A Structure Preserving Numerical Method for Solution of Stochastic Epidemic Model of Smoking Dynamics

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Abstract: In this manuscript, we consider a stochastic smoking epidemic model from behavioural sciences. Also, we develop a structure preserving numerical method to describe the dynamics of stochastic smoking epidemic model in a human population. The structural properties of a physical system include positivity, boundedness and dynamical consistency. These properties play a vital role in non-linear dynamics. The solution for nonlinear stochastic models necessitates the conservation of these properties. Unfortunately, the aforementioned properties of the model have not been restored in the existing stochastic methods. Therefore, it is essential to construct a structure preserving numerical method for a reliable analysis of stochastic smoking model. The usual explicit stochastic numerical methods are time-dependent and violate most of the structural properties. In this work, we have developed the implicitly driven explicit method for the solution of stochastic smoking model. It is also proved that the newly developed method sustains all the aforementioned properties of the system. Finally, the convergence analysis of the newly developed method and graphical illustrations are presented.

Keywords: Smoking model, stochastic numerical techniques, convergence.

1 Literature survey

In the 19th century, due to trade between Europe and China, the use of opium smoking became popular due to its medicinal properties. Later on, opium smoking spread in the elite class of Europe and gradually this habit spread all over the world. In World War II, the trend decreased in Europe, while in China, it became outdated during the traditional revolution. Zheng et al. [Zheng, Wang and Xia (2018)] investigated that 5 million people died in a year due to the use of tobacco and can be doubled by 2025. Jeong et al. [Jeong, Kuk and Kim

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Received: 29 April 2020; Accepted: 05 June 2020.

(2019)] investigated that the extensive use of tobacco, which leads to severe ailments like cardiovascular disease, chronic lungs disease and lungs cancer, is the main cause of such high death rate. A recent study in Saudi Arabia has revealed that the total number of smokers in the country is 6 million, which will be increased to 10 million by the end of 2020. These smokers spend 21 billion Saudi riyals annually on smoking. Zaman [Zaman (2011)] has presented first time, the smoking model by taking into account, the occasional smoker's compartment. Smoking is very common in Pakistan. According to a careful estimate, about one hundred and seventy-seven million cigarettes are smoked daily. A demographic health survey found that 46% of men and 4.5% of women smoke in Pakistan. This habit greatly attracted young generation and farmers of the country. Pakistan is one of the largest consumers of cigarettes in South Asia. The State Bank of Pakistan reported that in the financial year 2016, Pakistanis spent Rs 250 billion on approximately 64 billion cigarettes. It is reported that 90% of lungs diseases in Pakistan are caused by the use of tobacco, which kills 100,000 people annually. Bassiony [Bassiony (2009)] concluded on the basis of a survey conducted by sustainable development policy institute (SDPI) that the number of tobacco users in the country is over 3.95 million, out of which 125,000 suffered from lungs diseases every year due to the widespread of tobacco use. More than 80% of smokers belong to the third world or developing countries. Extensive use of tobacco in such countries leads to deaths and diseases. For instance, in Karachi, a large number of young people are being diagnosed with lungs' cancer due to tobacco use, and the number is day by day growing. Smoking causes asthma in adults and also damages their immune system. The smoking in developed countries has declined over the past few decades. Sharomi et al. [Sharomi and Gumel (2008)] have found that mathematical frameworks are major tools to understand the dynamics of diseases in humans, animals and plant populations. Zeb et al. [Zeb, Bano, Alzahrani et al. (2018)] presented dynamical analysis of smoking model with saturated bilinear rate. Selya et al. [Selya, Lvanov, Bachman et al. (2019)] presented the anti-smoking policies, by using a system of dynamics simulation in youth. Xiong et al. [Xiong, Yang, Zhao et al. (2017)] presented robust dynamics network traffic partitioning against malicious attacks of smoking. Chen et al. [Chen, Xu, Zuo et al. (2019)] found the fire recognition algorithm using dynamic feature fusion and IV-SVM classifier. Zeb et al. [Zeb and Zaman (2013)] discussed square root dynamics of the smoking model with deterministic analysis. Raza et al. [Raza, Arif and Rafiq (2019)] pointed out that some explicit numerical methods produce unexpected fluctuations for certain scenario. Arif et al. [Arif, Raza, Rafiq et al. (2019)] investigated that explicit techniques are less efficient and less reliable. No doubt, the stochastic differential equations (SDEs) have no explicit solutions. Pierret [Pierret (2015)] presented the structure preserving properties on Euler Maruyama method. Cresson et al. [Cresson and Pierret (2014)] studied the structure preserving dynamical properties for biological problems. These problems are solved by the stochastic numerical techniques. A convergence analysis is the main concern of the numerical techniques. There is not any stochastic numerical technique that preserves all the dynamical properties in the existing literature. The main focus of this paper is to introduce the stochastic implicitly driven explicit technique known as stochastic nonstandard finite difference method. Mickens [Mickens (2005)] designed construction rules of the nonstandard finite difference technique. Our paper plan is as follows: In Section 2, we shall consider deterministic smoking model and its equilibria. We will construct the stochastic smoking model in Section 3. In Section 4, we will use different stochastic numerical techniques. Finally, in Section 5 conclusion and directions will be discussed.

2 Deterministic smoking model

In this section, we consider the deterministic smoking model. The state variables involved in the model are described as follows: X_1 : Signifies the non-smokers (potential smokers), X_2: Signifies the light smokers (Occasional smokers), X_3: Signifies the heavy smokers, X_4: Signifies the smokers who left smoking temporarily, X_5: Signifies the smokers who left smoking forever. The flow chart of smoking in the human population is shown in Fig. 1.

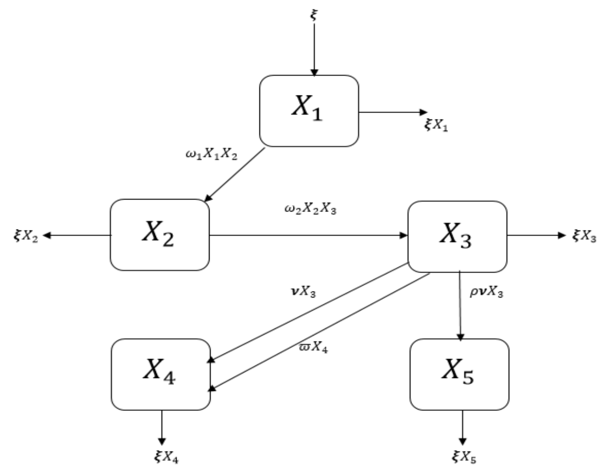


Figure 1: Flow diagram of the smoking model

The parameters of the model are described as follows: ϖ (denotes the contact rate between smokers and temporarily left smoking individuals), ν (denotes the rate at which the individuals stop smoking), ω_1 (denotes the contact rate between non-smokers and light smokers), ω_2 (denotes the contact rate between light smokers and heavy smokers), ξ (denotes the rate of recruitment of potential smokers) and ρ (denotes the remaining fraction who left smoking forever). The governing equations of the smoking model as follows:

$$X_1{}' = \xi - \xi X_1 - \omega_1 X_1 X_2 \,. \tag{1}$$

$$X_2{}' = -\xi X_2 + \omega_1 X_1 X_2 - \omega_2 X_2 X_3. \tag{2}$$

$$X_3{}' = -\xi X_3 - \nu X_3 + \omega_2 X_2 X_3 + \varpi X_4. \tag{3}$$

$$X_4{}' = -\xi X_4 - \varpi X_4 + \nu(1 - \sigma) X_3. \tag{4}$$

$$X_5{}' = -\xi X_5 + \rho \nu X_3. \tag{5}$$

with conditions $X_1 \geq 0, X_2 \geq 0, X_3 \geq 0, X_4 \geq 0\,, X_5 \geq 0$ and $X_1 + X_2 + X_3 + X_4 + X_5 = \mathrm{w}_1$, $\mathrm{w}_1$is represented as the whole human population.

The reduced form of Eqs. (1) to (5) is

$$X_1{}' = \xi - \xi X_1 - \omega_1 X_1 X_2. \tag{6}$$

$$X_2{}' = -\xi X_2 + \omega_1 X_1 X_2 - \omega_2 X_2 X_3. \tag{7}$$

$$X_3{}' = -\xi X_3 - \nu X_3 + \omega_2 X_2 X_3 + \varpi X_4. \tag{8}$$

$$X_4{}' = -\xi X_4 - \varpi X_4 + \nu(1-\sigma)X_3. \quad (9)$$

with conditions $X_1 \geq 0, X_2 \geq 0, X_3 \geq 0, X_4 \geq 0$ and $X_1 + X_2 + X_3 + X_4 \leq \mathrm{w}_1$.

2.1 Equilibria of smoking model

The two equilibria states of model are as follows:

Smoking free equilibrium is $\mathrm{D}_1 = (1, 0, 0, 0)$.

Smoking present equilibrium is $\mathrm{E}_1 = (X_1, X_2, X_3, X_4)$.

where, $X_1 = \frac{\xi\omega_2(\xi+\varpi)}{\omega_2\xi(\xi+\varpi)+\omega_2[\xi(\xi+\varpi)+\xi\nu+\varpi\nu\rho]}$, $X_2 = \frac{\xi(\xi+\varpi)+\xi\nu+\varpi\nu\rho}{\omega_2(\xi+\varpi)}$,

$X_3 = \frac{\xi}{\omega_2}\left[\frac{\omega_1\omega_2(\xi+\varpi)}{\omega_2\xi(\xi+\varpi)+\omega_1[\xi(\xi+\varpi)+\xi\nu+\varpi\nu\rho]} - 1\right]$ and $X_4 = \frac{\nu(1-\rho)X_3{}^{o}}{(\xi+\varpi)}$.

2.2 Force of infection

The reproduction number of the deterministic smoking model is as follows: $\mathrm{R}_0^{\mathrm{d}} = \frac{\beta}{\xi}$. This model has two states as the force of infection $\mathrm{R}_0^{\mathrm{d}} = \frac{\beta}{\xi} < 1$, which represents the smoking free population. If the force of infection $\mathrm{R}_0^{\mathrm{d}} = \frac{\beta}{\xi} > 1$ which represents smoking present in the human population.

3 Stochastic smoking model

Allen et al. [Allen, Allen and Arciniega (2008)] presented the parametric noise idea in which we shall introduce the stochastic environmental factors $\omega_1\mathrm{dt} = \omega_1\mathrm{dt} + \varpi_1\,\mathrm{dB}$ and $\varpi\mathrm{dt} = \varpi\mathrm{dt} + \varpi_2\mathrm{dB}$ in the Eqs. (6) to (9) as follows:

$$\mathrm{d}X_1 = (\xi - \xi X_1 - \omega_1 X_1 X_2)\mathrm{dt} - X_1 X_2 \varpi_1 \mathrm{dB}. \quad (10)$$

$$\mathrm{d}X_2 = (-\xi X_1 + \omega_1 X_1 X_2 - \omega_2 X_2 X_3)\mathrm{dt} + X_1 X_2 \varpi_1 \mathrm{dB}. \quad (11)$$

$$\mathrm{d}X_3 = (-\xi X_3 - \nu X_3 + \omega_2 X_2 X_3 + \varpi \mathrm{Q_t})\mathrm{dt} - \varpi_2 X_4 \mathrm{dB}. \quad (12)$$

$$\mathrm{d}X_4 = (-\xi X_4 - \varpi X_4 + \nu(1-\rho)X_3)\mathrm{dt} + \varpi_2 X_4 \mathrm{dB}. \quad (13)$$

where the Brownian motion is represented by B and ϖ_1, ϖ_2 are the random environmental effects in the contact rates of smokers. But $\frac{\mathrm{d}X_1}{\mathrm{dt}} + \frac{\mathrm{d}X_2}{\mathrm{dt}} + \frac{\mathrm{d}X_3}{\mathrm{dt}} + \frac{\mathrm{d}X_4}{\mathrm{dt}} \leq \xi - \xi(X_1 + X_2 + X_3 + X_4)$. Let $\mathrm{w}_1 = X_1 + X_2 + X_3 + X_4$, then $\mathrm{w}_1{}' \leq \xi - \xi\mathrm{w}_1$. The problem $\varphi' = \xi - \xi\varphi$ with $\varphi(0) = \mathrm{w}(0)$ has solution $\varphi(\mathrm{t}) = 1 - \mathrm{ce}^{-\xi\mathrm{t}}$, and $\lim_{\mathrm{n}\to\infty} \varphi(\mathrm{t}) = 1$. Therefore, $\mathrm{w}_1(\mathrm{t}) \leq \varphi(\mathrm{t})$ and $\lim_{\mathrm{t}\to\infty} \sup \mathrm{w}_1(\mathrm{t}) \leq 1$. So, the system from Eqs. (10) to (13) has feasible region i.e., $\Omega = \{X_1, X_1, X_1, X_1): X_1 + X_2 + X_3 + X_4 \leq 1, X_1 \geq 0, X_2 \geq 0, X_3 \geq 0, X_4 \geq 0\}$.

The solutions of this region will be nonnegative invariant. This region is also called the feasible region of Eqs. (10) to (13).

3.1 Stochastic threshold dynamics

For Eqs. (10) to (13) the infected individuals $X_2(\mathrm{t})$ is said to be extinct if $\lim_{\mathrm{t}\to\infty} X_2(\mathrm{t}) = 0$.

Theorem: If ${\varpi_1}^2 < \frac{\omega_1}{\xi}$ and $R_0^d < 1$, then the infected individuals in the model Eqs. (10) to (13) tend to zero exponentially.

Proof: Assume that (X_1, X_2, X_3, X_4) is a solution of the system from Eqs. (10) to (13) satisfying the initial value $(X_1(0), X_2(0), X_3(0), X_4(0)) \in {R_+}^4$ by Ito's lemma and $f(X_2) = \ln(X_2)$

$d\ln(X_2) = f'(X_2)dX_2 + \frac{1}{2}f''(X_2){X_2}^2({X_1}^2{X_2}^2{\varpi_1}^2)dt.$

$d\ln(X_2) = (\frac{1}{X_2}[(-\xi X_2 + \omega_1 X_1 X_2 - \omega_2 X_2 X_3)dt + \varpi_1 X_1 X_2\ dB] + \frac{1}{2}(\frac{-1}{{X_2}^2}){X_2}^2({X_1}^2{\varpi_1}^2)dt.$

$d\ln(X_2) = (-\xi + \omega_1 X_1 - \omega_2 X_3)dt + \varpi_1 X_1\ dB - \frac{1}{2}{X_1}^2{\varpi_1}^2 dt.$

$d\ln(X_2) = \left(-\xi + \omega_1 X_1 - \omega_2 X_3 - \frac{{X_1}^2{\varpi_1}^2}{2}\right)dt + \varpi_1 X_1\ dB.$

$\ln X_2(t) = \ln X_2(0) + \left(-\xi + \omega_1 X_1 - \omega_2 X_3 - \frac{{X_1}^2{\varpi_1}^2}{2}\right)t + \int_0^t \varpi_1 X_1 dB.$

where $M_1(t) = \int_0^t \varpi_1 X_1\ dB$ with $M(0) = 0.$

If ${\varpi_1}^2 < \frac{\omega_1}{\xi}$,

$\ln X_2(t) \le \left(-\xi + \omega_1 - \frac{{\varpi_1}^2}{2}\right)t + M_1(t) + \ln X_2(0).$

$\ln X_2(t) \le \xi\left(-1 + \frac{\omega_1}{\xi} - \frac{{\varpi_1}^2}{2\xi}\right)t + M_1(t) + \ln X_2(0).$

$\frac{\ln X_2(t)}{t} \le \xi\left(-1 + \frac{\omega_1}{\xi} - \frac{{\varpi_1}^2}{2\xi}\right)t + \frac{M_1(t)}{t} + \frac{\ln X_2(0)}{t}.$

if $\lim_{t\to\infty}\frac{M_1(t)}{t} = 0.$

$\lim_{t\to\infty}\sup\frac{\ln X_2(t)}{t} \le \xi({R_0}^s - 1)$, then when ${R_0}^s < 1$, we get

$\lim_{t\to\infty}\sup\frac{\ln X_2(t)}{t} < 0.$

$\lim_{t\to\infty} X_2(t) = 0$ almost sure.

$R_o^S = R_o^d - \frac{{\varpi_1}^2}{2\xi} < 1.$

Note that R_o^S is the stochastic threshold number, the human population will be smoking free if $R_o^S < 1$, and smoking will be present if $R_o^S > 1$.

4 Numerical simulations

In this section, we will study the discrete behavior of the continuous system. For this, we will use the existing explicit techniques and the proposed technique. We will prove the effectiveness of the proposed method with convergence analysis. Alkhudhari et al. [Alkhudhari, Sheikh and Tuwairqi (2011)] selected the values of parameters as shown in Tab. 1.

Table 1: Parameters values

Parameters	Values (values)
ξ	0.15
ρ	0.09
ω_1	SFE=0.04
	SPE=0.4
ω_2	0.3
ϖ_1	0.08
ϖ_2	0.07
ν	0.08
ϖ	0.25

4.1 Stochastic Euler method

The Eqs. (10) to (13) in this technique as follows:

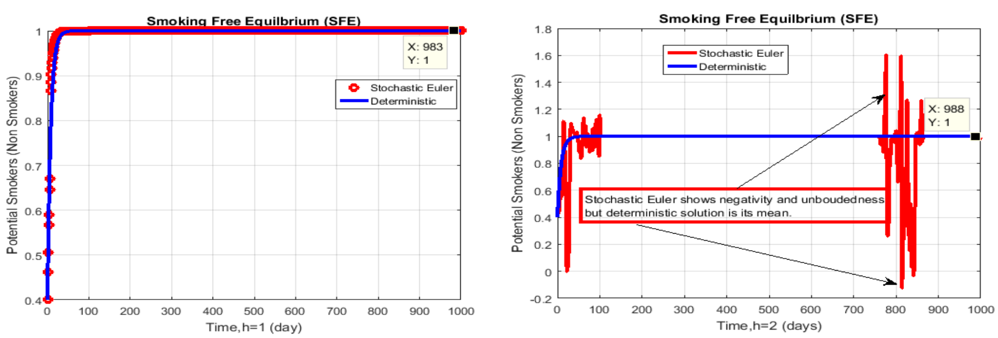
$$X_1^{n+1} = X_1^n + h[\xi - \xi X_1^n - \omega_1 X_1^n X_2^n - X_1^n X_2^n \varpi_1 \Delta B_n]. \quad (14)$$

$$X_2^{n+1} = X_2^n + h[\omega_1 X_1^n X_2^n - \omega_2 X_2^n X_3^n - \xi X_2^n + X_1^n X_2^n \varpi_1 \Delta B_n]. \quad (15)$$

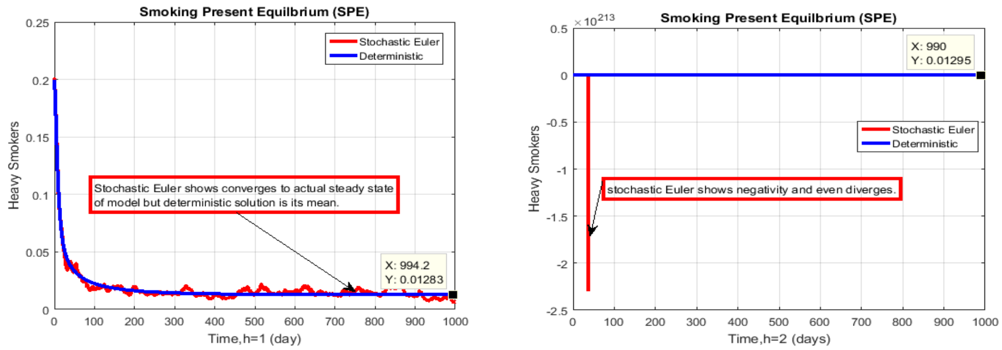
$$X_3^{n+1} = X_3^n + h[\omega_2 X_2^n X_3^n - (\xi + \nu) X_3^n + \varpi X_4^n + \varpi_2 Q_t^n \Delta B_n]. \quad (16)$$

$$X_4^{n+1} = X_4^n + h[\nu X_3^n - (\xi + \varpi) X_4^n - \varpi_2 X_4^n \Delta B_n]. \quad (17)$$

where 'h' represents the step size and ΔB_n means Brownian motion normally distributed in the feasible region Ω i.e., $\Delta B_n \sim N(0,1)$. Now, we use MATLAB database and parameters values presented in Tab. 1 for simulation of the Eqs. (14) to (17).



(a) **(b)**



(c) **(d)**

Figure 2: (a) Potential smokers at h=1 (b) Potential smokers at h=2 (c) Heavy smokers at h=1 (d) Heavy smokers at h=2

4.2 Stochastic Runge Kutta method

The Eqs. (10) to (13) in this technique is represented as follows:

Stage 1

$S_1 = h[(\xi - \xi^n X_1 - \omega_1 {X_1}^n {X_2}^n) - {X_1}^n {X_2}^n \varpi_1 \Delta B_n].$

$U_1 = h[(\xi - {X_2}^n + \omega_1 {X_1}^n {X_2}^n - \omega_2 {X_2}^n {X_3}^n) + {X_1}^n {X_2}^n \varpi_1 \Delta B_n].$

$V_1 = h[-(\xi + \nu){X_3}^n + \omega_2 {X_2}^n {X_3}^n + \varpi {X_4}^n + \varpi_2 {X_4}^n \Delta B_n].$

$W_1 = h[-(\xi + \varpi){X_4}^n + \nu {X_3}^n - \varpi_2 {X_4}^n \Delta B_n].$

Stage 2

$S_2 = h\left[\left(\xi - \xi\left({X_1}^n + \frac{S_1}{2}\right) - \omega_1\left({X_1}^n + \frac{S_1}{2}\right)\left({X_2}^n + \frac{U_1}{2}\right)\right) - \left({X_1}^n + \frac{S_1}{2}\right)\left({X_2}^n + \frac{U_1}{2}\right)\varpi_1 \Delta B_n\right].$

$U_2 = h\left[\left(-\xi\left({X_2}^n + \frac{U_1}{2}\right) + \omega_1\left({X_1}^n + \frac{S_1}{2}\right)\left({X_2}^n + \frac{U_1}{2}\right) - \omega_2\left({X_2}^n + \frac{U_1}{2}\right)\left({X_3}^n + \frac{V_1}{2}\right)\right) + \left({X_1}^n + \frac{S_1}{2}\right)\left({X_2}^n + \frac{U_1}{2}\right)\varpi_1 \Delta B_n\right].$

$V_2 = h\left[-(\xi + \nu)\left({X_3}^n + \frac{V_1}{2}\right) + \omega_2\left({X_2}^n + \frac{U_1}{2}\right)\left({X_3}^n + \frac{V_1}{2}\right) + \varpi\left({X_4}^n + \frac{W_1}{2}\right) + \varpi_2\left({X_4}^n + \frac{W_1}{2}\right)\Delta B_n\right].$

$W_2 = h\left[-(\xi + \varpi)\left({X_4}^n + \frac{W_1}{2}\right) + \nu\left({X_3}^n + \frac{V_1}{2}\right) - \varpi_2\left({X_4}^n + \frac{W_1}{2}\right)\Delta B_n\right].$

Stage 3

$S_3 = h\left[\left(\xi - \xi\left({X_1}^n + \frac{S_2}{2}\right) - \omega_1\left({X_1}^n + \frac{S_2}{2}\right)\left({X_2}^n + \frac{U_2}{2}\right)\right) - \left({X_1}^n + \frac{S_2}{2}\right)\left({X_2}^n + \frac{U_2}{2}\right)\varpi_1 \Delta B_n\right].$

$U_3 = h\left[\left(-\xi\left(X_2{}^n + \frac{U_2}{2}\right) + \omega_1\left(X_1{}^n + \frac{S_2}{2}\right)\left(X_2{}^n + \frac{U_2}{2}\right) - \omega_2\left(X_2{}^n + \frac{U_2}{2}\right)\left(X_3{}^n + \frac{V_2}{2}\right)\right) + \left(X_1{}^n + \frac{S_2}{2}\right)\left(X_2{}^n + \frac{U_2}{2}\right)\varpi_1 \Delta B_n\right].$

$V_3 = h\left[-(\xi + \nu)\left(X_3{}^n + \frac{V_2}{2}\right) + \omega_2\left(X_2{}^n + \frac{U_2}{2}\right)\left(X_3{}^n + \frac{V_2}{2}\right) + \varpi\left(X_4{}^n + \frac{W_2}{2}\right) + \varpi_2\left(X_4{}^n + \frac{W_2}{2}\right)\Delta B_n\right].$

$W_3 = h\left[-(\xi + \varpi)\left(X_4{}^n + \frac{W_2}{2}\right) + \nu\left(X_3{}^n + \frac{V_2}{2}\right) - \varpi_2\left(X_4{}^n + \frac{W_2}{2}\right)\Delta B_n\right].$

Stage 4

$S_4 = h\left[\left(\xi - \xi(X_1{}^n + S_3) - \omega_1(X_1{}^n + S_3)(X_2{}^n + U_3)\right) - (X_1{}^n + S_3)(X_2{}^n + U_3)\varpi_1 \Delta B_n\right].$

$U_4 = h\left[\left(-\xi(X_2{}^n + U_3) + \omega_1(X_1{}^n + S_3)(X_2{}^n + U_3) - \omega_2(X_2{}^n + U_3)(X_3{}^n + V_3)\right) + (X_1{}^n + S_3)(X_2{}^n + U_3)\varpi_1 \Delta B_n\right].$

$V_4 = h[-(\xi + \nu)(X_3{}^n + V_3) + \omega_2(X_2{}^n + U_3)(X_3{}^n + V_3) + \varpi(X_4{}^n + W_3) + \varpi_2(X_4{}^n + W_3)\Delta B_n].$

$W_4 = h[-(\xi + \varpi)(X_4{}^n + W_3) + \nu(X_3{}^n + V_3) - \varpi_2(X_4{}^n + W_3)\Delta B_n].$

Final stage

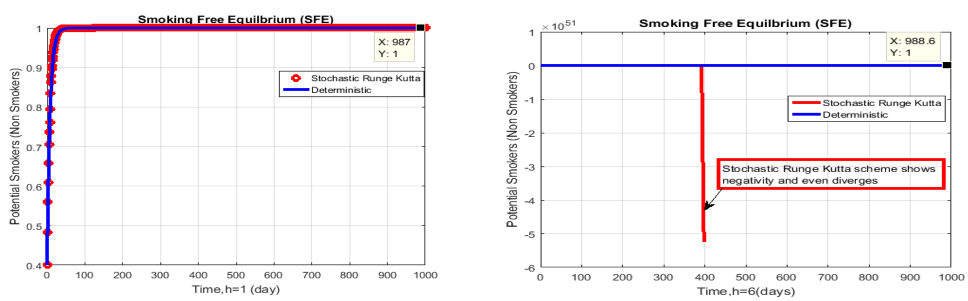
$$X_1{}^{n+1} = X_1{}^n + \frac{1}{6}[S_1 + 2S_2 + 2S_3 + S_4]. \quad (18)$$

$$X_2{}^{n+1} = X_2{}^n + \frac{1}{6}[U_1 + 2U_2 + 2U_3 + U_4]. \quad (19)$$

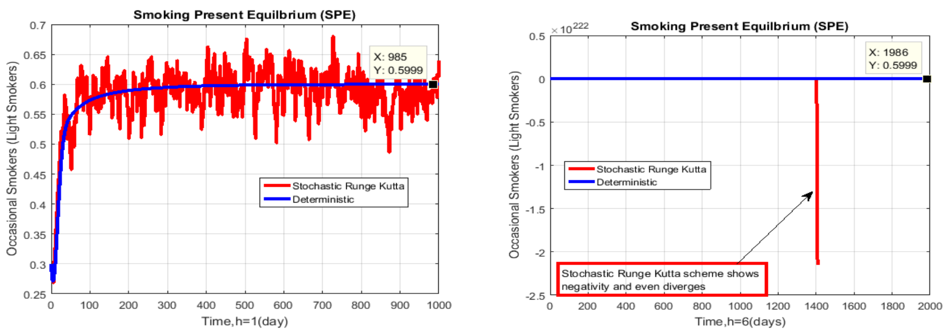
$$X_3{}^{n+1} = X_3{}^n + \frac{1}{6}[V_1 + 2V_2 + 2V_3 + V_4]. \quad (20)$$

$$X_4{}^{n+1} = X_4{}^n + \frac{1}{6}[W_1 + 2W_2 + 2W_3 + W_4]. \quad (21)$$

where 'h' represents the step size and $\Delta B_n \sim N(0,1)$. Now, we shall use MATLAB database and parameters values presented in Tab.1 for simulation of the Eqs. (18) to (21).



(e) **(f)**



(g) **(h)**

Figure 3: (e) Potential smokers at h=1 (f) Potential smokers at h=6 (g) Light smokers at h=1 (h) Light smokers at h=6

4.3 Stochastic NSFD method

In this technique, the Eqs. (10) to (13) is take the form,

$$X_1^{n+1} = \frac{X_1^n + h\xi}{1 + h\xi + h\omega_1 X_2^n + hX_2^n \varpi_1 \Delta B_n}. \quad (22)$$

$$X_2^{n+1} = \frac{X_2^n + h\omega_1 X_1^n X_2^n + hX_1^n X_2^n \varpi_1 \Delta B_n}{1 + h\omega_2 X_3^n + h\xi}. \quad (23)$$

$$X_3^{n+1} = \frac{X_3^n + h\omega_2 X_2^n X_3^n + h\varpi X_4^n + h\varpi_2 X_4^n \Delta B_n}{1 + h(\xi + \nu)}. \quad (24)$$

$$X_4^{n+1} = \frac{X_4^n + h\nu X_3^n}{1 + h(\xi + \varpi) + h\varpi_2 \Delta B_n}. \quad (25)$$

where ‘h’ represents the step size and $\Delta B_n \sim N(0,1)$.

4.4 Convergence analysis

In this section, we shall verify the following theorems:

Theorem: The Eqs. (22) to (25) has a unique positive solution $(S^n, I^n, R^n) \in R_+^3$ with n≥0, with any given initial value $(X_1^n(0), X_2^n(0), X_3^n(0), X_4^n(0)) \in R_+^4$.

Theorem: The section $\Omega = \{(X_1^n, X_2^n, X_3^n, X_4^n) \in R_+^4 : X_1^n \geq 0, X_2^n \geq 0, X_3^n \geq 0, X_4^n \geq 0, X_1^n + X_2^n + X_3^n + X_4^n \leq 1\}$ for all $n \geq 0$ is a translation-invariant for Eqs. (22) to (25).

Proof: Rewriting the Eqs. (22) to (25) as below:

$$\frac{X_1^{n+1} - X_1^n}{h} = \xi - \xi X_1^n - \omega_1 X_1^n X_2^n - X_1^n X_2^n \varpi_1 \Delta B_n.$$

$$\frac{X_2^{n+1} - X_2^n}{h} = \omega_1 X_1^n X_2^n - \omega_2 X_2^n X_3^n - \xi X_2^n + X_1^n X_2^n \varpi_1 \Delta B_n.$$

$$\frac{X_3^{n+1} - X_3^n}{h} = \omega_2 X_2^n X_3^n - (\xi + \nu) X_3^n + \varpi X_4^n + \varpi_2 X_4^n \Delta B_n.$$

$$\frac{X_4^{n+1} - X_4^n}{h} = \nu X_3^n - (\xi + \varpi) X_4^n - \varpi_2 X_4^n \Delta B_n.$$

$$\frac{X_1^{n+1} + X_2^{n+1} + X_3^{n+1} + X_4^{n+1}) - (X_1^n + X_2^n + X_3^n + X_4^n)}{h} = \xi - \xi(X_1^n + X_2^n + X_3^n + X_4^n).$$

$({X_1}^{\mathrm{n+1}} + {X_2}^{\mathrm{n+1}} + {X_3}^{\mathrm{n+1}} + {X_4}^{\mathrm{n+1}}) - ({X_1}^{\mathrm{n}} + {X_2}^{\mathrm{n}} + {X_3}^{\mathrm{n}} + {X_4}^{\mathrm{n}}) = \xi - \xi.$

$({X_1}^{\mathrm{n+1}} + {X_2}^{\mathrm{n+1}} + {X_3}^{\mathrm{n+1}} + {X_4}^{\mathrm{n+1}}) \leq 1 + 0 \times h.$

$({X_1}^{\mathrm{n+1}} + {X_2}^{\mathrm{n+1}} + {X_3}^{\mathrm{n+1}} + {X_4}^{\mathrm{n+1}}) \leq 1.$

Theorem: The Eqs. (22) to (25) has identical equilibria as that of the Eqs. (10) to (13) for all $n \geq 0$.

Proof: Resolving the Eqs. (22) to (25) as follows:

Smoking free equilibrium is $\mathrm{D}_3 = (1,0,0,0)$.

Smoking present equilibrium is $\mathrm{E}_3 = ({X_1}^{\mathrm{n}}, {X_2}^{\mathrm{n}}, {X_3}^{\mathrm{n}}, {X_4}^{\mathrm{n}})$.

where, ${X_1}^{\mathrm{n}} = \frac{\xi}{\xi + \omega_1 {X_2}^{\mathrm{n}} + {X_2}^{\mathrm{n}} \varpi_1 \Delta \mathrm{B_n}}, {X_3}^{\mathrm{n}} = \frac{(\omega_1 + \varpi_1 \Delta \mathrm{B_n}) {X_1}^{\mathrm{n}} - \xi}{\omega_2},$

${X_2}^{\mathrm{n}} = \frac{(\xi + \nu)(\xi + \varpi + \varpi_2 \Delta \mathrm{B_n} - \nu \varpi - \nu \varpi_2 \Delta \mathrm{B_n})}{\omega_2 (\xi + \varpi + \varpi_2 \Delta \mathrm{B_n})}$ and ${X_4}^{\mathrm{n}} = \frac{\nu {X_3}^{\mathrm{n}}}{\xi + \varpi + \varpi_2 \Delta \mathrm{B_n}}.$

Theorem: For stability of proposed stochastic NSFD, the eigen values of Eqs. (22) to (25) should lie in the unit circle.

Proof: We consider F, G, H and K from Eqs. (22) to (25) as follows:

$\mathrm{F} = \frac{X_1}{1 + \mathrm{h}\xi + \mathrm{h}\omega_1 X_2 + \mathrm{h} X_2 \varpi_1 \Delta \mathrm{B_n}}.$

$\mathrm{G} = \frac{X_2 + \mathrm{h}\omega_1 X_1 X_2 + \mathrm{h} X_1 X_2 \varpi_1 \Delta \mathrm{B_n}}{1 + \mathrm{h}\xi + \mathrm{h}\omega_2 X_2}.$

$\mathrm{H} = \frac{X_3 + \mathrm{h}\omega_2 X_2 X_3 + \mathrm{h} X_4 \varpi + \mathrm{h}\varpi_2 X_4 \Delta \mathrm{B_n}}{1 + \mathrm{h}(\xi + \nu)}.$

$\mathrm{K} = \frac{X_4 + \mathrm{h}\nu X_3}{1 + \mathrm{h}(\xi + \varpi) + \mathrm{h}\varpi_2 \Delta \mathrm{B_n}}.$

$\frac{\partial \mathrm{F}}{\partial X_1} = \frac{1}{1 + \mathrm{h}\xi + \mathrm{h}\omega_1 X_2 + \mathrm{h} X_2 \varpi_1 \Delta \mathrm{B_n}}, \frac{\partial \mathrm{F}}{\partial X_2} = \frac{-(X_1 + \mathrm{h}\xi)(\mathrm{h}\omega_1 + \mathrm{h}\varpi_1 \Delta \mathrm{B_n})}{(1 + \mathrm{h}\xi + \mathrm{h}\omega_1 X_2 + \mathrm{h} X_2 \varpi_1 \Delta \mathrm{B_n})^2}, \frac{\partial \mathrm{F}}{\partial X_3} = 0, \frac{\partial \mathrm{F}}{\partial X_4} = 0,$

$\frac{\partial \mathrm{G}}{\partial X_1} = \frac{\mathrm{h}\omega_1 X_2 + \mathrm{h} X_2 \varpi_1 \Delta \mathrm{B_n}}{1 + \mathrm{h}\xi + \mathrm{h}\omega_2 X_3}, \frac{\partial \mathrm{G}}{\partial X_2} = \frac{1 + \mathrm{h}\omega_1 X_1 + \mathrm{h} X_1 \varpi_1 \Delta \mathrm{B_n}}{1 + \mathrm{h}\xi + \mathrm{h}\omega_1 X_3}, \frac{\partial \mathrm{G}}{\partial X_3} = \frac{-[X_2 + \mathrm{h}\omega_1 X_1 X_2 + \mathrm{h} X_1 X_2 \varpi_1 \Delta \mathrm{B_n}](\mathrm{h}\omega_2)}{(1 + \mathrm{h}\xi + \mathrm{h}\omega_2)^2}$

$\frac{\partial \mathrm{G}}{\partial X_4} = 0, \frac{\partial \mathrm{H}}{\partial X_1} = 0, \frac{\partial \mathrm{H}}{\partial X_2} = \frac{\mathrm{h}\omega_2 X_3}{1 + \mathrm{h}(\xi + \nu)}, \frac{\partial \mathrm{H}}{\partial X_3} = \frac{1 + \mathrm{h}\omega_2 X_2}{1 + \mathrm{h}(\xi + \nu)}, \frac{\partial \mathrm{H}}{\partial X_4} = \frac{\mathrm{h}\varpi + \mathrm{h}\varpi_2 \Delta \mathrm{B_n}}{1 + \mathrm{h}(\xi + \nu)}, \frac{\partial \mathrm{K}}{\partial X_1} = 0, \frac{\partial \mathrm{K}}{\partial X_2} = 0,$

$\frac{\partial \mathrm{K}}{\partial X_3} = \frac{\mathrm{h}\nu}{1 + \mathrm{h}(\xi + \varpi) + \mathrm{h}\varpi_2 \Delta \mathrm{B_n}}, \frac{\partial \mathrm{K}}{\partial X_4} = \frac{1}{1 + \mathrm{h}(\xi + \varpi) + \mathrm{h}\varpi_2 \Delta \mathrm{B_n}}.$

The Jacobean matrix "J" is defined as

$$\mathrm{J} = \begin{bmatrix} \frac{\partial \mathrm{F}}{\partial X_1} & \frac{\partial \mathrm{F}}{\partial X_2} & \frac{\partial \mathrm{F}}{\partial X_3} & \frac{\partial \mathrm{F}}{\partial X_4} \\ \frac{\partial \mathrm{G}}{\partial X_1} & \frac{\partial \mathrm{G}}{\partial X_2} & \frac{\partial \mathrm{G}}{\partial X_3} & \frac{\partial \mathrm{G}}{\partial X_4} \\ \frac{\partial \mathrm{H}}{\partial X_1} & \frac{\partial \mathrm{H}}{\partial X_2} & \frac{\partial \mathrm{H}}{\partial X_3} & \frac{\partial \mathrm{H}}{\partial X_4} \\ \frac{\partial \mathrm{K}}{\partial X_1} & \frac{\partial \mathrm{K}}{\partial X_2} & \frac{\partial \mathrm{K}}{\partial X_3} & \frac{\partial \mathrm{K}}{\partial X_4} \end{bmatrix}$$

Put $\mathrm{D}_2 = (1,0,0,0)$ and $\mathrm{R_0^S} < 1$, we have

$$J = \begin{bmatrix} \frac{1}{1+h\xi} & \frac{-(h\omega_1+h\varpi_1\Delta B_n)}{(1+h\xi)} & 0 & 0 \\ 0 & \frac{1+h\omega_1+h\varpi_1\Delta B_n}{1+h\xi} & 0 & 0 \\ 0 & 0 & \frac{1}{1+h(\xi+\nu)} & \frac{h\varpi+h\varpi_2\Delta B_n}{1+h(\xi+\nu)} \\ 0 & 0 & \frac{h\nu}{1+h(\xi+\varpi)+h\varpi_2\Delta B_n} & \frac{1}{1+h(\xi+\varpi)+h\varpi_2\Delta B_n} \end{bmatrix}$$

The eigen values are $\lambda_1 = \frac{1}{1+h\xi} < 1,\ \lambda_2 = \frac{1+h\omega_1+h\varpi_1\Delta B_n}{1+h\xi} < 1$, if $R_o^S < 1$.

$$J = \begin{bmatrix} \frac{1}{1+h(\xi+\nu)} & \frac{h\varpi+h\varpi_2\Delta B_n}{1+h(\xi+\nu)} \\ \frac{h\nu}{1+h(\xi+\varpi)+h\varpi_2\Delta B_n} & \frac{1}{1+h(\xi+\varpi)+h\varpi_2\Delta B_n} \end{bmatrix}$$

$U_1 =$ Trace of J.

$U_2 =$ Determinant of J.

$$U_1 = \frac{1}{1+h(\xi+\nu)} + \frac{1}{1+h(\xi+\varpi)+h\varpi_2\Delta B_n}.$$

$$U_2 = \left(\frac{1}{1+h(\xi+\nu)}\right)\left(\frac{1}{1+h(\xi+\varpi)+h\varpi_2\Delta B_n}\right) - \frac{h^2\nu(\varpi+\varpi_2\Delta B_n)}{(1+h(\xi+\nu))(1+h(\xi+\varpi)+h\varpi_2\Delta B_n)}.$$

Lemma: For the quadratic equation $\lambda^2 - U_1\lambda + U_2 = 0$, $|\lambda_i| < 1, i = 1, 2$; if and only if subsequent situations are fulfilled as follows:

(i) $1 - U_1 + U_2 > 0$

(ii) $1 + U_1 + U_2 > 0$

(iii) $U_2 < 1$.

Proof: (i). $1 - U_1 + U_2 < 0$

$\big(1 + h(\xi + \nu)\big)(1 + h(\xi + \varpi) + h\varpi_2\Delta B_n) - \big(1 + h(\xi + \nu)\big) - (1 + h(\xi + \varpi) + h\varpi_2\Delta B_n) + h^2\nu(\varpi + \varpi_2\Delta B_n) < 0.$

$h^2\big(\nu(\varpi + \varpi_2\Delta B_n) + (\xi + \nu)(\xi + \varpi)\big) < 1.$

$h^2 < \frac{1}{(\nu(\varpi+\varpi_2\Delta B_n)+(\xi+\nu)(\xi+\varpi))}.$

$$h^2 - 2h\left[\frac{1}{2h\big(\nu(\varpi+\varpi_2\Delta B_n)+(\xi+\nu)(\xi+\varpi)\big)}\right] + \left[\frac{1}{2h\big(\nu(\varpi+\varpi_2\Delta B_n)+(\xi+\nu)(\xi+\varpi)\big)}\right]^2 < \left[\frac{1}{2h\big(\nu(\varpi+\varpi_2\Delta B_n)+(\xi+\nu)(\xi+\varpi)\big)}\right]^2.$$

$$\left[\frac{1}{2h\big(\nu(\varpi+\varpi_2\Delta B_n)+(\xi+\nu)(\xi+\varpi)\big)} - h\right]^2 < \left[\frac{1}{2h\big(\nu(\varpi+\varpi_2\Delta B_n)+(\xi+\nu)(\xi+\varpi)\big)}\right]^2.$$

This inequality always holds because all parameters are, always positive and $h > 0$, where "h" is any time step size.

(ii). $1 + U_1 + U_2 > 0$

Since $1 > 0$ and $U_1 > 0$ so it is enough to show $U_2 > 0$.

$$\left(\frac{1}{1+h(\xi+\nu)}\right)\left(\frac{1}{1+h(\xi+\varpi)+h\varpi_2\Delta B_n}\right)-\frac{h^2\nu(\varpi+\varpi_2\Delta B_n)}{(1+h(\xi+\nu))(1+h(\xi+\varpi)+h\varpi_2\Delta B_n)}>0.$$

$$\left(\frac{1}{1+h(\xi+\nu)}\right)\left(\frac{1}{1+h(\xi+\varpi)+h\varpi_2\Delta B_n}\right)>\frac{h^2\nu(\varpi+\varpi_2\Delta B_n)}{(1+h(\xi+\nu))(1+h(\xi+\varpi)+h\varpi_2\Delta B_n)}.$$

$$1>h^2\nu(\varpi+\varpi_2\Delta B_n)$$

$$h^2<\frac{1}{\nu(\varpi+\varpi_2\Delta B_n)}.$$

$$h^2-2h\left[\frac{1}{2h\nu(\varpi+\varpi_2\Delta B_n)}\right]+\left[\frac{1}{2h\nu(\varpi+\varpi_2\Delta B_n)}\right]^2<\left[\frac{1}{2h\nu(\varpi+\varpi_2\Delta B_n)}\right]^2.$$

$$[\frac{1}{2h\nu(\varpi+\varpi_2\Delta B_n)}-h]^2<\left[\frac{1}{2h\nu(\varpi+\varpi_2\Delta B_n)}\right]^2.$$

This inequality always holds because all parameters are certainly positive and $h>0$, where "h" is any time step size.

(iii). $U_2<1$

$$\left(\frac{1}{1+h(\xi+\nu)}\right)\left(\frac{1}{1+h(\xi+\varpi)+h\varpi_2\Delta B_n}\right)-\frac{h^2\nu(\varpi+\varpi_2\Delta B_n)}{(1+h(\xi+\nu))(1+h(\xi+\varpi)+h\varpi_2\Delta B_n)}<1.$$

$$1-h^2\nu(\varpi+\varpi_2\Delta B_n)<\big(1+h(\xi+\nu)\big)(1+h(\xi+\varpi)+h\varpi_2\Delta B_n).$$

$$h^2(\nu(\varpi+\varpi_2\Delta B_n)+(\xi+\nu)((\xi+\varpi)+\varpi_2\Delta B_n))+h(2\xi+\nu+\varpi+\varpi_2\Delta B_n)>0.$$

$$h^2+h(\frac{(2\xi+\nu+\varpi+\varpi_2\Delta B_n)}{(\nu(\varpi+\varpi_2\Delta B_n)+(\xi+\nu)((\xi+\varpi)+\varpi_2\Delta B_n))})>0.$$

$$h^2+2h\left[\frac{(2\xi+\nu+\varpi+\varpi_2\Delta B_n)}{2(\nu(\varpi+\varpi_2\Delta B_n)+(\xi+\nu)((\xi+\varpi)+\varpi_2\Delta B_n))}\right]+\left[\frac{(2\xi+\nu+\varpi+\varpi_2\Delta B_n)}{2(\nu(\varpi+\varpi_2\Delta B_n)+(\xi+\nu)((\xi+\varpi)+\varpi_2\Delta B_n))}\right]^2>\left[\frac{(2\xi+\nu+\varpi+\varpi_2\Delta B_n)}{2(\nu(\varpi+\varpi_2\Delta B_n)+(\xi+\nu)((\xi+\varpi)+\varpi_2\Delta B_n))}\right]^2.$$

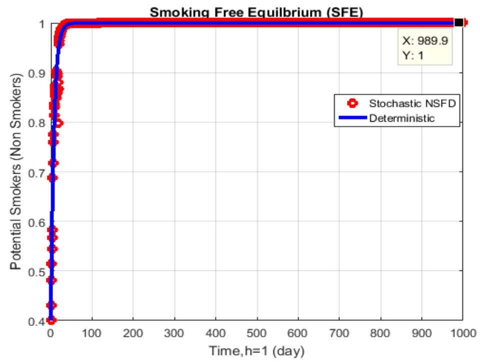
$$[h+\frac{(2\xi+\nu+\varpi+\varpi_2\Delta B_n)}{2(\nu(\varpi+\varpi_2\Delta B_n)+(\xi+\nu)((\xi+\varpi)+\varpi_2\Delta B_n))}]^2>\left[\frac{(2\xi+\nu+\varpi+\varpi_2\Delta B_n)}{2(\nu(\varpi+\varpi_2\Delta B_n)+(\xi+\nu)((\xi+\varpi)+\varpi_2\Delta B_n))}\right]^2.$$

$$h+\frac{(2\xi+\nu+\varpi+\varpi_2\Delta B_n)}{2(\nu(\varpi+\varpi_2\Delta B_n)+(\xi+\nu)((\xi+\varpi)+\varpi_2\Delta B_n))}>\frac{(2\xi+\nu+\varpi+\varpi_2\Delta B_n)}{2(\nu(\varpi+\varpi_2\Delta B_n)+(\xi+\nu)((\xi+\varpi)+\varpi_2\Delta B_n))}.$$

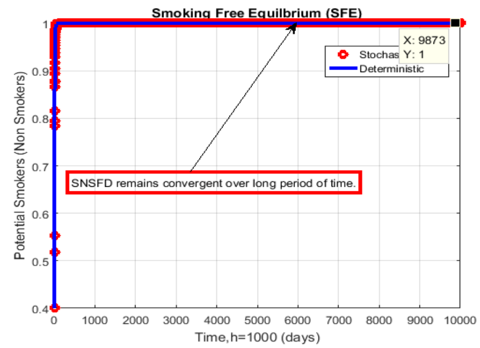
$h>0$, where "h" is any time step size.

The lemma is satisfied around the point D_2 and $R_o^S<1$. Thus, SNSFD is linearizable about equilibria of model.

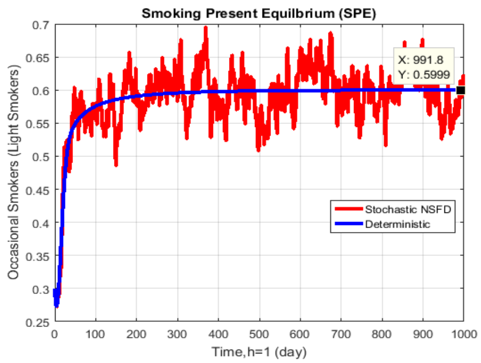
Now, we shall use MATLAB database and parameters values presented in Tab. 1 for simulation of the Eqs. (22) to (25).



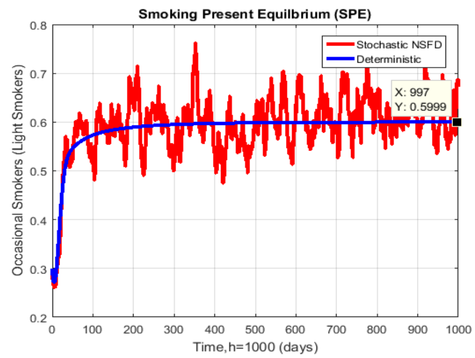
(i)



(j)



(k)

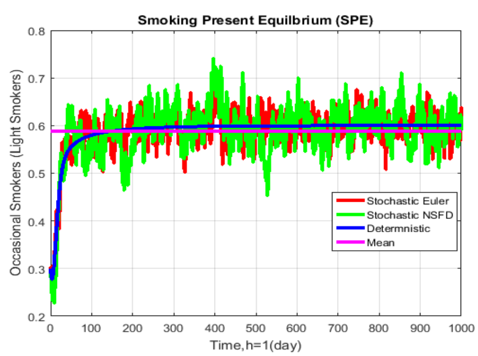


(l)

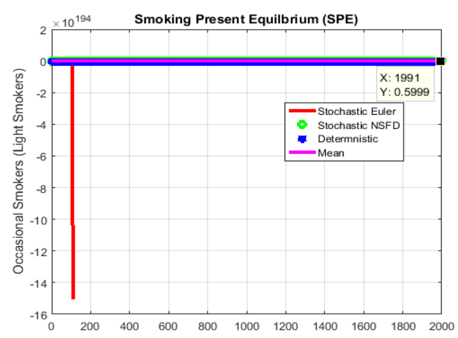
Figure 4: (i) Potential smokers at h=1 (j) Potential smokers at h=1000 (k) Light smokers at h=1 (l) Light smokers at h=1000

4.5 Contrast section

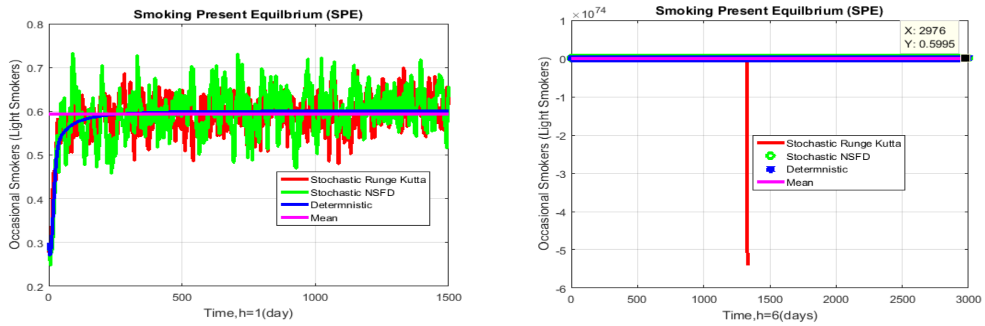
Now, we make the contrast between explicit techniques and our proposed technique as follows:



(m)



(n)



(p) (q)

Figure 5: Contrast in solutions of stochastic techniques and its mean (average) (m) Occasional smokers at h=1 (n) Occasional smokers at h=5 (p) Occasional smokers at h=1 (q) Occasional smokers at h=6

4.6 Covariance of smoking model

For this, we will calculate the correlation coefficients in the smoking model compartments. The result reported in the Tab. 2.

Table 2: Correlation coefficient

Sub-Populations	Correlation Coefficient (ρ)	Relationship
(X_1, X_2)	-0.9525	Inverse
(X_2, X_3)	-0.9734	Inverse
(X_3, X_4)	0.9687	Direct
(X_4, X_1)	0.7261	Direct
(X_1, X_3)	0.8581	Direct
(X_2, X_4)	-0.8988	Inverse

Results in Tab. 2, an inverse relationship between occasional smokers and others. This means that the number of occasional smokers increase with the decrease in other compartments of the model. Therefore, the model will achieve smoking free equilibrium (SFE). Heavy smokers are directly related to potential smokers and those who left smoking temporarily. There is also a direct relationship between potential smokers and those who quit smoking temporarily.

4.7 Results and discussion

In Fig. 2, we have developed the behaviour of a subpopulation of the smoking model using the stochastic Euler technique. The stochastic Euler technique converges towards the equilibria of the model under certain conditions over time steps. In Figs. 2(b) and 2(d), we can observe that this method is conditionally convergent. In Fig. 3, we have developed the behaviour of a subpopulation of the smoking model using the stochastic Runge Kutta technique. This method converges to the equilibria of the model under certain conditions over time steps. In Figs. 3(f) and 3(h), we can observe that the method is conditionally convergent. These explicit stochastic methods rely on time parameters and may lose dynamical properties of the model. In Fig. 4, we have developed the behaviour of a subpopulation of the smoking model using the proposed method. The proposed method converges towards equilibrium. In Fig. 5, contrast is planned between the stochastic methods of different compartments.

5 Conclusion and directions

The researchers of the paper have analysed that the study of stochastic differential equations (SDEs) is more realistic as compared to the ordinary differential equations (ODEs) for physical systems. The discrete stochastic and continues models have the same solutions under certain conditions. It is the first time when the new method named as stochastic nonstandard finite difference method is constructed for the smoking model. By using this method, we can study the smoking dynamics in human population over a long period of time. This method is unconditionally convergent unlike the other explicit stochastic methods. This scheme preserves all the dynamic properties of stochastic models such as, consistency, stability, positivity and boundedness. In future, we will extend this stochastic analysis to the other epidemiological models of humans, animals and plants. Also, we will expand this idea in the modelling of neural networking with fixed time intervals as presented in Yu et al. [Yu, Liu, Xiao et al. (2019)].

Funding Statement: We have no funding for this research article.

Conflicts of Interest: The authors declare that they have no conflicts of interest to report regarding the present study.

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