New Optimal Newton-Householder Methods for Solving Nonlinear Equations and Their Dynamics

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Abstract: The classical iterative methods for finding roots of nonlinear equations, like the Newton method, Halley method, and Chebyshev method, have been modified previously to achieve optimal convergence order. However, the Householder method has so far not been modified to become optimal. In this study, we shall develop two new optimal Newton-Householder methods without memory. The key idea in the development of the new methods is the avoidance of the need to evaluate the second derivative. The methods fulfill the Kung-Traub conjecture by achieving optimal convergence order four with three functional evaluations and order eight with four functional evaluations. The efficiency indices of the methods show that methods perform better than the classical Householder's method. With the aid of convergence analysis and numerical analysis, the efficiency of the schemes formulated in this paper has been demonstrated. The dynamical analysis exhibits the stability of the schemes in solving nonlinear equations. Some comparisons with other optimal methods have been conducted to verify the effectiveness, convergence speed, and capability of the suggested methods.

Keywords: Iterative method, householder method, simple root, optimal convergence, nonlinear equation.

1 Introduction

Many iterative schemes for finding roots of nonlinear equations $f(x)=0$ have been introduced. Traub presented the general theory of iterative schemes for solving nonlinear equations numerically [Kumar, Sharma and Argyros (2020)]. New ideas are continuously being developed for constructing better iterative schemes, see for example, Noor et al. [Noor, Waseem, Noor et al. (2015); Alharbi, Faisal, Shah et al. (2019); Argyros, Behl, Machado et al. (2019); Herceg and Herceg (2018); Kumar, Maroju, Behl et al. (2018); Solaiman, Karim and Hashim (2018); Solaiman and Hashim (2019); Waseem, Noor, Shah et al. (2018); Wang and Tao (2020)].

Newton's method is the most well-known method to find a simple root of a nonlinear equation, and it is given below:

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$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{1}$$

The convergence rate is quadratic provided that the initial guess is close enough to the real root. Householder [Abbasbandy (2003)] introduced an iterative scheme that reaches to the third order of convergence:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f''(x_n)f^2(x_n)}{2f'^3(x_n)} \tag{2}$$

Noor et al. [Noor, Aslam and Momani (2007)] suggested a two-step Householder scheme, which is the Newton method as a predictor and the Householder method as a corrector. This scheme has sixth order of convergence:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)} \\ x_{n+1} &= y_n - \frac{f(y_n)}{f'(y_n)} - \frac{f''(y_n)f^2(y_n)}{2f'^3(y_n)} \end{aligned} \tag{3}$$

Several years later, Nazeer et al. [Nazeer, Tanveer, Min et al. (2016)] presented a new two-step Householder scheme that is free from second derivatives. The scheme was modified to achieve the fifth order of convergence:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)} \\ x_{n+1} &= y_n - \frac{f(y_n)}{f'(y_n)}\left(1 - \frac{f'(y_n)f'(x_n)f(y_n) - f'^2(x_n)f(x_n)}{2f'^2(y_n)f(x_n)}\right) \end{aligned} \tag{4}$$

All of the methods mentioned above did not achieve optimal order of convergence in the sense of the Kung-Traub conjecture [Behl, Alshomrani and Magreñán (2019)], which states that an optimal iterative method without memory can reach the highest order of convergence at 2^{k-1} by k number of functional and derivative evaluations. The effectiveness of the iterative method can be evaluated using the Efficiency Index that was introduced by Ostrowski [Liu and Wang (2010)], which is

$$E_i = P^{\frac{1}{m}} \tag{5}$$

where P is the value of convergence order, and m is the total number of functional evaluations and derivatives per iteration.

Since the method presented by Nazeer et al. [Nazeer, Tanveer, Min et al. (2016)] cannot achieve optimal order, we suggest developing a new optimal Newton-Householder method with improved efficiency as compared to the previous works [Abbasbandy (2003); Nazeer, Tanveer, Min et al. (2016); Noor, Aslam and Momani (2007)]. The rate of convergence for our proposed method has been validated and supported by numerical experiments. Furthermore, we also explore their dynamic behavior in the complex plane, which provides us information about the convergence, divergence, and stability of the suggested methods to ensure the competitiveness of our methods.

2 New suggested methods

In this section, we shall modify the Newton-Householder method that was presented by Noor et al. [Noor, Aslam and Momani (2007)] to get two optimal schemes with a fewer number of function and derivative evaluations. The modification avoids the second derivative in the original method and reaches the optimal convergence order.

2.1 Optimal two-step fourth-order method

We consider a combination of the Newton method and Householder method which was suggested by Noor et al. [Noor, Aslam and Momani (2007)] as follows:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)} \\ x_{n+1} &= y_n - \frac{f(y_n)}{f'(y_n)} - \frac{f^2(y_n) f''(y_n)}{2f'^3(y_n)} \end{aligned} \tag{6}$$

Now, taking the approximations of the first and second derivatives in Eq. (6) as

$$\begin{aligned} f'(y_n) &\approx f'(x_n) \\ f''(y_n) &\approx \frac{10f(y_n)+4f(x_n)}{(y_n-x_n)^2} \end{aligned} \tag{7}$$

gives a new optimal order method with three functional and derivative evaluations in the two-step method:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)} \\ x_{n+1} &= y_n - \frac{f(y_n)}{f'(x_n)} - L(x_n, y_n)\frac{f^2(y_n)}{2f'^3(x_n)} \end{aligned} \tag{8}$$

where $L(x_n, y_n) = \dfrac{10f(y_n)+4f(x_n)}{(y_n-x_n)^2}$.

2.2 Optimal three-step eighth-order method

By utilizing the ideas from Eq. (8), we add one more step as the Newton method to achieve an optimal eight method:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}$$
$$z_n = y_n - \frac{f(y_n)}{f'(x_n)} - L(x_n, y_n)\frac{f^2(y_n)}{2f'^3(x_n)} \quad (9)$$
$$x_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)}$$

The scheme achieves optimal order, but it still does not fulfill the Kung-Traub conjecture. Hence, we take the following approximation:

$$f'(z_n) \approx \frac{f'(x_n)}{G(x)} \quad (10)$$

with

$$G(x) = G(s_n, t_n, u_n) = 1 + 2s_n + t_n + 4u_n + 6s_n^2 + 6s_n^3 \quad (11)$$

where $G: \mathbb{C}^3 \to \mathbb{C}$ is an analytical function in the neighborhood of $(0,0,0)$. Therefore, the final scheme of the eighth-order Newton-Householder is

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}$$
$$z_n = y_n - \frac{f(y_n)}{f'(x_n)} - L(x_n, y_n)\frac{f^2(y_n)}{2f'^3(x_n)} \quad (12)$$
$$x_{n+1} = z_n - G(s_n, t_n, u_n)\frac{f(z_n)}{f'(x_n)}$$

where $s_n = \frac{f(y_n)}{f(x_n)}$, $t_n = \frac{f(z_n)}{f(y_n)}$ and $u_n = \frac{f(z_n)}{f(x_n)}$.

3 Convergence analysis

In Theorems 1 and 2, we show the convergence orders of the schemes in Eqs. (8) and (12).

Theorem 1. Assume that the function $f: D \subset \mathbb{C} \to \mathbb{C}$ for an open interval D has a simple root $\alpha \in D$. Let $f(x)$ be sufficiently smooth in the interval D. Then, the order of convergence of the new method described by Eq. (8) is four and meets the following error equation:

$$e_{n+1} = -\frac{c_2 c_3}{c_1^2} e_n^4 + [e_n^5] \quad (13)$$

Proof. Let α be a simple root of $f(x)$, i.e., $f(\alpha) = 0$ and $f'(\alpha) \neq 0$. By using the Taylor series expansion, we have

$$f(x_n)=\sum_{m=1}^{\infty}\frac{f^{(m)}(\alpha)(x_n-\alpha)^m}{m!}=f(\alpha)+f'(\alpha)(x_n-\alpha)+\frac{f''(\alpha)}{2}(x_n-\alpha)^2+\ldots \tag{14}$$

or we can write $f(x_n)$ as

$$f(x_n)=f'(\alpha)(c_1e_n+c_2e_n^2+c_3e_n^3+c_4e_n^4+O[e_n^5]) \tag{15}$$

where $e_n=x_n-\alpha$, and $c_k=\frac{f^{(k)}(\alpha)}{k!f'(\alpha)}$ for $k\in$ ⁀. Expanding $f'(x_n)$ at α, we get

$$f'(x_n)=f'(\alpha)(c_1+2c_2e_n+3c_3e_n^2+4c_4e_n^3+O[e_n^4]) \tag{16}$$

By substituting Eqs. (15) and (16) into the first substep of the proposed method Eq. (8), we obtain:

$$y_n-\alpha=\frac{c_2}{c_1}e_n^2+\frac{2(-c_2^2+c_1c_3)}{c_1^2}e_n^3+\frac{(4c_2^3-7c_1c_2c_3+3c_1^2c_4)}{c_1^3}e_n^4+[e_n^5] \tag{17}$$

Now, using Eq. (17) and expanding it in the form of a Taylor series, we obtain:

$$f(y_n)=f'(\alpha)\left(c_2e^2+2\left(-\frac{c_2^2}{c_1}+c_3\right)e_n^3+\frac{5c_2^3-7c_1c_2c_3+3c_1^2c_4}{c_1^2}e_n^4+[e_n^5]\right) \tag{18}$$

Hence, from Eqs. (15)-(18) we get:

$$L=f'(\alpha)\left[\frac{4c_1}{e_n}+22c_2+4\left(\frac{c_2^2}{c_1}+10c_3\right)e_n+2\left(\frac{c_2(-2c_2^2+13c_1c_3)}{c_1^2}+29c_4\right)e_n^2+[e_n^3]\right] \tag{19}$$

Substituting Eqs. (15)-(19) into the second substep of the suggested method Eq. (8), we get the following error:

$$e_{n+1}=-\frac{c_2c_3}{c_1^2}e_n^4+[e_n^5] \tag{20}$$

This complete the proof.

Theorem 2. Assume that the function $f:Z\subset$ ⁀ $\to$ ⁀ for an open interval Z has a simple root $\alpha\in Z$. Let $f(x)$ be sufficiently smooth in the interval Z. Then, the order of convergence of the new method defined by Eq. (12) is eight and satisfies the following error equation:

$$e_{n+1}=\frac{c_2c_3(14c_2^2c_3-c_1c_3^2-c_1c_2c_4)}{c_1^5}e_n^8+O[e_n^9] \tag{21}$$

Proof. Let $e_n=x_n-\alpha$ and $c_m=\frac{f^{(m)}(\alpha)}{m!f'(\alpha)}$ for $m\in$ ⁀. By applying $f(\alpha)=0$, the Taylor expansion of $f(x_n)$ at α yields

$$f(x_n)=f'(\alpha)(c_1e_n+c_2e_n^2+c_3e_n^3+c_4e_n^4+c_5e_n^5+c_6e_n^6+c_7e_n^7+c_8e_n^8+O[e_n^9]) \tag{22}$$

Expanding $f'(x_n)$ using the Taylor series around α, we get

$$f'(x_n) = f'(\alpha)(c_1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + 5c_5e_n^4 + 6c_6e_n^5 + 7c_7e_n^6 + O[e_n^8]) \tag{23}$$

Substituting Eqs. (22) and (23) into the first substep of the iterative scheme in Eq. (12), we get

$$y_n - \alpha = \frac{c_2}{c_1}e_n^2 + \frac{2(-c_2^2 + c_1c_3)}{c_1^2}e_n^3 + \sum_{k=1}^{5}\beta_k e_n^{k+3} + [e_n^9] \tag{24}$$

where

$$\beta_1 = \tfrac{1}{c_1^3}(4c_2^3 - 7c_1c_2c_3 + 3c_1^2c_4)$$

$$\beta_2 = \tfrac{2}{c_1^4}(-4c_2^4 + 10c_1c_2^2c_3 - 5c_1^2c_2c_4 + c_1^2(-3c_3^2 + 2c_1c_5))$$

$$\beta_3 = \tfrac{1}{c_1^5}((16c_2^5 - 52c_1c_2^3c_3 + 28c_1^2c_2^2c_4 + c_1^2c_2(33c_3^2 - 13c_1c_5) + c_1^3(-17c_3c_4 + 5c_1c_6)))$$

$$\beta_4 = \tfrac{2}{c_1^6}(-16c_2^6 + 64c_1c_2^4c_3 - 36c_1^2c_2^3c_4 + 9c_1^2c_2^2(-7c_3^2 + 2c_1c_5) + 2c_1^3c_2(23c_3c_4 - 4c_1c_6)$$
$$+ c_1^3(9c_3^3 - 11c_1c_3c_5 + 3c_1(-2c_4^2 + c_1c_7)))$$

$$\beta_5 = \tfrac{1}{c_1^7}(64c_2^7 - 304c_1c_2^5c_3 + 176c_1^2c_2^4c_4 + 4c_1^2c_2^3(102c_3^2 - 23c_1c_5) + 4c_1^3c_2^2(-87c_3c_4 + 11c_1c_6)$$
$$+ c_1^3c_2(-135c_3^3 + 118c_1c_3c_5 + c_1(64c_4^2 - 19c_1c_7)) + c_1^4(75c_3^2c_4 - 27c_1c_3c_6$$
$$+ c_1(-31c_4c_5 + 7c_1c_8)))$$

Using Eq. (24) and the Taylor series expansion, we get the following equation:

$$f(y_n) = f'(\alpha)\left(c_2e^2 + 2\left(-\frac{c_2^2}{c_1} + c_3\right)e_n^3 + \sum_{k=1}^{5}\bar{\beta}_k e_n^{k+3} + [e_n^9]\right) \tag{25}$$

where $\bar{\beta}_k = \bar{\beta}_k(c_1, c_2, c_3, \ldots, c_8)$ are given in terms of $c_1, c_2, c_3, \ldots, c_8$. Using the expressions in Eqs. (22)-(25), we have:

$$L = f'(\alpha)\Big[\frac{4c_1}{e_n} + 22c_2 + 4\Big(\frac{c_2^2}{c_1} + 10c_3\Big)e_n + 2\Big(\frac{c_2(-2c_2^2 + 13c_1c_3)}{c_1^2} + 29c_4\Big)e_n^2$$
$$+4\Big(\frac{c_2^4}{c_1^3} - \frac{10c_2^2c_3}{c_1^2} + \frac{9c_3^2 + 11c_2c_4}{c_1} + 19c_5\Big)e_n^3 + \sum_{k=1}^{5}\omega_k e_n^{k+3} + [e_n^9]\Big] \tag{26}$$

where $\omega_k = \omega_k(c_1, c_2, c_3, \ldots, c_8)$ are given in terms of $c_1, c_2, c_3, \ldots, c_8$.

Inserting Eqs. (23)-(26) into the second substep of the iterative scheme yields:

$$z_n - \alpha = -\frac{c_2c_3}{c_1^2}e_n^4 + \sum_{k=1}^{4}\gamma_k e_n^{k+4} + [e_n^9] \tag{27}$$

where

$$\gamma_1 = \frac{2}{c_1^4}(-7c_2^4 - c_1c_2^2c_3 + c_1^2c_3^2 + c_1^2c_2c_4)$$

$$\gamma_2 = \frac{1}{c_1^5}(-140c_2^5 + 108c_1c_2^3c_3 + 3c_1^2c_2^2c_3 + 3c_1^2c_2^2c_4 - 7c_1^3c_3c_4 - 7c_1^2c_2(-2c_3^2 + c_1c_5))$$

$$\gamma_3 = \frac{2}{c_1^6}(430c_2^6 - 640c_1c_2^4c_3 + 82c_1^2c_2^3c_4 + 2c_1^2c_2^2(79c_3^2 + c_1c_5) + c_1^3(2c_3^3 - 3c_1c_4^2 - 5c_1c_3c_5)$$
$$- 2c_1^3c_2(-4c_3c_4 + c_1c_6))$$

$$\gamma_4 = \frac{1}{c_1^7}(-4165c_2^7 + 8974c_1c_2^5c_3 - 1872c_1^2c_2^4c_4 + c_1^2c_2^3(-4649c_3^2 + 219c_1c_5)$$
$$+ c_1^3c_2^2(961c_3c_4 + 5c_1c_6) + c_1^4(c_4(14c_3^2 - 17c_1c_5) - 13c_1c_3c_6)$$
$$+ c_1^3c_2(413c_3^3 + 20c_1c_3c_5 - 5c_1(-2c_4^2 + c_1c_7)))$$

We obtain the following expression by taking Eq. (27) and the Taylor series expansion:

$$f(z_n) = f'(\alpha)\Big(-\frac{c_2c_3}{c_1}e_n^4 - 2\frac{2(-7c_2^4 - c_1c_2^2c_3 + c_1^2c_3^2 + c_1^2c_2c_4)}{c_1^3}e_n^5 + \sum_{k=1}^{3}\bar{\omega}_k e_n^{k+5} + [e_n^9]\Big) \tag{28}$$

Substituting expressions in Eqs. (22), (25) and (28) into the expressions for s, t and u in the iterative scheme in Eq. (12) gives:

$$s = \frac{c_2}{c_1}e_n + \frac{-3c_2^2 + 2c_1c_3}{c_1^2}e_n^2 + \sum_{k=1}^{5}\lambda_k e_n^{k+2} + [e_n^8] \tag{29}$$

$$t = -\frac{c_3}{c_1}e_n^2 + \frac{14c_2^3 - 2c_1^2c_4}{c_1^3}e_n^3 + \sum_{k=1}^{3}\eta_k e_n^{k+3} + [e_n^7] \tag{30}$$

$$u = -\frac{c_2c_3}{c_1^2}e_n^3 + \frac{14c_2^4 + 3c_1c_2^2c_3 - 2c_1^2c_3^2 - 2c_1^2c_2c_4}{c_1^4}e_n^4 + \sum_{k=1}^{3}\theta_k e_n^{k+4} + [e_n^8] \tag{31}$$

where $\lambda_k = \lambda_k(c_1, c_2, \ldots, c_8)$, $\eta_k = \eta_k(c_1, c_2, \ldots, c_8)$ and $\theta_k = \theta_k(c_1, c_2, \ldots, c_8)$ are given in terms of $c_1, c_2, c_3, \ldots, c_8$.

Using Eqs. (22)-(31) in the third substep of the iterative scheme I Eq. (12), we will obtain the following error:

$$e_{n+1} = -\frac{c_2c_3(-14c_2^2c_3 + c_1c_3^2 + c_1c_2c_4)}{c_1^5}e_n^8 + [e_n^9] \tag{32}$$

This error proves that the iterative method presented in Eq. (12) reaches the eighth order of convergence.

4 Numerical analysis

In this section, we carry out some tests to verify the effectiveness and efficiency of the suggested methods. All the test functions are taken from the paper [Behl, Maroju and Motsa (2017)] and are related to applied problems such as the chemical equilibrium calculation and kinetic problem. To examine the effectiveness of the proposed method, we name our scheme in Eq. (8) as $\mathbf{NHM_4}$ and Eq. (12) as $\mathbf{NHM_8}$ and make comparisons with the existing schemes of previous researchers:

1. Chun [Chun (2007)] introduced a fourth-order method that is free from derivatives to solve nonlinear equations. We label this scheme as $\mathbf{CM_4}$.

$$
\begin{aligned}
y_n &= x_n - \frac{f(x_n)}{f'(x_n)} \\
x_{n+1} &= y_n - \frac{f(x_n)^2}{f(x_n)^2 - 2f(x_n)f(y_n) + 2f(y_n)^2}\frac{f(y_n)}{f'(x_n)}
\end{aligned} \tag{33}
$$

2. The second method we shall consider is the fourth-order method of Soleymani [Soleymani, Khattri and Vanani (2012)]. We label this scheme as $\mathbf{SM_4}$.

$$
\begin{aligned}
y_n &= x_n - \frac{2}{3}\frac{f(x_n)}{f'(x_n)} \\
x_{n+1} &= x_n - \frac{2f(x_n)}{f'(x_n)+f'(y_n)}\left[\left(1+\left(\frac{f(x_n)}{f'(x_n)}\right)^4\right)\left(2-\frac{7}{4}\frac{f'(y_n)}{f'(x_n)}+\frac{3}{4}\left(\frac{f'(y_n)}{f'(x_n)}\right)^2\right)\right]
\end{aligned} \tag{34}
$$

3. Next, we consider the fourth-order scheme of Maheshwari [Maheshwari (2009)]. The scheme is denoted as $\mathbf{MM_4}$.

$$
\begin{aligned}
y_n &= x_n - \frac{f(x_n)}{f'(x_n)} \\
x_{n+1} &= x_n + \frac{1}{f'(x_n)}\left[\frac{f(x_n)^2}{f(y_n)-f(x_n)} - \frac{f(y_n)^2}{f(x_n)}\right]
\end{aligned} \tag{35}
$$

4. Next, we consider the fourth-order method by Behl et al. [Behl, Maroju and Motsa (2017)] ($\mathbf{BM_4}$) that is free from the second derivative.

$$
\begin{aligned}
y_n &= x_n - \frac{f(x_n)}{f'(x_n)} \\
x_{n+1} &= y_n - \frac{f(y_n)(f(x_n)A + 2f(y_n)\gamma_2)}{4f'(x_n)(f(x_n)+f(y_n)\gamma_2)} \\
&\quad \times \frac{(4f(x_n)^2 + 2\gamma_2(\alpha-1)f(y_n)^2 + f(x_n)f(y_n)(B+\alpha A))}{(4f(x_n)^2 - 2f(y_n)^2\gamma_2 + f(x_n)f(y_n)B)}
\end{aligned} \tag{36}
$$

where $\gamma_1 = 2(\alpha-3)$, $\gamma_2 = 2(\alpha-2)$, $A = 2-\gamma_1+\gamma_2$, $B = -2+\gamma_1+3\gamma_2$ and $\alpha \in$ ⁀. Here, we take $\alpha = 2$.

5. Lee et al. [Lee, Kim and Neta (2016)] presented a family of eighth-order iterative schemes. We selected the following and label it as LM_8:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)} \\ z_n &= y_n - (1+2s)\frac{f(y_n)}{f'(x_n)} \\ x_{n+1} &= z_n - \left(\frac{1-2s+u}{1-4s+7s^2-6s^3}\right)\frac{f(z_n)}{f'(x_n)} \end{aligned} \tag{37}$$

where $s = \dfrac{f(y_n)}{f(x_n)}$ and $u = \dfrac{f(z_n)}{f(x_n)}$.

6. Lastly, we consider the eighth-order scheme of Maroju et al. [Maroju, Behl and Motsa (2016)] (**MRM_8**):

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)} \\ z_n &= y_n - \frac{f(x_n)+\beta f(y_n)}{f(x_n)+(\beta-2)f(y_n)}\frac{f(y_n)}{f'(x_n)} \\ x_{n+1} &= z_n - \frac{f(z_n)}{f'(x_n)}\left(1+4uv+u-\frac{(4\beta+1)v}{2(\beta^2-6\beta+6)}+\frac{a_2 v}{a_1 v+1}\right) \end{aligned} \tag{38}$$

where $u = \dfrac{f(z_n)}{f(y_n)}$, $v = \dfrac{f(y_n)}{f(x_n)}$, $a_1 = \dfrac{2(\beta^2-6\beta+6)}{2\beta-5}$, $a_2 = \dfrac{2\beta-5}{a_1}$ and $\beta = 0$.

We utilize multi-precision mathematical programming of Maple 18 to perform numerical computations on the test functions as listed in Tab. 1. We display the results of the tests which consist of the absolute difference between two consecutive iterations $|x_n - x_{n-1}|$, absolute residual error of the corresponding function $|f(x_n)|$, CPU time in milliseconds (ms) and computational order of convergence (COC) that was presented by Cordero et al. [Cordero and Torregrosa (2007)]:

$$\text{COC} \approx \frac{\ln(|x_{n+1}-x_n|/|x_n-x_{n-1}|)}{\ln(|x_n-x_{n-1}|/|x_{n-1}-x_{n-2}|)} \tag{39}$$

All the computational results presented in the form of $X(\pm Y)$, which stands for $X \times 10^{(\pm Y)}$ and up to 5000 significant digits to minimize the round-off error. Besides that, all the computational order convergence is calculated up to four significant digits (please see Tabs. 2-4 for the complete results).

Table 1: Test functions

Test Functions	x_0
$f_1(x)=(\sin x)^2-x^2+1$	6.0
$f_2(x)=x^6-10x^3+x^2-x+3$	0.5
$f_3(x)=8x^4-62.326x^3+117.956x^2+20.088x-13.392$	0.5
$f_4(x)=\frac{1}{x^2}\exp\left(\frac{21000}{x}\right)-1.11\times10^{11}$	555

Table 2: Numerical results for the function $f_1(x)$

Method	n	$\lvert x_n-x_{n-1}\rvert$	$\lvert f(x_n)\rvert$	COC	CPU Time
CM_4	1	5.6 (-1)	2.1 (0)	3.995	188.0
	2	2.5 (-2)	6.3 (-2)		
	3	4.7 (-7)	1.2 (-6)		
SM_4	1	2.6 (2)	6.7 (4)	Diverge	-
	2	1.4 (52)	2.5 (21)		
	3	2.4 (259)	2.0 (104)		
MM_4	1	5.7 (-1)	2.2 (0)	3.394	188.0
	2	2.7 (-2)	6.9 (-2)		
	3	8.9 (-7)	2.2 (-6)		
BM_4	1	5.1 (-1)	1.8 (0)	3.535	188.0
	2	1.8 (-2)	4.4 (-2)		
	3	1.2 (-7)	3.1 (-7)		
NHM_4	1	4.0 (-1)	1.3 (0)	4.414	188.0
	2	5.6 (-3)	1.4 (-2)		
	3	3.8 (-11)	9.5 (-11)		
LM_8	1	2.0 (-1)	5.7 (-1)	7.857	141.0
	2	4.4 (-6)	1.1 (-5)		
	3	1.1 (-42)	2.7 (-42)		
MRM_8	1	8.1 (-2)	2.2 (-1)	7.972	219.0
	2	1.8 (-9)	4.5 (-9)		
	3	1.8 (-70)	4.6 (-70)		
NHM_8	1	1.5 (-1)	4.3 (-1)	8.151	250.0
	2	1.4 (-7)	3.6 (-7)		
	3	1.1 (-56)	2.6 (-56)		

Table 3: Numerical results for the function $f_2(x)$

Method	n	$\lvert x_n - x_{n-1} \rvert$	$\lvert f(x_n) \rvert$	COC	CPU Time
CM_4	1	9.3 (-3)	1.13 (-1)		
	2	4.7 (-8)	5.6 (-7)	3.995	47.0
	3	3.2 (-29)	3.8 (-28)		
SM_4	1	1.9 (-2)	2.3 (-1)		
	2	8.8 (-7)	1.0 (-5)	3.987	47.0
	3	4.8(-24)	5.7 (-23)		
MM_4	1	2.3 (-2)	2.9 (-1)		
	2	2.2 (-6)	2.6 (-5)	3.982	47.0
	3	2.1 (-22)	2.5 (-21)		
BM_4	1	1.3 (-2)	1.6 (-1)		
	2	1.8 (-7)	2.1 (-6)	3.992	78.0
	3	6.9 (-27)	8.3 (-26)		
NHM_4	1	3.8 (-2)	4.3 (-1)		
	2	6.3 (-6)	7.6 (-5)	4.217	47.0
	3	7.7 (-22)	9.2 (-21)		
LM_8	1	2.4 (-4)	2.8 (-3)		
	2	3.3 (-27)	3.9 (-26)	8.000	62.0
	3	4.2 (-210)	5.0 (-209)		
MRM_8	1	1.0(-4)	1.2 (-3)		
	2	7.2 (-31)	8.5 (-30)	8.000	63.0
	3	3.7 (-240)	4.4 (-239)		
NHM_8	1	3.3 (-2)	4.2 (-1)		
	2	6.4 (-12)	7.6 (-11)	7.996	78.0
	3	1.2 (-89)	1.5 (-88)		

Table 4: Numerical results for the function $f_3(x)$

Method	n	$\lvert x_n - x_{n-1} \rvert$	$\lvert f(x_n) \rvert$	COC	CPU Time
CM_4	1	1.6 (-3)	1.2 (-1)		
	2	2.5 (-11)	1.8 (-9)	3.999	47.0
	3	1.4 (-42)	9.8 (-41)		
SM_4	1	1.4 (-3)	1.0 (-1)		
	2	1.6 (-11)	1.2 (-9)	3.999	47.0
	3	2.9 (-43)	2.1 (-41)		

MM_4	1	1.8 (-3)	1.3 (-1)	3.999	47.0
	2	4.4 (-11)	3.2 (-9)		
	3	1.7 (-41)	1.2 (-39)		
BM_4	1	1.6 (-3)	1.2 (-1)	3.999	63.0
	2	2.2 (-11)	1.6 (-9)		
	3	8.8 (-43)	6.3 (-41)		
NHM_4	1	1.1 (-3)	8.1 (-2)	4.001	47.0
	2	1.2 (-12)	8.4 (-11)		
	3	1.3 (-48)	9.6 (-47)		
LM_8	1	1.6 (-5)	1.2 (-3)	8.000	62.0
	2	4.2 (-37)	3.0 (-35)		
	3	8.3 (-290)	5.9 (-288)		
MRM_8	1	5.6 (-4)	6.2 (-2)	8.000	63.0
	2	1.2 (-23)	1.3 (-21)		
	3	5.2 (-181)	5.6 (-179)		
NHM_8	1	1.0 (-5)	7.2 (-4)	8.000	62.0
	2	7.9 (-40)	5.6 (-38)		
	3	1.1 (-312)	7.9 (-311)		

Table 5: Numerical results for the function $f_4(x)$

Method	n	$\|x_n - x_{n-1}\|$	$\|f(x_n)\|$	COC	CPU Time
CM_4	1	1.6 (-2)	1.3 (8)	4.000	94.0
	2	9.3 (-12)	7.5 (-2)		
	3	9.5 (-49)	7.6 (-39)		
SM_4	1	4.8 (1710)	1.1 (11)	Diverge	-
	2	3.2 (25714)	1.1 (11)		
	3	9.3 (385771)	1.1 (11)		
MM_4	1	3.0 (-2)	2.4 (8)	4.000	94.0
	2	1.5 (-10)	1.2 (0)		
	3	8.7 (-44)	7.0 (-34)		
BM_4	1	1.9 (-2)	1.5 (8)	4.000	110.0
	2	1.6 (-11)	1.3 (-1)		
	3	9.1 (-48)	7.3 (-38)		
NHM_4	1	2.3 (-2)	1.8 (8)	4.001	110.0
	2	1.0 (-11)	8.4 (-2)		
	3	4.6 (-49)	3.7 (-39)		
LM_8	1	2.9 (-5)	2.3 (5)	8.000	125.0
	2	1.3 (-45)	1.0 (-35)		
	3	1.6 (-368)	1.3 (-358)		

MRM_8	1	1.5 (-6)	1.2 (4)	8.000	141.0
	2	2.5 (-57)	2.0 (-47)		
	3	1.3 (-463)	1.1 (-453)		
NHM_8	1	2.7 (-4)	2.1 (6)	8.000	157.0
	2	1.8 (-38)	1.4 (-28)		
	3	7.7 (-312)	6.2 (-302)		

Figs. 1 to 6 show that the dynamics for NHM_4 have fewer divergence points than SM_4 and are almost the same as that of CM_4, MM_4, BM_4. Our suggested method NHM_8 has a wider region of convergence and is comparable to LM_8 and MRM_8. For the test function $p_3(z)$, NHM_4 and NHM_8 give better convergence regions than SM_4 and LM_4. We conclude that the proposed methods have fast convergence and greater stability based on the dynamical analysis.

5 Dynamical analysis

In this part, we plot the dynamical planes of the schemes CM_4, SM_4, MM_4, BM_4, NHM_4, LM_8, MRM_8, and NHM_8 using the ideas represented in the paper [Chicharro, Cordero and Torregrosa (2013)] for stability comparisons. Using Mathematica 12, we took a mesh of 400×400 points in the region of the complex plane $[-100,100] \times [-100,100]$ and specified a color (purple, turquoise, yellow, red, blue) to each point whose orbit converges to the simple root and used black for those points whose orbits diverge from the root. A black point in the figure represents the zeros. We set a maximum of 100 iterations with a tolerance of 10^{-3} as the stopping criterion for convergence. The standard dynamic test functions are listed in Tab. 6.

Table 6: Dynamics test functions and their roots

Dynamic Test Functions	List of Roots	Dynamics Figure
$p_1(z) = z^5 + 2z - 1$	$-0.945068 - 0.854518i$ $-0.945068 + 0.854518i$ 0.486389 $0.701874 - 0.879697i$ $0.701874 + 0.879697i$	1, 2
$p_2(z) = (z-1)^3 - 1$	$0.5 - 0.866025i$ $0.5 + 0.866025i$ 2.0	3, 4
$p_3(z) = z^2 - z - \frac{1}{z}$	1.46557 $-0.232786 - 0.792552i$ $-0.232786 + 0.792552i$	5, 6

Figs. 1 to 6 show that the dynamics for NHM_4 have fewer divergence points than SM_4 and are almost the same as that of CM_4, MM_4, BM_4. Our suggested method NHM_8 has a wider region of convergence and is comparable to LM_8 and MRM_8. For the test function $p_3(z)$, NHM_4 and NHM_8 give better convergence regions than SM_4 and LM_4. We conclude that the proposed methods have fast convergence and greater stability based on the dynamical analysis.

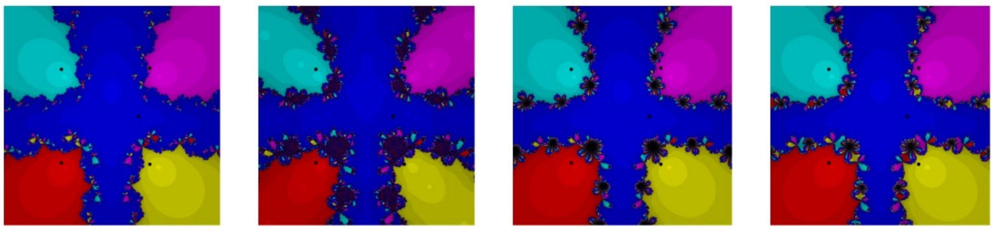

Figure 1: Dynamics of CM_4, SM_4, MM_4 and BM_4, respectively for $p_1(z)$

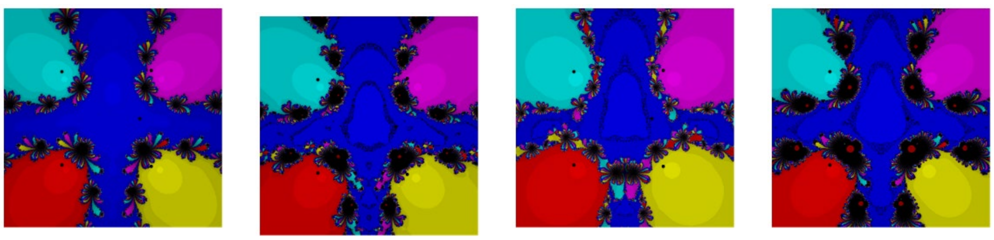

Figure 2: Dynamics of NHM_4, LM_8, MRM_8 and NHM_8, respectively for $p_1(z)$

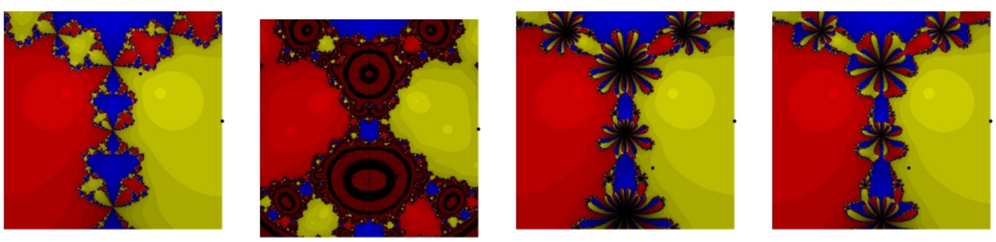

Figure 3: Dynamics of CM_4, SM_4, MM_4 and BM_4, respectively for $p_2(z)$

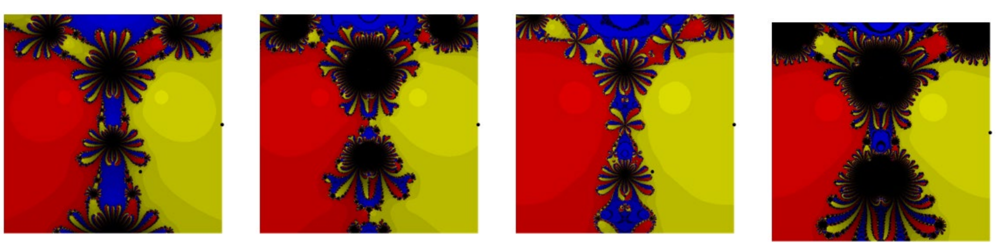

Figure 4: Dynamics of NHM_4, LM_8, MRM_8 and NHM_8, respectively for $p_2(z)$

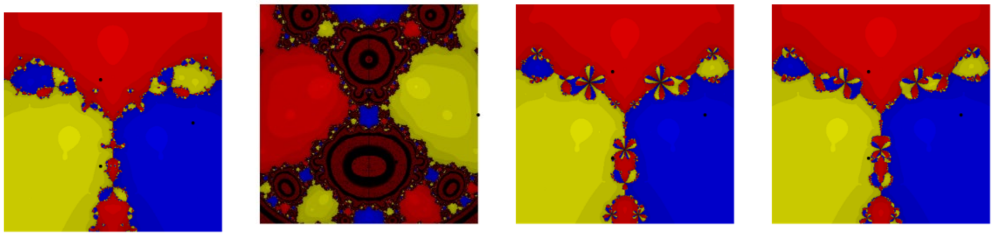

Figure 5: Dynamics of CM_4, SM_4, MM_4 and BM_4, respectively for $p_3(z)$

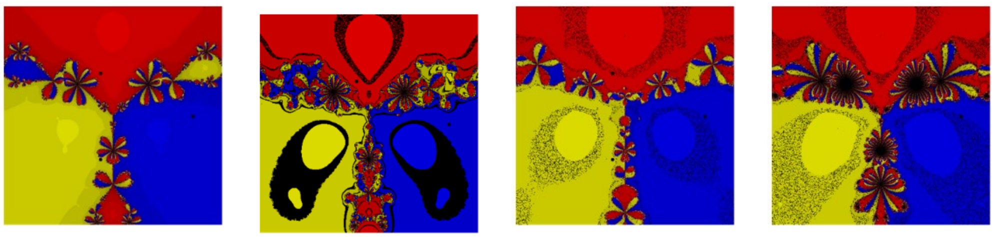

Figure 6: Dynamics of NHM_4, LM_8, MRM_8 and NHM_8, respectively for $p_3(z)$

5 Conclusion

In this research, we constructed two new optimal Newton-Householder methods to find the simple roots of nonlinear equations. Based on theoretical analysis and numerical tests, we found that the new Newton-Householder methods achieve convergence order four and eight with a greater efficiency index $E_i = 1.5874$ and $E_i = 1.6179$, respectively, compared to the original Householder method [Abbasbandy (2003)] ($E_i = 1.4423$), the methods of Noor et al. [Noor, Aslam and Momani (2007)]$(E_i = 1.4310)$ and Nazeer et al. [Nazeer, Tanveer, Min et al. (2016)]$(E_i = 1.4954)$. The dynamics also verified the convergence analysis and numerical analysis of the suggested methods and showed that the modified Newton-Householder methods could compete with the existing schemes. Thus, the newly proposed methods contribute to the improvement of the Householder method.

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