the residual corrected procedure to solve chaotic systems and the Lorenz system.

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