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Zet Theory

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ABSTRACT
The theory of Zets is presented and the standard techniques of set theory allows for the development of a rich algebra of Zets. It shows that Zets and fuzzy sets are essentially interchangeable. However, the fundamental manipulations, techniques, and definitions of Zets are simple and more amenable to analyze. For example, the extension principle is easy to define.

KEYWORDS
Zet theory; Fuzzy set theory; Lattice; Extension principle

1. Zadeh Set

A fuzzy set is a simple extension of a set. From an operational standpoint, a set is implemented with its characteristic function, which returns true (1) or false (0) depending on whether or not a candidate x of the universe of discourse is a member or is not a member of the set. Thus a classical set A has a characteristic function, χA that maps the universe U into the Boolean set $\{0, 1\}$. A fuzzy set maps a classical set A into the continuous interval $[0, 1]$ with a membership function m_A or, in a more modern style, A where $A{:}U \to [0, 1]$. The resultant structure is called a fuzzy set.

The two most important tools in the fuzzy system are the resolution identity and the extension principle. Both of these tools are developed in Zadeh's (1975) paper on linguistic variables. Yager (1986) generalizes the extension principle from functions to relations. Šešelja and Tepavčević, (1992, 1995) examine mapping the universe U directly to relations. Wierman (1997a), examines the extension of set functions. In Wierman (1999), requirements for a generalized extension of mappings between U and an abstract image set besides the unit interval. Araabi, Kehtarnavaz, and Lucas (2001) also examine the consequences of the extension principle.

1.1. Horizontal View

Given $\alpha \in [0, 1]$ the alpha-cut of A, $A\alpha$, is the cut-set (also called the alpha–cut).

$$A^{\alpha} = \{x \in U | A(x) \geq \alpha\} \tag{1}$$

In a fuzzy set theory this horizontal view is paramount. For example, let P be a property of a set theory that is true of every cut–set of A for every $\alpha > 0$. Then we say that A has property P, or that P is *cutworthy*. So if $U = \mathbb{R}$ and every cut set of A is convex then A is convex. Note that A is not necessarily convex considered as a set in $\mathbb{R}2$. Wierman (1997a) examines the requirements necessary for a cutworthy extension of arbitrary properties.

Given the family of all cuts of a fuzzy set A, $\mathcal{A}$, we can recapture the original fuzzy set by defining $A(x)$ to be the supremum of the set of alphas such that x is in the corresponding alpha level cut–set.

$$A(x) = \{\alpha | x \in A^{\alpha} \wedge A^{\alpha} \in \mathcal{A}\} \tag{2}$$

While Zadeh used resolution identity, this formula, or its equivalent, is often called the representation principle, because a fuzzy set is completely characterized (represented) by its cut sets. If U is the set of real numbers and alpha-cut of A is a bounded convex set for ever $\alpha > 0$ then A is considered to be a fuzzy number. The extension principal, discussed in Section 1.2, allows much of traditional mathematics to be extended into Zadeh's framework of fuzzy sets. For example, if $f(x1, x2) = x1 + x2$ then using the extension principal we can add fuzzy numbers.

Every engineer knows that even the most precise manufacturing process has a margin of error. If a widget is supposed to be 10 cm tall, it may be that the process has a margin of error of ±1 mm. This means that the actual height is in the interval of [9.99, 10.01] centimeters. What happens when we stack two widgets on top of each other? Error accumulates, the stack will be 20 cm tall ±2 mm, or in terms of an interval, the stack will have a height somewhere in the interval [19.98, 20.02] centimeters. Interval arithmetic captures this methodology, and there is a whole science of interval-based methods (Kearfott, 2016).

It turns out that all interval based techniques can be readily transported into Zadeh's methodology by taking alpha-cuts of fuzzy numbers, which are intervals, applying the appropriated interval technique to the intervals for a fixed value of alpha, and then constructing a fuzzy set from the resultant collection of intervals viewed as a family of alpha-cuts.

1.2. Vertical View

Definition 1 (extension principle). Let $C_1, C_2, C_3, \ldots, C_n$ be n fuzzy sets defined upon the universes $U_1, U_2, U_3, \ldots U_n$ respectively. Assume that f is a function with the domain $U = U_1 \times U_2 \times \ldots \times U_n$ and range Y. From the set theory, we recall the preimage of f,

$$f^{-1}(y) = \left\{\boldsymbol{x} \,| f(\boldsymbol{x}) = y\right\} \tag{3}$$

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where we note that f^{-1}, the inverse of f, is not necessarily a function. We can now define a fuzzy set D on Y via the extension principle:

$$D(y) = \sup_{x \in f^{-1}(y)} \min \left\{ C_i(x_i) \right\} \tag{4}$$

Where $i \in N_n = \{1, 2, 3 \ldots n\}$ and x_i is the i–th component of $x \in f^{-1}(y)$.

Example 1. Suppose we illustrate the definition with a simple example. Let

$\mathbb{R} = U1 = U2$ and let

$$f(x_1, x_2) = x_1 + x_2 \tag{5}$$

Equations (6) and (7) define two fuzzy numbers A and B.

$$A(x) = \begin{cases} x & 0 \leq x \leq 1 \\ 2 - x & 1 < x \leq 2 \end{cases} \tag{6}$$

and the Equation for fuzzy set A is

$$B(x) = \begin{cases} x - 1 & 1 \leq x \leq 2 \\ 3 - x & 2 < x \leq 3 \end{cases} \tag{7}$$

Examining a graph of A and B, see Figure 1, we see that the fuzzy numbers are represented by triangularly shaped functions made up of two line segments. Note also, that we suppressed mentioning the additional restriction that $A(x) = 0$ if x is outside the closed interval $[0, 2]$ and that $B(x) = 0$ if x is outside the closed interval $[1, 3]$.

The sum of A and B should be another fuzzy number C and we need a formula to define the membership grade of x in C based upon the definitions of A and B. In an application, $A(x)$ and $B(x)$ might represent the opinions of two different experts. Expert A thinks we need about one cup of milk and expert B thinks we need about two cups of flour. Therefore, our mixing bowl must fit about three cups of ingredients. The problem would be different if the experts gave us two different opinions on the amount of sugar.

The proper definition of the sum of fuzzy numbers is given by the formula

$$C(z) = \max_{x+y=z} \min \left[A(x), B(y) \right] \tag{8}$$

Which gives us the membership function of $C = A + B$. In this example we are assuming that x, y, and z are all ranging over the real numbers $\mathbb{R}$. It is important to remember that $C = A + B$ cannot be done by adding the two membership functions of A and B. It is more on the order of a convolution than a simple addition. The result turns out to be another triangular fuzzy number,

$$C(x) = \begin{cases} \frac{x-1}{2} & 1 \leq x \leq 3 \\ \frac{5-x}{2} & 3 < x \leq 5 \end{cases} \tag{9}$$

When we examine Equations (2) and (4), we note there is similarity in that both use the sup of a subset of the unit interval. In the case of Equation (2), since $\alpha < \beta$ necessitates $A\alpha \subseteq A\beta$ the set $\{\alpha | x \in A^\alpha \wedge A^\alpha \in \mathcal{A}\}$ must be an interval. In the case of Equation (4) it would be nice if the set $\bigcup_{x \in f^{-1}(y)} \left\{ C_i \right\}$ were an interval, which occurs if f^{-1} is continuous.

The easiest way to derive Eq. (9) is to utilize the fact that set of alpha-cuts, $\mathcal{A}$, completely characterize the Zet A. For a Zet-number, all the cuts-sets are intervals, and it turns out that the cut-sets of C are equal to the sum of the cut-sets of A and B performed using interval arithmetic, (see Kaufmann & Gupta, (1985) for a detailed discussion). At an alpha level of 0.3 the cut sets of A and B are $[0.3, 1.7]$ and $[1.3, 2.7]$ respectively. If we perform interval arithmetic, the sum of the intervals is an interval produced by adding the endpoints. Thus the cut set of C at alpha equal 0.3 is $[1.6, 4.4]$.

To get the formula in Equation (9) we need to reproduce the proceeding manipulations for an arbitrary alpha. Since alpha is plotted on the y-axis in the illustration, this turns out to be equivalent to solving the linear equations for x as a function of y (alpha). For an arbitrary alpha we get cut sets of A and B as $[\alpha, 2 - \alpha]$ and $[\alpha + 1, 3 - \alpha]$. Their sum is $[2\alpha + 1, 5 - 2\alpha]$ which is the cut set of C. Solving for alpha as a function of x gives the formula in Equation (9).

2. Zets

Both the vertical and horizontal view of a fuzzy set is based on manipulation of the points under the curve of the membership function. This paper presents a basis for fuzzy set theory using point sets and basic set theory. This approach was suggested by a series of papers that started examining the algebraic nature of the extension principle. A traditional fuzzy set A is defined by a function from a domain of interest U to the unit interval $[0, 1]$. It is well known that the fuzzy set theory can use an arbitrary poset, especially a lattice, as the image set Ω of the domain.

Work on the extension principle by Wierman (1999, 2004), shows that the axiomization of the fuzzy set theory needs only an image set Ω that has a relation r, which possesses three properties.

(1) For each $\omega \in \Omega$ the preimage of ω under the relation r, $r^{-1}(\omega) = \{\alpha | \alpha r \omega\}$, must be unique.
(2) The union of any set of preimages is also a preimage.
(3) The intersection of a countable set of preimages is a preimage.

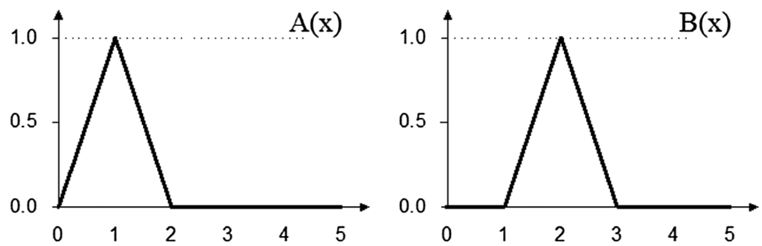

Figure 1. The Fuzzy Numbers $\underline{A}(\underline{x})$, About One and $\underline{B}(\underline{x})$, About Two

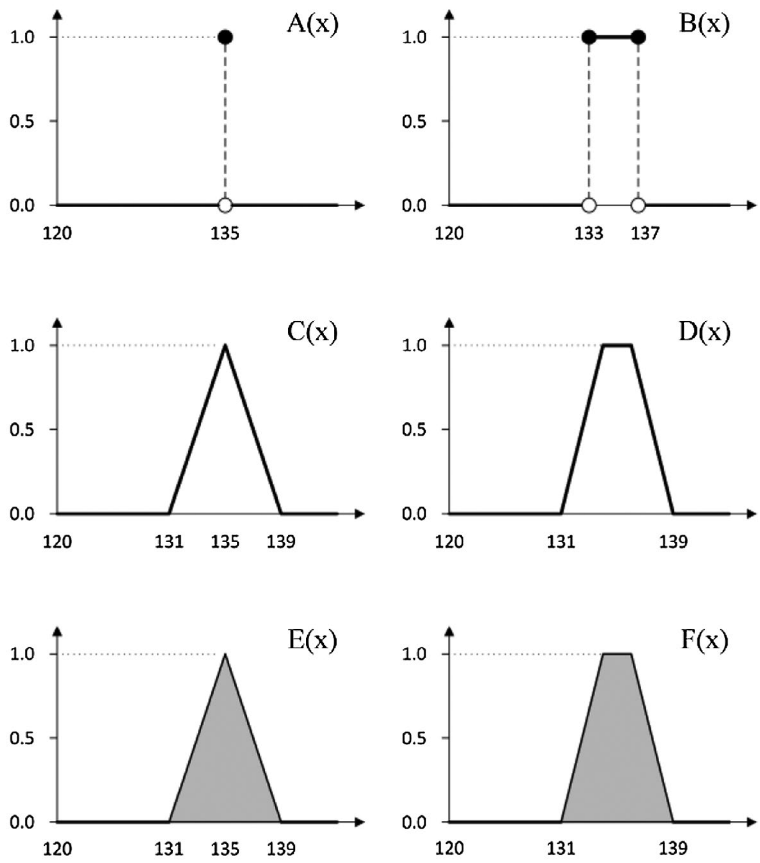

Figure 2. Sets, Fuzzy Sets, and Zets.

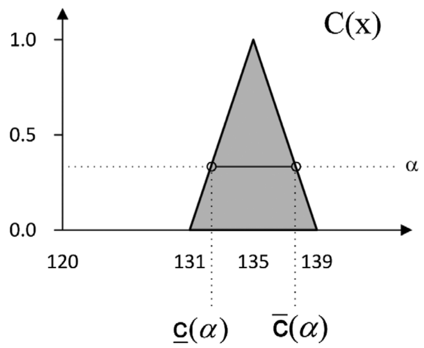

Figure 3. A Horizontal Slice, the Projection into $\underline{U}$ of the Intersection of the Zet $\underline{C}$ and the Line $\Omega = \omega$. An $\underline{\alpha}$–cut in Fuzzy Set Theory.

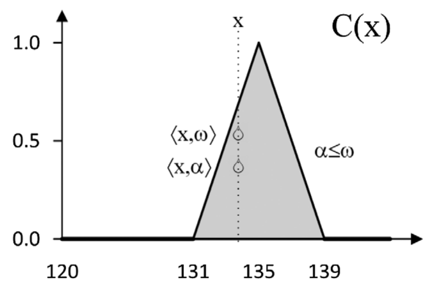

Figure 4. A Vertical Slice, the Projection into Ω of the Intersection of the Zet C and the Line $U = x$.

The preimages thus form a topology, if the empty set and universe are preimages. The preimages also form a lattice under set inclusion if the empty set is preimage. If we are going to replace the supremum with a measure however, it is best to assume that the set of preimages form a sigma algebra.

Thus, the best axiomatic definition of a fuzzy set uses $\Omega = [0, 1]$ considered as a poset with the order relation $\leq$.

A down–set (order–ideal) of a poset P is a set Q such that if Q contains an element ω and $\alpha \leq \omega$ then Q must contain α. Note that both open and closed intervals form lower sets, thus both $[0, 0.3]$ and $[0, 0.3)$ are lowersets of $[0, 1]$. In the theory of posets, the downset induced by ω is denoted $\downarrow \omega$ with definition

$$\downarrow \omega = \{\alpha \in P | \alpha \leq \omega\}. \tag{10}$$

This definition inspires the definition of a Zet (a contraction for Zadeh's set).

Definition 1. A subset C of $U \times \Omega$ is a Zet if $x, \omega \in C$ implies $\langle x, \alpha \rangle \in C$ whenever $\alpha < \omega$.

The interior of a Zet can be considered a Cloud (Neumaier, 2003) or a Cloud Set (Wierman, 2010), and this interpretation is what inspires the following nomenclature. A *Zet C* is a subset of $U \times \Omega$ that satisfies the *raindrop principle.* The raindrop principle says that if a point is an element of a Zet then all points below it are elements of that Zet, that is points *rain down* towards the U axis.

Remark 1. We note here that if C is a Zet then if we remove the boundary of the Zet to C to form its interior Cloud Set, then that resulting set still verifies the raindrop principle.

Example 2. Let $U = \mathbb{R}$, the real numbers, and let E be the interior of the triangle formed by the points $\langle 131, 0\rangle$, $\langle 135, 1\rangle$, and $\langle 139, 0\rangle$. Then E is a Zet. Let F be the trapezoid formed by the four points $\langle 131, 0\rangle$, $\langle 135, 1\rangle$, $\langle 137, 1\rangle$, and $\langle 139, 0\rangle$. Then F is also a Zet. See Figure 2.

Since Zets are point sets, it is natural to define the intersection and union of two Zets to be the intersection and union of the point sets.

Theorem 1. The union of two Zets is a Zet.

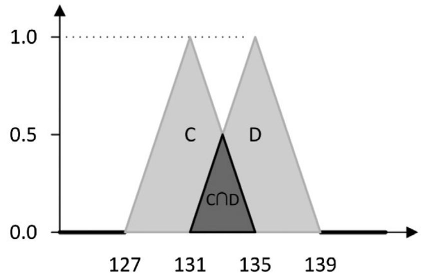

Figure 5. The Intersection of Zets.

Proof. Let E, F be Zets and let $G = E \cup F$. If $\langle x, \alpha \rangle$ is an element of G, then by the definition of set union x, α must have been an element of E or of F. If $\beta < \alpha$, then by the raindrop principle x, β must be in the same set that contains x, α and so x, β will be in G.

Theorem 2. The intersection of two Zets is a Zet.

Proof. Let C, D be Zets and let $E = C \cap D$. If $\langle x, \alpha \rangle$ is an element of E, then by the definition of set intersection $\langle x, \alpha \rangle$ must have been an element of both C and D. If $\beta < \alpha$, then by the raindrop principle x, β must be an element of both C and D so that x, β will be in E.

Example 3. If C is the triangle formed by the points $\langle 127, 0 \rangle$, $\langle 131, 1 \rangle$, and $\langle 135, 0 \rangle$, and D is the triangle formed by the points $\langle 131, 0 \rangle$, $\langle 135, 1 \rangle$, and $\langle 139, 0 \rangle$., then the intersection of C and D is the triangle formed by the points $\langle 131, 0 \rangle$, $\langle 133, 0.5 \rangle$, and $\langle 135, 0 \rangle$. See Figure 5. Note that this is a Venn diagram like situation. The intersection is just the overlap.

Definition 2. A Zet C is a subset of a Zet D if, as point sets, $C \subseteq D$.

We would like to emphasize the naturalness of the three definitions above. Union, intersection, and subsethood are simply the union, intersection, and subsethood of point sets, in this case Zets.

3. Projections and Slices

Here we review some basic definitions from set theory. Let U and Ω be crisp non-empty universal (sets). Let $E \subseteq U \times \Omega$. Then the marginal projection of E into $\boldsymbol{U}$, E^{Ω}, is defined as

$$E^{\Omega} = \{\boldsymbol{x} \mid \exists \omega \in \Omega \text{ such that } \langle \boldsymbol{x}, \omega \rangle \in E\} \tag{11}$$

The marginal projection of E into Ω, ${}^{U}E$ is defined as

$${}^{U}E = \{\omega \mid \exists x \in \Omega \text{ such that } \langle \boldsymbol{x}, \omega \rangle \in E\} \tag{12}$$

Given the definition of marginal projections, we can now define two types of Zet slices; vertical and horizontal. A vertical slice of a Zet picks out the values of $\boldsymbol{\Omega}$ (the range) that occurs at a particular value of U. A horizontal slice picks out the values of U (the domain) that has a particular Ω value. To do a slice we intersect the Zet, (which is a point set) with a line that is either vertical or horizontal, and then project the resulting interval(s) into the appropriate space.

Definition 3. [Horizontal Slice] If $E \subseteq U \times \Omega$ then a Ω–slice of E at ω, written E^{ω}, is the projection into U of the intersection of E and the line $\boldsymbol{\Omega} = \omega$.

$$E^{\omega} = \{\boldsymbol{x} \in \boldsymbol{U} \mid \langle \boldsymbol{x}, \omega \rangle \in E\} \tag{13}$$

Definition 4. [Vertical Slice] If $E \subseteq U \times \Omega$ then a U–slice of E at x, written ${}^{x}E$, is the projection into Ω of the intersection of E and the line $U = x$.

$${}^{x}E = \{\omega \in \Omega \mid \langle \boldsymbol{x}, \omega \rangle \in E\} \tag{14}$$

Note 12. Marginals and slices are definitions that work for any subset of $E \subseteq U \times \Omega$ and not just Zets. Figure 3 illustrates a vertical slice of a Zet and Figure 4 illustrates a horizontal slice of a Zet.

A Zet number is a Zet such that every horizontal slice for $\omega > 0$ is a bounded non-empty interval.

3.1. The Vertical View

We now list some trivial propositions, without proofs, that show the connection between operations on zets and operation on vertical or U–slices.

Proposition 1. If A and B are Zets, then the cuts of the union are the union of the cuts

$${}^{x}(A \cup B) = {}^{x}A \cup {}^{x}B \tag{15}$$

and the cuts of the intersection are the intersection of the cuts

$${}^{x}(A \cap B) = {}^{x}A \cap {}^{x}B \tag{16}$$

Definition 5. [Subset] If A and B are Zets, then A is a subset of B if every cut of A is a subset of the corresponding cut of B,

$$A \subset B \Leftrightarrow \forall x {}^{x}A \subset {}^{x}B \tag{17}$$

3.2. The Horizontal View

The following two theorems connect the horizontal (Ω–slice) and vertical (U–slice) views.

Theorem 4. We can reconstruct a Zet C from the class of all its Ω–slices. Thus

$$C = \bigcup_{\alpha \in \Omega} C^{\omega} \times \{\omega\} \tag{18}$$

We can reconstruct a Zet C from the class of all its U–slices. Thus

$$C = \bigcup_{x \in U} \{x\} \times {}^{x}C \tag{19}$$

In a slight abuse of notation, we will abbreviate $C^{\omega} \times \{\omega\}$ and $\{x\} \times {}^{x}C$ with $C^{\omega} \times \omega$ and ${}^{x} \times {}^{x}C$ in the following sections.

Theorem 5. If A and B are Zets then:

$$(A \cup B)^{\omega} = A^{\omega} \cup B^{\omega}$$

$$(A \cap B)^{\omega} = A^{\omega} \cap B^{\omega}$$

$$A = B \text{ iff } \forall \omega\, A^{\omega} = B^{\omega}$$

$$A \subset B \text{ iff } \forall \omega\, A^{\omega} \subset B^{\omega}$$

4. Zet Complement

Finally we define the complement of a Zet A defined on a universe. Geometrically, the concept is easy to visualize. Take the set complement of A in $U \times \Omega$ and flip it upside down. The only difficulty is the fact that the complement of a closed set will be half open. Therefore, we must take the complement of the interior (Cloud Set) to produce the closed set. To do this we need to introduce a new operator upon our image set $\Omega = [0, 1]$.

Let c be any order inverting operator from Ω onto Ω, so that $\alpha < \beta \Leftrightarrow c(\alpha) > c(\beta)$ for all $\alpha, \beta \in \Omega$. It is easy to show that $c(0) = 1$, $c(1) = 0$ and that c is 1 - 1.

Definition 6 (Complement). The complement of a Zet A, A^c, is the Zet

$$A^c = \{ \langle x, c(\omega) \rangle \mid \langle x, \omega \rangle \notin interior(A) \} \quad (20)$$

However, we do need to prove that the set $\{ \langle x, c(\omega) \rangle | \langle x, \omega \rangle \notin interior(A) \}$ is indeed a Zet.

Theorem 6. If A is a Zet the set $Ac = \{ \langle x, c(\omega) \rangle | \langle x, \omega \rangle \notin interior(A) \}$ is a Zet.

Proof. Let $= \{ \langle x, c(\omega) \rangle | \langle x, \omega \rangle \notin interior(A) \}$.We only need to show C verifies the raindrop principle. We noted in (4) that the interior of a Zet still verifies the raindrop principle. Since A is a Zet and c:$\Omega \to \Omega$ we know that C is thus a subset of $U \times \Omega$. Suppose that $\langle x, \beta' \rangle \in C$ and that $0 < \alpha' < \beta'$. Since we assume$x, \beta' \in C$ by the definition of, there is an $\langle x, \beta \rangle \notin interior(A)$ such that $\beta' = c(\beta)$. Since c is onto and monotone and $c(1) = 0$ it must map all the values between β and one to values between β' and zero. But this interval contains α' so $x, \alpha' \in C$ and C verifies the raindrop principle. Thus C is a Zet.

Of course, the standard involution operator is $c(\alpha) = 1 - \alpha$ and this produces the standard complement.

The Zet complement does not follow the rules of the complement in a normal set theory. For example, in general, $A \cap Ac \neq \emptyset$ and $A \cup Ac \neq U$.

5. Product Zet

Let U, and Y be universal sets with Zets A and B defined upon them. We shall define the product Zet $A \times B$ upon $U \times Y$ with the following definition, x, y, α is an element of $A \times B$ if x, α is an element of A and y, α is an element of B. If $U1, U2 \ldots Un$, are universes with Zets $A_1, A_2 \ldots A_n$ defined upon them, then we define $A_1 \times A_2 \ldots A_n$ upon $U_1 \times U_2 \ldots U_n$ with the following definition: $\langle x_1, x_2 \ldots x_n, \alpha \rangle$ is an element of $A_1 \times A_2 \ldots A_n$ if and only if, for all i with $1 \leq i \leq n$, we have that $\langle x_i, \alpha \rangle$ is an element of A_i. In vector notation with $A = A_1 \times A_2 \ldots A_n$, $U = U_1 \times U_2 \ldots U_n$, and $\langle x = x_1, x_2 \ldots x_n \rangle$ we say that x, α is an element of A if xi, α is an element of Ai for all $i, 1 \leq i \leq n$.

6. Modulators

There are three basic kinds of modulators that map Zets to Zets. The first kind modulates the membership values - the omega values, the second modulates the domain,—the x values, and the third kind allows for interactions between the x and omega values.

The first class of modulators are omega modulators, which map Ωn to Ω. Thus they operate upon values in Ω for a *fixed* value of x. Given a collection of n Zets $C_i \subseteq U \times \Omega$the first kind of modulator produces another Zet $D \subseteq U \times \Omega$. The values of $\omega \in \Omega$ assigned to a fixed $x \in U$ depends only on the values αi associated with x by the Zets C_i.

The second class of modulators operates upon values in U^n for a fixed value of ω. Given a collection of n Zets $C_i \subseteq U_i \times \Omega$ the second kind of modulator produces another Zet $D \subseteq Y \times \Omega$. While these calculations are done for a fixed $\omega \in \Omega$ the sets U_i are not necessarily distinct.

The third kind of modulators allow for functional interaction with both of the values x and ω of a point $\langle x, \omega \rangle \in A$. While they are not as prominent in the development of fuzzy se theory as the former two modulators they have their applications.

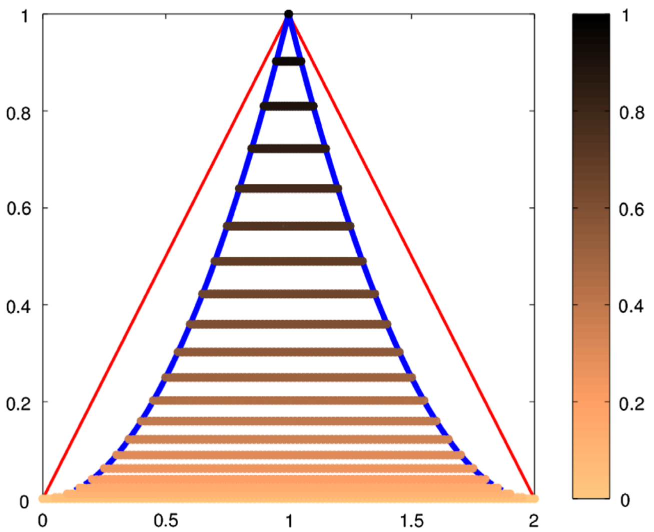

Figure 6. Modulated Alpha takes the Points under $\underline{A}$, in Red, and Creates $\Gamma(A)$ Where $\langle \underline{x}, \underline{\alpha} \rangle \rightarrow \langle \underline{x}, \underline{\alpha}^2 \rangle$.

6.1. Omega Modulators

Of interest in Zet theory, the functions on Ω, and Ω_n that preserve the raindrop principle as they operate upon a Zet. If C is a Zet and Γ is any continuous monotonic non-decreasing function that maps $[0, 1]$ onto itself then Γ will map a Zet C to a Zet D if we define

$$D = \Gamma(C) = \{ \langle x, \Gamma(\alpha) \rangle | \langle x, \alpha \rangle \in C \} \quad (21)$$

Similarly for any function Γ, Γ: $\Omega_n \to \Omega$, that is monotonic non-decreasing in all its arguments and onto we can produce a Zet D from Zets Ci with $1 \leq i \leq n$ by defining

$$\begin{aligned} D &= \Gamma(C_1, C_2, ..., C_n) \\ &= \{ \langle x, \Gamma(\alpha_1, \alpha_2, ..., \alpha_n) \rangle | \forall i \ \ \langle x_i, \alpha \rangle \in C_i \}. \end{aligned} \quad (22)$$

Example 4. Consider the Zet A, which is the points under the fuzzy set A given by Equation (6). Let $\Gamma(\alpha) = \alpha^2$. In linguistic terms (Zadeh, 1975) says this modulator intensifies a Zet. Since $\langle 1, 0.5 \rangle \in A$ we would have $\langle 1, \Gamma(0.5) \rangle = \langle 1, 0.25 \rangle \in \Gamma(A)$. The cut set of A at alpha equals 0.5 is the interval [0.5.1.5] so that the cut set of $\Gamma(A)$ at alpha equal 0.25 is [0.5.1.5]. This is illustrated in Figure 6. In English, the Zet number one, A, becomes very one, $\Gamma(A)$.

Example 5. Suppose $\Gamma(\alpha, \beta) = \min[\alpha, \beta]$ then the min of Zets C and D on U is the Zet E with

$$E = \{ \langle x, \ \min[\alpha, \beta] \rangle | \langle x, \alpha \rangle \in C \wedge (x, \beta) \in D \} \quad (23)$$

We continue Example (8) where C is the triangle formed by the points $\langle 127, 0 \rangle$, $\langle 131, 1 \rangle$, and $\langle 135, 0 \rangle$, and D is the triangle formed by the points $\langle 131, 0 \rangle$, $\langle 135, 1 \rangle$, and $\langle 139, 0 \rangle$.

Then the min of Zets C and D is the triangle formed by the points $\langle 131, 0 \rangle$, $\langle 133, 0.5 \rangle$, and $\langle 135, 0 \rangle$.

Remark 22. We note here that $\min[A, B] = A \cap B$ and that $\max[A, B] = A \cup B$. The proof is very simple and we omit it.

6.2. Domain Modulators

If we perform operations on domain values for fixed values of omega we get another class of modulators.

If C is a Zet, g is any function that maps U onto itself then g will map a Zet C to a Zet D if we define

$$D = g(C) = \{ \langle g(x), \alpha \rangle | \langle x, \alpha \rangle \in C \} \quad (24)$$

Similarly for any function g, $g{:}U_n \rightarrow U$, we can produce a Zet D from Zets C_i with $1 \leq i \leq n$ by defining

$$\begin{aligned} D &= g(C_1, C_2, ..., C_n) \\ &= \left\{\langle g(x_1, x_2, ..., x_n), \alpha\rangle | \forall i \;\; x_i, \alpha \in C_i\right\}. \end{aligned} \quad (25)$$

Example 6. Consider the Zet A, which is the points under the fuzzy set A given by Equation (6). Let $G(x) = x^2$. Now we have Zet algebra where we are squaring about one. Since $\langle 1.5, 0.5\rangle \in A$ we would have $\langle G(1, 5), 0.5\rangle = \langle 2.25, 0.5\rangle \in G(A)$. The cut set of A at alpha equal 0.5 is the interval [0.5.1.5] so that the cut set of $G(A)$ at alpha equal 0.5 is [0.25.2.25]. This is illustrated in Figure 7.

Example 7. Suppose $U = \mathbb{R}$ and $f(x, y) = x + y$ then the sum of Zets C and D on U is the Zet E with

$$E = \{\langle x + y, \alpha\rangle | \langle x, \alpha\rangle \in C \wedge \langle y, \alpha\rangle \in D\} \quad (26)$$

We continue Example (8) where C is the triangle formed by the points $\langle 127, 0\rangle$, $\langle 131, 1\rangle$, and $\langle 135, 0\rangle$, and $D$is the triangle formed by the points $\langle 131, 0\rangle$, $\langle 135, 1\rangle$, and $\langle 139, 0\rangle$. Then the s of C and D is the triangle formed by the points $\langle 258, 0\rangle$, $\langle 266, 1\rangle$, and $\langle 274, 0\rangle$.

7. Interactive Modulators

Let us take the simplest example. We might have two functions $f(x, \omega){:}U \times \Omega \rightarrow U$ and $g(x, \omega){:}U \times \Omega \rightarrow \Omega$ and the map $\langle x, \omega\rangle \rightarrow \langle f(x, \omega), g(x, \omega)\rangle$. When $U = \mathbb{R}$ since $\Omega \subseteq \mathbb{R}$ the function f has few restrictions. On the other hand, g must result in a value in Ω, which puts severe restrictions on the nature of g. In practice there are two major areas of research that involve modulators of this kind. Both these areas simplify things by looking at functions h that map U into Ω. Two important classes of such functions are fuzzy sets themselves and probability functions, both distributions and densities. To illustrate, let $p(x)$ be a probability distribution, then $\langle x, \omega\rangle \rightarrow \langle x, p(x) \cdot \omega\rangle$ is a kind of a Zet probability distribution (Irwin & Goodman, 1982; Nguyen, 2005).

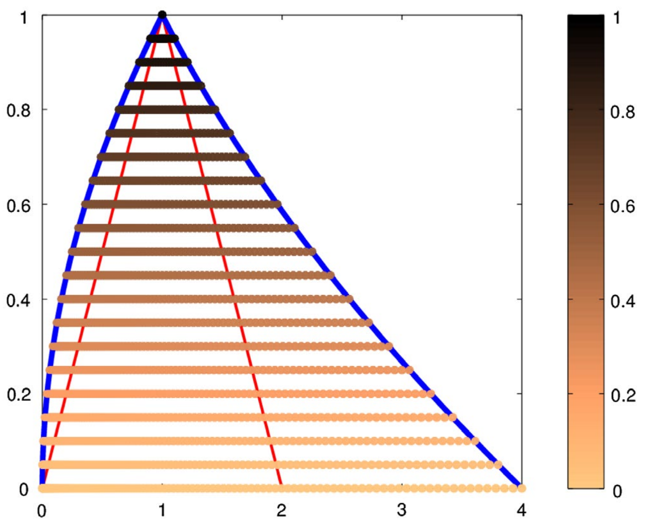

Figure 7. Modulated $\underline{x}$ takes the Points under $\underline{A}$, in Red, and Creates $G(A)$ where $\langle \underline{x}, \underline{\alpha}\rangle \rightarrow \langle \underline{x}^2, \underline{\alpha}\rangle$.

8. Assessors

An assessor (Wierman, 1996) is a function that maps a Zet to a real number. Typical examples of these types of functions are; measures of central value Wierman (Wierman (1997b) If $U = \mathbb{R}$, then the center of gravity is extremely useful in fuzzy controllers.

Example 8. Consider the Zet $A \times B$, which represents the rule if A then B in a fuzzy controller, see Figure 8. If we receive an input value x, then if we slice the pyramid at x, then ${}^{x}A \times B = \{\langle y, \omega\rangle | \langle x, y, \omega\rangle \in A \times B\}$ is a trapezoidal zet. If we have multiple rules firing in parallel we have multiple, possibly overlapping, conclusions, see Figure 9. The best estimate of the result is the center of gravity of the resulting 2D object.

9. Zets and Fuzzy Sets

Given a Zet C_i we can construct an equivalent fuzzy set $\boldsymbol{Ci}$ by setting

$$C_i(x) = \sup\{\alpha | \langle x, \alpha\rangle \in C_i\} \quad (27)$$

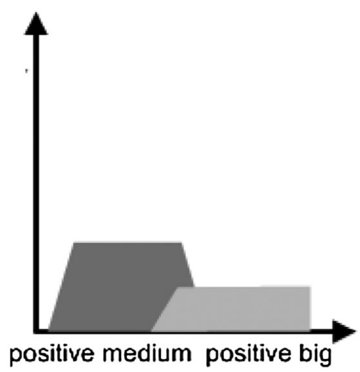

Figure 9. Multiple Zet Rules Produce Multiple Trapezoids. Assessment would Proceed with the Center of Gravity, or some other Method to Determine a Unique $\underline{Y}$.

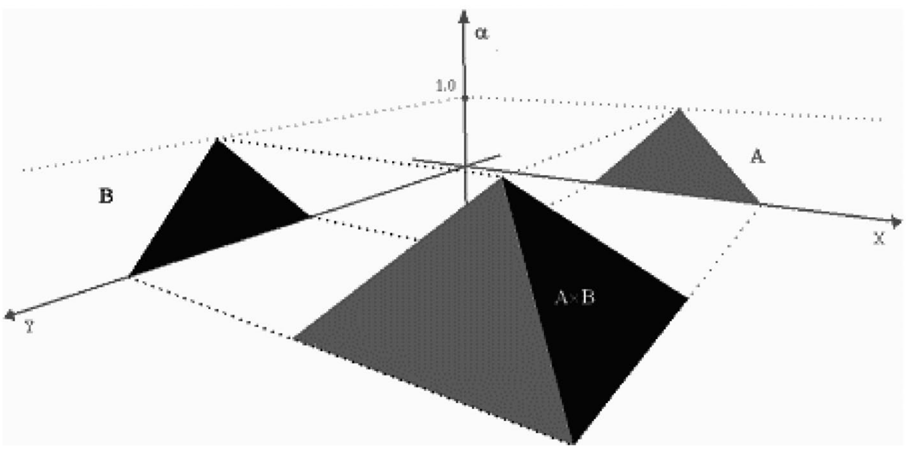

Figure 8. A Zet Rule for *if A then B*. An $\underline{x}$ Impulse Slices a Trapezoid out of the Pyramid.

Conversely, given a fuzzy set membership function C_i we can get a C_i by defining the Zet

$$C_i = \{\langle x, \alpha\rangle | C_i(x) \leq \alpha\} \tag{28}$$

And it is easy to see that we can go from zet to fuzzy set back to the original Zet and vice versa. We can easily prove the following equivalence.

Theorem 7. Equation (24) is equivalent to (4). That is, the fuzzy set function $D(y)$ is just the pointwise upper bound of the corresponding Zet, $D(y) = \sup\{{}^{y}D\}$.

Example (19) showed that min was equivalent to intersection. It is also obvious that max is equivalent to union. Thus, in Equation (4) the *sup* is over an uncountable set and the *inf* is over a countable set.

10. Discussion

Except for the complement, all the formulas and definitions of Zet theory are simple formulas of set theory. It is easy to see that Equation (24) is much simpler in form than Equation (4).

As another example, suppose that $\odot$ is an associative, commutative, monotone binary operator from Ω^2 onto Ω. Let

$$f(C, D) = C \odot D \tag{29}$$

If, for all C and D, we have

$$f(C, D) \subseteq C \cap D \tag{30}$$

Then f is a t–norm. If

$$C \cup D \subseteq f(C, D). \tag{31}$$

For all C and D, then f is a t–conorm Klir and Yuan (1995). This example also illustrates an important consequence of Zet thinking. Union and Intersection are the union and intersection of the Zets. Operators, such as t–norms, OWAs, etc., are all modulators. Many modulators are based on logical operators, but only *min* is equivalent to intersection. The product operator $g(\alpha, \beta) = \alpha \cdot \beta$ is a t–norm and is a *logical-and,* but it is sub-intersection.

Finally, let us take something that is incredibly involved in fuzzy set's original functional view. Let $F(x)$ be a function and $f(x)$ its derivative. Then $f(A)$ is the derivative of $F(A)$ when A is a Zet number, because for a fixed α, the distance from $F(x), \alpha$ to $F(x + \Delta X), \alpha$ is the distance from $F(x)$ to $F(x + \Delta x)$ and we know that the ratio of this distance divided by Δx converges to $f(x)$. We should be a little careful and use limits from the right and left at the endpoints of $A\alpha$, but the conclusion is sound. If something is correct for every alpha then it is correct for a Zet is just the horizontal view in another guise. This is actually a meta-property.

Disclosure statement

No potential conflict of interest was reported by the author.

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References

Araabi, B.N., Kehtarnavaz, N., & Lucas, C. (2001). Restrictions imposed by the fuzzy extension of relations and functions. *Journal of Intelligent & Fuzzy Systems, 11*, 9-22.

Irwin R., & Goodman, I.R. (1982). Fuzzy sets as equivalence classes of random sets. In R.R. Yager (Ed.), *Fuzzy set and possibility theory* (pp. 327–342). Pergamon Press.

Kearfott, R.B. (2016). Interval computations: Introduction, uses, and resources. Retrieved from http://interval.louisiana.edu/preprints/survey.pdf

Klir, G.J., & Yuan, B. (1995). *Fuzzy sets and fuzzy logic*. Prentice Hall, Upper Saddle River, NY: Theory and applications.

Kaufmann, A., & Gupta, M.M. 1985, Introduction to Fuzzy Arithmetic, New York, NY: Reinhold

Neumaier, A. (2003). Clouds, fuzzy sets and probability intervals. *Reliable Computing, 10*, 249–272.

Nguyen, H.T. (2005). Fuzzy and random sets. *Fuzzy Sets and Systems, 156*, 349–356.

Šešelja, B., & Tepavčević, A. (1992). Relational valued fuzzy sets. *Fuzzy Sets and Systems, 52*, 217–222.

Šešelja, B., & Tepavčević, A. (1995). Partially ordered and relational valued fuzzy relations I. *Fuzzy Sets and Systems, 72*, 205–213.

Wierman, M.J. (1996). Assessing fuzzy sets and the potential of possibility theory. *Intelligent Systems, 88*, 247–261.

Wierman, M. J. (1997a). Central values of fuzzy numbers – Defuzzification. *Information Sciences, 100*, 207–215.

Wierman, M.J. (1997b). Extending set functions and relations. *International Journal of General Systems, 26*, 91–96.

Wierman, M.J. (1999). Generalized representation and extension. *Fuzzy Sets and Systems, 106*, 387–392.

Wierman, M. J. 2004. Yet another axiomization of fuzzy set theory. In *IEEE Annual Meeting of the Fuzzy Information, 2004. Processing NAFIPS '04*, (Vol., 1. pp. 265–269).

Wierman, M.J. (2010). Cloud sets as a measure theoretic basis for fuzzy set theory. In M.J. Wierman (Ed.), *NAFIPS 2010* (pp. 1–5).

Yager, R.R. (1986). A characterization of the extension principle. *Fuzzy Sets and Systems, 18*, 205–217.

Zadeh, L.A. (1965). Fuzzy sets. *Information and Control, 8*, 338–353.

Zadeh, L.A. (1975). The concept of a linguistic variable and its application to approximate reasoning, Part I. *Information Science, 8*, 199–249.