

Generalized Marshall Olkin Inverse Lindley Distribution with Applications

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Abstract: In this article, a new generalization of the inverse Lindley distribution is introduced based on Marshall-Olkin family of distributions. We call the new distribution, the generalized Marshall-Olkin inverse Lindley distribution which offers more flexibility for modeling lifetime data. The new distribution includes the inverse Lindley and the Marshall-Olkin inverse Lindley as special distributions. Essential properties of the generalized Marshall-Olkin inverse Lindley distribution are discussed and investigated including, quantile function, ordinary moments, incomplete moments, moments of residual and stochastic ordering. Maximum likelihood method of estimation is considered under complete, Type-I censoring and Type-II censoring. Maximum likelihood estimators as well as approximate confidence intervals of the population parameters are discussed. A comprehensive simulation study is done to assess the performance of estimates based on their biases and mean square errors. The notability of the generalized Marshall-Olkin inverse Lindley model is clarified by means of two real data sets. The results showed the fact that the generalized Marshall-Olkin inverse Lindley model can produce better fits than power Lindley, extended Lindley, alpha power transmuted Lindley, alpha power extended exponential and Lindley distributions.

Keywords: Generalized Marshall-Olkin family, inverse Lindley distribution, maximum likelihood estimation.

1 Introduction

In the last decade, the general method of adding a shape parameter to expand a family of distributions was introduced by Marshall et al. [Marshall and Olkin (1997)]. This family is called the Marshall-Olkin (MO)-G class. The cumulative distribution function (cdf) and the probability density function (pdf) of the MO-G class are defined as follows

$$F_{MO}(x) = W(x) / (1 - \bar{\alpha}(\bar{W}(x))), \quad (1)$$

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Received: 03 April 2020; Accepted: 30 April 2020.

and,

$$f_{MO}(x) = \alpha w(x) / \left[1 - \bar{\alpha}(\bar{W}(x))\right]^2, \quad (2)$$

where, $\alpha > 0$, $\bar{\alpha} = 1 - \alpha$, $\bar{W}(x) = 1 - W(x)$ is the survival function and $W(x)$ is the baseline distribution. The parameter $\bar{\alpha}$ is known as a tilt parameter and interpreted $\bar{\alpha}$ in terms of the behavior of the hazard rate function (hrf) of X . This ratio is increasing in X for $\bar{\alpha} \geq 1$ and decreasing in X for $\bar{\alpha} \in (0, 1)$ (see Nanda et al. [Nanda and Das (2012)]).

The generalization of the MO-G family proposed by Jayakumar et al. [Jayakumar and Mathew (2008)] through Lehmann alternative 1 approach by exponentiating the MO survival function (sf) as

$$\bar{F}_{GMO}(x) = \left[\frac{\alpha(\bar{W}(x))}{1 - \bar{\alpha}(\bar{W}(x))} \right]^b, \quad (3)$$

where $b > 0$, is an additional shape parameter. When $b = 1$, $\bar{F}_{GMO}(x) = \bar{F}_{MO}(x)$. The pdf corresponding to Eq. (3) is given by

$$f_{GMO}(x) = \frac{b\alpha^b [\bar{W}(x)]^{b-1} w(x)}{\left[1 - \bar{\alpha}(\bar{W}(x))\right]^{b+1}}. \quad (4)$$

For more information about Marshall-Olkin family see Barakat et al. [Barakat, Ghitany and AL-Hussaini (2009); Barreto-Souza, Lemonte and Cordeiro (2013); Cordeiro, Lemonte and Ortega (2014); Alizadeh, Tahir, Cordeiro et al. (2015); Handique, Chakraborty and Hamedani (2017)].

Recently, one parameter Lindley distribution has attracted the researchers for its use in modeling lifetime data. A mixture of exponential distributions $(2, \theta)$ and (θ)

with mixing proportion $\left\{\frac{\theta}{1+\theta}\right\}$ in the context of Fiducial and Bayesian statistics. It has the following pdf and cdf

$$w(y; \theta) = \frac{\theta^2}{1+\theta} (1+y) e^{-\theta y}, \quad y, \theta > 0, \quad (5)$$

and,

$$W(y; \theta) = 1 - \left\{1 + \frac{\theta}{1+\theta} y\right\} e^{-\theta y}, \quad y, \theta > 0. \quad (6)$$

The Lindley distribution and its applications have been discussed by Ghitany et al. [Ghitany, Atieh and Nadarajah (2008)] and showed that the Lindley distribution is a better fit than the exponential distribution based on the waiting time at the bank for service.

Sharma et al. [Sharma, Singh, Singh et al. (2015)] proposed the inverse Lindley (IL) distribution by using the transformation $X = 1/Y$, where Y has the pdf in Eq. (5) and cdf in Eq. (6), with the following pdf

$$w(x; \theta) = \frac{\theta^2}{1 + \theta} \left(\frac{1+x}{x^3} \right) e^{-\frac{\theta}{x}}, \quad x, \theta > 0. \tag{7}$$

The cdf related to Eq. (7) is as follows

$$W(x; \theta) = \left(1 + \frac{\theta}{1 + \theta} \frac{1}{x} \right) e^{-\frac{\theta}{x}}, \quad x, \theta > 0. \tag{8}$$

Sharma et al. [Sharma, Singh, Singh et al. (2015)] mentioned that the IL distribution has the upside-down bathtub-shaped hrf. They discussed the estimation of stress strength reliability using classical and Bayesian approaches. Sharma et al. [Sharma, Singh, Singh et al. (2016)], introduced another two-parameter IL distribution called the generalized IL distribution as a new statistical inverse model with upside-down bathtub survival data. The power IL distribution has been introduced by Barco et al. [Barco, Mazucheli and Janeiro (2017)]. An extension of the inverse power Lindley distribution using the MO-G family has been introduced and discussed by Hibatullah et al. [Hibatullah, Widyaningsih and Abdullah (2018)]. Exponentiated inverse power Lindley distribution has been proposed by Jan et al. [Jan, Jan and Ahmad (2018)].

In this article, we provide a new three-parameter model as an interesting extension for the IL distribution. We are motivated to study the generalized Marshall-Olkin inverse Lindley (GMOIL) distribution because (i) it includes the Marshall-Olkin inverse Lindley (MOIL) and the IL as sub-models; (ii) it gives more flexibility to model various types of data (iii) it outperforms some lifetime distributions in regard to two real data examples. This paper can be sorted as follows. Section 2 gives the structure of the pdf, cdf and hrf of the GMOIL distribution. Main properties of the GMOIL model appear in Section 3. Estimation of the population parameters are derived in Section 4 based on complete, Types I and II censored sampling schemes. Simulation study is carried out to illustrate theoretical results in Section 5. Section 6 provides application to real data and the article ends with concluding remarks.

2 Generalized marshel olkin inverse lindley distribution

In this section, the GMOIL distribution with parameters α , b and θ are proposed based on pdf in Eq. (3). The GMOIL distribution is specified according to the following definition.

Definition: Let X be a random variable having pdf in Eq. (7) and cdf in Eq. (8), then the random variable X is said to follow the GMOIL distribution with the following pdf and cdf

$$f(x; \kappa) = b\alpha^b \frac{\theta^2}{1 + \theta} \left(\frac{1+x}{x^3} \right) e^{-\frac{\theta}{x}} \left\{ 1 - \left(1 + \frac{\theta}{1 + \theta} \frac{1}{x} \right) e^{-\frac{\theta}{x}} \right\}^{b-1} \left[1 - \bar{\alpha} \left\{ 1 - \left(1 + \frac{\theta}{1 + \theta} \frac{1}{x} \right) e^{-\frac{\theta}{x}} \right\} \right]^{-b-1}, \tag{9}$$

and,

$$F(x; \kappa) = 1 - \left\{ \alpha \left(1 - \left(1 + \frac{\theta}{1 + \theta} \frac{1}{x} \right) e^{-\frac{\theta}{x}} \right) \right\}^b \left[1 - \bar{\alpha} \left\{ 1 - \left(1 + \frac{\theta}{1 + \theta} \frac{1}{x} \right) e^{-\frac{\theta}{x}} \right\} \right]^{-b}, \tag{10}$$

where, $\kappa = (\alpha, b, \theta)$ is a set of parameters.

For $b = 1$, the GMOIL distribution reduces to MOIL distribution.

For $b = \alpha = 1$, the GMOIL distribution reduces to IL (see Ghitany et al. [Ghitany, Atieh and Nadadrajah (2008)]).

The sf and hrf of the GMOIL distribution are respectively, given by

$$\bar{F}(x; \kappa) = \left\{ 1 - \alpha \left(1 - \left(1 + \frac{\theta}{1 + \theta x} \right) e^{-\frac{\theta}{x}} \right) \right\}^b \left[1 - \bar{\alpha} \left\{ 1 - \left(1 + \frac{\theta}{1 + \theta x} \right) e^{-\frac{\theta}{x}} \right\} \right]^{-b}, \quad (11)$$

and,

$$h(x; \kappa) = \frac{b\alpha\theta^2}{1 + \theta} \left(\frac{1 + x}{x^3} \right) e^{-\frac{\theta}{x}} \left\{ 1 - \left[1 + \frac{\theta}{1 + \theta x} \right] e^{-\frac{\theta}{x}} \right\}^{b-1} \left[1 - \bar{\alpha} \left\{ 1 - \left[1 + \frac{\theta}{1 + \theta x} \right] e^{-\frac{\theta}{x}} \right\} \right]^{-b-1}. \quad (12)$$

Plots of the pdf and hrf of the GMOIL distribution are displayed in Fig. 1, for different values of parameters. As seen from Fig. 1, the shapes of the pdf take different forms. Also, it is clear that the shapes of the hrf are decreasing, increasing and up-side down shaped at some selected values of parameters.

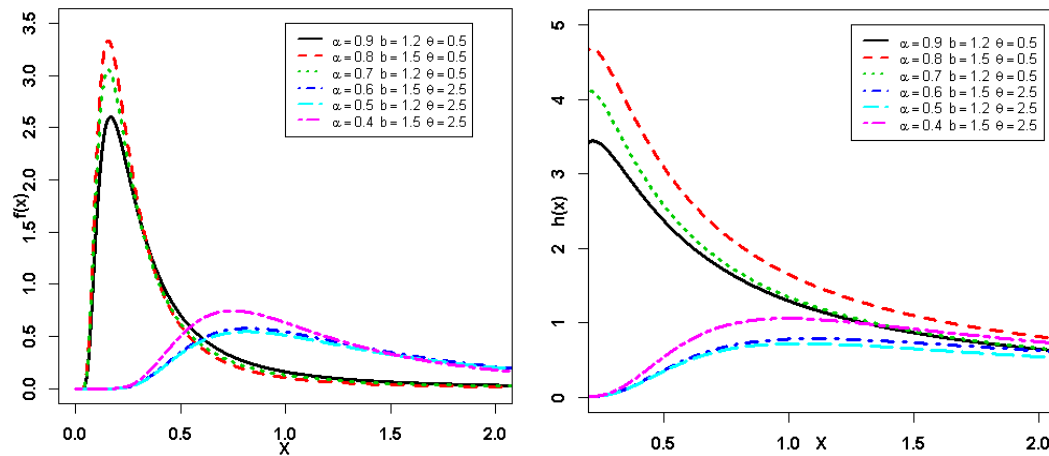


Figure 1: The pdf and hrf of the GMOIL model at selected values of parameters

3 Main properties

In this section, we obtain some important statistical properties of the GMOIL distribution such as quantile function, ordinary and incomplete moments, moment generating function, moments of the residual and reversed residual lives and stochastic ordering.

3.1 Quantile function

Quantiles are essential for estimation and simulation. The quantile function, say $Q(u) = F^{-1}(u)$, where $u \in (0,1)$, is obtained by inverting Eq. (10) as follows

$$u = 1 - \left[\frac{\alpha \left\{ 1 - \left(1 + \frac{\theta}{1 + \theta} \frac{1}{Q(u)} \right) e^{\frac{-\theta}{Q(u)}} \right\}}{1 - \bar{\alpha} \left\{ 1 - \left(1 + \frac{\theta}{1 + \theta} \frac{1}{Q(u)} \right) e^{\frac{-\theta}{Q(u)}} \right\}} \right]^b, \tag{13}$$

which yields;

$$\left(1 + \frac{\theta}{1 + \theta} \frac{1}{Q(u)} \right) e^{\frac{-\theta}{Q(u)}} = \frac{\alpha - \alpha(1-u)^{1/b}}{\alpha + \bar{\alpha}(1-u)^{1/b}}. \tag{14}$$

Multiply both sides by $(1 + \theta)e^{-(1+\theta)}$, then we have the Lambert equation

$$-[1 + \theta + (\theta/Q(u))]e^{-\frac{\theta}{Q(u)} + 1 + \theta} = -(1 + \theta)e^{-(1+\theta)} \left\{ \alpha - \alpha(1-u)^{1/b} / \left(\alpha + \bar{\alpha}(1-u)^{1/b} \right) \right\}. \tag{15}$$

Hence, we have the negative Lambert W function of the real argument

$$-(1 + \theta)e^{-(1+\theta)} \left\{ \alpha - \alpha(1-u)^{1/b} / \left(\alpha + \bar{\alpha}(1-u)^{1/b} \right) \right\}, \tag{16}$$

i.e.,

$$Q(u) = \left\{ -\frac{1}{\theta} - 1 - \frac{1}{\theta} W_{-1} \left[-(1 + \theta)e^{-(1+\theta)} \left\{ \alpha - \alpha(1-u)^{1/b} / \left(\alpha + \bar{\alpha}(1-u)^{1/b} \right) \right\} \right] \right\}^{-1}, \tag{17}$$

where $u \in (0,1)$, and $W_{-1}(\cdot)$ is the negative Lambert W function.

3.2 Moments and incomplete moments

The s^{th} moment about zero for the GMOIL distribution is derived. To obtain the s^{th} moment, firstly explicit expression for the pdf in Eq. (9) is obtained. Since, the binomial expansion, for real non-integer value of m , is given by

$$(1 - y)^{-m} = \sum_{j=0}^{\infty} \frac{\Gamma(m + j)y^j}{\Gamma(m)j!}, \quad |y| < 1, m > 0. \tag{18}$$

Then by employ Eq. (18) in Eq. (9), then

$$f(x; \kappa) = \sum_{j=0}^{\infty} \frac{\alpha^b b \theta^2}{1 + \theta} \frac{\Gamma(b + 1 + j) \bar{\alpha}^j}{j! \Gamma(b + 1)} \left(\frac{1 + x}{x^3} \right) e^{\frac{-\theta}{x}} \left\{ 1 - \left(1 + \frac{\theta}{1 + \theta} \frac{1}{x} \right) e^{\frac{-\theta}{x}} \right\}^{b+j-1}. \tag{19}$$

Apply the generalized binomial expansion, in Eq. (19)

$$f(x; \kappa) = \sum_{i,j=0}^{\infty} (-1)^i \binom{b + j - 1}{i} \frac{\alpha^b b \theta^2}{1 + \theta} \frac{\Gamma(b + 1 + j) \bar{\alpha}^j}{j! \Gamma(b + 1)} \left(\frac{1 + x}{x^3} \right) e^{\frac{-(i+1)\theta}{x}} \left(1 + \frac{\theta}{1 + \theta} \frac{1}{x} \right)^i. \tag{20}$$

Again, using the binomial expansion in Eq. (20)

$$f(x; \kappa) = \sum_{i,j=0}^{\infty} \sum_{m=0}^i E_{i,j,m} \left(\frac{1 + x}{x^{3+m}} \right) e^{\frac{-(i+1)\theta}{x}}, \tag{21}$$

where,

$$E_{i,j,m} = (-1)^i \binom{b+j-1}{i} \frac{b\alpha^b \theta^{2+m} \Gamma(b+1+j) \bar{\alpha}^j}{(1+\theta)^m j! \Gamma(b+1)}.$$

Hence, the s^{th} moment of the GMOIL distribution is obtained as follows

$$\mu'_s = \sum_{i,j=0}^{\infty} \sum_{m=0}^i E_{i,j,m} \left\{ \int_0^{\infty} x^{s-m-3} e^{-\frac{(i+1)\theta}{x}} dx + \int_0^{\infty} x^{s-m-2} e^{-\frac{(i+1)\theta}{x}} dx \right\}, \tag{22}$$

which leads to

$$\mu'_s = \sum_{i,j=0}^{\infty} \sum_{m=0}^i E_{i,j,m} \left\{ \frac{\Gamma(m-s+2)}{((i+1)\theta)^{m-s+2}} + \frac{\Gamma(m-s+1)}{((i+1)\theta)^{m-s+1}} \right\}. \tag{23}$$

The s^{th} central moment (μ_s) of X is given by

$$\mu_s = E(X - \mu'_1)^s = \sum_{i=0}^s (-1)^i \binom{s}{i} (\mu'_1)^i \mu'_{s-i}. \tag{24}$$

Recall the Taylor's series expansion of the function e^{tx} , that is $e^{tx} = \sum_{s=0}^{\infty} \frac{(tx)^s}{s!}$, so the moment generating function of the GMOIL distribution for $|t| < 1$, is given by

$$M_x(t) = \sum_{i,j,s=0}^{\infty} \sum_{m=0}^i E_{i,j,m} \frac{t^s}{s!} \left\{ \frac{\Gamma(m-s+2)}{((i+1)\theta)^{m-s+2}} + \frac{\Gamma(m-s+1)}{((i+1)\theta)^{m-s+1}} \right\}. \tag{25}$$

Numerical values of the first four-moments, variance (σ^2), skewness (SW) and kurtosis (KU) of the GMOIL distribution are displayed in Tab. 1. Some choice values of b and α are selected as follows (1) ($\alpha = 1.5, b = 4$), (2) ($\alpha = 1.5, b = 6$), (3) ($\alpha = 2, b = 4$), (4) ($\alpha = 2, b = 6$), (5) ($\alpha = 0.5, b = 4$), (6) ($\alpha = 0.5, b = 6$) for $\theta = 1$.

Table 1: First four-moments, σ^2 , SK and KU of X for some choices of parameters values

μ'_s	(1)	(2)	(3)	(4)	(5)	(6)
μ'_1	0.546	0.423	0.644	0.488	0.327	0.273
μ'_2	0.462	0.235	0.688	0.327	0.134	0.086
μ'_3	0.787	0.18	1.609	0.319	0.079	0.032
μ'_4	74.781	0.211	234.259	0.506	0.985	0.015
σ^2	0.164	0.056	0.273	0.089	0.028	0.011
SW	5.357	2.57	5.7	2.768	3.753	1.77
KU	2740	20.223	3102	22.875	1233	11.313

We conclude from Tab. 1 that the values of the μ'_1 and σ^2 of the GMOIL distribution get larger as the values of α increase for fixed value of b. Also, the distribution can be right skewed and leptokurtic.

Moreover, the s^{th} incomplete moment, say $\eta_s(t)$ is defined by

$$\eta_s(t) = \int_{-\infty}^t x^s \mathcal{F}(x) dx. \tag{26}$$

Hence, the s^{th} moment of the GMOIL distribution is derived by substituting Eq. (21) in Eq. (26) as follows

$$\eta_s(t) = \sum_{i,j=0}^{\infty} \sum_{m=0}^i E_{i,j,m} \left\{ \frac{\Gamma(m-s+2, (i+1)\theta t^{-1})}{((i+1)\theta)^{m-s+2}} + \frac{\Gamma(m-s+1, (i+1)\theta t^{-1})}{((i+1)\theta)^{m-s+1}} \right\}, \tag{27}$$

where $\Gamma(k, t) = \int_t^{\infty} x^{k-1} e^{-x} dx$ is the upper incomplete gamma function. Bonferroni and

Lorenz curves measures of in-equality are widely used in various fields such as survival analysis, demography and insurance. These measures are the main applications of the first incomplete moment.

3.3 Residual and reversed residual life functions

Here, the k^{th} moment of the residual lifetime (MRL) of a random variable X is defined as follows

$$\varpi_k(t) = \frac{1}{\bar{F}(t)} \int_t^{\infty} (x-t)^k f(x) dx. \tag{28}$$

Employing the binomial expansion and Eq. (21) in Eq. (28), then the k^{th} MRL of the GMOIL distribution is derived as follows

$$\varpi_k(t) = \frac{1}{\bar{F}(t)} \sum_{r=0}^k \sum_{i,j=0}^{\infty} \sum_{m=0}^i E_{i,j,m} (-1)^{k-r} \binom{k}{r} t^{k-r} \int_t^{\infty} x^r \left(\frac{1+x}{x^{3+m}} \right) e^{-\frac{(i+1)\theta}{x}} dx. \tag{29}$$

After simplification, the k^{th} MRL of the GMOIL distribution is given by

$$\varpi_k(t) = \frac{1}{\bar{F}(t)} \sum_{r=0}^k \sum_{i,j=0}^{\infty} \sum_{m=0}^i E_{i,j,m} \phi_k \left[\frac{\gamma[m-r+2, ((i+1)\theta/t)]}{((i+1)\theta)^{m-r+2}} + \frac{\gamma[m-r+1, ((i+1)\theta/t)]}{((i+1)\theta)^{m-r+1}} \right]. \tag{30}$$

where $\phi_k = (-1)^{k-r} \binom{k}{r} t^{k-r}$ and $\gamma(.,t)$ is the lower incomplete gamma function. The

mean residual life plays an important tool in different areas like; life insurance, industrial reliability, biomedical science, and demography. So, the mean residual life of the GMOIL distribution is obtained by substituting $k=1$ in Eq. (30).

On the other hand, the k^{th} moment of reversed residual life (RRL) of a random variable X is defined as follows

$$\upsilon_k(t) = \frac{1}{F(t)} \int_0^t (x-t)^k f(x) dx. \tag{31}$$

Again, we employ the binomial expansion and pdf in Eq. (21) in Eq. (31), then the k^{th} moment of RRL of the GMOIL will be

$$\nu_k(t) = \frac{1}{F(t)} \sum_{r=0}^k \sum_{i,j=0}^{\infty} \sum_{m=0}^i E_{i,j,m} \phi_k \left[\frac{\Gamma[m-r+2, ((i+1)\theta/t)]}{((i+1)\theta)^{m-r+2}} + \frac{\Gamma[m-r+1, ((i+1)\theta/t)]}{((i+1)\theta)^{m-r+1}} \right]. \quad (32)$$

For $k=1$ in Eq. (32), we obtain the mean of RRL or the mean waiting time of the GMOIL distribution, which represents the waiting time elapsed since the failure of an item on condition that this failure had occurred.

3.4 Stochastic ordering

Let X and Y are independent random variables with cdfs F_X and F_Y respectively, then according to Shaked et al. [Shaked and Shanthikumar (1994)], X is said to be less than Y if the following ordering holds;

Stochastic order ($X \leq_{sr} Y$) if $F_X(x) \geq F_Y(x)$ for all x

Likelihood ratio order ($X \leq_{lr} Y$) if $f_X(x)/f_Y(x)$ is decreasing in x

Hazard rate order ($X \leq_{hr} Y$) if $h_X(x) \geq h_Y(x)$ for all x

Mean residual life order ($X \leq_{mrl} Y$) if $\bar{F}_X(x) \geq \bar{F}_Y(x)$ for all x

We have the following chain of implications among the various partial orderings mentioned above:

$$\begin{array}{ccc} X \leq_{lr} Y \Rightarrow X & \begin{array}{c} \leq_{hr} \\ \Downarrow \\ \leq_{sr} \end{array} & Y \Rightarrow X \leq_{mrl} Y \\ & & \end{array}$$

Theorem: Let $X \sim \text{GMOIL}(\alpha_1, \beta_1, \theta_1)$ and $Y \sim \text{GMOIL}(\alpha_2, \beta_2, \theta_2)$ if $\alpha_1 > \alpha_2, \beta_1 > \beta_2$ and $\theta_1 = \theta_2 = \theta$, then $X \leq_{lr} Y$, $X \leq_{hr} Y$, $X \leq_{mrl} Y$, and $X \leq_{sr} Y$.

Proof: It is sufficient to show $f_X(x)/f_Y(x)$ is a decreasing function of x ; the likelihood ratio is

$$\frac{f_X(x)}{f_Y(x)} = \frac{\alpha_1^{\beta_1} \beta_1 \{D\}^{\beta_1-1} [1 - \bar{\alpha}_1 D]^{-\beta_1-1}}{\alpha_2^{\beta_2} \beta_2 \{1-D\}^{\beta_2-1} [1 - \bar{\alpha}_2 D]^{-\beta_2-1}}, \quad (33)$$

where, $D = 1 - \left(1 + \frac{\theta}{1+\theta} \frac{1}{x}\right) e^{-\frac{\theta}{x}}$. Therefore,

$$\frac{d}{dx} \log \frac{f_X(x)}{f_Y(x)} = \frac{(b_1 - b_2) dD/dx}{D} + \frac{\bar{\alpha}_1 (b_1 + 1) (dD/dx)}{1 - \bar{\alpha}_1 D} - \frac{\bar{\alpha}_2 (b_2 + 1) (dD/dx)}{1 - \bar{\alpha}_2 D} < 0, \quad (34)$$

where, $\frac{dD}{dx} = \left(\frac{-\theta^2(1+x)}{x^3(1+\theta)}\right) e^{-\frac{\theta}{x}}$. Thus, $f_X(x)/f_Y(x)$ is decreasing in x and hence

$X \leq_{hr} Y$. Similarly, we can conclude for $X \leq_{lr} Y$, $X \leq_{mrl} Y$, and $X \leq_{sr} Y$.

4 Parameter estimation

In view of the cost and time constraints, censoring is used in the statistical analysis of reliability characteristics for a system or device even with a loss in efficiency. There are several types of censoring schemes which are employed in life-testing and reliability studies. Two types of censoring are generally recognized, Type-I censoring (TIC) and Type-II censoring (TIIC). In TIC scheme, the experiment continues until a pre-assigned time τ , and failures that occur after τ are not observed. In contrast, in TIIC scheme the experiment decides to terminate after a pre-assigned number of failures observed, say $r, r \leq n$.

In this section, the point and approximate confidence intervals (CIs) estimators of the GMOIL model parameters, under TIC and TIIC schemes, are obtained using maximum likelihood (ML) method.

4.1 ML estimators based on TIC

Let $X_{(1)}, X_{(2)}, \dots, X_{(n)}$, be the observed TIC sample of size r whose life time's has the GMOIL distribution with Eq. (9) are placed on a life test and the test is terminated at specified time τ before all n items have failed. The log-likelihood function, based on TIC, of vector of parameters κ is given by:

$$\ln l_1 = \ln C + r \ln b + r b \ln \alpha + 2r \ln \theta - r \ln(1 + \theta) + \sum_{i=1}^r \ln(1 + x_{(i)}) - 3 \sum_{i=1}^r \ln x_{(i)} - \sum_{i=1}^r \frac{\theta}{x_{(i)}} + (b-1) \sum_{i=1}^r \ln(1 - D_i) - (b+1) \sum_{i=1}^r \ln[1 - \bar{\alpha}(1 - D_{(i)})] + (n-r) \ln \left[(\alpha(1 - D_{(\tau)}))^b (1 - \bar{\alpha}(1 - D_{(\tau)}))^{-b} \right], \tag{35}$$

where, $C = \frac{n!}{(r-1)!(n-r)!}$, $D_i = \left(1 + \frac{\theta}{1 + \theta x_i}\right) e^{-\frac{\theta}{x_i}}$, $D_\tau = \left(1 + \frac{\theta}{1 + \theta \tau}\right) e^{-\frac{\theta}{\tau}}$, also, for simplicity we write x_i instead of $x_{(i)}$. Hence the partial derivatives of the log-likelihood function with respect to b, α and θ components of the score vector $U(\kappa) = \partial \ln l_1 / \partial \kappa = (U(\alpha) = 0, U(b) = 0, U(\theta) = 0)^T$ can be obtained as follows

$$U(b) = r/b + r \ln \alpha + \sum_{i=1}^r \ln(1 - D_i) - \sum_{i=1}^r \ln[1 - \bar{\alpha}(1 - D_i)] + (n-r) \ln(\alpha(1 - D_\tau)) - (n-r) \ln(1 - \bar{\alpha}(1 - D_\tau)), \tag{36}$$

$$U(\alpha) = br/\alpha - (b+1) \sum_{i=1}^r \frac{1 - D_i}{(1 - \bar{\alpha}(1 - D_i))} + \frac{(n-r)b}{\alpha} - \frac{(n-r)b(1 - D_\tau)}{(1 - \bar{\alpha}(1 - D_\tau))}, \tag{37}$$

$$U(\theta) = 2r/\theta - r/(1 + \theta) - \sum_{i=1}^r \frac{1}{x_i} - (b-1) \sum_{i=1}^r \frac{\partial D_i / \partial \theta}{(1 - D_i)} - (b+1) \sum_{i=1}^n \frac{\bar{\alpha} \partial D_i / \partial \theta}{[1 - \bar{\alpha}(1 - D_i)]} - (n-r)b \frac{\partial D_\tau}{\partial \theta} \left[\frac{1}{\alpha(1 - D_\tau)} + \frac{\bar{\alpha}}{(1 - \bar{\alpha}(1 - D_\tau))} \right], \tag{38}$$

where, $\frac{\partial D_i}{\partial \theta} = \left[\frac{x_i^2 - x_i(1 + \theta) - \theta(1 + \theta)}{(1 + \theta)^2 x_i^2} \right] e^{-\frac{\theta}{x_i}}$, $\frac{\partial D_\tau}{\partial \theta} = \left[\frac{\tau^2 - \tau(1 + \theta) - \theta(1 + \theta)}{(1 + \theta)^2 \tau^2} \right] e^{-\frac{\theta}{\tau}}$.

The ML estimators of the model parameters are determined by solving the Eqs. (36)-(38) after setting them with zeros. These equations cannot be solved analytically and statistical software can be used to solve them numerically via iterative technique.

4.2 ML estimators based on TIIC

Let $X_{(1)}, X_{(2)}, \dots, X_{(n)}$, be the observed TIIC sample of size r whose life time's has the GMOIL distribution with Eq. (9) are placed on a life test and the test is terminated when the r^{th} item fails for some fixed values of r . The log-likelihood function, based on TIIC, of vector of parameters κ is given by:

$$\begin{aligned} \ln l_2 = & \ln C + r \ln b + br \ln \alpha + 2r \ln \theta - r \ln(1 + \theta) + \sum_{i=1}^r \ln(1 + x_{(i)}) - 3 \sum_{i=1}^r \ln x_{(i)} - \sum_{i=1}^r \frac{\theta}{x_{(i)}} \\ & + (b-1) \sum_{i=1}^r \ln(1 - D_i) - (b+1) \sum_{i=1}^r \ln[1 - \bar{\alpha}(1 - D_{(i)})] \\ & + (n-r) \ln \left[\left(\alpha(1 - D_{(r)}) \right)^b \left(1 - \bar{\alpha}(1 - D_{(r)}) \right)^{-b} \right], \end{aligned} \quad (39)$$

where, $D_r = \left(1 + \frac{\theta}{1 + \theta} \frac{1}{x_r} \right) e^{-\frac{\theta}{x_r}}$, also for simplicity we write x_i instead of $x_{(i)}$. Hence the partial derivatives of the log-likelihood function with respect to b, α and θ components of the score vector $U(\kappa) = \partial \ln l_2 / \partial \kappa = (U(\alpha), U(b), U(\theta))^T$ can be obtained as follows

$$\begin{aligned} U(b) = & r/b + r \ln \alpha + \sum_{i=1}^r \ln(1 - D_i) - \sum_{i=1}^r \ln[1 - \bar{\alpha}(1 - D_i)] + (n-r) \ln(\alpha(1 - D_r)) \\ & - (n-r) \ln(1 - \bar{\alpha}(1 - D_r)), \end{aligned} \quad (40)$$

$$U(\alpha) = br/\alpha - (b+1) \sum_{i=1}^r \frac{1 - D_i}{(1 - \bar{\alpha}(1 - D_i))} + \frac{(n-r)b}{\alpha} - \frac{(n-r)b(1 - D_r)}{(1 - \bar{\alpha}(1 - D_r))}, \quad (41)$$

$$\begin{aligned} U(\theta) = & 2r/\theta - r/(1 + \theta) - \sum_{i=1}^r \frac{1}{x_i} - (b-1) \sum_{i=1}^r \frac{\partial D_i / \partial \theta}{(1 - D_i)} - (b+1) \sum_{i=1}^n \frac{\bar{\alpha} \partial D_i / \partial \theta}{[1 - \bar{\alpha}(1 - D_i)]} \\ & - (n-r)b \frac{\partial D_r}{\partial \theta} \left[\frac{1}{\alpha(1 - D_r)} + \frac{\bar{\alpha}}{(1 - \bar{\alpha}(1 - D_r))} \right], \end{aligned} \quad (42)$$

where, $\frac{\partial D_r}{\partial \theta} = \left[\frac{x_r^2 - x_r(1 + \theta) - \theta(1 + \theta)}{(1 + \theta)^2 x_r^3} \right] e^{-\frac{\theta}{x_r}}$. The ML estimators of the model parameters are

determined by solving Eqs. (40)-(42) after setting them with zeros. These equations cannot be solved analytically and statistical software can be used to solve them numerically via iterative technique.

Note that, for $r=n$, we obtain the ML estimators of the model parameters in case of complete sample.

4.3 Approximate confidence intervals

In this subsection, approximate CIs of the model parameters for the GMOIL distribution are obtained.

The 3×3 observed information matrix $I(\kappa) = \{I_{mn}\}$ for $(m, n) = (\alpha, b, \theta)$, are determined. The known asymptotic properties of the ML method, under the regularity conditions, guarantee that:

$\sqrt{n}(\hat{\kappa} - \kappa) \xrightarrow{d} N_3(0, I^{-1}(\kappa))$ as $n \rightarrow \infty$ where \xrightarrow{d} means the convergence in distribution, with mean $\underline{0} = (0, 0, 0)^T$ and 3×3 variance-covariance matrix $I^{-1}(\kappa)$ then, the approximate $100(1-v)\%$ two-sided CIs for (α, b, θ) are respectively, given by:

$$\hat{\alpha} \pm Z_{v/2} \sqrt{\text{var}(\hat{\alpha})}, \hat{b} \pm Z_{v/2} \sqrt{\text{var}(\hat{b})}, \hat{\theta} \pm Z_{v/2} \sqrt{\text{var}(\hat{\theta})}. \tag{43}$$

Here, $Z_{v/2}$ is the upper $v/2$ th percentile of the standard normal distribution and $\text{var}(\cdot)$'s denote the diagonal elements of $I^{-1}(\kappa)$ corresponding to the model parameters.

5 Simulation study

The behavior of the estimators is assessed for some selected parameter values through a simulation. Measures include mean square error (MSE), bias, lower bound (LB) of the CIs, upper bound (UB) of the CIs, and average length (AL) of 90% and 95% are calculated. The following algorithm is utilized, in case of complete, TIC, TIIC via Mathematica 9, as follows:

- 1000 random samples of size $n=100$ and 200 are generated from the GMOIL distribution.
- Selected sets of parameters (Ps) are considered as I= $(\alpha=2, b=0.5, \theta=0.8)$, II= $(\alpha=2, b=0.5, \theta=1.2)$, III= $(\alpha=2, b=1.2, \theta=1.2)$, and IV= $(\alpha=2, b=1, \theta=1)$.
- The termination time is selected as $\tau=80$ and 100 under TIC. Three levels of censoring are chosen as $r=70\%$, 90% (TIIC) and 100% (complete sample).
- The MSE, bias, LB, UB and AL for all selected sets of the parameters are calculated.
- Numerical outcomes of the previous measures are listed in Tabs. 2 to 5 based on TIC, while Tabs. 6 to 9 contain the numerical results in case of complete and TIIC.

From these tables we conclude the following

- Bias, MSE and AL of all parameters decrease as the sample size increases.
- As the value of τ increases, the bias, MSE and AL of all parameters decrease.
- As the value of r increases, the bias, MSE and AL of all parameters decrease.
- At $\alpha=2, \theta=1.2$ and as the value of b increases, the bias, MSE and AL of all parameters increase.
- The AL of the CIs increases as the confidence levels increase from 90% to 95% .

Table 2: Estimate, Bias, MSE, LB, UB and AL of the GMOIL distribution for I=($\alpha=2$, $b=0.5$, $\theta=0.8$) under TIC

n	τ	Ps	ML	Bias	MSE	90%			95%		
						LB	UB	AL	LB	UB	AL
100	80	α	2.5075	0.5075	3.1254	-1.126	6.1418	7.2686	-1.822	6.8377	8.6605
		b	0.5483	0.0483	0.0080	0.3866	0.7099	0.3234	0.3556	0.7409	0.3853
		θ	0.8209	0.0209	0.0354	0.3957	1.2460	0.8503	0.3143	1.3274	1.0131
100	100	α	2.2024	0.2024	1.7623	-0.613	5.0177	5.6307	-1.152	5.5568	6.7089
		b	0.4928	-0.007	0.0045	0.3475	0.6380	0.2905	0.3197	0.6658	0.3461
		θ	0.9031	0.1031	0.0765	0.4760	1.3302	0.8542	0.3942	1.4120	1.0178
200	80	α	2.4661	0.4661	1.0949	0.2207	4.7116	4.4909	-0.209	5.1415	5.3508
		b	0.5054	0.0054	0.0040	0.4002	0.6106	0.2104	0.3801	0.6307	0.2506
		θ	0.7682	-0.031	0.0248	0.4799	1.0565	0.5765	0.4247	1.1117	0.6869
200	100	α	2.3687	0.3686	1.6390	0.1726	4.5647	4.3921	-0.247	4.9852	5.2332
		b	0.5049	0.0049	0.0019	0.4021	0.6076	0.2055	0.3824	0.6273	0.2449
		θ	0.8241	0.0241	0.0493	0.5273	1.1210	0.5937	0.4704	1.1778	0.7074

Table 3: Estimate, Bias, MSE, LB, UB and AL of the GMOIL distribution for II=($\alpha=2$, $b=0.5$, $\theta=1.2$) under TIC

n	τ	Ps	ML	Bias	MSE	90%			95%		
						LB	UB	AL	LB	UB	AL
100	80	α	1.8740	-0.126	1.1472	-0.476	4.2248	4.7016	-0.927	4.6749	5.6019
		b	0.4956	-0.004	0.0023	0.3411	0.6501	0.3090	0.3115	0.6797	0.3682
		θ	1.5250	0.3250	0.3248	0.8103	2.2397	1.4294	0.6734	2.3765	1.7031
100	100	α	2.4938	0.4938	1.1442	-0.711	5.6989	6.4102	-1.325	6.3126	7.6377
		b	0.5052	0.0052	0.0055	0.3492	0.6612	0.3121	0.3193	0.6911	0.3718
		θ	1.1915	-0.008	0.0969	0.5562	1.8268	1.2706	0.4345	1.9485	1.5140
200	80	α	1.7100	-0.290	0.2763	0.2362	3.1837	2.9475	-0.046	3.4659	3.5119
		b	0.4925	-0.007	0.0019	0.3848	0.6002	0.2153	0.3642	0.6208	0.2566
		θ	1.3346	0.1346	0.0642	0.8723	1.7969	0.9246	0.7838	1.8855	1.1017
200	100	α	1.6235	-0.376	0.7488	0.1997	3.0472	2.8475	-0.073	3.3199	3.3928
		b	0.4712	-0.028	0.0036	0.3721	0.5704	0.1983	0.3531	0.5893	0.2362
		θ	1.3021	0.1021	0.0655	0.8586	1.7456	0.8870	0.7737	1.8306	1.0568

Table 4: Estimate, Bias, MSE, LB, UB and AL of the GMOIL distribution for III=($\alpha=2$, $b=1.2$, and $\theta=1.2$) under TIC

n	τ	Ps	ML	Bias	MSE	90%			95%		
						LB	UB	AL	LB	UB	AL
100	80	α	2.7279	0.7279	2.6718	-0.370	5.8262	6.1966	-0.963	6.4195	7.3832
		b	1.4061	0.2061	0.1891	0.7753	2.0370	1.2618	0.6545	2.1578	1.5034
		θ	1.1740	-0.026	0.0600	0.7477	1.6002	0.8525	0.6661	1.6818	1.0157
	100	α	1.9758	-0.024	0.9445	-0.153	4.1048	4.2581	-0.561	4.5125	5.0735
		b	1.1792	-0.020	0.0800	0.7059	1.6525	0.9466	0.6153	1.7431	1.1278
		θ	1.2663	0.0663	0.0723	0.8241	1.7084	0.8843	0.7394	1.7931	1.0537
200	80	α	2.0285	0.0285	0.3018	0.4837	3.5733	3.0896	0.1879	3.8691	3.6812
		b	1.2333	0.0333	0.0295	0.8766	1.5899	0.7132	0.8084	1.6582	0.8498
		θ	1.2764	0.0764	0.0166	0.9526	1.6002	0.6476	0.8906	1.6623	0.7717
	100	α	1.7239	-0.276	0.2685	0.4151	3.0327	2.6177	0.1644	3.2834	3.1189
		b	1.1288	-0.071	0.0255	0.8161	1.4415	0.6254	0.7562	1.5014	0.7451
		θ	1.2746	0.0746	0.0227	0.9557	1.5935	0.6378	0.8946	1.6546	0.7599

Table 5: Estimate, Bias, MSE, LB, UB and AL of the GMOIL distribution for IV=($\alpha=2$, $b=1$, $\theta=1$) under TIC

n	τ	Ps	ML	Bias	MSE	90%			95%		
						LB	UB	AL	LB	UB	AL
100	80	α	1.9697	-0.030	1.7754	-0.229	4.1691	4.3990	-0.651	4.5903	5.2413
		b	1.0305	0.0305	0.0757	0.6319	1.4291	0.7972	0.5556	1.5054	0.9499
		θ	1.1395	0.1395	0.0878	0.7365	1.5426	0.8061	0.6593	1.6198	0.9605
	100	α	2.3510	0.3510	3.5458	-0.255	4.9575	5.2130	-0.754	5.4566	6.2112
		b	1.0357	0.0357	0.0576	0.6521	1.4193	0.7672	0.5786	1.4927	0.9141
		θ	1.0985	0.0985	0.0734	0.6923	1.5047	0.8123	0.6145	1.5824	0.9679
200	80	α	1.6054	-0.394	0.3990	0.3933	2.8175	2.4242	0.1612	3.0496	2.8884
		b	0.9307	-0.069	0.0161	0.6996	1.1618	0.4622	0.6554	1.2061	0.5507
		θ	1.0773	0.0773	0.0192	0.7973	1.3573	0.5600	0.7437	1.4109	0.6672
100	α	2.4850	0.4850	1.6030	0.5011	4.4690	3.9679	0.1211	4.8489	4.7277	

b	1.0211	0.0211	0.0067	0.7616	1.2807	0.5191	0.7119	1.3304	0.6185
θ	0.9935	-0.006	0.0251	0.7084	1.2786	0.5702	0.6538	1.3332	0.6794

Table 6: Estimate, Bias, MSE, LB, UB and AL of the GMOIL distribution for I=($\alpha=2$, $b=0.5$, $\theta=0.8$) under TIIC

n	x_r	Ps	ML	Bias	MSE	90%			95%			
						LB	UB	AL	LB	UB	AL	
100	80%	α	2.4216	0.4216	2.0883	-1.074	5.9176	6.9920	-1.743	6.5871	8.3309	
		b	0.5059	0.0059	0.0063	0.3219	0.6899	0.3680	0.2867	0.7252	0.4385	
		θ	0.8059	0.0059	0.0562	0.3856	1.2261	0.8405	0.3051	1.3066	1.0014	
	90%	α	2.7311	0.7311	3.1689	-0.886	6.3487	7.2351	-1.579	7.0414	8.6206	
		b	0.5527	0.0527	0.0103	0.3779	0.7274	0.3495	0.3445	0.7609	0.4164	
		θ	0.8404	0.0404	0.0746	0.4247	1.2561	0.8313	0.3451	1.3357	0.9905	
	100%	α	2.0118	0.0118	2.0404	-0.618	4.6424	5.2611	-1.122	5.1461	6.2685	
		b	0.5183	0.0183	0.0087	0.3815	0.6552	0.2737	0.3553	0.6814	0.3261	
		θ	1.0131	0.2131	0.1826	0.5514	1.4748	0.9234	0.4630	1.5633	1.1002	
	200	80%	α	2.2503	0.2503	1.4083	0.0801	4.4205	4.3404	-0.335	4.8361	5.1715
			b	0.4870	-0.013	0.0037	0.3690	0.6049	0.2359	0.3464	0.6275	0.2811
			θ	0.8118	0.0118	0.0177	0.5090	1.1146	0.6055	0.4511	1.1725	0.7215
90%		α	1.9954	-0.004	0.7632	0.1799	3.8109	3.6310	-0.167	4.1586	4.3263	
		b	0.5058	0.0058	0.0022	0.4002	0.6114	0.2112	0.3800	0.6316	0.2516	
		θ	0.9030	0.1030	0.0563	0.5824	1.2236	0.6412	0.5210	1.2849	0.7639	
100%	α	2.0663	0.0663	0.7215	0.2780	3.8545	3.5765	-0.064	4.1970	4.2614		
	b	0.4985	-0.001	0.0016	0.4077	0.5893	0.1816	0.3903	0.6067	0.2163		
		θ	0.8654	0.0654	0.0581	0.5644	1.1664	0.6020	0.5068	1.2240	0.7172	

Table 7: Estimate, Bias, MSE, LB, UB and AL of the GMOIL distribution for II=($\alpha=2$, $b=0.5$, $\theta=1.2$), under TIIC

n	x_r	Ps	ML	Bias	MSE	90%			95%		
						LB	UB	AL	LB	UB	AL
100	80%	α	2.2612	0.2612	2.2824	-0.804	5.3272	6.1319	-1.391	5.9143	7.3062
		b	0.5358	0.0358	0.0201	0.3347	0.7369	0.4022	0.2962	0.7755	0.4792
		θ	1.4793	0.2793	0.3203	0.7747	2.1839	1.4092	0.6398	2.3189	1.6791
	90%	α	2.4616	0.4616	1.8287	-0.716	5.6398	6.3565	-1.325	6.2484	7.5737

		b	0.5385	0.0385	0.0120	0.3710	0.7060	0.3350	0.3389	0.7381	0.3992	
		θ	1.2395	0.0395	0.1022	0.6038	1.8751	1.2712	0.4821	1.9968	1.5147	
		α	2.1653	0.1653	1.4852	-0.456	4.7866	5.2426	-0.958	5.2885	6.2465	
	100%	b	0.5027	0.0027	0.0060	0.3708	0.6345	0.2637	0.3456	0.6598	0.3142	
		θ	1.2906	0.0906	0.1231	0.6655	1.9157	1.2503	0.5458	2.0354	1.4897	
		α	2.5052	0.5052	1.0483	0.0940	4.9164	4.8224	-0.367	5.3781	5.7458	
	80%	b	0.4969	-0.003	0.0058	0.3745	0.6194	0.2448	0.3511	0.6428	0.2917	
		θ	1.0874	-0.112	0.0369	0.6536	1.5212	0.8676	0.5705	1.6043	1.0338	
		α	1.6211	-0.378	0.9512	0.2197	3.0226	2.8030	-0.048	3.2910	3.3397	
	200	90%	b	0.4760	-0.024	0.0046	0.3764	0.5756	0.1992	0.3573	0.5946	0.2373
		θ	1.3660	0.1660	0.0991	0.9158	1.8161	0.9003	0.8296	1.9023	1.0727	
		α	2.2344	0.2344	1.3488	0.3661	4.1027	3.7366	0.0083	4.4604	4.4521	
	100%	b	0.4926	-0.007	0.0045	0.4030	0.5822	0.1792	0.3858	0.5994	0.2135	
		θ	1.2331	0.0331	0.0221	0.7855	1.6807	0.8952	0.6998	1.7664	1.0667	

Table 8: Estimate, Bias, MSE, LB, UB and AL of the GMOIL distribution for III=($\alpha=2$, $b=1.2$, and $\theta=1.2$) under TIIC

n	x_r	Ps	ML	Bias	MSE	90%			95%			
						LB	UB	AL	LB	UB	AL	
		α	1.9598	-0.040	3.2120	-1.545	5.4647	7.0098	-2.216	6.1358	8.3521	
	80%	b	1.2367	0.0367	0.3814	0.1389	2.3345	2.1956	-0.071	2.5447	2.6160	
		θ	1.3722	0.1722	0.0885	0.8670	1.8775	1.0105	0.7702	1.9742	1.2040	
		α	2.7134	0.7134	2.7892	-0.927	6.3538	7.2808	-1.624	7.0509	8.6750	
	100	90%	b	1.3083	0.1083	0.1493	0.5494	2.0673	1.5178	0.4041	2.2126	1.8085
		θ	1.1602	-0.039	0.0613	0.7111	1.6092	0.8981	0.6251	1.6952	1.0700	
		α	2.0990	0.0990	1.1530	-0.180	4.3783	4.5586	-0.616	4.8148	5.4315	
	100%	b	1.2142	0.0142	0.0433	0.7266	1.7017	0.9751	0.6333	1.7951	1.1618	
		θ	1.2841	0.0841	0.0441	0.8233	1.7450	0.9217	0.7350	1.8332	1.0982	
		α	3.1361	1.1361	4.4934	-0.561	6.8338	7.3955	-1.269	7.5419	8.8117	
	80%	b	1.3506	0.1506	0.1913	0.5634	2.1377	1.5743	0.4127	2.2884	1.8757	
		θ	1.1202	-0.079	0.0647	0.7935	1.4469	0.6534	0.7309	1.5094	0.7785	
	200	α	2.1020	0.1020	1.2252	0.1409	4.0630	3.9221	-0.234	4.4385	4.6731	
	90%	b	1.2022	0.0022	0.0496	0.7336	1.6707	0.9371	0.6439	1.7605	1.1166	
		θ	1.2280	0.0280	0.0384	0.9086	1.5475	0.6390	0.8474	1.6087	0.7613	

	α	1.4262	-0.573	0.5781	0.3414	2.5110	2.1696	0.1337	2.7187	2.5851
100%	b	1.1144	-0.085	0.0382	0.8025	1.4262	0.6237	0.7428	1.4859	0.7431
	θ	1.3834	0.1834	0.0390	1.0554	1.7115	0.6560	0.9926	1.7743	0.7817

Table 9: Estimate, Bias, MSE, LB, UB and AL of the GMOIL distribution for IV=($\alpha=2$, $b=1$, $\theta=1$) under THIC

n	x_r	Ps	ML	Bias	MSE	90%			95%		
						LB	UB	AL	LB	UB	AL
80%		α	2.4729	0.4729	3.9227	-1.682	6.6281	8.3104	-2.478	7.4238	9.9018
		b	1.2564	0.2564	0.4592	0.1787	2.3341	2.1555	-0.027	2.5405	2.5682
		θ	1.1680	0.1680	0.1408	0.7195	1.6166	0.8972	0.6336	1.7025	1.0690
100	90%	α	2.4432	0.4432	2.0767	-0.729	5.6162	6.3460	-1.337	6.2238	7.5612
		b	1.0918	0.0918	0.1583	0.5256	1.6580	1.1324	0.4171	1.7664	1.3493
		θ	0.9950	-0.005	0.0233	0.5974	1.3925	0.7951	0.5213	1.4687	0.9474
100%		α	2.6344	0.6344	3.8963	-0.433	5.7018	6.1347	-1.020	6.2891	7.3095
		b	1.0410	0.0409	0.0792	0.6566	1.4253	0.7687	0.5830	1.4989	0.9159
		θ	1.0107	0.0107	0.0966	0.6112	1.4102	0.7990	0.5347	1.4867	0.9520
80%		α	2.5879	0.5879	2.8297	-0.122	5.2981	5.4203	-0.641	5.8170	6.4583
		b	1.0750	0.0750	0.0652	0.6057	1.5443	0.9386	0.5158	1.6342	1.1184
		θ	1.0329	0.0329	0.0746	0.7300	1.3357	0.6057	0.6720	1.3937	0.7217
200	90%	α	1.8454	-0.154	0.3432	0.2696	3.4213	3.1518	-0.032	3.7231	3.7553
		b	0.9904	-0.009	0.0308	0.6712	1.3096	0.6385	0.6100	1.3708	0.7607
		θ	1.0397	0.0397	0.0071	0.7584	1.3211	0.5628	0.7045	1.3750	0.6705
100%		α	3.1733	1.1733	3.8377	0.5580	5.7887	5.2308	0.0571	6.2896	6.2324
		b	1.0986	0.0986	0.0238	0.8075	1.3898	0.5823	0.7517	1.4456	0.6938
		θ	0.9256	-0.074	0.0403	0.6486	1.2026	0.5540	0.5955	1.2556	0.6601

6 Applications to real data

Two real data sets are analyzed to characterize the behavior of the GMOIL distribution in practice. The first data set is picked from Linhart et al. [Linhart and Zucchini (1986)]. These data were discussed by Hassan et al. [Hassan, Elgarhy, Mohamd et al. (2019); Jamal, Elbatal, Chesneau et al. (2019)]. The second data are collected from Aarset [Aarset (1987)] which represent the failure times of 50 devices. These data were handled by Hassan et al. [Hassan and Assar (2019)]. In both data, the results of the fits are compared with the power Lindley (PL) by Ghitany et al. [Ghitany, Al-Mutairi, Balakrishnan et al. (2013)], extended Lindley (EL) by Bakouch et al. [Bakouch, Al-

Zahrani, Al-Shomrani et al. (2012)], Lindley(L), alpha power transmuted L (APTL) see (Dey et al. [Dey, Ghosh and Kumar (2019)]), alpha power extended exponential (APEE) (Hassan et al. [Hassan, Mohamd, Elgarhy et al. (2019)]) and IL distributions. The pdfs of the APTL, APEE and EL are given by;

$$f_{APTL}(x; \alpha, \theta) = \frac{\log \alpha \theta^2 (1+x) e^{-\theta x} \alpha^{1-e^{-\theta x} [1+\frac{\theta x}{\theta+1}]}}{\alpha-1 \theta+1}, x, \alpha, \theta > 0, \alpha \neq 1$$

$$f_{APEE}(x; \alpha, \beta, \gamma) = \frac{\log \alpha \gamma^2 (1+\beta x) e^{-\gamma x} \alpha^{1-e^{-\gamma x} [1+\frac{\gamma \beta x}{\gamma+\beta}]}}{\alpha-1 \gamma+\beta}, x, \alpha, \beta, \gamma > 0, \alpha \neq 1$$

and,

$$f_{EL}(x; \alpha, \beta, \theta) = \frac{\theta(1+\theta+\theta x)^{\alpha-1} e^{-(\theta x)^\beta}}{(\theta+1)^\alpha} [\beta(1+\theta+\theta x)(\theta x)^{\beta-1} - \alpha], x, \alpha, \beta, \theta > 0,$$

Statistics measures like; minus log-likelihood (-log L), Kolmogorov-Smirnov (KS) test statistic, Akaike information criterion (AIC), corrected AIC (CAIC), Bayesian information criterion (BIC) and Hannan-Quinn information criterion (HQIC) are obtained. These measures are applied to test the superiority of the GMOIL distribution in comparison to some other models.

Data 1: Linhart and zucchini data

The first data set represents a sample of 30 failure times of air-conditioned system of an airplane. The data are: 23, 261, 87, 7, 120, 14, 62, 47, 225, 71, 246, 21, 42, 20, 5, 12, 120, 11, 3, 14, 71, 11, 14, 11, 16, 90, 1, 16, 52, 95. The ML estimates, standard errors (SEs) of parameters and the above statistics measures are given in Tab. 10.

Table 10: Analytical outcomes of the GMOIL and competing models for Linhart and Zucchini data

Model	ML Estimates (SE)	-Log L	AIC	BIC	CAIC	HQIC	KS
	$\hat{\alpha} = 79.48 (171.565)$						
GMOIL	$\hat{b} = 2.893 (2.517)$ $\hat{\theta} = 2.147 (2.633)$ $\hat{\alpha} = 0.161 (0.282)$	151.53	309.061	307.492	309.984	310.405	0.1379
APEE	$\hat{\beta} = 2.01 \times 10^{-4} (0.024)$ $\hat{\gamma} = 0.011 (0.022)$	176.631	359.262	357.694	360.186	360.607	0.14683
APTL	$\hat{\alpha} = 0.1 (0.1037)$ $\hat{\gamma} = 0.011 (0.024)$ $\hat{\alpha} = 0.634 (1.995)$	183.415	370.83	369.784	371.274	371.727	0.2803
EL	$\hat{\beta} = 0.194 (0.161)$ $\hat{\theta} = 104.392 (149.921)$	217.635	441.27	439.701	442.193	442.615	0.86825

PL	$\hat{\beta} = 1.525 (0.155)$ $\hat{\theta} = 2.63 \times 10^{-3} (2.058 \times 10^{-3})$	195.999	395.999	394.953	396.443	396.895	0.4927
L	$\hat{\theta} = 0.033 (0.0043)$	161.637	325.274	324.751	325.416	325.722	0.3453
IL	$\hat{\theta} = 12.037 (2.052)$	159.267	320.533	320.011	323.422	320.982	0.234

From Tab. 10, we conclude that the GMOIL model is the best fitted model compared with APTL, APEE, EL, PL, L and IL models. The estimated pdfs, cdfs, sfs and pp plots for the fitted models are displayed in Fig. 2. We conclude that the GMOIL model offers a better fit for the Linhart and Zucchini data.

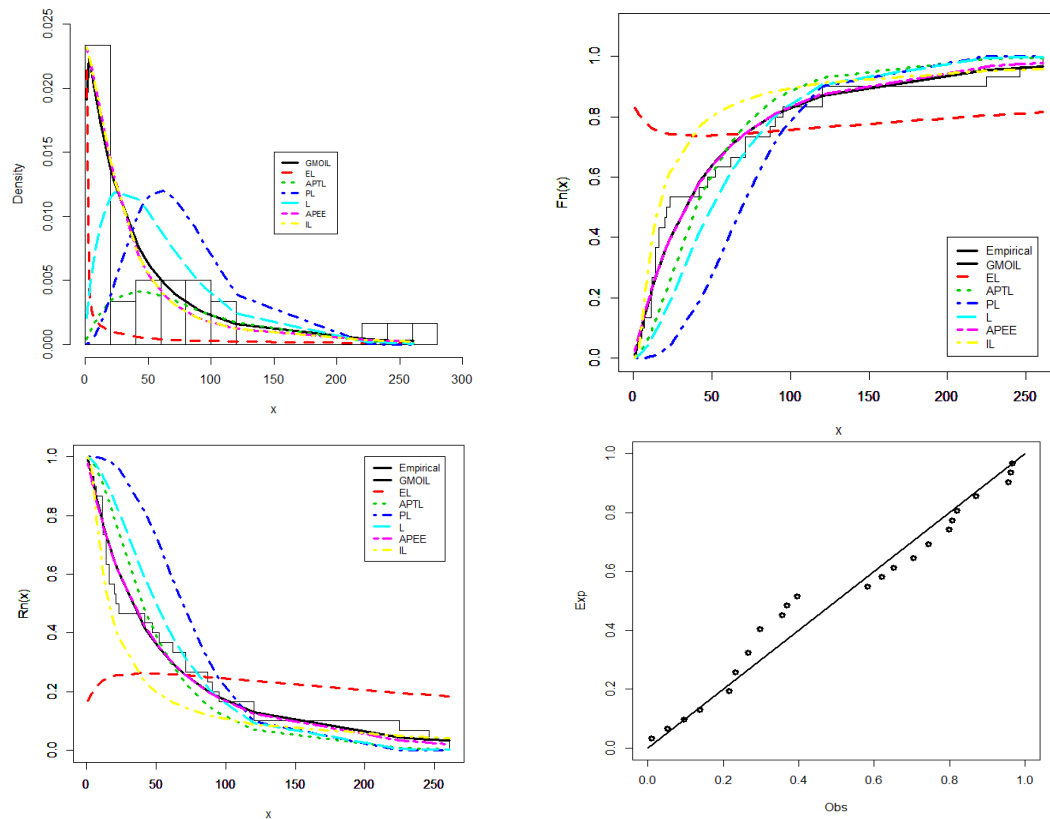


Figure 2: Estimated pdf, cdf, sf and pp plots of the GMOIL and other competing distributions for Linhart and Zucchini data

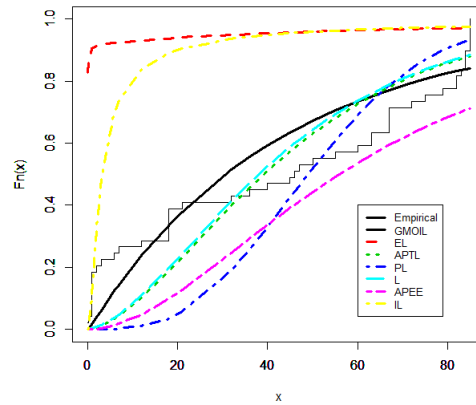
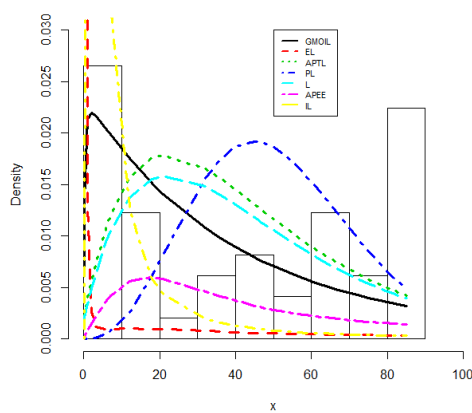
Data 2: Aarset data

The second data represent 50 failure times of devices. The data are: 0.1, 0.2, 1, 1, 1, 1, 2, 3, 6, 7, 11, 12, 18, 18, 18, 18, 18, 21, 32, 36, 40, 45, 46, 47, 50, 55, 60, 63, 63, 67, 67, 67, 72, 75, 79, 82, 82, 83, 84, 84, 84, 85, 85, 85, 85, 85, 86, 86. The ML estimates, SEs of the model parameters and analytical measures are provided in Tab. 11.

Table 11: Analytical results of the GMOIL and competing models for Aarset data

Model	ML Estimates (SE)	-Log L	AIC	BIC	CAIC	HQIC	KS
	$\hat{\alpha} = 3472 (2808.291)$						
GMOIL	$\hat{b} = 11.563 (9.145)$ $\hat{\theta} = 0.453(0.182)$	244.093	494.186	493.282	495.109	496.37	0.1944
APEE	$\hat{\alpha} = 0.0606(3.44 \times 10^{-10})$ $\hat{\beta} = 1400(2.653 \times 10^6)$ $\hat{\gamma} = 0.016(1.658 \times 10^{-3})$	304.783	615.566	614.663	616.49	617.751	0.19985
APTL	$\hat{\alpha} = 1.258(0.9024)$ $\hat{\gamma} = 0.044(0.0063)$	293.083	590.165	589.563	590.61	591.621	0.1951
EL	$\hat{\alpha} = 1.068(17.205)$ $\hat{\beta} = 0.217(2.128 \times 10^{-3})$ $\hat{\theta} = 168.806(6.413 \times 10^{-3})$	348.441	702.883	701.98	703.405	705.067	0.86086
PL	$\hat{\beta} = 1.753(0.196)$ $\hat{\theta} = 1.825 \times 10^{-3} (1.535 \times 10^{-3})$	305.67	615.34	614.738	615.595	616.796	0.3249
L	$\hat{\theta} = 0.043(0.0043)$	251.43	504.861	504.56	505.003	505.589	0.199
IL	$\hat{\theta} = 2.846(0.334)$	324.041	650.082	649.781	650.168	650.811	0.6288

Tab. 11 shows that the GMOIL model can be more adequate model for explaining the provided data than the other models. More details are shown in Fig. 3.



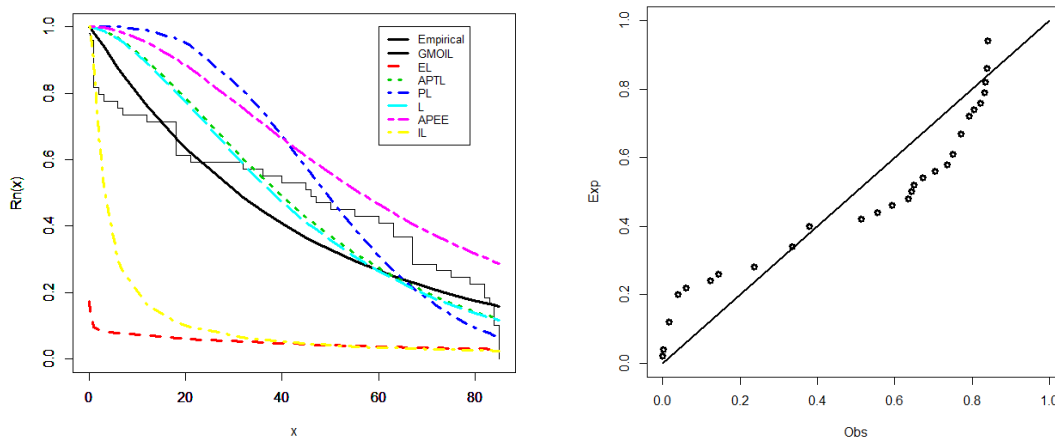


Figure 3: Estimated pdf, cdf, sf and pp plots of the GMOIL and other competing distributions for Aarset data

We conclude that the GMOIL distribution is more appropriate than the other models (see Fig. 3).

7 Conclusions

A new three-parameter extended form of the inverse Lindley distribution related to Marshall-Olkin-G class is proposed. The new distribution is named as the generalized Marshall-Olkin inverse Lindley distribution. Some main properties of the new model are given. Estimation of the model parameters is approached by maximum likelihood method for complete and censored samples. Point and approximate confidence interval estimators of the model parameters are obtained. Simulation study is designed to evaluate the performance of the estimates. Two applications explain that the proposed distribution provides consistently better fits than the other competitive models.

Acknowledgment: This project was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah, under grant No. (DF-279-150-1441). The authors, therefore, gratefully acknowledge DSR technical and financial support.

Funding Statement: This project was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah, under grant No. (DF-279-150-1441). The authors, therefore, gratefully acknowledge DSR technical and financial support.

Conflicts of Interest: The authors declare that they have no conflicts of interest to report regarding the present study.

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