

Four-Step Iteration Scheme to Approximate Fixed Point for Weak Contractions

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Abstract: Fixed point theory is one of the most important subjects in the setting of metric spaces since fixed point theorems can be used to determine the existence and the uniqueness of solutions of such mathematical problems. It is known that many problems in applied sciences and engineering can be formulated as functional equations. Such equations can be transferred to fixed point theorems in an easy manner. Moreover, we use the fixed point theory to prove the existence and uniqueness of solutions of such integral and differential equations. Let X be a non-empty set. A fixed point for a self-mapping T on X is a point $e \in X$ that satisfying $T e=e$. One of the most challenging problems in mathematics is to construct some iterations to faster the calculation or approximation of the fixed point of such problems. Some mathematicians constructed and generated some new iteration schemes to calculate or approximate the fixed point of such problems such as Mann et al. [Mann (1953); Ishikawa (1974); Sintunavarat and Pitea (2016); Berinde (2004b); Agarwal, O'Regan and Sahu (2007)]. The main purpose of the present paper is to introduce and construct a new iteration scheme to calculate or approximate the fixed point within a fewer number of steps as much as we can. We prove that our iteration scheme is faster than the iteration schemes given by Sintunavarat et al. [Sintunavarat and Pitea (2016); Agarwal, O'Regan and Sahu (2007); Mann (1953); Ishikawa (1974)]. We give some numerical examples by using MATLAB to compare the efficiency and effectiveness of our iterations scheme with the efficiency of Mann et al. [Mann (1953); Ishikawa (1974); Sintunavarat and Pitea (2016); Abbas and Nazir (2014); Agarwal, O'Regan and Sahu (2007)] schemes. Moreover, we introduce a problem raised from Newton's law of cooling as an application of our new iteration scheme. Also, we support our application with a numerical example and figures to illustrate the validity of our iterative scheme.

Keywords: Weak contraction, fixed point, iteration scheme, mean value theorem.

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1 Introduction

Numerical analysis is one of the most important subjects in science because scientists used numerical analysis to approximate solutions of such important problems in our life. Somayeh et al. [Somayeh and Tofigh (2018)] obtained numerical solution of linear regression based on Z-numbers by improved neural network. Liu et al. [Liu, Liu, Luo et al. (2019)] proposed a new algorithm with good optimization performance to enhance exploitation of artificial bee colony algorithm and can improve both the accuracy and the convergence speed. Very recently, Arif et al. [Arif, Raza, Shatanawi et al. (2019)] employed numerical analysis to present numerical analysis for stochastic SLBR model of computer virus over the internet.

Numerical analysis played a major role in constructing some iterations to approximate the existence of fixed points of such functions in fewer number of iterations as much as we can. The fixed point theory, which is basically used to study the existence and uniqueness of fixed point for functions under some conditions, has many applications in various scientific science fields such as physics, economics, some engineering topics, and many mathematical branches. In the rest of this paper, \mathcal{R} denotes to the set of real numbers, I denotes to an interval in \mathcal{R} , and $C(I)$ denotes to the set of all continuous real-valued functions on I .

In mathematics, some problems can be formulated to fixed point problems, for example the solution of the integral equation

$$w(t) = w_0 + \int_{t_0}^t H(r, w(r))dr, \quad (1)$$

where $H: I \times \mathcal{R} \rightarrow \mathcal{R}$ is a continuous mapping and $w \in C(I)$ is equivalent to the fixed point of the mapping $J: C(I) \rightarrow C(I)$ which defined by

$$Jw(t) = w_0 + \int_{t_0}^t H(r, w(r))dr. \quad (2)$$

So, the attraction of a high number of researchers to study fixed point theory is understandable.

It is worth mentioning that the Banach contraction theorem [Banach (1922)] is the first result in the subject of the fixed point theory. The Banach contraction theorem has been extended in many directions by modifying contractive conditions or extending the usual metric space to new forms such as G-metric spaces, partial metric spaces, cone metric spaces, Ω -distance mappings etc.

Kannan [Kannan (1968)] extended the Banach contraction to a new type of contraction, called Kannan contraction. Moreover, Kannan utilized his contraction to generate and prove many existing results in the frame of the concept of metric spaces.

In 1974, Ciric [Ciric (1974)] presented a generalization of the Banach contraction theorem by presenting a new type of contractive conditions.

Kikkawa et al. [Kikkawa and Suzuki (2008)] studied three fixed point theorems for generalized contractions with constants in complete metric spaces as a generalization of Banach contraction theorem. While, Suzuki [Suzuki (2008)] studied a generalization of the Banach contraction principle that characterizes metric completeness.

Aydi et al. [Aydi, Shatanawi and Vetro (2011)] generalized the Banach contraction theorem by generating a weak contraction in G-metric spaces.

Very recently, Mukheimer et al. [Mukheimer, Mlaiki, Abodayeh et al. (2019)] introduced new contractions and presented some theorems as a generalization of Banach fixed point theorem.

Some authors generalized the Banach contraction theorem by generating a new contractions on Ω -distance mappings, for example see Abu-Irwaq et al. [Abu-Irwaq, Shatanawi, Bataihah et al. (2019); Nuseir, Shatanawi, Abu-Irwaq et al. (2017); Shatanawi, Maniu, Bataihah et al. (2017)].

Aydi et al. [Aydi, Postolache and Shatanawi (2012)] introduced a new contractive condition based on a pair of functions (ψ, Φ) , and studied some new fixed point theorems. Also, Aydi et al. [Aydi, Shatanawi, Postolache et al. (2012)] utilized Boyd-Wong-type contractions to introduce some results in ordered metric spaces as a generalization of Banach contraction theorem.

For more modifications of Banach contraction theorem, see Aydi et al. [Aydi, Karapinar and Shatanawi (2012); Bataihah, Shatanawi and Tallafha (2020); Choudhury and Kundu (2010); Shatanawi (2018); Shatanawi and Abodayeh (2019); Shatanawi and Postolache (2013)]. Berinde [Berinde (2004a)] generalized the Banach contraction theorem by introducing a weak contraction on the concept of metric spaces. The weak contraction in sense of Berinde is given as follows:

Definition 1

Let (X, p) be a metric space and $T: X \rightarrow X$ be a self-mapping. Then T is called a weak contraction if there exist $a \in [0,1)$ and $b \geq 0$ such that

$$p(Ts, Te) \leq a p(s, e) + b p(s, Te), \text{ for all } s, e \in X. \tag{3}$$

Moreover, Berinde showed that every weak contraction T on a complete metric space has a fixed point.

One of the most attractive subjects in mathematics is to approximate the fixed point by using iterations and construct new schemes to faster the approximation of the fixed point of such problems in fewer number of steps. Recently, many authors introduced some iterations to speed the rate of convergence of the fixed point.

In this subject, Berinde [Berinde (2004b)] introduced the following important definitions regarding the rate of convergence.

Definition 2

Let (α_n) and (β_n) be two sequences of real numbers. Also, let $\alpha, \beta, l \in \mathcal{R}$ be such that

$$\lim_{n \rightarrow \infty} \alpha_n = \alpha \text{ and } \lim_{n \rightarrow \infty} \beta_n = \beta. \text{ Suppose that } \lim_{n \rightarrow \infty} \frac{|\alpha_n - \alpha|}{|\beta_n - \beta|} = l.$$

1. If $l=0$, then it can be said that (α_n) converges to α faster than (β_n) to β .
2. If $0 < l < \infty$, then it can be said that (α_n) and (β_n) have the same rate of convergence.

Definition 3

Let $(X, \|\cdot\|)$ be a normed linear space and let (p_n) and (q_n) be two sequences in X . Suppose that (p_n) and (q_n) converge to a point $z \in X$ and the error estimates

$\|p_n - z\| \leq \alpha_n$ and $\|q_n - z\| \leq \beta_n$ are available, where (α_n) and (β_n) are non-negative real sequences that converging to zero. If (α_n) converges to zero faster than (β_n) , then it can be said that (p_n) converges to z faster than (q_n) .

From now on, Y represents to a normed linear space, C represents to a convex subset of Y and (a_n) , (b_n) and (c_n) represent real sequences in the interval $[0, 1]$.

In 1953, Mann [Mann (1953)] presented an iteration process (M_n) by the sequence (x_n) to approximate a fixed point, which is defined as follows:

$$\begin{cases} x_0 \in Y \\ x_{n+1} = (1 - a_n)x_n + a_nTx_n. \end{cases} \quad (4)$$

While, Ishikawa [Ishikawa (1974)] presented an iteration process (I_n) by the sequence (x_n) which is defined as follows:

$$\begin{cases} x_0 \in Y \\ y_n = (1 - b_n)x_n + b_nTx_n \\ x_{n+1} = (1 - a_n)x_n + a_nTy_n. \end{cases} \quad (5)$$

It is worth mentioning that the iteration process of Ishikawa is faster than the iteration process of Mann.

In the last decade, Agarwal et al. [Agarwal, O'Regan and Sahu (2007)] presented an iteration process (ARS_n) by the sequence (s_n) which is defined as follows:

$$\begin{cases} s_0 \in Y \\ y_n = (1 - b_n)s_n + b_nTs_n \\ s_{n+1} = (1 - a_n)Ts_n + a_nTy_n. \end{cases} \quad (6)$$

It is worth noting that the iteration scheme of Agarwal et al. is faster than Mann [Mann (1953)] and Ishikawa [Ishikawa (1974)] iteration schemes.

Recently, Abbas et al. [Abbas and Nazir (2014)] introduced the following iteration scheme to speed the rate of convergence as much as they can. Also, they proved that their iteration scheme is faster than the iteration scheme given by Agarwal et al. [Agarwal, O'Regan and Sahu (2007)]:

$$\begin{cases} s_0 = s \in Y \\ s_{n+1} = (1 - b_n)Ty_n + b_nTz_n \\ y_n = (1 - a_n)Ts_n + a_nTz_n \\ z_n = (1 - c_n)s_n + a_nTs_n. \end{cases} \quad (7)$$

In 2016, Sintunavarat et al. [Sintunavarat and Pitea (2016)] presented an iteration scheme (S_n) by the sequence (t_n) as follows:

$$\begin{cases} t_0 \in C \\ y_n = (1 - b_n)t_n + b_nTt_n \\ z_n = (1 - c_n)t_n + c_ny_n \\ t_{n+1} = (1 - a_n)Tz_n + a_nTy_n. \end{cases} \quad (8)$$

The aim of the present paper is to introduce a new scheme to speed the approximation of a fixed point of such problems as much as we can. Also, we present some numerical examples to show the efficiency and effectiveness of our new scheme.

2 New iterative scheme with analytic proof

In this section, we introduce a four-step iterative scheme to approximate a fixed point for contraction mappings of weak type. Let C be a nonempty closed convex subset of a Banach space Y and T be a self-mapping on C . We define the iteration scheme (SBT_n) by the sequence (x_n) as follows:

$$\begin{cases} x_0 \in C \\ y_n = (1 - b_n)x_n + b_nTx_n \\ z_n = (1 - c_n)x_n + c_ny_n \\ w_n = (1 - a_n)Tz_n + a_nTy_n \\ x_{n+1} = (1 - d_n)Tz_n + d_nTw_n. \end{cases} \tag{9}$$

Theorem 1

Let $(Y, \|\cdot\|)$ be a Banach space. Suppose that C is a closed convex subset of Y and $T: C \rightarrow C$ be a self-mapping. Assume that T satisfies condition (3) and u is a fixed point of T . Suppose that (x_n) is the sequence in Y , defined by the iteration process (SBT_n) . Also, assume that $(a_n), (b_n), (c_n)$ and (d_n) are sequences in $[a, 1 - a], [b, 1 - b], [c, 1 - c]$ and $[d, 1 - d]$ respectively, where $a, b, c, d \in (0, \frac{1}{2})$. If $a < \frac{c}{1-c}$, then the iteration process (SBT_n) converges to u faster than (S_n) .

Proof: For each natural number n , (SBT_n) implies that

$$\begin{aligned} \|x_{n+1} - u\| &= \|(1 - d_n)Tz_n + d_nTw_n - u\| \\ &\leq (1 - d_n)\|Tz_n - u\| + d_n\|Tw_n - u\| \\ &\leq k(1 - d_n)\|z_n - u\| + kd_n\|w_n - u\|. \end{aligned}$$

So,

$$\|x_{n+1} - u\| \leq k(1 - d_n)\|z_n - u\| + kd_n\|w_n - u\|. \tag{10}$$

Now, a

$$\begin{aligned} \|z_n - u\| &= \|(1 - c_n)x_n + c_ny_n - u\| \\ &= \|(1 - c_n)(x_n - u) + c_n(y_n - u)\| \\ &\leq (1 - c_n)\|x_n - u\| + c_n\|y_n - u\|. \end{aligned}$$

Also,

$$\begin{aligned} \|y_n - u\| &= \|(1 - b_n)x_n + b_nTx_n - u\| \\ &= \|(1 - b_n)(x_n - u) + kb_n(x_n - u)\| \\ &\leq (1 - b_n)\|x_n - u\| + kb_n\|x_n - u\|. \end{aligned}$$

Thus,

$$\|y_n - u\| \leq (1 - (1 - k)b_n)\|x_n - u\| \tag{11}$$

Hence,

$$\|z_n - u\| \leq (1 - (1 - k)b_nc_n)\|x_n - u\| \tag{12}$$

Now,

$$\begin{aligned} \|w_n - u\| &= \|(1 - a_n)Tz_n + a_nTy_n - u\| \\ &= \|(1 - a_n)(Tz_n - u) + a_nT(y_n - u)\| \\ &\leq k(1 - a_n)\|z_n - u\| + ka_n\|y_n - u\|. \end{aligned}$$

By (10) and (11), we have

$$\begin{aligned}\|w_n - u\| &\leq \{k(1 - a_n)(1 - (1 - k)b_n c_n) + k a_n(1 - (1 - k)b_n)\} \|x_n - u\| \\ &= k\{(1 - a_n)(1 - (1 - k)b_n c_n) + a_n(1 - (1 - k)b_n)\} \|x_n - u\|.\end{aligned}$$

Hence,

$$\|w_n - u\| \leq \{1 - (1 - k)b_n(c_n + a_n(1 - c_n))\} \|x_n - u\|. \quad (13)$$

Therefore,

$$\begin{aligned}\|x_n - u\| &\leq \left\{ (1 - d_n)(1 - (1 - k)b_n c_n) + d_n \left(1 - (1 - k)b_n(c_n + a_n(1 - c_n)) \right) \right\} \\ &\quad \|x_n - u\| \\ &= \left\{ (1 - (1 - k)b_n c_n) - a_n b_n c_n ((1 - k)(1 - c_n)) \right\} \|x_n - u\| \\ &= \left\{ 1 - (1 - k)b_n(c_n + a_n d_n(1 - c_n)) \right\} \|x_n - u\| \\ &\leq \left\{ 1 - (1 - k)b(c + ad(1 - c)) \right\} \|x_n - u\|.\end{aligned}$$

So,

$$\|x_n - u\| \leq \left\{ 1 - (1 - k)b(c + ad(1 - c)) \right\}^n \|x_0 - u\|. \quad (14)$$

On the other hand, the iterative process S_n gives that

$$\|t_n - u\| \leq \left\{ 1 - (1 - k)b(c + a(1 - c)) \right\}^n \|t_0 - u\|. \quad (15)$$

Let $\alpha_n = \left\{ 1 - (1 - k)b(c + ad(1 - c)) \right\}^n \|x_0 - u\|$ and

$$\beta_n = \left\{ 1 - (1 - k)b(c + ad(1 - c)) \right\}^n \|x_0 - u\|.$$

Then, we have

$$\lim_{n \rightarrow \infty} \|x_n - u\| \leq \lim_{n \rightarrow \infty} \left\{ 1 - (1 - k)b(c + ad(1 - c)) \right\}^n \|x_0 - u\| = 0 \text{ and}$$

$$\lim_{n \rightarrow \infty} \|t_n - u\| \leq \lim_{n \rightarrow \infty} \left\{ 1 - (1 - k)b(c + a(1 - c)) \right\}^n \|t_0 - u\| = 0.$$

Since $\alpha_1 < \beta_1$, we get

$$\lim_{n \rightarrow \infty} \left\{ \frac{1 - (1 - k)b(c + ad(1 - c))}{1 - (1 - k)b(c + a(1 - c))} \right\}^n = 0.$$

Hence our result is satisfied.

3 Some numerical examples

Next, we give some numerical examples to illustrate our result.

Example 1

Let $Y = \mathcal{R}$ with the usual norm. Take $C = [0, 50]$. Define $T: C \rightarrow C$ by $Tx = \sqrt[3]{5x^2 - 2x + 48}$. Let $a = c = \frac{2}{5}$, and $b = d = \frac{1}{4}$. Also, let $a_n = c_n = \frac{1}{2}$, and $b_n = d_n = \frac{3}{4} - \frac{1}{4+n}$ for $n = 0, 1, 2, \dots$. The Mean Value Theorem implies that T satisfies condition (3). Also, note that $(a_n), (b_n), (c_n), (d_n), a, b, c$, and d satisfy the conditions of Theorem 2.1. Thus the iteration scheme $(SBTn)$ is faster than the iteration schemes (Sn) , $(ARSn)$, (In) and (Mn) . For more details, see the table below which gives the results to approximate the fixed point of the function T with start point $x_0 = 50$.

Table 1: comparative results of example 1

Step	M_n	I_n	ARS_n	S_n	SBT_n
1	36.587858878398400	34.180084789163100	20.767943667561600	19.598381679944100	11.729717300628600
2	27.705002368667000	24.347196887740000	11.961294173244300	11.106098557586700	7.094325635858250
3	21.675723790983000	18.121337795550400	8.609859836200890	8.070779153023590	6.217873743369290
4	17.488989111022200	14.103460100492300	7.176767916711330	6.858841543103770	6.043075029792180
⋮	⋮	⋮	⋮	⋮	⋮
20	6.149975887561380	6.017364383359820	6.000003535729840	6.000000533210640	6.000000000000110
21	6.115237489175310	6.011811992843360	6.000001586743620	6.000000215759680	6.000000000000020
22	6.088549543658810	6.008032810428890	6.000000711785820	6.000000087243640	6.000000000000000
23	6.068044636341280	6.005461356839480	6.000000319168790	6.000000035254280	6.000000000000000
24	6.052289287574550	6.003712186247620	6.000000143064610	6.000000014237210	6.000000000000000
25	6.040182784523080	6.002522678812220	6.000000064105560	6.000000005746330	6.000000000000000

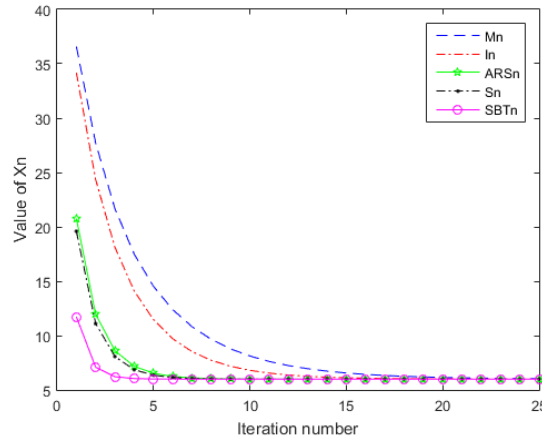


Figure 1: Behavior of iteration processes (M_n), (I_n), (ARS_n), (S_n) and (SBT_n) for the mapping T in Example 1

Example 2

Let $Y = \mathcal{R}$ with the usual norm. Take $C = [0,2]$. Define $T: C \rightarrow C$ by $Tx = e^{\sin(x)-\frac{3}{2}}$. Let $a = c = \frac{2}{5}$, and $b = d = \frac{1}{4}$. Also, let $a_n = c_n = \frac{1}{2}$ and $b_n = d_n = \frac{3}{4} - \frac{1}{4+n^2}$ for $n = 0, 1, 2, \dots$. The Mean Value Theorem implies that T satisfies condition (3). Also, note that $(a_n), (b_n), (c_n), (d_n), a, b, c$ and d satisfy the conditions of Theorem 2.1. Thus the iteration scheme (SBT_n) is faster than the iteration schemes (S_n), (ARS_n), (I_n) and (M_n). For more details, see the below table which gives the results to approximate the fixed point of the function T with start point $x_0 = 2$. Note that by using MATLAB we can see that L has the fixed point $u \sim 0.299786903282166$.

Table 2: comparative results of example 2

Step	Mn	In	$ARSn$	Sn	$SBTn$
1	1.276968983016460	1.283816457008920	0.560785440025372	0.586931029365896	0.364564361469188
2	0.929027181994992	0.877663221914339	0.361887859036987	0.358475878853376	0.302691056742270
3	0.713064099004039	0.636844645186341	0.313556053203331	0.310719114006616	0.299905328156244
⋮	⋮	⋮	⋮	⋮	⋮
10	0.319901011198184	0.306462794885001	0.299787171230084	0.299786958532239	0.299786903282178
11	0.312742990634971	0.303575580337322	0.299786959709464	0.299786912822620	0.299786903282166
12	0.308128080816175	0.301936338319421	0.299786915158052	0.299786904927771	0.299786903282166
13	0.305155222571262	0.301006059899413	0.299786905780421	0.299786903565765	0.299786903282166
14	0.303241179889617	0.300478287296089	0.299786903807509	0.299786903331007	0.299786903282166
15	0.302009273268062	0.300178935277693	0.299786903392603	0.299786903290572	0.299786903282166
16	0.301216578008352	0.300009172001756	0.299786903305376	0.299786903283612	0.299786903282166
17	0.300706576196817	0.299912911441948	0.299786903287043	0.299786903282415	0.299786903282166
18	0.300378483491956	0.299858334656172	0.299786903283191	0.299786903282209	0.299786903282166
19	0.300167428667533	0.299827393890902	0.299786903282381	0.299786903282173	0.299786903282166
20	0.300031666999588	0.299809854113763	0.299786903282211	0.299786903282167	0.299786903282166

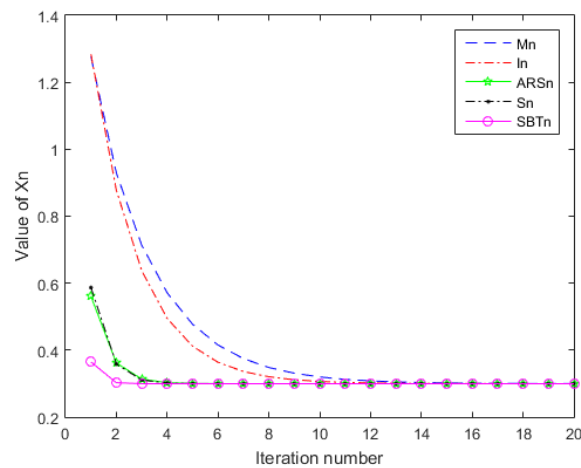


Figure 2: Behavior of iteration processes (Mn), (In), ($ARSn$), (Sn) and ($SBTn$) for the mapping T in Example 2

4 Applications

In this section, we shall prove that the initial value problem:

$$w'(t) = H(t, w(t)), w(t_0) = w_0 \tag{16}$$

has a unique solution. Also, we will give some examples to show the validity of our iterative scheme.

In order to prove that, we need the following lemma

Lemma 1

[Plaat (1971)] $w(t)$ is a solution for the initial value problem

$$w'(t) = H(t, w(t)), w(t_0) = w_0 \tag{17}$$

if and only if

$$w(t) = w_0 + \int_{t_0}^t H(r, w(r))dr. \tag{18}$$

Let $\| \cdot \|_\infty$ be the superior norm on $C(I)$ which is defined by $\| u \|_\infty = \sup_{t \in I} u(t)$. Now, we

have the following theorem:

Theorem 2

Let $H: I \times \mathcal{R} \rightarrow \mathcal{R}$ be a continuous function on $I \times \mathcal{R}$ and let t_0 be an interior point of I . Suppose that there is $l_0 > 0$ such that H satisfies the following condition:

$$|H(t, x_1) - H(t, x_2)| \leq l_0 |x_1 - x_2|. \tag{19}$$

For all $x_1, x_2 \in \mathcal{R}$ and for all $t \in I$. Then the initial value problem (16) has a unique solution $w \in C(I)$.

Proof: Let $\epsilon \geq 0$ be a real number such that $\epsilon l_0 < 1$. Define the mapping $T: C(I) \rightarrow C(I)$ by $Tw(t) = w_0 + \int_{t_0}^t H(r, w(r))dr$. Now, we show that T satisfies condition (3) on the interval $C_0 = [t_0, t_0 + \epsilon]$.

Let $u, v \in C(I)$. Then we have

$$\begin{aligned} \| Tu - Tv \|_\infty &= \sup_{t \in C_0} |Tu(t) - Tv(t)| \\ &= \sup_{t \in C_0} \left| \int_{t_0}^t H(r, u(r)) - H(r, v(r))dr \right| \\ &\leq \sup_{t \in C_0} \int_{t_0}^t |H(r, u(r)) - H(r, v(r))|dr \\ &\leq \sup_{t \in C_0} l_0 |u(t) - v(t)| \int_{t_0}^t dr \\ &\leq \epsilon l_0 \| u - v \|_\infty. \end{aligned}$$

Therefore, condition (3) is satisfied and hence the result is satisfied.

Newton's law of cooling is a differential equation that foresees the cooling of somebody that placed in a colder environment which may be written as follows:

$x'(t) = -l(x(t) - x_\epsilon)$, where $x(t)$ represents to the temperature of the object at the time t , x_ϵ represents to the temperature of the environment, and l represents to the proportionality constant.

If $x(t_0) = x_0$, then we have the following initial value problem

$$x'(t) = -l(x(t) - x_\epsilon), x(t_0) = x_0. \tag{20}$$

Let $H(t, x) = -l(x(t) - x_\epsilon)$. Then, one can show that H satisfies condition (19). By Theorem 4.2, the initial value problem (20) has a unique solution.

In fact, the exact solution of (20) is $x(t) = x_\epsilon + (x_0 - x_\epsilon)e^{-l(t-t_0)}$.

Now, we introduce the following example to illustrate the usability of our iterative scheme ($SBTn$).

Example 3

An orange pie was taken out of the oven at a temperature of 170° . Then it was left to cool down with an air temperature of 20° . If the temperature of the pie decreases initially at a rate of $3/\text{minute}$, find the relation between the time and the temperature of the pie.

Suppose that the pie temperature obeys Newton's law of cooling. Then, we get the following initial value problem:

$$x'(t) = -l(x(t) - x_\epsilon), x(0) = 170, x'(0) = -3. \quad (21)$$

One can easily find that $l = 0.02$. So, the exact solution is

$$x(t) = 20 + 150e^{-0.02t}.$$

Now, let $Tx(t) = 170 + \int_0^t -0.02(x(r) - 20)dr$, and

$H(t, x) = 0.02(x(r) - 20)$. Then, H satisfies condition (19), and so by Theorem 2, T has a unique fixed point.

The following figures illustrate the result of approximating the fixed point of T with the initial point $x_0(t) = t \sin t + t^2 \cos t$.

We use MATLAB to get the solution obtained from the above iteration schemes at the 10^{th} iteration:

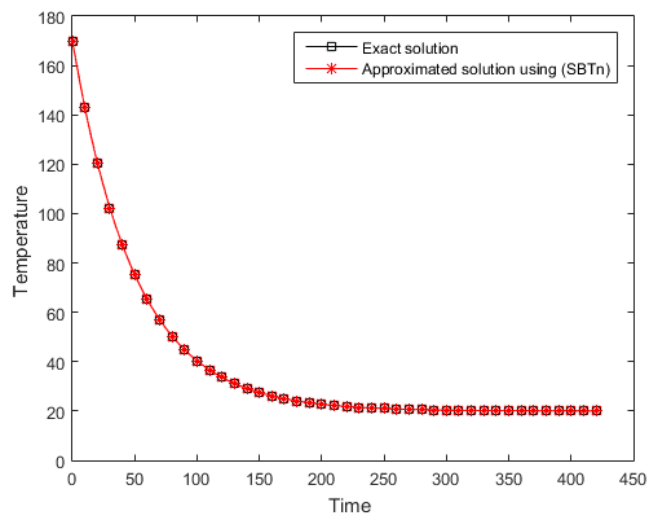


Figure 3: The solution obtained by ($SBTn$) at 10th iteration vs. exact solution of IVP (9)

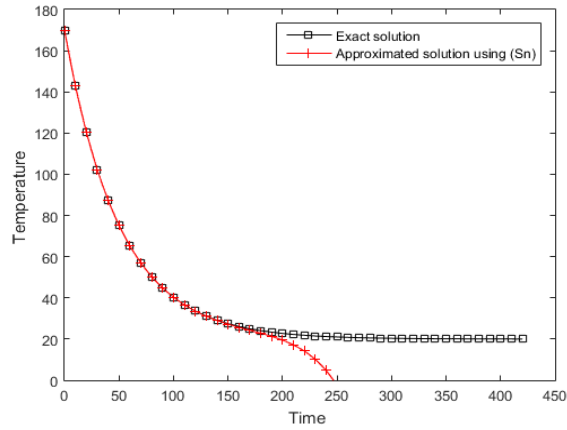


Figure 4: The solution obtained by (S_n) at 10th iteration vs. exact solution of IVP (9)

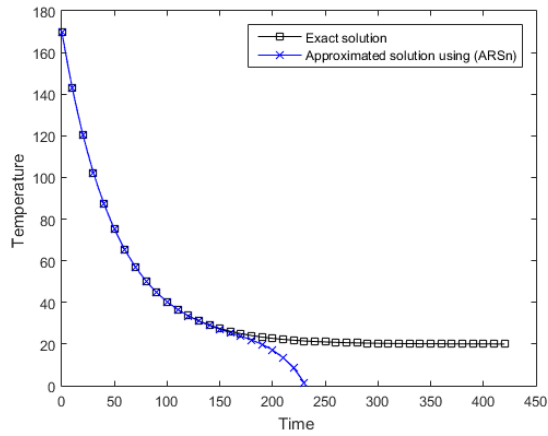


Figure 5: The solution obtained by $(AR S_n)$ at 10th iteration vs. exact solution of IVP (9)

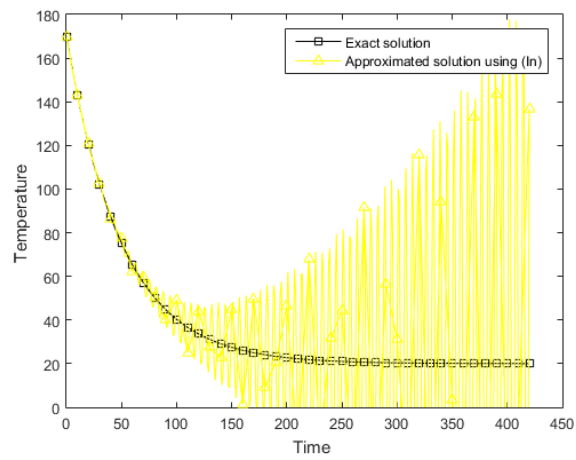


Figure 6: The solution obtained by (I_n) at 10th iteration vs. exact solution of IVP (9)

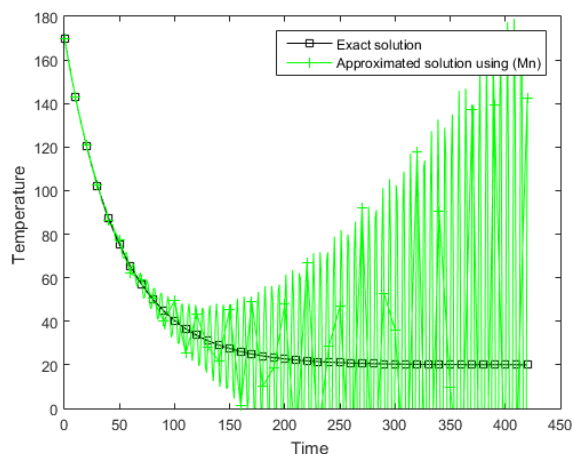


Figure 7: The solution obtained by (Mn) at 10th iteration vs. exact solution of IVP (9)

5 Conclusion

We introduced a new scheme to speed the approximation of a fixed point of such problems. We applied our scheme to some numerical examples. Also, we compared our scheme with some known schemes such as Mann's scheme [Mann (1953)], Ishikawa's scheme [Ishikawa (1974)], Agarwal et al.'s scheme [Agarwal, O'Regan and Sahu (2007)] and Sintunavarat et al. scheme [Sintunavarat and Pitea (2016)] to show the efficiency and effectiveness of our new scheme. Also, we gave an application that raised from Newton's law of cooling to show the applicability of our new scheme.

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