

Crank-Nicolson ADI Galerkin Finite Element Methods for Two Classes of Riesz Space Fractional Partial Differential Equations

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Abstract: In this paper, two classes of Riesz space fractional partial differential equations including space-fractional and space-time-fractional ones are considered. These two models can be regarded as the generalization of the classical wave equation in two space dimensions. Combining with the Crank-Nicolson method in temporal direction, efficient alternating direction implicit Galerkin finite element methods for solving these two fractional models are developed, respectively. The corresponding stability and convergence analysis of the numerical methods are discussed. Numerical results are provided to verify the theoretical analysis.

Keywords: Fractional partial differential equations, Galerkin approximation, alternating direction implicit method, stability, convergence.

1 Introduction

In this paper, we mainly consider the development of the efficient alternating direction implicit (ADI) Galerkin finite element methods for solving the two-dimensional Riesz space fractional partial differential equations including space-fractional and space-time-fractional ones. Let $T > 0$ be a fixed time. The space-fractional partial differential equation is described by

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = K_x \frac{\partial^{2\alpha_1} u}{\partial |x|^{2\alpha_1}} + K_y \frac{\partial^{2\alpha_2} u}{\partial |y|^{2\alpha_2}} + f(x, y, t), & (x, y, t) \in \Omega \times (0, T], \\ u = 0, & (x, y, t) \in \partial\Omega \times (0, T], \\ u(x, y, 0) = \phi_0(x, y), \quad u_t(x, y, t)|_{t=0} = \phi_1(x, y), & (x, y) \in \Omega, \end{cases} \quad (1)$$

in which $\frac{1}{2} < \alpha_1, \alpha_2 < 1$, $K_x, K_y > 0$, and $\Omega = (a, b) \times (c, d)$. The functions f , ϕ_0 , and ϕ_1 are assumed to be suitably smooth. The fractional operator $\frac{\partial^{2\alpha_1}}{\partial |x|^{2\alpha_1}}$ is in the Riesz sense, that is

$$\frac{\partial^{2\alpha_1} u}{\partial |x|^{2\alpha_1}} = -c_1 ({}_{RL}D_{a,x}^{2\alpha_1} u + {}_{RL}D_{x,b}^{2\alpha_1} u),$$

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where $c_1 = \frac{1}{2 \cos(\alpha_1 \pi)}$. Here, for $n - 1 < \beta < n$, $n \in \mathbb{N}$, the Riemann-Liouville derivatives ${}_{RL}D_{a,x}^\beta$ and ${}_{RL}D_{x,b}^\beta$ are defined respectively by

$$\begin{aligned} {}_{RL}D_{a,x}^\beta u(x, y, t) &= \frac{1}{\Gamma(n - \beta)} \frac{\partial^n}{\partial x^n} \int_a^x (x - s)^{n-\beta-1} u(s, y, t) ds, \\ {}_{RL}D_{x,b}^\beta u(x, y, t) &= \frac{(-1)^n}{\Gamma(n - \beta)} \frac{\partial^n}{\partial x^n} \int_x^b (s - x)^{n-\beta-1} u(s, y, t) ds. \end{aligned}$$

The fractional operator $\frac{\partial^{2\alpha_2}}{\partial |y|^{2\alpha_2}}$ is also in the Riesz sense, and can be given similarly.

Replacing the second order partial derivative in time in (1) with Caputo derivative operator ${}_C D_{0,t}^\beta$ with order $\beta \in (1, 2)$, we obtain the following space-time-fractional partial differential equation in two space dimensions,

$${}_C D_{0,t}^\beta u(x, y, t) = K_x \frac{\partial^{2\alpha_1} u}{\partial |x|^{2\alpha_1}} + K_y \frac{\partial^{2\alpha_2} u}{\partial |y|^{2\alpha_2}} + f(x, y, t), \quad (2)$$

in which the Caputo derivative operator ${}_C D_{0,t}^\beta$ is defined by

$${}_C D_{0,t}^\beta u(x, y, t) = \frac{1}{\Gamma(2 - \beta)} \int_0^t (t - s)^{1-\beta} \frac{\partial^2 u(x, y, s)}{\partial s^2} ds.$$

The initial and boundary conditions for (2) are imposed as same with the Eq. (1).

When $\alpha_1 = \alpha_2 = 1$ and $K_x = K_y$ (and $\beta = 2$ for (2)), then both the Eqs. (1) and (2) reduce to the classical partial differential wave equations in two spatial dimensions. So, these two fractional models (1) and (2) can be regarded as the generalization of the classical wave equations.

In recent years, fractional approach has been received many attentions as a powerful modeling methodology, and it is widely applied in materials and mechanics, anomalous diffusion, turbulence, wave propagation, etc. In particular, one of the resulting equations may be the space-fractional model (1) or the space-time-fractional model (2). Recently, the fractional model which involves the Riesz derivative in space or the Caputo derivative in time has attracted many researchers' attention. Li et al. [Li, Yi and Kurths (2018)] introduced the fractional convection operator with Riesz derivative. By employing a continuous time random walks (CTRWs) scheme on one-dimensional lattice and choosing certain kind of two-sided power-law jump length distribution, Li et al. [Li and Yi (2019)] obtained a fractional convection equation containing Riesz derivative. Cai et al. [Cai and Li (2019a)] further studied the properties of Riesz derivative and clarified the relationship between Riesz derivative and fractional Laplacian. In addition, they also investigated the regularity of the solution to a corresponding fractional differential equation [Cai and Li (2019b)]. Aman et al. [Aman, Al-Mdallal and Khan (2018)] investigated the effect of second order slip on magnetohydrodynamic flow of a fractional Maxwell fluid by considering the Caputo derivative in time. Al-Mdallal et al. [Al-Mdallal, Abro and Khan (2018)] also considered a kind of non-Newtonian models in a porous medium, namely fractional Walter's liquid

model by fractional approach in . Bira et al. [Bira, Raja Sekhar and Zeidan (2018)] studied some nonlinear time-fractional evolution equations arising in some important physical phenomena.

It is well known that the analytical solutions of fractional partial differential equations are not always readily obtained, so seeking efficient and reliable numerical treatments is of great importance in real applications. Many numerical methods have been proposed for solving fractional partial differential equations in one and several space variables [Li and Zeng (2015); Li and Cai (2019)]. Moreover, one can refer to the review paper [Li and Chen (2018)] for more details. For the Galerkin finite element methods, recently Li et al. [Li and Wang (2020)] established the discontinuous Galerkin finite element method for a Caputo-type nonlinear conservation law. However, the numerical discretized strategy can not be directly extended to deal with the space-fractional models. About 2006, Ervin et al. developed a rigorous theoretical framework for the wellposedness of a Galerkin weak formulation to fractional elliptic differential equations with constant diffusivity coefficient [Ervin and Roop (2006, 2007)]. Subsequently, many researchers extended the theoretical framework to other numerical methods for numerically solving different space-fractional models, see Zeng et al. [Zeng, Liu, Li et al. (2014)] and the references therein.

It seems that the numerical study for fractional models (1) or (2) is scarce, especially using the finite element methods. Besides, solving a space-fractional partial differential equation always leads to expensive computational cost due to the nonlocal properties of Riesz derivative, that is, one needs to calculate the inverse of the dense coefficient matrix with computational cost of N_x^3 and storage of N_x^2 , which N_x is the stepsize number. The computational cost will increase more rapidly if we solve the space-time one, let alone solve the high-dimensional one. Although some algorithms have been proposed to deal with such problem, such as high-order algorithms and fast algorithms (cf. [Wang and Basu (2012)]), more proper and efficient methods with rigorous analysis need to be further developed. It is known that ADI methods have proved to be valuable techniques for solving classical partial differential equations in several space dimensions. Their effectiveness is to rely on the fact that they reduce the solution of a multidimensional problem to the solution of sets of independent one-dimensional problems.

In this paper, we aim to apply the ADI technique with finite element methods to solving the multidimensional fractional problems (1) and (2). Especially, we propose a Crank-Nicolson ADI Galerkin finite element scheme, that is, we use the finite element method in space, and Crank-Nicolson method in time, then combine the alternating direction implicit method to obtain a numerical scheme for (1). Based on the same discretized technique in space, we further apply the modified L1 method for Caputo derivative in (2), then obtain an efficient discretized scheme for (2). It is worthy to mention that the modified L1 method is different from the classical L1 one. For the classical L1 method, the discretization of the Caputo derivative is considered on the grid points of the form $\{t_0, t_1, t_2, \dots\}$, while the modified L1 method is on $\{t_0, t_{1/2}, t_{3/2}, \dots\}$. So the two proposed methods naturally lead to the Crank-Nicolson schemes since they are derived on the direct discretization for the first-order/Caputo derivative at the half grid point in time.

The rest of this paper is constructed as follows. We first introduce the preliminaries in

Section 2, then we derive the Galerkin ADI finite element schemes in Section 3. The corresponding theoretical analyses, including stability analysis and error estimate of the derived schemes, are presented in Section 4. In Section 5, numerical results are demonstrated to verify the effectiveness of the numerical schemes. Some conclusions and further discussions are given in Section 6. As a notation convention, we use the notation C to denote a generic positive constant, which may change at different occurrences, but is always independent of the mesh spacing.

2 Preliminaries

In this section, we introduce some notations and lemmas that are useful in the following numerical study.

Let $\Omega \subseteq \mathbb{R}^2$ be a convex domain and satisfy $\Omega = I_x \times I_y = (a, b) \times (c, d)$. Denote the inner product on the space $L^2(\Omega)$ by (\cdot, \cdot) , and the associated norm is $\|u\|_{L^2(\Omega)}$. The fractional derivative spaces and some related lemmas are introduced as follows, see Ervin et al. [Ervin and Roop (2007); Zeng, Liu, Li et al. (2014)] for more details.

Definition 2.1. For $\mu > 0$, define the semi-norm

$$|u|_{J_L^\mu(\Omega)} = \left(\|_{RL}D_{a,x}^\mu u\|_{L^2(\Omega)}^2 + \|_{RL}D_{c,y}^\mu u\|_{L^2(\Omega)}^2 \right)^{1/2}$$

and the norm

$$\|u\|_{J_L^\mu(\Omega)} = \left(\|u\|_{L^2(\Omega)}^2 + |u|_{J_L^\mu(\Omega)}^2 \right)^{1/2},$$

and denote $J_L^\mu(\Omega)$ (or $J_{L,0}^\mu(\Omega)$) as the closure of $C^\infty(\Omega)$ (or $C_0^\infty(\Omega)$) with respect to $\|\cdot\|_{J_L^\mu(\Omega)}$.

Definition 2.2. For $\mu > 0$, define the semi-norm

$$|u|_{J_R^\mu(\Omega)} = \left(\|_{RL}D_{x,b}^\mu u\|_{L^2(\Omega)}^2 + \|_{RL}D_{y,d}^\mu u\|_{L^2(\Omega)}^2 \right)^{1/2}$$

and the norm

$$\|u\|_{J_R^\mu(\Omega)} = \left(\|u\|_{L^2(\Omega)}^2 + |u|_{J_R^\mu(\Omega)}^2 \right)^{1/2},$$

and denote $J_R^\mu(\Omega)$ (or $J_{R,0}^\mu(\Omega)$) as the closure of $C^\infty(\Omega)$ (or $C_0^\infty(\Omega)$) with respect to $\|\cdot\|_{J_R^\mu(\Omega)}$.

Definition 2.3. For $\mu \neq n - 1/2$, $n \in \mathbb{N}$, define the semi-norm

$$|u|_{J_S^\mu(\Omega)} = \left(|(_{RL}D_{a,x}^\mu u, _{RL}D_{x,b}^\mu u)_{L^2(\Omega)}| + |(_{RL}D_{c,y}^\mu u, _{RL}D_{y,d}^\mu u)_{L^2(\Omega)}| \right)^{1/2}$$

and the norm

$$\|u\|_{J_S^\mu(\Omega)} = \left(\|u\|_{L^2(\Omega)}^2 + |u|_{J_S^\mu(\Omega)}^2 \right)^{1/2},$$

and denote $J_S^\mu(\Omega)$ (or $J_{S,0}^\mu(\Omega)$) as the closure of $C^\infty(\Omega)$ (or $C_0^\infty(\Omega)$) with respect to $\|\cdot\|_{J_S^\mu(\Omega)}$.

The fractional Sobolev space $H^\mu(\Omega)$ can be defined through the Fourier transform.

Definition 2.4. For $\mu > 0$, define the semi-norm

$$|u|_{H^\mu(\Omega)} = \|\omega^\mu \mathcal{F}(u)(\omega)\|_{L^2(\Omega)}$$

and the norm

$$\|u\|_{H^\mu(\Omega)} = \left(\|u\|_{L^2(\Omega)}^2 + |u|_{H^\mu(\Omega)}^2 \right)^{1/2},$$

and denote $H^\mu(\Omega)$ (or $H_0^\mu(\Omega)$) as the closure of $C^\infty(\Omega)$ (or $C_0^\infty(\Omega)$) with respect to $\|\cdot\|_{H^\mu(\Omega)}$. Here, $\mathcal{F}(u)(\omega)$ is the Fourier transformation of function $u(x, y)$.

The above mentioned spaces are equivalent which are shown in the lemma below.

Lemma 2.1. For $\mu > 0$, $\mu \neq n-1/2$, $n \in \mathbb{N}$. Then the spaces $J_{L,0}^\mu(\Omega)$, $J_{R,0}^\mu(\Omega)$, $J_{S,0}^\mu(\Omega)$, and $H_0^\mu(\Omega)$ are equivalent, with equivalent seminorms and norms.

Lemma 2.2. Let $\beta \in (1, 2)$, Then $\forall u \in H_0^\beta(\Omega)$ and $\forall v \in H_0^{\beta/2}(\Omega)$, we have

$$({}_{RL}D_{a,x}^\beta u, v) = ({}_{RL}D_{a,x}^{\beta/2} u, {}_{RL}D_{x,b}^{\beta/2} v), \quad ({}_{RL}D_{x,b}^\beta u, v) = ({}_{RL}D_{x,b}^{\beta/2} u, {}_{RL}D_{a,x}^{\beta/2} v).$$

Similarly, the results for $({}_{RL}D_{c,y}^\beta u, v)$ and $({}_{RL}D_{y,d}^\beta u, v)$ also hold.

Lemma 2.3. Let $\mu > 0$ and denote \hat{u} as the extension of u by zero outside Ω . Then $\forall u \in J_{L,0}^\mu(\Omega) \cap J_{R,0}^\mu(\Omega)$, we have

$$({}_{RL}D_{a,x}^\mu u, {}_{RL}D_{x,b}^\mu u) = \cos(\mu\pi) \|{}_{RL}D_{-\infty,x}^\mu \hat{u}\|_{L^2(\mathbb{R}^2)}^2 = \cos(\mu\pi) \|{}_{RL}D_{x,\infty}^\mu \hat{u}\|_{L^2(\mathbb{R}^2)}^2.$$

Similarly, the results for $({}_{RL}D_{c,y}^\mu u, {}_{RL}D_{y,d}^\mu u)$ also hold.

Lemma 2.4. Let $\mu_1, \mu_2 > 0$ and denote \hat{u} as the extension of u by zero outside Ω . Then $\forall u \in J_{L,0}^{\max\{\mu_1, \mu_2\}}(\Omega) \cap J_{R,0}^{\max\{\mu_1, \mu_2\}}(\Omega)$, we have

$$\begin{aligned} ({}_{RL}D_{a,x}^{\mu_1} {}_{RL}D_{c,y}^{\mu_2} u, {}_{RL}D_{x,b}^{\mu_1} {}_{RL}D_{y,d}^{\mu_2} u) &= \cos(\mu_1\pi) \cos(\mu_2\pi) \|{}_{RL}D_{-\infty,x}^{\mu_1} {}_{RL}D_{-\infty,y}^{\mu_2} \hat{u}\|_{L^2(\mathbb{R}^2)}^2, \\ ({}_{RL}D_{a,x}^{\mu_1} {}_{RL}D_{y,d}^{\mu_2} u, {}_{RL}D_{x,b}^{\mu_1} {}_{RL}D_{c,y}^{\mu_2} u) &= \cos(\mu_1\pi) \cos(\mu_2\pi) \|{}_{RL}D_{-\infty,x}^{\mu_1} {}_{RL}D_{-\infty,y}^{\mu_2} \hat{u}\|_{L^2(\mathbb{R}^2)}^2. \end{aligned}$$

3 The ADI Galerkin finite element schemes

3.1 Space-fractional partial differential equation

We first rewrite the initial-boundary problem (1) as a first-order one by setting $\phi = \partial u / \partial t$. That is,

$$\begin{cases} \frac{\partial \phi}{\partial t} = (L_x + L_y)u + f(x, y, t), & (x, y, t) \in \Omega \times (0, T], \quad T > 0, \\ \frac{\partial u}{\partial t} = \phi, \end{cases} \quad (3)$$

with $L_x u = K_x \frac{\partial^{2\alpha_1} u}{\partial |x|^{2\alpha_1}}$ and $L_y u = K_y \frac{\partial^{2\alpha_2} u}{\partial |y|^{2\alpha_2}}$.

For the finite difference of temporal direction, let $t_n = n\tau$ for $n = 0, 1, \dots, n_T$, where $\tau = T/n_T$ is the time step size and n_T is given positive integer. Denote $t_{n+1/2} = (t_n + t_{n+1})/2$

for $n = 0, 1, \dots, n_T - 1$. Suppose that $u(x, y, t)$ is suitably smooth with respect to the variable t , then we readily obtain

$$\|u(x, y, t_{n+1/2}) - u^{n+1/2}\|_{L^2(\Omega)} \leq C \max_{0 \leq t \leq T} \|\frac{\partial^2 u}{\partial t^2}\|_{L^2(\Omega)} \tau^2, \tag{4}$$

and

$$\|\frac{\partial u(x, y, t_{n+1/2})}{\partial t} - \delta_t u^{n+1/2}\|_{L^2(\Omega)} \leq C \max_{0 \leq t \leq T} \|\frac{\partial^3 u}{\partial t^3}\|_{L^2(\Omega)} \tau^2, \tag{5}$$

where $u^{n+1/2} = \frac{u^{n+1} + u^n}{2}$ and $\delta_t u^{n+1/2} = \frac{u^{n+1} - u^n}{\tau}$.

Now we consider (3) on the time level $(x, y, t_{n+1/2})$, namely,

$$\begin{cases} \frac{\partial \phi(x, y, t_{n+\frac{1}{2}})}{\partial t} = (L_x + L_y)u(x, y, t_{n+\frac{1}{2}}) + f(x, y, t_{n+\frac{1}{2}}), \\ \frac{\partial u(x, y, t_{n+\frac{1}{2}})}{\partial t} = \phi(x, y, t_{n+\frac{1}{2}}). \end{cases}$$

Using the approximations (4) and (5), we have

$$\begin{cases} \delta_t \phi^{n+1/2} = (L_x + L_y)u^{n+1/2} + f(x, y, t_{n+\frac{1}{2}}) + O(\tau^2), \\ \delta_t u^{n+1/2} = \phi^{n+1/2} + O(\tau^2). \end{cases} \tag{6}$$

Note that $u^{n+1/2} = \frac{\tau}{2} \phi^{n+1/2} + u^n$, then (6) can be rewritten as

$$\begin{cases} \delta_t \phi^{n+1/2} = \frac{\tau}{2}(L_x + L_y)\phi^{n+1/2} + (L_x + L_y)u^n + f(x, y, t_{n+\frac{1}{2}}) + O(\tau^2), \\ \delta_t u^{n+1/2} = \phi^{n+1/2} + O(\tau^2). \end{cases} \tag{7}$$

Adding a small term $\frac{\tau^4}{16}L_xL_y\delta_t\phi^{n+1/2} = O(\tau^2)$ to the left-hand side of the first equation in (7), one has

$$\begin{cases} \delta_t \phi^{n+1/2} + \frac{\tau^4}{16}L_xL_y\delta_t\phi^{n+1/2} = \frac{\tau}{2}(L_x + L_y)\phi^{n+1/2} \\ \hspace{10em} + (L_x + L_y)u^n + f(x, y, t_{n+\frac{1}{2}}) + O(\tau^2), \\ \delta_t u^{n+1/2} = \phi^{n+1/2} + O(\tau^2). \end{cases}$$

That is,

$$\begin{cases} (1 - \frac{\tau^2}{4}L_x)(1 - \frac{\tau^2}{4}L_y)\phi^{n+1} = (1 + \frac{\tau^2}{4}L_x)(1 + \frac{\tau^2}{4}L_y)\phi^n \\ \hspace{10em} + \tau(L_x + L_y)u^n + \tau f(x, y, t_{n+\frac{1}{2}}) + O(\tau^3), \\ \delta_t u^{n+1/2} = \phi^{n+1/2} + O(\tau^2). \end{cases}$$

Define the finite element space X_h^r as the set of piecewise polynomials with degree at most r ($r \geq 1$). And denote the interpolation operator by I_h in X_h^r . Here, h denote the maximal length of the sides of each finite element on Ω , so h is a parameter which decreases as the element is made finer. X_h^r is obvious the finite dimensional subspace of the space V with

$$V = \left\{ u \mid u|_{\partial\Omega}, u, \frac{\partial^{\alpha_1} u}{\partial|x|^{\alpha_1}}, \frac{\partial^{\alpha_2} u}{\partial|y|^{\alpha_2}}, \frac{\partial^{\alpha_1}}{\partial|x|^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial|y|^{\alpha_2}} u \in L^2(\Omega) \right\}.$$

Suppose that $u_h^n \in X_h^r$ is the numerical solution of u^n and the initial values are approximated by $u_h^0 = \Pi_h^{1,0} \phi_0$ and $\phi_h^0 = \Pi_h^{1,0} \phi_1$, where $\Pi_h^{1,0}$ denotes an appropriate projection of ϕ_0 and ϕ_1 onto X_h^r . Thus, we obtain the ADI Galerkin finite element scheme as follows: Find $\phi_h^{n+1} \in X_h^r$ for $n = 0, 1, 2, \dots, n_T - 1$ such that

$$\begin{cases} (\phi_h^{n+1}, v) + \frac{\tau^2}{4} A(\phi_h^{n+1}, v) + \frac{\tau^4}{16} B(\phi_h^{n+1}, v) \\ = (\phi_h^n, v) - \frac{\tau^2}{4} A(\phi_h^n, v) + \frac{\tau^4}{16} B(\phi_h^n, v) \\ - \tau A(u_h^n, v) + \tau (I_h f(x, y, t_{n+\frac{1}{2}}), v), \quad v \in X_h^r, \\ u_h^0 = \Pi_h^{1,0} \phi_0, \quad \phi_h^0 = \Pi_h^{1,0} \phi_1, \end{cases} \quad (8)$$

from which we obtain the following approximation to u :

$$u_h^{n+1} = u_h^n + \tau \phi_h^{n+1/2}. \quad (9)$$

Here the bilinear form A is defined by

$$\begin{aligned} A(u, v) = & c_1 K_x \left[(RLD_{a,x}^{\alpha_1} u, RLD_{x,b}^{\alpha_1} v) + (RLD_{x,b}^{\alpha_1} u, RLD_{a,x}^{\alpha_1} v) \right] \\ & + c_2 K_y \left[(RLD_{c,y}^{\alpha_2} u, RLD_{y,d}^{\alpha_2} v) + (RLD_{y,d}^{\alpha_2} u, RLD_{c,y}^{\alpha_2} v) \right], \end{aligned} \quad (10)$$

and the bilinear form B in the perturbation term is defined by

$$\begin{aligned} B(u, v) = & c_1 c_2 K_x K_y \left[(RLD_{a,x}^{\alpha_1} RLD_{c,y}^{\alpha_2} u, RLD_{x,b}^{\alpha_1} RLD_{y,d}^{\alpha_2} v) \right. \\ & + (RLD_{a,x}^{\alpha_1} RLD_{y,d}^{\alpha_2} u, RLD_{x,b}^{\alpha_1} RLD_{c,y}^{\alpha_2} v) \\ & + (RLD_{x,b}^{\alpha_1} RLD_{c,y}^{\alpha_2} u, RLD_{a,x}^{\alpha_1} RLD_{y,d}^{\alpha_2} v) \\ & \left. + (RLD_{x,b}^{\alpha_1} RLD_{y,d}^{\alpha_2} u, RLD_{a,x}^{\alpha_1} RLD_{c,y}^{\alpha_2} v) \right]. \end{aligned} \quad (11)$$

Eq. (8) present the ADI Galerkin method in inner product form. We can rewrite them as the matrix form. To this end, suppose $X_h^r = X_{h,x}^r \otimes X_{h,y}^r$, where $X_{h,x}^r$ and $X_{h,y}^r$ are finite-dimensional subspaces of $H_0^{\alpha_1}(I_x)$ and $H_0^{\alpha_2}(I_y)$, respectively. And their corresponding bases are denoted as $\{\varphi_i(x)\}_{i=1}^{N_x}$ and $\{\vartheta_j(y)\}_{j=1}^{N_y}$. Here, N_x, N_y are the positive integers. The uniform partitions of I_x and I_y are then given by

$$\begin{aligned} a &= x_0 < x_1 < \dots < x_{N_x-1} < x_{N_x} < x_{N_x+1} = b, \\ c &= y_0 < y_1 < \dots < y_{N_y-1} < y_{N_y} < y_{N_y+1} = d. \end{aligned}$$

Let

$$u_h^n(x, y) = \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} u_{i,j}^n \varphi_i(x) \vartheta_j(y), \quad \phi_h^n(x, y) = \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} \phi_{i,j}^n \varphi_i(x) \vartheta_j(y),$$

and denote

$$\begin{aligned} (M_x)_{k,m} &= (\varphi_k(x), \varphi_m(x)), & (S_x)_{k,m} &= (RLD_{a,x}^{\alpha_1} \varphi_k(x), RLD_{x,b}^{\alpha_1} \varphi_m(x)), \\ (M_y)_{k,m} &= (\vartheta_k(y), \vartheta_m(y)), & (S_y)_{k,m} &= (RLD_{c,y}^{\alpha_2} \vartheta_k(y), RLD_{y,d}^{\alpha_2} \vartheta_m(y)). \end{aligned}$$

Choosing $v = \varphi_k(x)\vartheta_m(y)$, $k = 1, 2, \dots, N_x$, $m = 1, 2, \dots, N_y$, then by Lemma 2.2 we obtain the following matrix form for (8).

$$(M_x + \frac{\tau^2}{4}c_1K_x(S_x + S_x^T))\Phi^{n+1}(M_y + \frac{\tau^2}{4}c_2K_y(S_y + S_y^T))^T = RHS^n,$$

where

$$\begin{aligned} RHS^n &= (M_x - \frac{\tau^2}{4}c_1K_x(S_x + S_x^T))\Phi^n(M_y - \frac{\tau^2}{4}c_2K_y(S_y + S_y^T))^T \\ &\quad - \tau[c_1K_x(S_x + S_x^T)U^nM_y^T + c_2K_yM_xU^n(S_y + S_y^T)^T] + \tau F^{n+\frac{1}{2}}, \\ (F^{n+\frac{1}{2}})_{k,m} &= (f^{n+\frac{1}{2}}, \varphi_k\vartheta_m). \end{aligned}$$

Here, $\Phi^{n+1}, RHS^n, U^n, F^{n+\frac{1}{2}} \in \mathbb{R}^{N_x \times N_y}$.

Therefore, denoting the intermediate variable $\Phi_*^{n+1} \in \mathbb{R}^{N_x \times N_y}$, the alternating direction Galerkin finite element scheme can be executed as follows. First we solve

$$(M_x + \frac{\tau^2}{4}c_1K_x(S_x + S_x^T))\Phi_*^{n+1} = RHS^n,$$

then

$$(M_y + \frac{\tau^2}{4}c_2K_y(S_y + S_y^T))(\Phi^{n+1})^T = (\Phi_*^{n+1})^T,$$

from which we obtain the approximation to U^{n+1} :

$$U^{n+1} = U^n + \tau\Phi^{n+\frac{1}{2}}.$$

3.2 Space-time-fractional partial differential equation

We firstly present the modified L1 scheme of the Caputo derivative ${}_C D_{0,t}^\beta$ with $\beta \in (1, 2)$ below,

$$\begin{aligned} {}_C D_{0,t}^\beta g(t)|_{t=t_{n+\frac{1}{2}}} &= \frac{\tau^{1-\beta}}{\Gamma(3-\beta)} \left[b_0\delta_t g^{n+\frac{1}{2}} - \sum_{k=1}^n (b_{n-k} - b_{n-k+1})\delta_t g^{k-\frac{1}{2}} \right. \\ &\quad \left. - (b_n - B_n)\delta_t g^{\frac{1}{2}} - B_n g'(t_0) \right] + R^{n+\frac{1}{2}}, \end{aligned} \tag{12}$$

where the truncation error

$$|R^{n+\frac{1}{2}}| \leq C\tau^{3-\beta} \max_{0 \leq t \leq T} |g'''(t)|. \tag{13}$$

The coefficients b_n and B_n are defined by

$$\begin{cases} b_k = (k+1)^{2-\beta} - k^{2-\beta}, & k = 0, 1, 2, \dots, n, \\ B_n = 2\left((n+\frac{1}{2})^{2-\beta} - n^{2-\beta}\right). \end{cases} \tag{14}$$

One can readily observe that the coefficients (14) have the following properties,

$$\left\{ \begin{array}{l} B_n > 0, \\ b_{n-1} - B_n > 0, \\ B_n - b_n > 0, \\ b_{n-k} - b_{n-k+1} > 0. \end{array} \right.$$

Now considering the Eq. (2) on the time level $(x, y, t_{n+1/2})$, and using the approximations (12) and (4), we get

$$\begin{aligned} \frac{1}{\mu} \left[b_0 \delta_t u^{n+\frac{1}{2}} - \sum_{k=1}^n (b_{n-k} - b_{n-k+1}) \delta_t u^{k-\frac{1}{2}} \right. \\ \left. - (b_n - B_n) \delta_t u^{\frac{1}{2}} - B_n \phi_1 \right] = (L_x + L_y) u^{n+\frac{1}{2}} + f^{n+\frac{1}{2}} + O(\tau^{3-\beta}), \end{aligned} \tag{15}$$

where $\mu = \tau^{\beta-1} \Gamma(3 - \beta)$.

Denoting $\tilde{u}_h^{n+1} = u_h^{n+1} - u_h^n$, then multiplying by $\mu\tau$ on both sides of (15), one has

$$\begin{aligned} \tilde{u}^{n+1} - \frac{\mu\tau}{2} (L_x + L_y) \tilde{u}^{n+1} \\ = \sum_{k=1}^n (b_{n-k} - b_{n-k+1}) \tilde{u}^k + (b_n - B_n) \tilde{u}^1 + \tau B_n \phi_1 \\ + \mu\tau (L_x + L_y) u^n + \mu\tau f^{n+\frac{1}{2}}, \quad n \geq 1, \end{aligned} \tag{16}$$

and

$$\tilde{u}^1 - \frac{1}{2} \frac{\mu\tau}{B_0} (L_x + L_y) \tilde{u}_h^1 = \frac{1}{B_0} \left[\tau B_0 \phi_1 + \tau\mu (L_x + L_y) u^0 + \tau\mu f^{\frac{1}{2}} \right], \quad n = 0. \tag{17}$$

Adding two small terms $\frac{(\tau\mu)^2}{4} L_x L_y \tilde{u}^{n+1}$ and $\frac{(\tau\mu)^2}{4B_0^2} L_x L_y \tilde{u}^1$ to the left-hand side of (16) and (17), respectively, we obtain the following ADI Galerkin scheme for (2) in inner product form: Find $\tilde{u}_h^{n+1} \in X_h^r$ for $n = 0, 1, 2, \dots, n_T - 1$ such that

$$\left\{ \begin{array}{l} (\tilde{u}_h^1, \chi) + \frac{1}{2} \frac{\mu\tau}{B_0} A(\tilde{u}_h^1, \chi) + \frac{1}{4} \left(\frac{\mu\tau}{B_0} \right)^2 B(\tilde{u}_h^1, \chi) \\ = \frac{1}{B_0} \left[\tau B_0 (\phi_h^0, \chi) - \tau\mu A(u_h^0, \chi) + \tau\mu (I_h f(t_{\frac{1}{2}}), \chi) \right], \quad n = 0, \\ (\tilde{u}_h^{n+1}, \chi) + \frac{\mu\tau}{2} A(\tilde{u}_h^{n+1}, \chi) + \left(\frac{\mu\tau}{2} \right)^2 B(\tilde{u}_h^{n+1}, \chi) \\ = \sum_{k=1}^n (b_{n-k} - b_{n-k+1}) (\tilde{u}_h^k, \chi) + (b_n - B_n) (\tilde{u}_h^1, \chi) + \tau B_n (\phi_h^0, \chi) \\ - \mu\tau A(u_h^n, \chi) + \mu\tau (I_h f(t_{n+\frac{1}{2}}), \chi), \quad n \geq 1, \end{array} \right. \tag{18}$$

with initial values $u_h^0 = \Pi_h^{1,0} \phi_0$ and $\phi_h^0 = \Pi_h^{1,0} \phi_1$. From (18) we can obtain the following approximation to u :

$$u_h^{n+1} = \tilde{u}_h^{n+1} + u_h^n, \quad n \geq 0.$$

Here the bilinear forms A and B are defined by (10) and (11), respectively. The matrix form of (18) can be formulated similarly to (8), we omit the details here.

4 Stability and convergence

In this section, we first study the stability and convergence of the fully discrete scheme (8). To this end, we rewrite (8) as

$$(\delta_t \phi_h^{n+\frac{1}{2}}, v) + A(u_h^{n+\frac{1}{2}}, v) + \frac{\tau^4}{16} B(\delta_t \phi_h^{n+\frac{1}{2}}, v) = (I_h f(t_{n+\frac{1}{2}}), v), \quad v \in X_h^r. \quad (19)$$

Denote $\alpha = (\alpha_1, \alpha_2)$, $\alpha_{\max} = \max\{\alpha_1, \alpha_2\}$. The seminorm $|\cdot|_\alpha$ and norm $\|\cdot\|_\alpha$ are defined respectively by

$$|u|_\alpha = \left(K_x (RLD_{a,x}^{\alpha_1} u, RLD_{x,b}^{\alpha_1} u) + K_y (RLD_{c,y}^{\alpha_2} u, RLD_{y,d}^{\alpha_2} u) \right)^{\frac{1}{2}}, \quad \|u\|_\alpha = (\|u\|^2 + |u|_\alpha^2)^{\frac{1}{2}}.$$

From the definition of the bilinear form A , we can readily obtain $|u|_\alpha \leq C \sqrt{A(u, u)}$.

The seminorm $|\cdot|_\alpha$ and norm $\|\cdot\|_\alpha$ are equivalent if $u \in H_0^{\alpha_1} \cap H_0^{\alpha_2}$, see [Zeng, Liu, Li et al. (2014)]. This result is demonstrated by the following lemma.

Lemma 4.1. If $u \in H_0^{\alpha_1} \cap H_0^{\alpha_2}$, then the seminorm $|\cdot|_\alpha$ and norm $\|\cdot\|_\alpha$ are equivalent, that is, there exists positive constants $C_1 < 1$ and C_2 independent of u , such that

$$C_1 \|u\|_\alpha \leq |u|_\alpha \leq \|u\|_\alpha \leq C_2 |u|_{H^{\alpha_{\max}}(\Omega)}.$$

Define the operator $\Pi_h : V \rightarrow X_h^r$. Suppose that $u \in H^l(\Omega)$, $0 < l < r + 1$, $0 \leq s \leq l$, then the following approximation property holds,

$$\|u - \Pi_h u\|_s \leq C h^{l-s} \|u\|_l, \quad (20)$$

where the constant C is only dependent on Ω .

Define the orthogonal projection operator $\Pi_h^{\alpha,0} : V \rightarrow X_h^r$ by

$$A(u - \Pi_h^{\alpha,0} u, v) = 0, \quad u \in V, \quad \forall v \in X_h^r. \quad (21)$$

We have the following important lemma for $\Pi_h^{\alpha,0}$ (cf. [Zeng, Liu, Li et al. (2014)]), for which we sketch a proof for completeness.

Lemma 4.2. Let $r \in \mathbb{Z}^+$, $1 \leq m \leq r + 1$, and $u \in H^m(\Omega) \cap V$. If $\alpha_i \neq 1/2$, $i = 1, 2$, then there exists a positive constant C independent of mesh spacing such that

$$|u - \Pi_h^{\alpha,0} u|_\alpha \leq h^{m-\alpha_{\max}} \|u\|_m.$$

Proof. By Cauchy-Schwarz inequality and Lemma 2.3 for the definition of A , we obtain

$$A(u, v) \leq C |u|_\alpha |v|_\alpha, \quad A(u, u) \geq |u|_\alpha^2.$$

It follows that for $\forall u_h \in X_h^r$, one has

$$\begin{aligned} |u - \Pi_h^{\alpha,0} u|_\alpha^2 &= A(u - \Pi_h^{\alpha,0} u, u - \Pi_h^{\alpha,0} u) \\ &\leq A(u - \Pi_h^{\alpha,0} u, u - u_h) + A(u - \Pi_h^{\alpha,0} u, u_h - \Pi_h^{\alpha,0} u) \\ &= A(u - \Pi_h^{\alpha,0} u, u - u_h) \quad (\text{Using (21)}) \\ &\leq C |u - \Pi_h^{\alpha,0} u|_\alpha |u - u_h|_\alpha. \end{aligned}$$

That is,

$$|u - \Pi_h^{\alpha,0} u|_\alpha \leq C|u - u_h|_\alpha.$$

Taking u_h as $\Pi_h u$, and utilizing Lemma 4.1 and the estimate (20), we obtain the desired result.

Lemma 4.3. If $w \in H^m \cap H_0^{\alpha_1} \cap H_0^{\alpha_2}$ and $1 \leq m \leq r + 1$, then we have

$$A(w^{n+\frac{1}{2}}, \delta_t w^{n+\frac{1}{2}}) = \frac{1}{2\tau} (|w^{n+1}|_\alpha^2 - |w^n|_\alpha^2),$$

and

$$B(\delta_t w^{n+\frac{1}{2}}, w^{n+\frac{1}{2}}) = \frac{1}{2\tau} [B(w^{n+1}, w^{n+1}) - B(w^n, w^n)]. \tag{22}$$

Proof. One can readily derive the above results from the symmetry of these bilinear forms A and B defined by (10) and (11). Thus we complete the proof.

Combining Lemma 2.4 with the above proof for equality (22), we readily obtain the following result.

Lemma 4.4. If $w \in H^m \cap H_0^{\alpha_1} \cap H_0^{\alpha_2}$ and $1 \leq m \leq r + 1$, then we have

$$B(w, w) \geq 0.$$

We shall need the following version of Gronwall’s inequality (cf. [Zeng, Cao and Li (2013)]).

Lemma 4.5. Assume that $\{\kappa_n\}$ and $\{p_n\}$ are nonnegative sequences, and the sequence $\{\rho_n\}$ satisfies

$$\rho_0 \leq g_0, \quad \rho_n \leq g_0 + \sum_{k=0}^{n-1} p_k + \sum_{k=0}^{n-1} \kappa_k \rho_k, \quad n \geq 1, \tag{23}$$

where $g_0 \geq 0$. Then the sequence $\{\rho_n\}$ satisfies

$$\rho_n \leq \left(g_0 + \sum_{k=0}^{n-1} p_k \right) \exp \left(\sum_{k=0}^{n-1} \kappa_k \right), \quad n \geq 1.$$

Now, we are in the position to present the stability for the fully discrete scheme (8).

Theorem 4.1. The fully discrete scheme (8) associated with (9) is unconditionally stability in the sense that

$$\|\phi_h^m\|^2 + |u_h^m|_\alpha^2 \leq C \left[\|\phi_h^0\|^2 + |u_h^0|_\alpha^2 + \frac{\tau^4}{16} B(\phi_h^0, \phi_h^0) + \tau \sum_{k=0}^{m-1} (\|f^{k+\frac{1}{2}}\|^2 + |\rho^{k+\frac{1}{2}}|_\alpha^2) \right],$$

where C is a positive constant independent of mesh spacing, B is a bilinear form defined by (11), and $\rho^{n+1/2}$ is the perturbation term for (9), that is

$$\delta_t u^{n+\frac{1}{2}} = \phi^{n+\frac{1}{2}} + \rho^{n+\frac{1}{2}}.$$

Proof. Let $v = \phi_h^{n+\frac{1}{2}}$ or, equivalently, $v = \delta_t u_h^{n+1/2} - \rho^{n+1/2}$ in the equivalent form (19) of (8), then we obtain

$$\begin{aligned} & (\delta_t \phi_h^{n+\frac{1}{2}}, \phi_h^{n+\frac{1}{2}}) + A(u_h^{n+\frac{1}{2}}, \delta_t u_h^{n+\frac{1}{2}}) + \frac{\tau^4}{16} B(\delta_t \phi_h^{n+\frac{1}{2}}, \phi_h^{n+\frac{1}{2}}) \\ &= (I_h f(t_{n+\frac{1}{2}}), \phi_h^{n+\frac{1}{2}}) + A(u_h^{n+\frac{1}{2}}, \rho^{n+\frac{1}{2}}). \end{aligned} \quad (24)$$

Note that

$$(\delta_t \phi_h^{n+\frac{1}{2}}, \phi_h^{n+\frac{1}{2}}) = \frac{1}{2\tau} [\|\phi_h^{n+1}\|^2 - \|\phi_h^n\|^2].$$

So, it follows from Lemma 4.3 that (24) can be rewritten as

$$\begin{aligned} & \|\phi_h^{n+1}\|^2 - \|\phi_h^n\|^2 + |u_h^{n+1}|_\alpha^2 - |u_h^n|_\alpha^2 + \frac{\tau^4}{16} B(\phi_h^{n+1}, \phi_h^{n+1}) - \frac{\tau^4}{16} B(\phi_h^n, \phi_h^n) \\ &= 2\tau (I_h f(t_{n+\frac{1}{2}}), \phi_h^{n+\frac{1}{2}}) + 2\tau A(u_h^{n+\frac{1}{2}}, \rho^{n+\frac{1}{2}}). \end{aligned} \quad (25)$$

Using Cauchy-Schwarz inequality and the equivalence of $|\cdot|_\alpha$ and $\|\cdot\|_\alpha$, one has

$$\begin{aligned} & 2A(u_h^{n+\frac{1}{2}}, \rho^{n+\frac{1}{2}}) \\ &= A(u_h^{n+1}, \rho^{n+\frac{1}{2}}) + A(u_h^n, \rho^{n+\frac{1}{2}}) \\ &\leq |u_h^{n+1}|_\alpha |\rho^{n+\frac{1}{2}}|_\alpha + |u_h^n|_\alpha |\rho^{n+\frac{1}{2}}|_\alpha \\ &\leq C [|u_h^{n+1}|_\alpha^2 + |u_h^n|_\alpha^2 + |\rho^{n+\frac{1}{2}}|_\alpha^2]. \end{aligned}$$

Similarly, we have

$$\begin{aligned} & 2(I_h f(t_{n+\frac{1}{2}}), \phi_h^{n+\frac{1}{2}}) \\ &= (I_h f(t_{n+\frac{1}{2}}), \phi_h^{n+1}) + (I_h f(t_{n+\frac{1}{2}}), \phi_h^n) \\ &\leq C [\tau \|(I_h f(t_{n+\frac{1}{2}}))\|^2 + \|\phi_h^{n+1}\|^2 + \|\phi_h^n\|^2]. \end{aligned}$$

For sufficiently small τ , we can derive from (25) that

$$\begin{aligned} & \|\phi_h^{n+1}\|^2 - \|\phi_h^n\|^2 + |u_h^{n+1}|_\alpha^2 - |u_h^n|_\alpha^2 + \frac{\tau^4}{16} B(\phi_h^{n+1}, \phi_h^{n+1}) - \frac{\tau^4}{16} B(\phi_h^n, \phi_h^n) \\ &\leq C [\tau \|(I_h f(t_{n+\frac{1}{2}}))\|^2 + |\rho^{n+\frac{1}{2}}|_\alpha^2 + \tau (\|\phi_h^n\|^2 + |u_h^n|_\alpha^2)]. \end{aligned} \quad (26)$$

Then summing up n from 0 to $m-1$ for (26) gives

$$\begin{aligned} & \|\phi_h^m\|^2 + |u_h^m|_\alpha^2 + \frac{\tau^4}{16} B(\phi_h^m, \phi_h^m) \\ &\leq C [\|\phi_h^0\|^2 + |u_h^0|_\alpha^2 + \frac{\tau^4}{16} B(\phi_h^0, \phi_h^0) \\ &+ \tau \sum_{n=0}^{m-1} (\|(I_h f(t_{n+\frac{1}{2}}))\|^2 + |\rho^{n+\frac{1}{2}}|_\alpha^2) + \tau \sum_{n=0}^{m-1} (\|\phi_h^n\|^2 + |u_h^n|_\alpha^2)]. \end{aligned} \quad (27)$$

From Lemma 4.4, the third term on the left-hand side of (27) is nonnegative. Removing the nonnegative term, we get

$$\begin{aligned} & \|\phi_h^m\|^2 + |u_h^m|_\alpha^2 \\ & \leq C \left[\|\phi_h^0\|^2 + |u_h^0|_\alpha^2 + \frac{\tau^4}{16} B(\phi_h^0, \phi_h^0) \right. \\ & \quad \left. + \tau \sum_{n=0}^{m-1} (\|I_h f(t_{n+\frac{1}{2}})\|^2 + |\rho^{n+\frac{1}{2}}|_\alpha^2) + \tau \sum_{n=0}^{m-1} (\|\phi_h^n\|^2 + |u_h^n|_\alpha^2) \right]. \end{aligned} \tag{28}$$

Comparing each term in the above inequality with that in (23), then applying Lemma 4.5, we can conclude that

$$\begin{aligned} & \|\phi_h^m\|^2 + |u_h^m|_\alpha^2 \\ & \leq C \left[\|\phi_h^0\|^2 + |u_h^0|_\alpha^2 + \frac{\tau^4}{16} B(\phi_h^0, \phi_h^0) + \tau \sum_{n=0}^{m-1} (\|I_h f(t_{n+\frac{1}{2}})\|^2 + |\rho^{n+\frac{1}{2}}|_\alpha^2) \right]. \end{aligned}$$

This completes the proof.

Next, we study the convergence analysis for (8).

Theorem 4.2. Suppose that $r \geq 1$, u and u_h^{n+1} ($0 \leq n \leq n_T - 1$) are the solutions of (1) and the fully discrete scheme (8), respectively. If $u \in C^3([0, T]; H^m) \cap V$, $\frac{\partial^{\alpha_1}}{\partial |x|^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial |y|^{\alpha_2}} \partial_{tt} u \in L^\infty(0, T; H^m(\Omega))$, $m \geq r + 1$, then there exists a positive constant C independent of τ and h such that

$$|u_h^{n+1} - u(t_{n+1})|_\alpha \leq C(\tau^2 + h^{r+1-\alpha_{\max}}).$$

Proof. It is convenient to use the equivalent forms of (8) and (9) below.

$$\begin{cases} (\delta_t \phi_h^{n+\frac{1}{2}}, v) + A(u_h^{n+\frac{1}{2}}, v) + \frac{\tau^4}{16} B(\delta_t \phi_h^{n+\frac{1}{2}}, v) = (I_h f(t_{n+\frac{1}{2}}), v), \\ \delta_t u_h^{n+\frac{1}{2}} = \phi_h^{n+\frac{1}{2}}. \end{cases} \tag{29}$$

On the other hand, we consider the inner form for (3) at $t = t_{n+1/2}$, that is,

$$\begin{cases} \left(\frac{\partial \phi}{\partial t}, v \right) \Big|_{t=t_{n+1/2}} = ((L_x + L_y)u, v) \Big|_{t=t_{n+1/2}} + (f(x, y, t_{n+1/2}), v), \\ \frac{\partial u}{\partial t} \Big|_{t=t_{n+1/2}} = \phi, \end{cases}$$

which can be written as

$$\begin{cases} (\delta_t \phi^{n+\frac{1}{2}}, v) + A(u^{n+\frac{1}{2}}, v) + \frac{\tau^4}{16} B(\delta_t \phi^{n+\frac{1}{2}}, v) \\ \quad = (f(t_{n+1/2}), v) + ([\delta_t \phi^{n+\frac{1}{2}} - \partial_t \phi|_{t=t_{n+1/2}}], v) + \frac{\tau^4}{16} B(\delta_t \phi^{n+\frac{1}{2}}, v), \\ \delta_t u^{n+\frac{1}{2}} = \phi^{n+\frac{1}{2}} + \delta_t u^{n+\frac{1}{2}} - \frac{\partial u}{\partial t} \Big|_{t=t_{n+1/2}}. \end{cases} \tag{30}$$

Let $u_* = \Pi_h^{\alpha,0}u$, and denote $e = u_* - u_h, \eta = u - u_*, \hat{e} = \frac{\partial u_*}{\partial t} - \phi_h$, and $\hat{\eta} = \phi - \frac{\partial u_*}{\partial t}$. Then $u - u_h = e + \eta, \phi - \phi_h = \hat{e} + \hat{\eta}$. Subtracting (29) from (30) gives the following error equations.

$$\begin{cases} (\delta_t \hat{e}^{n+\frac{1}{2}}, v) + A(e^{n+\frac{1}{2}}, v) + \frac{\tau^4}{16} B(\delta_t \hat{e}^{n+\frac{1}{2}}, v) \\ \quad = (R^{n+1/2}, v) + \frac{\tau^4}{16} B(\delta_t(\phi^{n+\frac{1}{2}} - \hat{\eta}^{n+\frac{1}{2}}), v), \\ \delta_t u^{n+\frac{1}{2}} = \hat{e}^{n+\frac{1}{2}} + \rho^{n+\frac{1}{2}}. \end{cases}$$

where

$$\begin{aligned} R^{n+1/2} &= \delta_t \phi^{n+\frac{1}{2}} - \partial_t \phi|_{t=t_{n+1/2}} - \delta_t \hat{\eta}^{n+\frac{1}{2}} + f(t_{n+1/2}) - I_h f(t_{n+1/2}), \\ \rho^{n+\frac{1}{2}} &= \hat{\eta}^{n+\frac{1}{2}} + \delta_t u^{n+\frac{1}{2}} - \frac{\partial u}{\partial t}|_{t=t_{n+1/2}} - \delta_t \eta^{n+\frac{1}{2}}. \end{aligned}$$

By Theorem 4.1, we can derive that

$$\begin{aligned} & \|\hat{e}^m\|^2 + |e^m|_\alpha^2 \\ & \leq C \left[\|\hat{e}^0\|^2 + |e^0|_\alpha^2 + \tau^4 B(\hat{e}^0, \hat{e}^0) + \tau \sum_{k=0}^{m-1} (\|R^{k+1/2}\|^2 + |\rho^{k+\frac{1}{2}}|_\alpha^2) \right. \\ & \quad \left. + \tau^5 \sum_{k=0}^{m-1} \left\| \frac{\partial^{\alpha_1}}{\partial |x|^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial |y|^{\alpha_2}} (\delta_t(\phi^{k+\frac{1}{2}} - \hat{\eta}^{k+\frac{1}{2}})) \right\|^2 \right], \end{aligned} \tag{31}$$

Next, we estimate the right-hand side of the above inequality.

For the initial errors e^0 and \hat{e}^0 , we have

$$\begin{aligned} |e^0|_\alpha &= |\Pi_h^{\alpha,0} \phi_0 - \Pi_h^{1,0} \phi_0|_\alpha \leq |\Pi_h^{\alpha,0} \phi_0 - \phi_0|_\alpha + |\phi_0 - \Pi_h^{1,0} \phi_0|_\alpha \leq Ch^{r+1-\alpha_{\max}}, \\ \|\hat{e}^0\| &= \|\Pi_h^{\alpha,0} \phi_1 - \Pi_h^{1,0} \phi_1\| \leq \|\Pi_h^{\alpha,0} \phi_1 - \phi_1\| + \|\phi_1 - \Pi_h^{1,0} \phi_1\| \leq Ch^{r+1-\alpha_{\max}}. \end{aligned}$$

The third term on the right-hand side of (31) has the following estimate.

$$\tau^4 B(\hat{e}^0, \hat{e}^0) \leq C\tau^4.$$

Since

$$\begin{aligned} \|\delta_t \phi^{n+\frac{1}{2}} - \partial_t \phi|_{t=t_{n+1/2}}\| &\leq C\tau^2, \\ \|f(t_{n+1/2}) - I_h f(t_{n+1/2})\| &\leq Ch^{r+1}, \\ \|\delta_t u^{n+\frac{1}{2}} - \frac{\partial u}{\partial t}|_{t=t_{n+1/2}}\| &\leq C\tau^2, \end{aligned}$$

we obtain

$$\begin{aligned} & \tau \sum_{k=0}^{m-1} \|R^{k+1/2}\|^2 \\ & \leq C \left[\tau^4 + h^{2(r+1)} + \tau \sum_{k=0}^{m-1} \|\delta_t \hat{\eta}^{n+\frac{1}{2}}\|^2 \right] \\ & \leq C \left[\tau^4 + h^{2(r+1)} + \left\| \frac{\partial \hat{\eta}}{\partial t} \right\|_{L^2(0,T;L^2(\Omega))}^2 \right] \\ & \leq C \left[\tau^4 + h^{2(r+1)} \right], \end{aligned}$$

and

$$\begin{aligned} & \tau \sum_{k=0}^{m-1} |\rho^{n+\frac{1}{2}}|_{\alpha}^2 \\ & \leq C \left[\tau^4 + \tau \sum_{k=0}^{m-1} |\hat{\eta}^{n+\frac{1}{2}}|_{\alpha}^2 + \tau \sum_{k=0}^{m-1} |\delta_t \eta^{n+\frac{1}{2}}|_{\alpha}^2 \right] \\ & \leq C \left[\tau^4 + h^{2(r+1-\alpha_{\max})} \right]. \end{aligned}$$

For the last term in the right-hand side of (31), we can obtain

$$\begin{aligned} & \tau^5 \sum_{k=0}^{m-1} \left\| \frac{\partial^{\alpha_1}}{\partial |x|^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial |y|^{\alpha_2}} (\delta_t (\phi^{k+\frac{1}{2}} - \hat{\eta}^{k+\frac{1}{2}})) \right\|^2 \\ & \leq \tau^5 \sum_{k=0}^{m-1} \left[\left\| \frac{\partial^{\alpha_1}}{\partial |x|^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial |y|^{\alpha_2}} \delta_t \phi^{k+\frac{1}{2}} \right\|^2 + \left\| \frac{\partial^{\alpha_1}}{\partial |x|^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial |y|^{\alpha_2}} \delta_t \hat{\eta}^{k+\frac{1}{2}} \right\|^2 \right] \leq C \tau^4. \end{aligned}$$

Therefore, from (31), we have

$$|e^{n+1}|_{\alpha} \leq C(\tau^2 + h^{r+1-\alpha_{\max}}).$$

It follows that

$$|u_h^{n+1} - u(t_{n+1})|_{\alpha} = |e^{n+1} + \eta^{n+1}|_{\alpha} \leq |e^{n+1}|_{\alpha} + |\eta^{n+1}|_{\alpha} \leq C(\tau^2 + h^{r+1-\alpha_{\max}}).$$

All this completes the proof.

In the following, we demonstrate the stability for the ADI Galerkin scheme (18).

Theorem 4.3. Suppose that u_h^n is the solution of the ADI Galerkin scheme (18) and $f \in C([0, T]; L^2(\Omega))$. Then it holds that

$$|u_h^n|_{\alpha}^2 \leq C \left[|u_h^0|_{\alpha}^2 + \|\phi_h^0\|^2 + \tau \sum_{k=1}^n \|f^{k-\frac{1}{2}}\|^2 \right], \tag{32}$$

where C is a positive constant and independent of τ and h .

Proof. Notice that the properties of the coefficients B_n and b_n are nice, one can readily derive the above stability result (32) by using the idea in the proof of Theorem 4.1. Thus we omit the details here.

Finally, we present the error estimate for the ADI Galerkin scheme (18).

Theorem 4.4. Suppose that $r \geq 1$, u and u_h^{n+1} ($0 \leq n \leq n_T - 1$) are the solutions of (2) and the fully discrete scheme (18), respectively. If $u \in C^3([0, T]; H^m) \cap V$, ${}_C D_{0,t}^\beta \in L^\infty(0, T; H^m)$, $\frac{\partial^{\alpha_1}}{\partial |x|^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial |y|^{\alpha_2}} \partial_{tt} u \in L^\infty(0, T; H^m(\Omega))$, $m \geq r + 1$, then there exists a positive constant C independent of τ and h such that

$$|u_h^{n+1} - u(t_{n+1})|_\alpha \leq C(\tau^{\min\{3-\beta, \frac{1+\beta}{2}\}} + h^{r+1-\alpha_{\max}}). \quad (33)$$

Proof. We first consider the case $n \geq 1$ in (18). We use its equivalent form which is shown below,

$$\begin{aligned} & (\delta_t u_h^{n+\frac{1}{2}}, \chi) + \mu A(u_h^{n+\frac{1}{2}}, \chi) + \frac{(\mu\tau)^2}{4} B(\delta_t u_h^{n+\frac{1}{2}}, \chi) \\ &= \sum_{k=1}^n (b_{n-k} - b_{n-k+1})(\delta_t u_h^{k-\frac{1}{2}}, \chi) + (b_n - B_n)(\delta_t u_h^{\frac{1}{2}}, \chi) + B_n(\phi_h^0, \chi) \\ & \quad + \mu(f^{n+\frac{1}{2}}, \chi). \end{aligned} \quad (34)$$

The weak form of the Eq. (2) on the time level $t_{n+\frac{1}{2}}$ is

$$({}_C D_{0,t}^\beta u(x, y, t_{n+\frac{1}{2}}), v) + A(u(x, y, t_{n+\frac{1}{2}}), v) = (f(x, y, t_{n+\frac{1}{2}}), v), \quad v \in V,$$

which can be written as

$$\begin{aligned} & \frac{\tau^{1-\beta}}{\Gamma(3-\beta)} \left[b_0(\delta_t u^{n+\frac{1}{2}}, v) - \sum_{k=1}^n (b_{n-k} - b_{n-k+1})(\delta_t u^{k-\frac{1}{2}}, v) \right. \\ & \quad \left. - (b_n - B_n)(\delta_t u^{\frac{1}{2}}, v) - B_n(\phi_1, v) \right] = -A(u^{n+\frac{1}{2}}, v) + (f^{n+\frac{1}{2}}, v) + (R^{n+\frac{1}{2}}, v). \end{aligned} \quad (35)$$

Here $R^{n+\frac{1}{2}}$ is the form as denoted in (13), with $\partial^3 u / \partial t^3$ instead of $g^{(3)}(t)$. Multiplying by μ and adding the small term on both side of (35), we obtain

$$\begin{aligned} & (\delta_t u^{n+\frac{1}{2}}, v) + \mu A(u^{n+\frac{1}{2}}, v) + \frac{(\mu\tau)^2}{4} B(\delta_t u^{n+\frac{1}{2}}, v) \\ &= \sum_{k=1}^n (b_{n-k} - b_{n-k+1})(\delta_t u^{k-\frac{1}{2}}, v) + (b_n - B_n)(\delta_t u^{\frac{1}{2}}, v) + B_n(\phi_1, v) \\ & \quad + \mu(f^{n+\frac{1}{2}}, \chi) \left] + \mu(R^{n+\frac{1}{2}}, v) + \frac{(\mu\tau)^2}{4} B(\delta_t u^{n+\frac{1}{2}}, v). \end{aligned} \quad (36)$$

Let $u_* = \Pi_h^{\alpha,0} u$, $e = u_* - u_h$, $\eta = u - u_*$, $\hat{e} = \frac{\partial u_*}{\partial t} - \phi_h$, $\hat{\eta} = \phi - \frac{\partial u_*}{\partial t}$, and $v = \chi$ in (36).

Subtracting (34) from (36) and using the properties of u_* , we obtain

$$\begin{aligned} & (\delta_t e^{n+\frac{1}{2}}, \chi) + \mu A(e^{n+\frac{1}{2}}, \chi) + \frac{(\mu\tau)^2}{4} B(\delta_t e^{n+\frac{1}{2}}, \chi) \\ &= \sum_{k=1}^n (b_{n-k} - b_{n-k+1})(\delta_t e^{k-\frac{1}{2}}, \chi) \\ & \quad + (b_n - B_n)(\delta_t e^{\frac{1}{2}}, \chi) + B_n(\hat{e}^0, \chi) \Big] + \mu(\rho^n + r^n, \chi) + \frac{(\mu\tau)^2}{4} B(\sigma^n, \chi). \end{aligned}$$

Here

$$\rho^n = \frac{1}{\mu} \left[\sum_{k=1}^n (b_{n-k} - b_{n-k+1}) \delta_t \eta^{k-\frac{1}{2}} + (b_n - B_n) \delta_t \eta^{\frac{1}{2}} + B_n \hat{\eta}^0 - \delta_t \eta^{n+\frac{1}{2}} \right],$$

$$\sigma^n = \delta_t (u^{n+\frac{1}{2}} - \eta^{n+\frac{1}{2}}), \text{ and } r^n = R^{n+\frac{1}{2}}.$$

From (32) we can derive that

$$|e^n|_\alpha^2 \leq C \left[\|\hat{e}^0\|^2 + |e^0|_\alpha^2 + \tau \sum_{k=0}^{n-1} \|\rho^k + r^k\|^2 + \mu\tau^3 \sum_{k=0}^{n-1} \left\| \frac{\partial^{\alpha_1}}{\partial |x|^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial |y|^{\alpha_2}} \sigma^k \right\|^2 \right].$$

It follows that

$$\begin{aligned} |u^n - u_h^n|_\alpha^2 &\leq C \left[|\eta^n|_\alpha^2 + |e^n|_\alpha^2 \right] \\ &\leq C \left[|\eta^n|_\alpha^2 + \|\hat{e}^0\|^2 + |e^0|_\alpha^2 + \tau \sum_{k=0}^{n-1} \|\rho^k + r^k\|^2 \right. \\ & \quad \left. + \mu\tau^3 \sum_{k=0}^{n-1} \left\| \frac{\partial^{\alpha_1}}{\partial |x|^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial |y|^{\alpha_2}} \sigma^k \right\|^2 \right]. \end{aligned} \tag{37}$$

In the following, we estimate the terms on the right-hand side of (37). First, by Lemma 4.2, we obtain

$$|\eta^n|_\alpha \leq Ch^{r+1-\alpha_{\max}}. \tag{38}$$

Second, we have

$$\tau \sum_{k=0}^{n-1} \|\rho^k\|^2 \leq C\tau \sum_{k=0}^{n-1} \|CD_{0,t}^\alpha \eta(t_{k+1/2})\|^2 \leq C \|CD_{0,t}^\alpha \eta\|_{L^\infty(L^2)}^2 \leq Ch^{2(r+1-\alpha_{\max})}, \tag{39}$$

and

$$\tau \sum_{k=0}^{n-1} \|r^k\|^2 \leq C\tau^{6-2\beta}. \tag{40}$$

For the last term on the right-hand side of equation (37), we have the following estimate.

$$\begin{aligned} & \mu\tau^3 \sum_{k=0}^{n-1} \left\| \frac{\partial^{\alpha_1}}{\partial|x|^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial|y|^{\alpha_2}} \sigma^k \right\|^2 \\ & \leq \mu\tau^3 \sum_{k=0}^{n-1} \left[\left\| \frac{\partial^{\alpha_1}}{\partial|x|^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial|y|^{\alpha_2}} \delta_t u^{k+\frac{1}{2}} \right\|^2 + \left\| \frac{\partial^{\alpha_1}}{\partial|x|^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial|y|^{\alpha_2}} \delta_t \eta^{k+\frac{1}{2}} \right\|^2 \right] \leq C\tau^{1+\beta}. \end{aligned} \tag{41}$$

So using (38)-(41) gives (33). All this ends the proof.

Remark 4.1. It seems that the convergence order in time of the ADI Galerkin scheme (18) will be destroyed when $\beta < \frac{5}{3}$. However we observe that the convergence order in time always be $3 - \beta$ in the numerical tests. The disagreement probably means that the theoretical analysis in (33) may be improved. One possible way to remove the term $O(\tau^{\frac{1+\beta}{2}})$ is by introducing a much smaller disturbance term to construct the ADI Galerkin scheme, however it seems that the resulting scheme does not improve the numerical results in practical computation but only cause more complicated computations.

5 Numerical results

In this section, we present numerical results to verify the theoretical analysis. For sake of computational convenience, in our numerical tests, we fix the coefficients $K_x = K_y = 1/2$, and let $\Omega = (0, 1) \times (0, 1)$ and $T = 1$. All the tests are done by using the linear element.

Example 5.1. Consider the following initial conditions in the two-dimensional fractional model (1):

$$u(x, y, 0) = (x - x^2)(y - y^2), \quad \frac{\partial u}{\partial t} \Big|_{t=0} = -(x - x^2)(y - y^2).$$

The source term f is given by

$$\begin{aligned} f(x, y, t) = & e^{-t}(x - x^2)(y - y^2) \\ & + c_1 K_x e^{-t}(y - y^2) \left[\frac{x^{1-2\alpha_1} + (1 - x)^{1-2\alpha_1}}{\Gamma(2 - 2\alpha_1)} - 2 \frac{x^{2-2\alpha_1} + (1 - x)^{2-2\alpha_1}}{\Gamma(3 - 2\alpha_1)} \right] \\ & + c_2 K_y e^{-t}(x - x^2) \left[\frac{y^{1-2\alpha_2} + (1 - y)^{1-2\alpha_2}}{\Gamma(2 - 2\alpha_2)} - 2 \frac{y^{2-2\alpha_2} + (1 - y)^{2-2\alpha_2}}{\Gamma(3 - 2\alpha_2)} \right]. \end{aligned}$$

The corresponding exact solution is

$$u(x, y, t) = e^{-t}(x - x^2)(y - y^2).$$

From the ADI Galerkin scheme (8), we obtain the numerical results of Tabs. 1-2 for Eq. (1). All the numerical errors are computed at $t = 1$ for different values of α_1 and α_2 . From Tabs. 1-2, we can see that convergent orders in space and in time are $(2 - \alpha_{\max})$ -order and second-order accuracy in seminorm $|\cdot|_\alpha$ respectively, which is in line with the theoretical analysis.

Table 1: The $|\cdot|_\alpha$ errors in space for Eq. (1) and $\tau = 1/128$

$(\alpha_1, \alpha_2) \setminus 1/h$	4	8	16	32	64
(0.7, 0.8)	1.8091E-02	6.2876E-03	2.4139E-03	9.8037E-04	4.0946E-04
	-	1.5247	1.3811	1.3000	1.2596
(0.8, 0.6)	1.6880E-02	5.7873E-03	2.2043E-03	8.9343E-04	3.7385E-04
	-	1.5443	1.3926	1.3029	1.2569
(0.9, 0.9)	2.5674E-02	1.1559E-02	5.6330E-03	2.8015E-03	1.3968E-03
	-	1.1513	1.0370	1.0077	1.0041

Table 2: The $|\cdot|_\alpha$ errors in time for Eq. (1) and $h = \tau^{2/(2-\alpha_{\max})}$

$(\alpha_1, \alpha_2) \setminus 1/\tau$	8	10	12	14	16
(0.7, 0.8)	9.0185E-04	5.7691E-04	4.0031E-04	2.8875E-04	2.2066E-04
	-	2.00	2.00	2.12	2.01
(0.8, 0.6)	8.2156E-04	5.2625E-04	3.6580E-04	2.6436E-04	2.0239E-04
	-	2.00	1.99	2.11	2.00
(0.9, 0.9)	1.3025E-03	8.3332E-04	5.7845E-04	4.2427E-04	3.2614E-04
	-	2.00	2.00	2.01	1.97

Example 5.2. Consider the following homogeneous initial conditions in the two-dimensional space-time fractional model (2):

$$u(x, y, 0) = 0, \quad \frac{\partial u}{\partial t} \Big|_{t=0} = 0.$$

The source term f is given by

$$\begin{aligned} f(x, y, t) = & \frac{6t^{3-\beta}}{\Gamma(4-\beta)}(x-x^2)(y-y^2) \\ & + c_1 K_x t^3 (y-y^2) \left[\frac{x^{1-2\alpha_1} + (1-x)^{1-2\alpha_1}}{\Gamma(2-2\alpha_1)} - 2 \frac{x^{2-2\alpha_1} + (1-x)^{2-2\alpha_1}}{\Gamma(3-2\alpha_1)} \right] \\ & + c_2 K_y t^3 (x-x^2) \left[\frac{y^{1-2\alpha_2} + (1-y)^{1-2\alpha_2}}{\Gamma(2-2\alpha_2)} - 2 \frac{y^{2-2\alpha_2} + (1-y)^{2-2\alpha_2}}{\Gamma(3-2\alpha_2)} \right]. \end{aligned}$$

Then we obtain the exact solution below

$$u(x, y, t) = t^3(x-x^2)(y-y^2).$$

Using the ADI Galerkin scheme (18), we obtain the numerical results of Tabs. 3-4. Here we let $\tau = 1/128$ when test the convergence order in space (see Tab. 3), and let $h = \tau^{(3-\beta)/(2-\alpha_{max})}$ when test the convergence order in time (see Tab. 4). The numerical errors are also obtained at $t = 1$. From Tabs. 3-4, we observe that the convergence orders are $2 - \alpha_{max}$ and $3 - \beta$ in space and in time, respectively, which show better results than that in Theorem 4.4.

Table 3: The $|\cdot|_\alpha$ errors in space for Eq. (2) and $\tau = 1/128$

$(\beta, \alpha_1, \alpha_2) \setminus 1/h$	4	8	16	32	64
(1.2, 0.6, 0.6)	2.2963E-02	8.2403E-03	3.0516E-03	1.1438E-03	4.3074E-04
	-	1.4785	1.4331	1.4157	1.4089
(1.5, 0.7, 0.8)	3.3901E-02	1.4005E-02	5.9388E-03	2.5407E-03	1.0916E-03
	-	1.2754	1.2377	1.2250	1.2188
(1.8, 0.8, 0.9)	4.3564E-02	1.9590E-02	8.9962E-03	4.1585E-03	1.9362E-03
	-	1.1530	1.1227	1.1132	1.1028

Example 5.3. Now, we consider the following case for the space-time fractional model (2) in which the exact solution is unknown: $f = 1$. The corresponding initial conditions are provided by $u(x, y, 0) = 0$ and $\frac{\partial u}{\partial t} \Big|_{t=0} = 0$.

Since the exact solution is unknown, we determine the convergence order by the ratios of differences between different numerical solutions which are computed by the consecutive halved stepsizes. That is, we compute the spatial convergence order by adopting $\log_2(e(h)/e(h/2))$ with the error $e(h) = |u_h - u_{h/2}|_\alpha$. By applying the numerical scheme

Table 4: The $|\cdot|_\alpha$ errors in time for Eq. (2) and $h = \tau^{(3-\beta)/(2-\alpha_{\max})}$

$(\beta, \alpha_1, \alpha_2) \setminus 1/\tau$	4	8	16	32	64
(1.1, 0.6, 0.7)	9.6692E-03	2.5979E-03	6.6892E-04	1.7573E-04	4.6300E-05
	-	1.8960	1.9574	1.9285	1.9243
(1.3, 0.7, 0.8)	1.4540E-02	4.5724E-03	1.4354E-03	4.3194E-04	1.3093E-04
	-	1.6690	1.6715	1.7326	1.7220
(1.9, 0.9, 0.9)	5.2136E-02	2.4952E-02	1.1715E-02	5.4774E-03	2.5577E-03
	-	1.0631	1.0909	1.0968	1.0986

(18), we obtain the numerical results which are shown on Tab. 5. From the numerical results, it can be seen that the $|\cdot|_\alpha$ errors will gradually decrease as step sizes shrink at the rate of $1/2$. However, the convergence orders are both less than the theoretical ones. Similar numerical results are also observed for the temporal direction which are not presented here. One possible reason is that the right-hand side function f is constant, which leads to the issue that the regularity of analytical solution in (2) does not meet the requirements in Theorem 4.4.

Table 5: The $|\cdot|_\alpha$ errors in space for Eq. (2) with $f = 1, \tau = 1/512$

$(\beta, \alpha_1, \alpha_2) \setminus 1/h$	4	8	16	32	64
(1.5, 0.6, 0.7)	6.3901E-02	2.8252E-02	1.3456E-02	8.8215E-03	-
	1.1775	1.0701	0.6091	-	-
(1.5, 0.7, 0.8)	5.9511E-02	2.4521E-02	1.0241E-02	5.1653E-03	-
	1.2792	1.2597	0.9874	-	-
(1.5, 0.51, 0.99)	4.6506E-02	1.8758E-02	8.2999E-03	5.0017E-03	-
	1.3099	1.1763	0.7307	-	-

The corresponding theoretical analyses of the less regularity issue are not available in this paper and need to be further studied in the future. However from the above numerical study in Example 5.3, one still can get a glimpse of the behavior of the numerical solution numerically with both the orders α_1 and α_2 tend to one. For simplicity of presentation, we focus on the case $\beta = 2$ here, that is the fractional model (1), and compute the numerical solution with various α_1 and α_2 by the numerical scheme (8). The numerical results are presented in Fig. 1. From Fig. 1, we can observe that the numerical solution is more closer to that of the integer case as the orders α_1 and α_2 tend to one. We also observe similar phenomenon for the case $\beta \in (1, 2)$ in which numerical results are not demonstrated here. From this perspective, we may conclude that these two fractional models considered in this paper can be regarded as the generalization of the corresponding classical integer-order model.

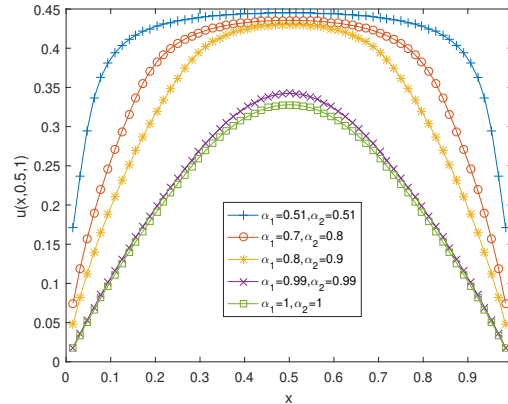


Figure 1: Numerical comparison at $t = 1$ for the fractional model (1) with $f = 1$, $u(x, y, 0) = 0$ and $\frac{\partial u}{\partial t}|_{t=0} = 0$, computed by the scheme (8) with $\tau = 1/512$ and $h = 1/64$

6 Conclusion

In this work, we studied two classes of Riesz space-fractional partial differential equations including space-fractional and space-time fractional derivatives. Our main contributions in this work are to develop the efficient ADI Galerkin finite element schemes for these two considered models and establish their stability and convergence. Numerical tests are presented to verify the convergence theories. For sake of convenience, the current study relies crucially on the high regularity of the solution. It would be interesting to extend the argument to the less regularity issue, for example the weak singularity of the solution at $t = 0$. However it seems that the extension is not an easy job and requires further study, since most of the existed results only deal with the time-fractional problems, see Li et al. [Li and Chen (2018)] and the references therein.

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References

- Cai, M.; Li, C.** (2019): On Riesz derivative. *Fractional Calculus and Applied Analysis*, vol. 22, no. 2, pp. 287-301.
- Cai, M.; Li, C.** (2019): Regularity of the solution to Riesz-type fractional differential equation. *Integral Transforms and Special Functions*, vol. 30, no. 9, pp. 711-742.

- Aman, S.; Al-Mdallal, Q.; Khan, I.** (2018): Heat transfer and second order slip effect on MHD flow of fractional Maxwell fluid in a porous medium. *Journal of King Saud University-Science*, pp. 1-9.
- Al-Mdallal, Q.; Abro, K. A.; Khan, I.** (2018): Analytical solutions of fractional Walter's B fluid with applications. *Complexity*, vol. 2018, pp. 1-10.
- Bira, B.; Raja Sekhar, T.; Zeidan, D.** (2018): Exact solutions for some time-fractional evolution equations using Lie group theory. *Mathematical Methods in the Applied Sciences*, vol. 41, pp. 6717-6725.
- Ervin, V. J.; Roop, J. P.** (2006): Variational formulation for the stationary fractional advection dispersion equation. *Numerical Methods for Partial Differential Equations*, vol. 22, no. 3, pp. 558-576.
- Ervin, V. J.; Roop, J. P.** (2007): Variational solution of fractional advection dispersion equations on bounded domains in \mathbb{R}^d . *Numerical Methods for Partial Differential Equations*, vol. 23, no. 2, pp. 256-281.
- Li, C.; Cai, M.** (2019): *Theory and Numerical Approximations of Fractional Integrals and Derivatives*. Society for Industrial and Applied Mathematics, Philadelphia.
- Li, C.; Chen, A.** (2018): Numerical methods for fractional partial differential equations. *International Journal of Computer Mathematics*, vol. 95, no. 6-7, pp. 1048-1099.
- Li, C.; Wang, Z.** (2020): The discontinuous Galerkin finite element method for Caputo-type nonlinear conservation law. *Mathematics and Computers in Simulation*, vol. 169, pp. 51-73.
- Li, C.; Yi, Q.** (2019): Modeling and computing of fractional convection equation. *Communications on Applied Mathematics and Computation*, pp. 1-31.
- Li, C.; Yi, Q.; Kurths, J.** (2018): Fractional convection. *Journal of Computational and Nonlinear Dynamics*, vol. 13, no. 1, 011004.
- Li, C.; Zeng, F.** (2015): *Numerical Methods for Fractional Calculus*, vol. 24. Chapman and Hall/CRC Press, Boca Raton, USA.
- Wang, H.; Basu, T. S.** (2012): A fast finite difference method for two-dimensional space-fractional diffusion equations. *SIAM Journal on Scientific Computing*, vol. 34, no. 5, pp. A2444-A2458.
- Zeng, F.; Cao, J.; Li, C.** (2013): Gronwall inequalities. *Recent Advances in Applied Nonlinear Dynamics with Numerical Analysis* (Li, C., Wu, Y. and Ye, R., eds.), pp. 49-68. World Scientific, Singapore.
- Zeng, F.; Liu, F.; Li, C.; Burrage, K.; Turner, I. et al.** (2014): A Crank–Nicolson ADI spectral method for a two-dimensional Riesz space fractional nonlinear reaction-diffusion equation. *SIAM Journal on Numerical Analysis*, vol. 52, no. 6, pp. 2599-2622.