Fractional Optimal Control of Navier-Stokes Equations

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Abstract: In this paper, the non-stationary incompressible fluid flows governed by the Navier-Stokes equations are studied in a bounded domain. This study focuses on the time-fractional Navier-Stokes equations in the optimal control subject, where the control is distributed within the domain and the time-fractional derivative is proposed as Riemann-Liouville sort. In addition, the control object is to minimize the quadratic cost functional. By using the Lax-Milgram lemma with the assistance of the fixed-point theorem, we demonstrate the existence and uniqueness of the weak solution to this system. Moreover, for a quadratic cost functional subject to the time-fractional Navier-Stokes equations, we prove the existence and uniqueness of optimal control. Also, via the variational inequality upon introducing the adjoint linearized system, some inequalities and identities are given to guarantee the first-order necessary optimality conditions. A direct consequence of the results obtained here is that when $\alpha \to 1,$ the obtained results are valid for the classical optimal control of systems governed by the Navier-Stokes equations.

Keywords: Optimal control, lax-milgram lemma, fixed point theorem, navier-stokes equations.

1 Introduction

This discussion provides the necessary conditions of the optimal control for the time-fractional Navier-Stokes equations in a bounded domain $\Omega \subset \mathbb{R}^n$ with $n \leq 4$ and smooth boundary $\partial\Omega$. For a fixed $T > 0$, we set $Q = \Omega \times (0, T)$ and $\Sigma = \partial\Omega \times (0,\ T)$. The time-fractional Navier-Stokes equations can be described by the system of equations:

$$\begin{cases} D_+^\alpha y - \mu \Delta y + (y.\nabla)y + \nabla p = f & \text{in } Q, \\ \nabla \cdot y = 0 & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ I_+^{1-\alpha} y(\cdot\,,0) = 0 & \text{in } \Omega, \end{cases} \tag{1}$$

where D_+^α and $I_+^{1-\alpha}$ denote respectively, the Riemann-Liouville time-fractional order derivative and integral, y is the velocity of the fluid, p represents the pressure, f stands the

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given external body forces and μ is a constant.

The idea of fractional calculus deals with derivatives and integrals of any orders. Fractional calculus has gained importance, mainly due to its demonstrated applications in many areas of physics, economics, and engineering. The fractional calculus has been occurring in many physical problems such as damping law, perfusion processes and a motion of a large thin plate in the Newtonian fluid. For more specifics on the scientific applications of fractional calculus, see Kilbas et al. [Kilbas, Srivastava and Trujillo (2006); Ghany and Hyder (2013); Mophou (2011); Zhou (2014); Xie, Jin and Luo (2014); Soliman and Hyder (2020)]. Many engineers and mathematicians have concentrated their efforts on the fulfillment of the Navier-Stokes equations and wave equations see Ghany et al. [Ghany, Hyder and Zakarya (2017); Hyder and Zakarya (2016); Vu-Huu, Le-Thanh, Nguyen-Xuan et al. (2018, 2019, 2019, 2020); Hyder and Barakat (2020)]. Let us introduce a brief review of some results of optimal control for Navier-Stokes equations. The optimal control of classical Navier-Stokes equations has been studied by many authors see Galdi et al. [Galdi (2011); Chowdhury and Ramaswamy (2013); Kien, Lee and Son (2016)].

The optimal control of phenomena governed by fractional models is more precise than the optimal control for models, which has an integer-order derivative. Thus, many researchers are attracted to them see Biswas et al. [Biswas and Sen (2014); El-Nabulsi and Torres (2007); Frederico and Torres (2008); Hyder and El-Badawy (2019); Mophou (2011)].

The main novelty of this paper is the consideration of time-fractional Navier-Stokes equations in the subject of optimal control where the time-fractional derivative is proposed as Riemann-Liouville sort. The fractional optimal control of the Navier-Stokes models has already been examined in Zhou et al. [Zhou and Peng (2016)], this work deals with the optimal control of Navier-Stokes equations with Caputo time-fractional derivative. In that work, the existence and uniqueness of the weak solution are proved via the Galerkin approximations method, and hence, the optimality condition is obtained and leads to approximately optimal control. While in our paper, the existence and uniqueness of the weak solution are proved by a general analytical method and from this result the optimality condition is obtained. Thereby, we can find more and more accurate optimal control.

This methodology used in this paper based on Lax-Milgram lemma, fixed-point theorem, and fractional calculus. The merit of this methodology, atop other alternative methods, appears in the fact that it is an analytical method to obtain the optimality conditions, and hence, we can more accurately evaluate optimal control, whereas most of the alternative methods are approximated methods, like the Galerkin approximations method and finite element methods. Please see Djilali et al. [Djilali and Rougirel (2018); Zhou and Peng (2016)].

This article is arranged as follows. Section 2 introduces some functional spaces to represent the time-fractional Navier-Stokes equations. In Section 3, we formulate the parabolic initial-boundary value problem and the existence of variational form to the initial-boundary value problem is derived. In Section 4, the issue of optimal control with satisfying the existence and uniqueness of that control is created. Moreover, identities and inequalities that describe the optimal control are obtained. Section 5, is devoted to summary and discussion. Section 6, has been specified to open problems for fractional Navier-Stokes equations.

2 Fractional calculus and auxiliary results

The weak solution of the Navier-Stokes equation is based totally on the variational formulation and hence, the use of Sobolev spaces is necessary for the mathematical treatment of the variational formulation of this model. This part is divided into two subparts. In subpart 2.1, we present a brief overview of fractional calculus, and in subpart 2.2, we identify the spaces of our problem with its embedding properties.

2.1 *Overview on fractional calculus*

Definition 2.1. Kilbas et al. [Kilbas, Srivastavam and Trujillo (2006); Mophou (2011); Zhou (2014)] Let $f: \mathbb{R}^+ \to \mathbb{R}$ be a continuous function and $t \in [0,T] \subset \mathbb{R}^+$. The term $I_+^\alpha f(t)$ is called the fractional left Riemann-Liouville integral of $f(t)$ of order $\alpha > 0$ and has the form:

$$I_+^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\xi)^{\alpha-1} f(\xi)\, d\xi, \qquad t > 0. \tag{2}$$

Definition 2.2. Kilbas et al. [Kilbas, Srivastavam and Trujillo (2006); Mophou (2011); Zhou (2014)] Let $f: \mathbb{R}^+ \to \mathbb{R}$ be a continuous function and $t \in [0,T] \subset \mathbb{R}^+$. The fractional left and right Riemann-Liouville derivatives of $f(t)$ of order $\alpha \in (0,1)$ have the forms:

$$D_+^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{f(\xi)}{(t-\xi)^\alpha} d\xi, \quad t > 0, \tag{3}$$

$$D_-^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^T \frac{f(\xi)}{(\xi-t)^\alpha} d\xi, \quad t > 0. \tag{4}$$

Definition 2.3. Kilbas et al. [Kilbas, Srivastavam and Trujillo (2006); Mophou (2011); Zhou (2014)] Let $f: \mathbb{R}^+ \to \mathbb{R}$ be a continuous function and $t \in [0,T] \subset \mathbb{R}^+$. The expressions $D_+^{c,\alpha} f(t)$ and $D_-^{c,\alpha} f(t)$ are called the fractional left and right Caputo derivatives of $f(t)$ of order $\alpha \in (0,1)$ respectively, which have the forms:

$$D_+^{c,\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(\xi)}{(t-\xi)^\alpha} d\xi, \quad t > 0, \tag{5}$$

$$D_-^{c,\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_t^T \frac{f'(\xi)}{(\xi-t)^\alpha} d\xi, \quad t > 0. \tag{6}$$

For a Hilbert space V, the linear map $D(0,T) \to V$ and $\varphi \mapsto -\int_0^T f(t) D_-^\alpha \varphi(t) dt$ forms a distribution with order 1 (at most). The collection of distributions with values in V is denoted by $D^*(0,T;V)$. This permits us to formulate the following notation.

Definition 2.4. Djilali et al. [Djilali and Rougirel (2018)] Let $\alpha \in (0,1)$ and $f \in L^2(0,T;V)$. Then the weak derivative of order α of f is the vector-valued distribution, denoted by $D_+^\alpha f$, and defined for all $\varphi \in D(0,T)$ by

$$\langle D_+^\alpha f, \varphi \rangle_{D^*(0,T;V),D(0,T)} = -\int_0^T f(t) D_-^\alpha \varphi(t) dt. \tag{7}$$

Proposition 2.1. Djilali et al. [Djilali and Rougirel (2018)] Let $\alpha \in (0,1)$, V be a real Banach space and $f \in L^2(0,T;V^*)$. We assume that f admits a derivative of order α in $L^2(0,T;V^*)$. Then, each φ in V, $\langle f,\varphi\rangle_{V^*,V}$ provides a derivative of order α in $L^2(0,T)$ and

$$\langle D_+^\alpha f,\varphi\rangle_{V^*,V} = D_+^\alpha\{\langle f,\varphi\rangle_{V^*,V}\}. \tag{8}$$

2.2 *Function spaces and auxiliary results*

We denote by $\mathcal{D}(\Omega)$ the space of infinitely differentiable functions with compact support in Ω, and $\mathcal{D}^*(\Omega)$ is its dual. We consider the space of divergence free functions defined by $C_{0,div}^\infty(\Omega) = \{v \in \mathcal{D}(\Omega); \nabla.v = 0\}$. We introduce the spaces $V = \{v \in H_0^1(\Omega); \nabla.v = 0\}$ and $H = \{v \in L^2(\Omega); \nabla.v = 0, v.n|_{\partial\Omega} = 0\}$, n is the unit outer normal to the fluid boundary $\partial\Omega$. Where, V is the complete closure of $C_{0,div}^\infty(\Omega)$ for the $H_0^1(\Omega)$-norm, while H is the complete closure of $C_{0,div}^\infty(\Omega)$ for the L^2-norm, the dual of V is denoted by V^*. We denote by $(\cdot,\cdot)$ and $|\cdot|$ the inner product and the norm in H respectively, while we denote by $\|\cdot\|$ the norm in V, and the duality pairing between V and V^* is denoted by $\langle\cdot,\cdot\rangle$. As then, V and H are Hilbert spaces, V is a dense subset of H and $V \subset H \subset V^*$. Also,

$$L^2(0,T;V) \subset L^2(0,T;H) \subset L^2(0,T;V^*). \tag{9}$$

Then, we can construct the function space for our problem, which is given by:

$$W^\alpha(0,T;V,V^*) = \{y: y \in L^2(0,T;V), D_+^\alpha y(x,t) \in L^2(0,T;V^*)\}, \tag{10}$$

where $D_+^\alpha y(x,t)$ is understood in the distribution sense.

Also, $W^\alpha(0,T;V,V^*)$ is Hilbert space with the norm equipped by:

$$\|y\|_{W^\alpha} := \left(\|y\|_{L^2(0,T;V)}^2 + \|D_+^\alpha y\|_{L^2(0,T;V^*)}^2\right)^{\frac{1}{2}}. \tag{11}$$

If $\alpha \to 1$, $W^1(0,T;V,V^*)$ is the standard Sobolev space used for solving some partial differential equations (PDEs) [Tröltzsch (2010)].

Lemma 2.1 Mophou [Mophou (2011)] (Fractional integration by parts) Let $\alpha \in (0,1)$, then for any $\phi \in C^\infty(\bar{Q})$, we have

$$\begin{aligned}\int_0^T\int_\Omega (D_+^\alpha y(x,t) - \Delta y(x,t))\phi(x,t)dxd\mathrm{t} &= \int_\Omega \phi(x,T)I_+^{1-\alpha}y(x,T)dx - \int_\Omega \phi(x,0)I_+^{1-\alpha}y(x,0)dx \\ &- \int_0^T\int_\Gamma \frac{\partial y}{\partial \nu}\phi d\sigma dt + \int_0^T\int_\Gamma y\frac{\partial \phi}{\partial \nu}d\sigma dt \\ &+ \int_0^T\int_\Omega y(x,t)(-D_-^{c,\alpha}\phi(x,t) - \Delta\phi(x,t))dxdt.\end{aligned} \tag{12}$$

From Lemma 2.1, we conclude the following result.

Lemma 2.2 Let $\alpha \in (0,1)$. Then for any $\phi \in C_{0,div}^\infty(\bar{Q})$, such that $\phi(x,T) = 0$ in Ω and $\phi(x,t)=0$ on Σ, we have

$$\int_0^T \int_\Omega \big(D_+^\alpha y(x,t) - \Delta y(x,t)\big)\phi(x,t)dxdt$$
$$= -\int_\Omega \phi(x,0) I_+^{1-\alpha} y(x,0)dx$$
$$+ \int_0^T \int_\Omega y(x,t)\big(-D_-^{c,\alpha}\phi(x,t) - \Delta\phi(x,t)\big)dxdt. \tag{13}$$

3 Variational formulation of the problem

Multiplying the first equation in (1) by a test function $\varphi(x) \in V$, integrating over Ω and then applying integration by parts, gives the next formulation:

Definition 3.1 Galdi et al. [Galdi (2011); Robinson, Rodrigo and Sadowski (2016)] Let V and H be the spaces defined in subpart 2.2. We consider a forcing term $f \in L^2\ (0,T;V^*)$. We say that $(x,t) \mapsto y(x,t)$ is a weak solution to the Navier-Stokes equation, if $y \in L^2(0,T;V) \cap L^\infty(0,T;H)$ and

$$\int_\Omega D_+^\alpha y.\varphi\, dx + \int_\Omega (y.\nabla)y.\varphi\, dx + \mu\int_\Omega \nabla y{:}\nabla\varphi\, dx = \int_\Omega f.\varphi\, dx \quad \forall\varphi \in V. \tag{14}$$

From the Eq. (14), we notice that the pressure vanishes, since

$$(\nabla p,\varphi) = \int_\Omega p\varphi.nds - (p,\nabla.\varphi) = 0, \tag{15}$$

because $\varphi.n = 0$ on $\partial\Omega$ and $\nabla.\varphi = 0$.

Now, define a bilinear form $a : V \times V \to \mathbb{R}$, for each $t \in (0,T)$ by:

$$a(y,\varphi) = (Ay,\varphi) = \int_\Omega \nabla y{:}\nabla\varphi dx, \quad \forall\varphi \in V, \tag{16}$$

where the linear operator $A \in \mathcal{L}(V,V^*)$ is defined by:

$$Ay = -\mu\Delta y. \tag{17}$$

Lemma 3.1 Galdi et al. [Galdi (2011); Chowdhury and Ramaswamy (2013)] The bilinear form defined in Eq. (16) is continuous, symmetric, and coercive over $V \times V$.

On the other hand, take the trilinear form $b{:}\, V \times V \times V \to \mathbb{R}$,

$$b(y,y,\varphi) = \int_\Omega (y.\nabla)y.\varphi dx. \tag{18}$$

Next, define the linear and continuous functional $F{:}\, V \to \mathbb{R}$, $F(\varphi) = (f,\varphi)$, $\forall\varphi \in V$, where

$$\parallel F \parallel_{V^*} = sup_{\varphi\in V}\frac{|(f,\varphi)|}{\parallel \varphi \parallel}. \tag{19}$$

And hence, the weak formulation (14) is tantamount to the abstract form

$$(D_+^\alpha y,\varphi) + a(y,\varphi) + b(y,y,\varphi) = F(\varphi), \qquad \forall\varphi \in V. \tag{20}$$

Lemma 3.2 Galdi et al. [Galdi (2011); Chowdhury and Ramaswamy (2013)] For $n \le 4$, and $C_1 > 0$, the trilinear form $b(y,z,w)$ is continuous on $V = H^1(\Omega)$ and

$$|b(y,z,w)| \leq C_1 \parallel y \parallel_V \parallel z \parallel_V \parallel \|w\|_V. \tag{21}$$

Lemma 3.3 Galdi et al. [Galdi (2011); Chowdhury and Ramaswamy (2013)] Consider $y, z, w \in V$. Then we have:

$$b(y,z,w) = -b(y,w,z), \tag{22}$$

$$b(y,z,z) = 0. \tag{23}$$

Theorem 3.1 The weak formulation (20) admits only one solution in $W^{\alpha}(0,T;V,V^{*})$.

Proof. For each $y \in V$, we rewrite Eq. (20) in the form:

$$(D_{+}^{\alpha}\varphi,\psi) + a(\varphi,\psi) + b(y;\varphi,\psi) = F(\psi), \qquad \forall \varphi,\psi \in V, \tag{24}$$

and take the bilinear form $\pi_y(\varphi,\psi)$ as following:

$$\pi_y(\varphi,\psi) = a(\varphi,\psi) + b(y;\varphi,\psi), \quad \forall \varphi,\psi \in V. \tag{25}$$

Define the map $\mathcal{B}: V \to V$ by $\mathcal{B}(y) = \varphi$ where $y \in V$ and φ is the unique solution to (25). From lemmas 3.1 and 3.2, we guarantee the continuity of the form $\pi_y(\varphi,\psi)$. On the other hand, due to the coerciveness of the form $a(\varphi,\psi)$ and the skew-symmetric property of the form $b(y;\varphi,\psi)$, we deduce the coerciveness of $\pi_y(\varphi,\psi)$, i.e.,

$$\pi_y(\varphi,\varphi) \geq C_2 \parallel \varphi \parallel_V^2, \quad C_2 > 0. \tag{26}$$

Then, via Lax-Milgram lemma, the mapping $\mathcal{B}(y) = \varphi$ is well defined. Moreover, the fixed point of $\mathcal{B}$ forms a solution of Eq. (20). If we replace ψ by φ in Eq. (24), then we deduce:

$$\parallel \varphi \parallel_V + \parallel D_{+}^{\alpha}\varphi \parallel_{V^*} \leq C_3 \parallel F \parallel_{V^*}, \qquad C_3 > 0. \tag{27}$$

Hence, $\mathcal{B}: K \to K$, where $K = \{\varphi \in V: \parallel \varphi \parallel_V + \parallel D_{+}^{\alpha}\varphi \parallel_{V^*} \leq C_3 \parallel F \parallel_{V^*}\}$ is a bounded, closed, and convex subset of V. Furthermore, $\mathcal{B}$ is also a contraction mapping. In fact, if $\mathcal{B}(y_1) = \varphi_1$, $\mathcal{B}(y_2) = \varphi_2$ we have, $\forall y_1, y_2 \in K$

$$(D_{+}^{\alpha}\varphi_1,\psi) + a(\varphi_1,\psi) + b(y_1;\varphi_1,\psi) = F(\psi), \qquad \forall \varphi_1,\psi \in V, \tag{28}$$

$$(D_{+}^{\alpha}\varphi_2,\psi) + a(\varphi_2,\psi) + b(y_2;\varphi_2,\psi) = F(\psi), \qquad \forall \varphi_2,\psi \in V. \tag{29}$$

Hence, we obtain

$$(D_{+}^{\alpha}(\varphi_1-\varphi_2),\psi) + a(\varphi_1-\varphi_2,\psi) + b(y_1;\varphi_1,\psi) - b(y_2;\varphi_2,\psi) = 0, \tag{30}$$

by replacing ψ by $\varphi_1 - \varphi_2$, we have

$$(D_{+}^{\alpha}(\varphi_1-\varphi_2),\varphi_1-\varphi_2) + a(\varphi_1-\varphi_2,\varphi_1-\varphi_2) + b(y_1;\varphi_1,\varphi_1-\varphi_2) - b(y_2;\varphi_2,\varphi_1-\varphi_2) = 0. \tag{31}$$

Take again

$$\pi_{y_2}(\varphi_1-\varphi_2,\varphi_1-\varphi_2) = a(\varphi_1-\varphi_2,\varphi_1-\varphi_2) + b(y_2;\varphi_1-\varphi_2,\varphi_1-\varphi_2), \tag{32}$$

then, (31) transformed to

$$(D_{+}^{\alpha}(\varphi_1-\varphi_2),\varphi_1-\varphi_2) + \pi_{y_2}(\varphi_1-\varphi_2,\varphi_1-\varphi_2) = b(y_2-y_1;\varphi_1,\varphi_1-\varphi_2), \tag{33}$$

and hence, by using (26) and lemma 3.2, we can derive

$$C_2 \parallel \varphi_1-\varphi_2 \parallel^2 + \parallel D_{+}^{\alpha}(\varphi_1-\varphi_2) \parallel \leq C_1 \parallel y_2-y_1 \parallel \parallel \varphi_1 \parallel \parallel \varphi_1-\varphi_2 \parallel, \tag{34}$$

and hence

$$C_2 \parallel \varphi_1-\varphi_2 \parallel^2 \leq C_1 \parallel y_2-y_1 \parallel \parallel \varphi_1 \parallel \parallel \varphi_1-\varphi_2 \parallel, \tag{35}$$

since $\varphi_1 \in K$, we have

$$\| \varphi_1 - \varphi_2 \| \leq \frac{C_1}{C_2} \| y_2 - y_1 \| (C_3 \| F \| - \| D_+^\alpha \varphi \|). \tag{36}$$

If we take $\frac{C_1}{C_2}(C_3 \| f \| - \| D_+^\alpha \varphi \|) < 1$, then, the mapping $\mathcal{B}$ is a contraction mapping, and the mapping $\mathcal{B}(y) = \varphi$ has a unique fixed point. Thereby the proof is complete.

4 Formulation of the control problem

This part is the main topic of this paper. We derive the adjoint state for our problem. Also, the necessary first-order optimality conditions are deduced. This problem drives us to build the minimization of the quadratic cost functional:

$$J(v) = \| y(v) - z_d \|_{L^2(Q)}^2 + (Nv, v)_{L^2(Q)}, \quad \forall v \in \mathcal{U}_{ad}, \tag{37}$$

subject to;

$$\begin{cases} D_+^\alpha y - \mu \Delta y + (y.\nabla) y + \nabla p = f + u & \text{in } Q, \\ \nabla \cdot y = 0 & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ I_+^{1-\alpha} y(\cdot, 0) = 0 & \text{in } \Omega. \end{cases} \tag{38}$$

Consider $\mathcal{U} = L^2(Q)$ as the control space. For the control $u \in \mathcal{U}$, the solution $y(u) \in L^2(0, T; V)$ of the system is given by Eq. (38). The observation identity is given by $z(u) = y(u)$. For a given $z_d \in L^2(Q)$, the quadratic cost functional is got by Eq. (37), where $N \in \mathcal{L}(L^2(Q), L^2(Q))$ is a definite and positive Hermitian operator such that:

$$(Nu, u) \geq \gamma \| u \|_{L^2(Q)}^2, \quad \gamma > 0. \tag{39}$$

Let $\mathcal{U}_{ad}$ is a convex closed subset of $L^2(Q)$, then the problem of the control is to find:

$$\begin{cases} u \in \mathcal{U}_{ad}, \\ \text{such that } J(u) \leq J(v), \quad \forall v \in \mathcal{U}_{ad}. \end{cases} \tag{40}$$

We may then define the operator $B(y, y): V \times V \to V^*$, $(B(y, y), \varphi) = b(y, y, \varphi)$, $\forall y, \varphi \in V$, and hence $B: V \longrightarrow V^*, B(y) = B(y, y)$. The operator $B(y)$ is strongly continuous for more specifics [Robinson, Rodrigo and Sadowski (2016)].

For each $y \in V$, define the operator $B'(y): V \longrightarrow V^*$ by:

$$(B'(y)z, w) = b(y, z, w) + b(z, y, w), \quad \forall z, w \in V, \tag{41}$$

and its adjoint operator $[B'(y)]^*: V \to V^*$ by:

$$([B'(y)]^* z, w) = b(y, w, z) + b(w, y, w), \quad \forall z, w \in V. \tag{42}$$

On the other hand, the weak formulation of (38) leads to the evolution problem of the operator form

$$\begin{cases} D_+^\alpha y + Ay + B(y) = f + u, \\ I_+^{1-\alpha} y(\cdot, 0) = 0. \end{cases} \tag{43}$$

Lemma 4.1 Galdi et al. [Galdi (2011); Robinson, Rodrigo and Sadowski (2016)] The mapping $u \mapsto y(u)$ from $L^2(0, T; H)$ into $L^2(0, T; V)$ has a Gâteaux derivatives $\frac{\partial y(u)}{\partial u}.w$ in any direction $w \in L^2(0, T; H)$. Moreover $\frac{\partial y(u)}{\partial u}.w = \psi(w)$ is a solution of the linearized problem:

$$\begin{cases} D_+^{\alpha}\psi + A\psi + B'(y(u))\psi = w, \\ I_+^{1-\alpha}\psi(\cdot\,,0) = 0. \end{cases} \tag{44}$$

Lemma 4.2 Let $w_1 \in L^2(0,T;H)$ and let $\psi(w_1)$ be the solution to the system (44), then for every $w_2 \in L^2(0,T;H)$, we have

$$\int_0^T \int_\Omega w_2\psi(w_1)dxdt = \int_0^T \int_\Omega w_1\phi(w_2)dxdt, \tag{45}$$

where $\phi(w_2)$ is a solution of the adjoint linearized problem

$$\begin{cases} -D_-^{c,\alpha}\phi + A\phi + [B'(y(u))]^*\phi = w_2, \\ \phi(\cdot,T) = 0. \end{cases} \tag{46}$$

Proof. For every $w_2 \in L^2(0,T;H)$, we have

$$\int_0^T \int_\Omega w_1\phi(w_2)dxdt = \int_0^T \int_\Omega \left[D_+^{\alpha}\psi + A\psi + B'\big(y(u)\big)\psi\right]\phi dxdt. \tag{47}$$

By using lemma 2.3, with the property of the self-adjoint of the operator A and the adjoint of the operator B defined above, we have

$$\int_0^T \int_\Omega w_1\phi(w_2)dxdt = \int_0^T \int_\Omega [-D_-^{c,\alpha}\phi + A(t)\phi + [B'(y(u)]^*\phi)]\psi dxdt - \int_\Omega \phi(x,0)I_+^{1-\alpha}\psi(x,0)dx = \int_0^T \int_\Omega w_2\psi(w_1)dxdt. \tag{48}$$

Theorem 4.1. If the functional cost is given by Eqs. (37) and (39) is satisfied, then the optimal control $u \in \mathcal{U}_{\text{ad}}$ exists and is unique. Moreover, the equations and inequalities that describes the optimal control are given as the following:

$$\begin{cases} -D_-^{c,\alpha}p + Ap + [B'(y(u)]^*p = y(u) - z_d & \text{in } Q, \\ p(x,T;u) = 0 & \text{in } Q, \end{cases} \tag{49}$$

with

$$\int_0^T \int_\Omega (p(u) + Nu)(v-u)dxdt \geq 0 \quad \forall v \in \mathcal{U}_{ad}, u \in \mathcal{U}_{ad}, \tag{50}$$

where $p \in L^2(0,T;V)$ is the adjoint state.

Proof. The control $u \in \mathcal{U}_{ad}$ is optimal iff.

$$J'(u).(u-v) \geq 0, \quad \forall v \in \mathcal{U}_{ad}, \tag{51}$$

which is equivalent to the following inequality [Tröltzsch (2010)]:

$$\big(y(u) - z_d, y(v) - y(u)\big) + (Nu, v-u) \geq 0, \quad \forall v \in \mathcal{U}_{ad}. \tag{52}$$

Taking the adjoint state $p(v)$ by:

$$\begin{cases} -D_-^{c,\alpha}p + Ap + [B'(y(u)]^*p = y(u) - z_d & \text{in } Q, \\ p(x,T;u) = 0 & \text{in } Q. \end{cases} \tag{53}$$

Multiplying the two sides of the first Eq. (53) by $y(v) - y(u)$ and integrating over Q then, applying lemma 2.3, we deduce

$$\begin{aligned}\int_0^T \int_\Omega (y(u) - z_d)(y(v) - y(u))dxdt \\ &= \int_0^T \int_\Omega (-D_-^{c,\alpha} p + Ap + [B'(y(u)]^* p)(y(v) - y(u))dxdt \\ &= -\int_\Omega p(x,0)\mathrm{I}_+^{1-\alpha}(y(v;x,0) - y(u;x,0))dx \\ &+ \int_0^T \int_\Omega p(u)(D_+^\alpha + A + B'(y(u)))(y(v) - y(u))dxdt.\end{aligned} \tag{54}$$

Since by using Eq. (44), we obtain

$$\int_0^T \int_\Omega (y(u) - z_d)(y(v) - y(u))dxdt = \int_0^T \int_\Omega p(v-u)dxdt, \tag{55}$$

and hence Eq. (52) is equivalent to

$$\int_0^T \int_\Omega p(v-u)dxdt + (Nu, v-u) \geq 0, \tag{56}$$

which is scaled down to

$$\int_0^T \int_\Omega (p(u) + Nu)(v-u)dxdt \geq 0, \quad \forall v \in \mathcal{U}_{ad}. \tag{57}$$

This ends the proof.

5 Summary and conclusion

This paper provides a new blueprint to study the problem of optimal control to the fractional differential models. We are focused on the time-fractional Navier-Stokes equations, which are formulated in the system (1). This planner is totally based on the variational formulation, the Lax-Milgram lemma with a fixed point theorem, and the adjoint problem mechanism. By using the Lax-Milgram lemma with the assist of the fixed point theorem, we prove the existence and uniqueness of the weak solution to the time-fractional Navier-Stokes equation. By exploiting the adjoint problem, the optimal control was distinguished. Besides, the identities and inequalities which provide the optimality necessary conditions are obtained. Also, If $\alpha \to 1$, our results tend to traditional optimal control theory in Chowdhury et al. [Chowdhury and Ramaswamy (2013); Tröltzsch (2010)].

6 Open problems

I) By a similar manner, we can also study the time-fractional optimal control of Navier-Stokes equations, where the time derivative is considered as the left Atangana-Baleanu fractional derivative in Riemann-Liouville sense as the following:

$$\begin{cases} {}^{ABR}_{0}D_t^{\alpha} y - \mu\Delta y + (y.\nabla)y + \nabla p = f & \text{in } Q, \\ \nabla \cdot y = 0 & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ I_+^{1-\alpha} y(\cdot,0) = 0 & \text{in } \Omega, \end{cases} \tag{58}$$

where ${}^{ABR}_{0}D_t^{\alpha} y$ is the left Atangana-Baleanu fractional derivative in the sense of Riemann-Liouville.

II) The problem can be extended to the time-space fractional derivative as the following:

$$\begin{cases} {}^{ABR}_{0}D_t^{\alpha} y + \mu(-\Delta)^{\beta} y + (y.\nabla)y + \nabla p = f & \text{in } Q, \\ \nabla \cdot y = 0 & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ I_+^{1-\alpha} y(\cdot,0) = 0 & \text{in } \Omega, \end{cases} \tag{59}$$

where $(-\Delta)^{\beta}$ is the fractional Laplacian for $\beta \in (0,1)$.

III) The methodology used in this paper based on Lax-Milgram lemma, fixed-point theorem, and fractional calculus. This methodology is analytical. That is, it investigates the weak solution for the problem analytically. Most of the alternative techniques are approximate techniques depend on very long and hard steps, such as the Galerkin approximations. The question here can the analytical results for our problem compared with approximated results.

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