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# Practical Application of Fractional Order Controllers to a Delay Thermal System

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This paper provides an application of Fractional Model Predictive Control (FMPC) and fractional-order Proportional Integral controller ($PI^{\lambda}$) on a thermal system with time delay.The first controller is based on Grünwald-Letnikov's method to predict the future dynamic behavior of the system. This method consists in replacing the non-integer derivation operator of the adopted system representation by a discrete approximation. Therefore, this controller is developed on the basis of a fractional order model. However, the second controller is founded on an extended version of Hermite-Biehler theorem to determine the complete set stabilizing $PI^{\lambda}$ parameters Experiment results onto a time delay thermal system are given to illustrate the effectiveness of the developed strategies.

Keywords: Fractional Model Predictive Control, Grünwald-Letnikov's method, fractional order PI controller, Hermite–Biehler theorem, Delaythermal system

## 1. INTRODUCTION

Fractional order calculus is a mathematical discipline with a 300-years-old history [1]. In recent years, it has attracted the attention of researchers in several fields such as engineering, biology, economics [2-6]. The non-integer order system appears also in the process industries, in particular through control application [7, 8]. The idea of using the fractional order regulator to control the dynamic systems was proposed by Oustaloup in 1988 [9, 10]. In 1994, Podlubny proposed the fractional order PID controller using mainly integrals and derivatives of non-integer order [1]. In [11], the authors proposed an approximation method for fractional order controllers using the state-space realization. In [12], the optimal control problems of non-integer order systems were proposed. Vinagre et al. used the fractional calculus in classic model reference adaptive control [13] In [14], two methods of fractional order proportional integral controllers of non-integer systems were studied. It was found that many physical systems have shown a dynamic behavior of fractional order, the first dynamic physical system to be widely recognized is the thermal systems. In Malti et al. [15], a thermal system is identified using a fractional order model. In [16], authors proposed an approach to the modeling of thermal systems and their identification by a fractional order model. Moreover, the presence of the integer derivation operator in the thermal system model can lead to instability of the controller or to poor closed-loop performances. Therefore, Stéphane et al. [17], have presented a robust path tracking using flatness for fractional thermal systems.

Moreover, the systems with time delay are widely encountered in the industrial processes [18, 19]. Therefore, several tuning methods have been developed for the setting of $PI^{\lambda}$ controller. In [20], the authors propose anew $PI^{\lambda}$ tuning method for first order systems with time delay. Some results on the control of integrating systems with time delay using fractional order PD controllers were obtained. Recently, Monje et al. [21] give a

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new tuning method called F-MIGO for $PI^\lambda$ extended from the MIGO method. These tuning rules are used to determine the best fractional and the best $PI^\lambda$ gains. In [22], the authors propose two sets of tuning rules for fractional PID similar to those of the first set of Ziegler–Nichols. A frequency approach for the auto-tuning of fractional-order PID is proposed in [23], where PI is used to cancel the slope of the curve phase of a position servo system with time delay around a frequency point and the $PI^\lambda D^\mu$ controller is designed to fulfill the specifications of gain crossover frequency.

Therefore, the Model Predictive Control (MPC) has become a mature control strategy over the last few years because it can take in account explicitly different types of constraints on input and output signals. It can handle a large class of systems especially the delayed systems [24]. Consequently, there are recently works which applied the predictive control to the fractional order system [25, 26, 27]. The MPC is a control technique that optimizes a cost function by using a model to predict the future behavior of process output. Indeed, the presence of the model is necessary for the development of the predictive control.

The originality of this work lies in applying both fractional order controllers to a thermal system with time delay. The first controller is the Fractional Model Predictive Control (FMPC), which is based on Grünwald-Letnikov's method to predict the future dynamic behavior of the system. The second controller is the Fractional Order proportional integral ($PI^\lambda$), this controller is founded on an extended version of Hermite-Biehler theorem to determine the complete set stabilizing$PI^\lambda$ parameters.

The outline of this paper is organized as follows. In section 2, a problem formulation and some definitions of fractional order systems are introduced, and the G-L definition used to approximate the fractional order system is detailed. The steps needed to find the optimal control law of the FMPC to fractional systems are introduced in section 3. The section 4 is reserved to focuses on the necessary steps in finding the design method proposed for the $PI^\lambda$. Experimental results on a thermal system are exhibited in section 4 to illustrate the effectiveness of the both controller proposed. Finally, a conclusion is given.

## 2. PRELIMINARY AND PROBLEM FORMULATION

The fractional calculus is a generalization of integration and derivation to the fractional order fundamental operators ${}_{t0}D_t^\alpha$ which is defined as:

$$ {}_{t0}D_t^\alpha = \begin{cases} \frac{d^\alpha}{dt^\alpha} & \alpha > 0 \\ 1 & \alpha = 0 \\ \int_{t0}^{t} (d\tau)^\alpha & \alpha < 0 \end{cases} \tag{1} $$

where $\alpha \in R$ is the order, $t_0$ and $t$ are the limitations. The value $\alpha$ can be negative or positive, corresponding to the integration and differentiation respectively.

In the development of fractional order calculus, there are several definitions of non-integer order [28]. The Grünwald-Letnikov's definition is the most known one for the fractional order control and its application [29-30], it has defined as:

$$ {}_{t0}D_t^\alpha f(t) = \lim_{h\to 0} \frac{1}{h^\alpha} \sum_{i=0}^{(t-t_0)/h} (-1)^i \binom{\alpha}{i} f(t-ih) \tag{2} $$

with $h$ being the sampling period and $\binom{\alpha}{i}$ meaning:

$$ \binom{\alpha}{i} = \frac{\alpha(\alpha-1)\dots(\alpha-i+1)}{i} $$

The relation (2) may be used to numerically evaluate the integral or the derivative of the non-integer order using some suitably chosen value of the sampling rate as follows [30].

$$ {}_{t_0}D_t^\alpha f(t) = \frac{1}{h^\alpha} \sum_{i=0}^{(t-t_0)/h} (-1)^i \binom{\alpha}{i} f(t-ih) \tag{3} $$

As $\binom{\alpha}{i}$ does not converge rapidly when $\alpha$ is fractional, the fractional operators are known to have a long memory behavior. For real implementation, by using the short memory principle [1], expression (3) can be rewritten using only the recent past values of $f(t)$ as:

$$ {}_{t_0}D_t^\alpha f(t) = \frac{1}{h^\alpha} \sum_{i=0}^{N} (-1)^i \binom{\alpha}{i} f(t-ih) \tag{4} $$

where $N$ is an integer.

Generally, a fractional model can be described by a fractional differential equation characterized by the following form:

$$ \sum_{l=0}^{L} a_l D_t^{\alpha_{a_l}} y(t) = \sum_{m=0}^{M} b_m D_t^{\alpha_{b_m}} u(t) \tag{5} $$

where $(a_l, b_m) \in R^2$, and $(\alpha_{al}, \alpha_{bm}) \in R_+^2$
The use of the numerical approximation (4), allows rewriting equation (5) as follows [31].

$$ \begin{aligned} y(k) = {} & \frac{1}{\sum_{l=0}^{L} \frac{a_l}{h^{\alpha_{a_l}}}} \sum_{m=0}^{M} \frac{b_m}{h^{\alpha_{b_m}}} \sum_{i=0}^{N} (-1)^i \binom{\alpha_{b_m}}{i} u(k-i) \\ & - \frac{1}{\sum_{l=0}^{L} \frac{a_l}{h^{\alpha_{a_l}}}} \sum_{l=0}^{L} \frac{a_l}{h^{\alpha_{a_l}}} \sum_{i=1}^{N} (-1)^i \binom{\alpha_{a_l}}{i} y(k-i) \end{aligned} \tag{6} $$

In reality, this equation is described with time delay, which means that the dynamic behavior of many industrial plants can be mathematically described by first order time delay systems [19].

The presence of time delay in the fractional order system model can lead the controller to be unstable or to have poor closed loop performances. In order to robustify the controller against the system with time delay and to handle a large class of systems, we will propose both fractional order controllers, which are the FMPC and $PI^\lambda$.

## 3. FRACTIONAL MODEL PREDICTIVE CONTROL DESIGN

In this section, we introduce the steps needed to find the optimal control law using the new proposed approach of FMPC for the fractional order systems with time delay. Consequently, the Grünwald-Letnikov's method of fractional order system represented in section 2 will be used to obtain the fractional order model. Therefore, it is assumed that this fractional system is described by the model given by relation (6).

The principle of predictive control is based on the following calculations, which are performed at each sampling instant:

- Using the model to calculate the predictor $\hat{y}(k+j/k),\ j \in [1, Hp]$, $Hp$ is the prediction horizon.
- Calculation of the future controls sequence $u(k+j), j \in [0,\ Hc-1]$. $Hc$ is the control horizon.
- Only the first control signal $u(k)$ of the optimized sequence is applied to the process.

There are several approaches of model predictive control employing different models to represent the relation between output and input of the system. Predictive control involves optimization of a cost function which indicates how well the process follows the desired trajectory. This function can be expressed by the future errors between setpoints and output signals, and the future incremental control signals. The cost function is given by:

$$J = \sum_{j=H_i}^{Hp} (\hat{y}(k+j/k) - yc(k+j))^2 + \lambda \sum_{i=0}^{Hc-1} \Delta u(k+i)^2 \tag{7}$$

When $H_i$ is the initial horizon, the predicted outputs $\hat{y}(k+j/k)$ are expressed in terms of the future control sequence, system output and control past measurements, the future control sequence is obtained by minimizing the cost function defined above:

$$\min J(\Delta U) \tag{8}$$

The initial horizon is chosen such that the product $(H_i * h)$ is equal to the delay of the system. Indeed, if the system has a delay $d$, it is useless to choose $H_i$ less than this delay, since the output only begins to react after this time [32].

For obvious reasons and without loss of generality, we will express $y(k)$ in terms of $u(k-1)$, depending on the input deviation. So, the expression (6) becomes:

$$\Delta y(k) = \frac{1}{\sum_{l=0}^{L} \frac{a_l}{h^{\alpha_{a_l}}}} \sum_{m=0}^{M} \frac{b_m}{h^{\alpha_{b_m}}} \sum_{i=0}^{N} (-1)^i \binom{\alpha_{b_m}}{i} \Delta u(k-1-i) - \frac{1}{\sum_{l=0}^{L} \frac{a_l}{h^{\alpha_{a_l}}}} \sum_{l=0}^{L} \frac{a_l}{h^{\alpha_{a_l}}} \sum_{i=1}^{N} (-1)^i \binom{\alpha_{a_l}}{i} \Delta y(k-i) + e(k) \tag{9}$$

where $\Delta = 1 - q^{-1}$ is an integral action introduced in order to obtain, in a closed loop, a nil steady state error.

By using the relation (9), we obtain the predicted output of the system in $k+1$:

$$\hat{y}(k+1/k) = y_l(k+1) + \alpha_1 \Delta u(k) \tag{10}$$

where: $\alpha_1 = \frac{1}{\sum_{l=0}^{L} \frac{a_l}{h^{\alpha_{a_l}}}} \sum_{m=0}^{M} \frac{b_m}{h^{\alpha_{b_m}}}$ and $y_l(k+1)$ is the free response of the system:

$$y_l(k+1) = y(k) + s_1 - s_2 \tag{11}$$

$$s_1 = \frac{1}{\sum_{l=0}^{L} \frac{a_l}{h^{\alpha_{a_l}}}} \left( \sum_{m=0}^{M} \frac{b_m}{h^{\alpha_{b_m}}} \sum_{i=1}^{N} (-1)^i \binom{\alpha_{b_m}}{i} \Delta u(k-i) \right)$$

$$s_2 = \frac{1}{\sum_{l=0}^{L} \frac{a_l}{h^{\alpha_{a_l}}}} \left( \sum_{l=0}^{L} \frac{a_l}{h^{\alpha_{a_l}}} \sum_{i=1}^{N} (-1)^i \binom{\alpha_{a_l}}{i} \Delta y(k+1-i) \right)$$

The 2-step ahead predictor is given by:

$$\hat{y}(k+2/k) = y(k+1) + \alpha_1 \Delta u(k+1) + \beta_1 \Delta u(k) + \beta_2 \Delta y(k+1) + s_3 - s_4 \tag{12}$$

where:

$$\beta_1 = \frac{-1}{\sum_{l=0}^{L} \frac{a_l}{h^{\alpha_{a_l}}}} \sum_{m=0}^{M} \frac{b_m}{h^{\alpha_{b_m}}} \alpha_{b_m};\ \beta_2 = \frac{1}{\sum_{l=0}^{L} \frac{a_l}{h^{\alpha_{a_l}}}} \sum_{l=0}^{L} \frac{a_l}{h^{\alpha_{a_l}}} \alpha_{a_l}$$

$$s_3 = \frac{1}{\sum_{l=0}^{L} \frac{a_l}{h^{\alpha_{a_l}}}} \left( \sum_{m=0}^{M} \frac{b_m}{h^{\alpha_{b_m}}} \sum_{i=2}^{N} (-1)^i \binom{\alpha_{b_m}}{i} \Delta u(k+1-i) \right)$$

$$s_4 = \frac{1}{\sum_{l=0}^{L} \frac{a_l}{h^{\alpha_{a_l}}}} \left( \sum_{l=0}^{L} \frac{a_l}{h^{\alpha_{a_l}}} \sum_{i=2}^{N} (-1)^i \binom{\alpha_{a_l}}{i} \Delta y(k+2-i) \right)$$

as: $\Delta y(k+1) = y(k+1) - y(k)$

then:

$$\hat{y}(k+2/k) = (1+\beta_2) y(k+1) + \alpha_1 \Delta u(k+1) + \beta_1 \Delta u(k) - \beta_2 y(k) + s_3 - s_4 \tag{13}$$

If were place $\hat{y}(k+1/k)$ by its expression (10), we obtain:

$$\hat{y}(k+2/k) = (1+\beta_2)\, y_l(k+1) + \alpha_2 \Delta u(k) + \alpha_1 \Delta u(k+1) - \beta_2 y(k) + s_3 - s_4 \tag{14}$$

where: $\alpha_2 = ((1+\beta_2)\,\alpha_1 + \beta_1)$
We set: $y_l(k+2) = (1+\beta_2)\, y_l(k+1) - \beta_2 y(k) + s_3 - s_4$

then: $$\hat{y}(k+2/k) = y_l(k+2) + \alpha_1 \Delta u(k+1) + \alpha_2 \Delta u(k) \tag{15}$$

Consequently, the expression of the j-step ahead predictor $\hat{y}(k+j/k)$, $(j \geq 1)$ is as follows:

$$\hat{y}(k+j/k) = \sum_{i=1}^{j} \alpha_{j-i+1} \Delta u(k+i-1) + y_l(k+j) \tag{16}$$

The output sequence on $Hp$ prediction horizon can be written as follows:

$$Y = G \Delta U + Y_l$$
$$Y = [\widehat{y}(k+1/k), \ldots, \hat{y}(k+Hp/k]^T \tag{17}$$

where: $\Delta U = [\Delta u(k), \ldots, \Delta u(k + Hc - 1)]^T$

$$Y_l = [y_l(k + 1), \ldots, y_l(k + Hp)]^T$$

The G matrix is illustrated as follows:

$$G = \begin{bmatrix} \alpha_1 & & \\ \alpha_2 \; \alpha_1 & & \\ \vdots & & \\ \alpha_{Hp} \cdots \alpha_{Hp-Hc+1} & & \end{bmatrix}; \; \dim(G) = (Hp, Hc)$$

Hence, the cost function of equation (7) is equivalent to:

$$J = (G\Delta U + Y_l - Y_c)^T (G\Delta U + Y_l - Y_c) + \lambda \Delta U^T \Delta U \quad (18)$$

where $\lambda$ is the weighting factor, $Y_c$ is the sequence of set-points on $Hp$ prediction horizon:

Minimizing the equation (18), we obtain the optimal control sequence.

$$\Delta U = [G^T G + \lambda I]^{-1} G^T [Y_c - Y_l] \quad (19)$$

## 4. FRACTIONAL-ORDER PI DESIGN

The aim of this section is to present the system which will be controlled by a fractional order PI controller and to present the design of the fractional controller. The first order time delay systems can be described by:

$$G(s) = \frac{K}{1 + Ts} e^{-Ls} \quad (20)$$

Our tuning strategy, is based on Hermite-Biehler theorem and the Pontryagin condition to determine the $k_p$ and $k_i$ parameters.

The fractional $PI^\lambda$ controller transfer function C(s) is given by the following equation:

$$C(s) = K_p + \frac{K_i}{s^\lambda} \quad (21)$$

The control input of the $PI^\lambda$ controller is:

$$u(t) = K_p (r(t) - y(t)) + K_i D_t^{-\alpha} (r(t) - y(t)) \quad (22)$$

Where r(t) is the reference input or the setpoint signal, e(t) is the error, u(t) is the control, y(t) is the output signal and $D_t^{-\alpha}$ is the fractional differential/integral operators.

The control design method proposed in this paper is based on a Hermite-Biehler and Pontryagin theorem which consist on interlacement property of the real roots of the polynomial characteristic.

The closed-loop characteristic polynomial of a first order time delay system is given by:

$$\begin{aligned} \delta^*(s) &= e^{Ls}\delta(s) \\ &= (KK_i + KK_p s^\lambda) + (1 + Ts)s^\lambda e^{Ls} \end{aligned}$$

Replacing the term "Ls" in the previous expression by "z" we obtain:

$$\delta^*(z) = KK_i + KK_p \left(\frac{z}{L}\right)^\lambda + \left(1 + T\left(\frac{z}{L}\right)\right)\left(\frac{z}{L}\right)^\lambda e^z \quad (23)$$

We consider: $z = jw$

$$\begin{aligned} \delta^*(j\omega) &= KK_i + KK_p(\frac{j\omega}{L})^\lambda + \left(1 + T\left(\frac{j\omega}{L}\right)\right)\left(\frac{j\omega}{L}\right)^\lambda \\ &\quad * (\cos(\omega) + j\sin(\omega)) \\ \delta^*(j\omega) &= \left(KK_i + KK_p\left(\frac{j\omega}{L}\right)^\lambda\right) + \left(\frac{j\omega}{L}\right)^\lambda \\ &\quad * \left(\cos(\omega) - \frac{T}{L}\omega\sin(\omega) + j\left(\frac{T}{L}\omega\cos(\omega) + \sin(\omega)\right)\right) \end{aligned}$$

This expression can be rewritten by:

$$\delta^*(j\omega) = \delta_r^*(\omega) + j\delta_i^*(\omega) \quad (24)$$

Where:

$$\begin{aligned} \delta_r^*(\omega) &= KK_i + \left(KK_p + \cos(\omega) - \frac{T}{L}\omega\sin(\omega)\right)\frac{|\omega|^\lambda}{(L)^\lambda} \\ &\quad - \left(\frac{T}{L}\omega\cos(\omega) + \sin(\omega)\right)\left|\mathrm{Im}\{(j)^\lambda\}\right|\frac{|\omega|^\lambda}{(L)^\lambda} sign(\omega) \\ \delta_i^*(\omega) &= \left(KK_p + \cos(\omega) - \frac{T}{L}\omega\sin(\omega)\right)\left|\mathrm{Im}\{(j)^\lambda\}\right| \\ &\quad \times \frac{|\omega|^\lambda}{(L)^\lambda} sign(\omega) + \left(\frac{T}{L}\omega\cos(\omega) + \sin(\omega)\right) \\ &\quad \times \left|real\{(j)^\lambda\}\right|\left(\frac{|\omega|}{L}\right)^\lambda \end{aligned}$$

Clearly, the controller parameter $k_p$ only affects the imaginary part of $\delta_i^*(\omega)$ whereas both parameters $k_p$ and $k_i$ appear in the real part $\delta_r^*(\omega)$. In order to solve our stabilization problem, we need first to determine the range of $k_p$ for which a solution to the $PI^\lambda$ stabilization problem of a closed-loop stable plant is given. According to Pontryagin Theorem, $\delta_i^*(\omega)$ has only real roots for every $K_p \in [K_{u\_}, K_{u_+}]$
where $K_{u\_}$ and $K_{u_+}$ are respectively the lower and the upper bound of $k_p$ range.

The successive step is to establish the ranges of the values of $k_p$ and $k_i$ that fulfill the interlacing condition between the roots of $\delta_i^*(\omega)$ and $\delta_r^*(\omega)$.

However, we present our theorem [33], which is useful to compute the stability region of a first order system with time delay. Based on the first property of Hermite-Biehler [34] which consist that all the roots of the polynomial characteristic of the closed loop equation are real.

**Theorem 1 (33)** We consider a first order plant given by the following transfer function:

$$G(s) = \frac{K}{1 + Ts} e^{-Ls}$$

where the parameters T, L and K are positive.

We can determine the set of all stabilizing $(k_p, k_i)$ values for the given plant using the fractional order controller $PI^\lambda$

$$C(s) = K_p + \frac{K_i}{s^\lambda}$$

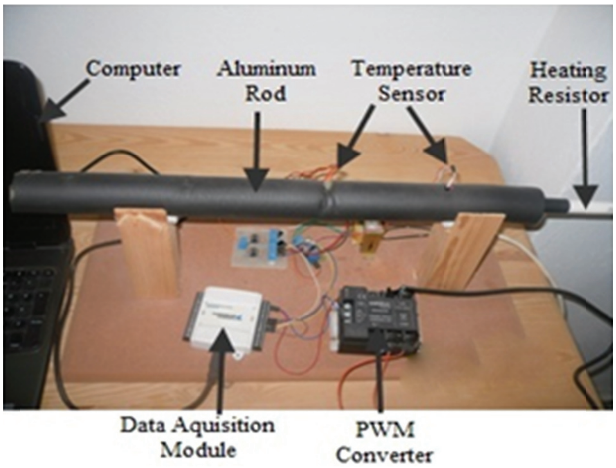


**Figure 1** Real schema of thermal system.

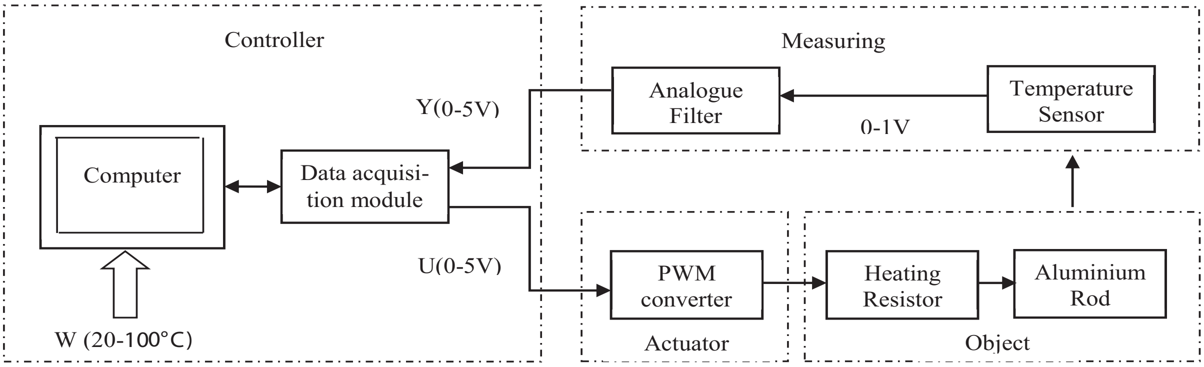


**Figure 2** Synoptic schema of thermal system.

The stabilizing set of parameters $k_p$ values for a closed-loop stable plant is given by:

$$\max\left(-\frac{1}{K}, K_{u_-}\right) < K_p < K_{u_+}$$

Where:

$$K_{u_-} = \frac{1}{K}\left(\left(\frac{T}{L}\alpha_1\cos(\alpha_1)+\sin(\alpha_1)\right)\frac{real\{(j)^\lambda\}}{\mathrm{Im}\{(j)^\lambda\}} - \left(\cos(\alpha_1)-\frac{T}{L}\alpha_1\sin(\alpha_1)\right)\right)$$

$$K_{u_+} = -\frac{1}{K}\left(\left(\frac{T}{L}\alpha_1\cos(\alpha_1)+\sin(\alpha_1)\right)\frac{real\{(j)^\lambda\}}{\mathrm{Im}\{(j)^\lambda\}} + \left(\cos(\alpha_1)-\frac{T}{L}\alpha_1\sin(\alpha_1)\right)\right)$$

$\alpha_1 \in [-\pi; 0]$ and $\alpha_2 \in [0; \pi]$ are respectively the solutions of the two previous equations:

$$\tan(\alpha_1) = -\frac{\frac{T}{L}\alpha_1\left|\mathrm{Im}\{(j)^\lambda\}\right| + (1+\frac{T}{L})\left|real\{(j)^\lambda\}\right|}{(1+\frac{T}{L})\left|\mathrm{Im}\{(j)^\lambda\}\right| - \frac{T}{L}\alpha_1\left|real\{(j)^\lambda\}\right|}$$

$$\tan(\alpha_2) = \frac{-\frac{T}{L}\alpha_2\left|\mathrm{Im}\{(j)^{\frac{a}{b}}\}\right| + (1+\frac{T}{L})\left|real\{(j)^{\frac{a}{b}}\}\right|}{(1+\frac{T}{L})\left|\mathrm{Im}\{(j)^{\frac{a}{b}}\}\right| + \frac{T}{L}\alpha_2\left|real\{(j)^{\frac{a}{b}}\}\right|}$$

Once the $k_p$ range established, we determine $k_i$ as follows:

$$\max_{j=0,2,4,\cdots}\{-m_jK_p - b_j\} < K_i < \min_{j=1,3,5,\cdots}\{-m_jK_p - b_j\}$$

Where:

$$m_j = m(\omega_j) = -\left|real\{(j)^\lambda\}\right|\frac{|\omega_j|^\lambda}{(L)^\lambda}$$

$$b_j = b(\omega_j) = -(\cos(\omega_j) - \frac{T}{L}\omega_j\sin(\omega_j))\left|real\{(j)^\lambda\}\right|\frac{|\omega_j|^\lambda}{(L)^\lambda} + \left(\frac{T}{L}\omega_j\cos(\omega_j)+\sin(\omega_j)\right)\left|\mathrm{Im}\{(j)^\lambda\}\right|\frac{|\omega_j|^\lambda}{(L)^\lambda}sign(\omega_j)$$

$w_j,\ j = 1, 2, 3\ldots$ are the roots, arranged in ascending order of magnitude, of $\delta_i^*(\omega)$.

## 5. EXPERIMENT RESULTS

In order to illustrate the effectiveness and performances of the both fractional order controllers developed in this paper, we have considered a thermal system depicted in figure 1. The synoptic schema of thermal system is depicted in figure 2. Indeed, the thermal flux throughout a metallic rod can be defined with the fractional order model [27-35].

**Table 1** Parameters of open loop system.

| u (v) | 1.5 | 2 | 2.5 | 3 | 4 |
|---|---|---|---|---|---|
| K | 0.312 | 0.518 | 0.718 | 0.77 | 0.712 |
| T(sec) | 2250 | 1800 | 2260 | 2150 | 2050 |
| D(sec) | 120 | 110 | 105 | 100 | 70 |

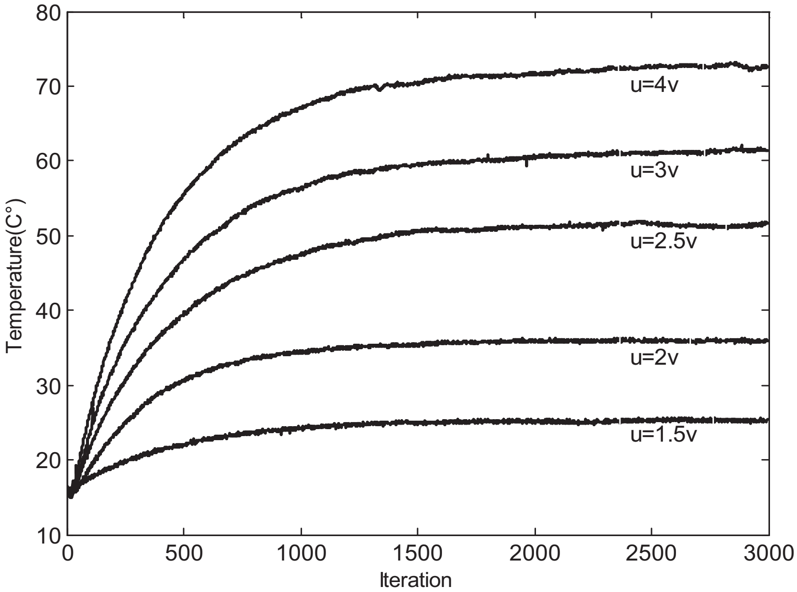


**Figure 3** Open loop step responses.

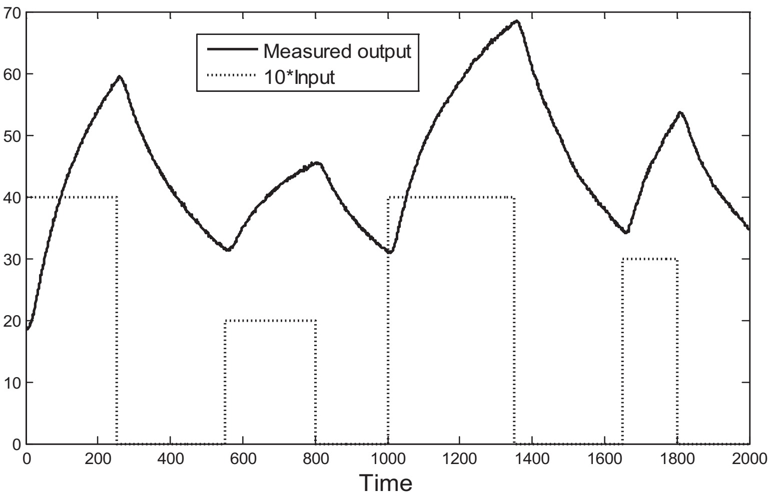


**Figure 4** Identification data.

## 5.1 Modeling and Identification

The fractional thermal system is composed by an metallic rod of 2 cm section and 41 cm length. The input signal of this thermal system is a thermal flux (Q) which is generated by a heating resistor. The output of this system is the cylinder temperature measured with a distance ‘d’ from the heated surface by an LM35DZ sensor. The sensor signal is amplified to obtain an output voltage varying from 0 to 5v. Several researchers are modeling the thermal system by a fractional order model Cois [36].

To determine the thermal system model, we have applied various step input signals to the thermal system with different amplitudes. The open loop step responses obtained are depicted in figure 3.

The table 1 shows the response time ($T$), the time delay ($D$) and the static gain ($K$) for each step input signal. The ambient temperature of 15.4°C, corresponding to 0.77v.

Based on the results presented on table 1, we deduce that the time delay is about 100sec and the time response is about 2000sec.

To estimate the fractional order model of the thermal system, we have applied to the heating resistor the input sequence given by the figure 4. This last depicts also the evolution of the temperature, at distance of d = 15 cm of the cylinder extremity. For displaying reasons, we multiplied the input by 10. Based

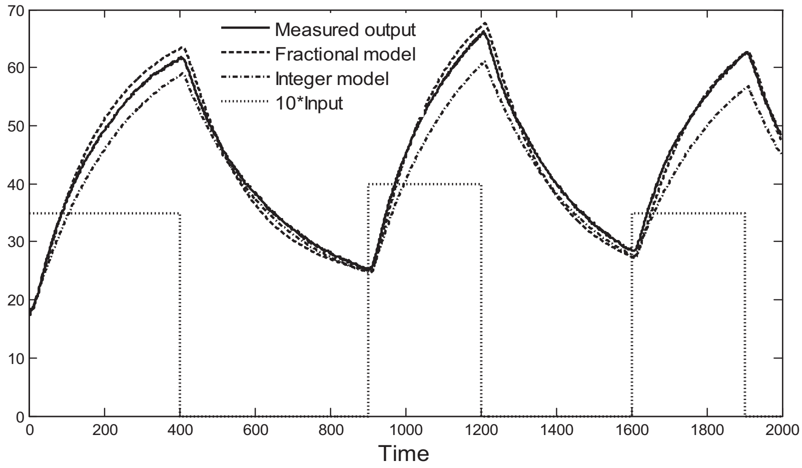


**Figure 5** Validation data.

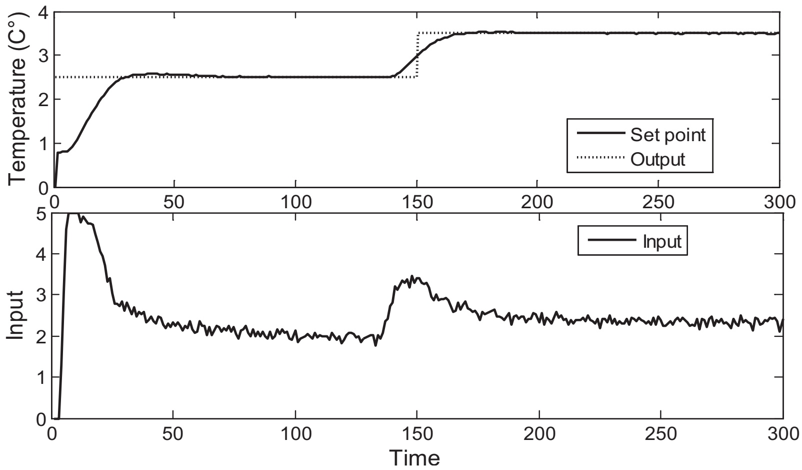


**Figure 6** Closed-loop results obtained with FMPC.

on this figure and the table 1, we notice that the thermal system having a delay of 100sec.

The fractional model is established from the data identification using the simplified refined instrumental variable for continuous-time fractional models (SRIVCF) method [37]:

$$H_1(s) = \frac{0.8623}{506.2843s^{1.5} + 135.3925s + 6.3598s^{0.5} + 1}e^{-100s} \tag{25}$$

For comparison purposes, we have identified the thermal system by an integer order model. Hence, by using the toolbox ident of Matlab, we were able to determine this model which is given by:

$$H_2(s) = \frac{0.712}{30.77s^2 + 236.84s + 1}e^{-100s} \tag{26}$$

In order to test the both models performances, we have performed another input excitation sequence and we have measured the corresponding temperature. As represented in the validation data of Figure 5, we deduce that the identified fractional model is closer to the measured output than to the integer model. The Normalized Mean Squared Error (NMSE) [38] computed on validation data for the fractional order model is $NMSE(H_1) = 8.8 \times 10^{-4}$, whereas for the integer model is $NMSE(H_2) = 5.7 \times 10^{-3}$. Therefore, the fractional identification is more adapted than integer one with this thermal system.

## 5.2 Controller Design

The goal is to maintain the temperature of the system measured at 15cm from the heated surface by adjustment the power of the heating resistor obtained with application of the FMPC and PI developed in this paper.

In all experiences, we have used the fractional model given by equation (24), the sample time is equal to 20sec and the control signal is limited between the following values:

$$0 \le u(k) \le 5v$$

In the case of FMPC proposed in this paper, the j-step ahead prediction is expressed by equation (9) with the following parameters:

$$\begin{cases} h = 0.1;\ L = 3;\ M = 0;\ b_0 = 0.8623 \\ a_0 = 1;\ a_1 = 6.3598; a_2 = 135.3925;\ a_3 = 506.2843 \\ \alpha_{b_0} = 0;\ \alpha_{a_0} = 0;\ \alpha_{a_1} = 0.5;\ \alpha_{a_2} = 1;\ \alpha_{a_3} = 1.5 \end{cases}$$

Figure 6 exhibits the measured temperature, the setpoint and the control signal when the proposed controller is designed with the following parameters: $Hp = 15,\ Hc = 1$ and $\lambda = 0.5$.

Based on practical results shown in figure 6 it is clear that the temperature follows the desired set points. Consequently, these results show good performances of the FMPC approach.

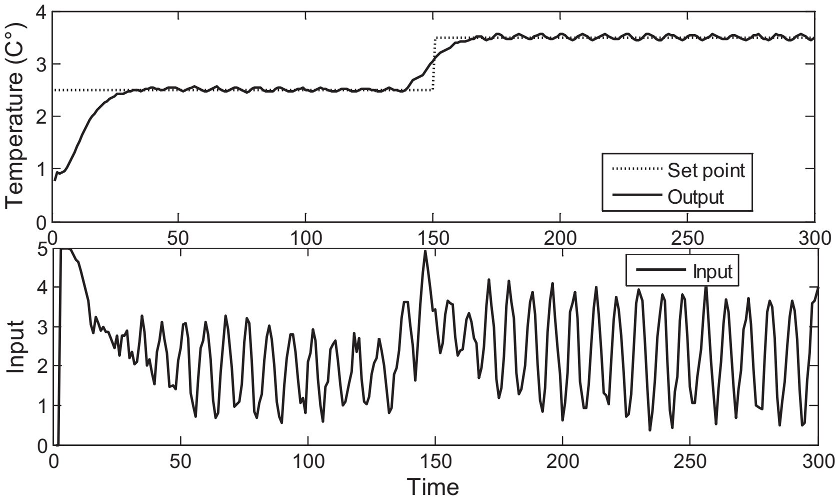


**Figure 7** Closed-loop results obtained with MPC.

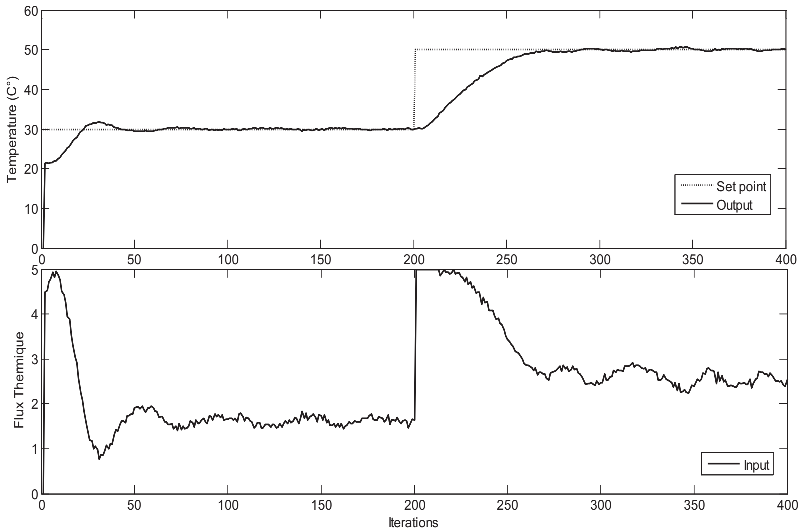


**Figure 8** Closed-loop results obtained with $PI^{\lambda}$

A comparison of the closed-loop performances of the proposed approach is established with a classic MPC based on the integer order model given by equation (26). Both predictive controllers are designed with the following parameters:

$$Hp = 15,\ Hc = 1 \text{ and } \lambda = 0.5$$

Figure 7 presents the outputs and control signals evaluation obtained by the predictive controller based on the integer order model. Based on this figure, we deduce that the measured temperature and the control signal present many fluctuations. Comparing the results obtained by the proposed FMPC and the MPC, we deduce that the first controller reaches the desired reference, whereas the second controller presents oscillations at the setpoint variations. We have also remarked that the control law obtained by the RFMPC is smoother the one obtained by FMPC.

To implement the fractional-order $PI^{\lambda}$ of the thermal system we have used the model given by equation (25). So, the thermal system is defined as a first order system with time delay. Therefore, we proceed the design of the controller by exploiting the approach exposed in section 4.

The designed $PI^{\lambda}$ parameters are fixed as follows:

$$K_p = 2,\ K_i = 0.6 \text{ and } \lambda = 0.9$$

The evolutions of the setpoint, the measured temperature (output signal) and the thermal flux (input signal) with $PI^{\lambda}$ controller are represented in Figure 8. Based on these results, we note that the measured temperature meets the desired requirements and the control signal obtained provides a small variation.

Comparing the results obtained by the FMPC and the PI fractional controller, we deduce that the both controller exhibits a good performance. But, we have also remark that the control law obtained by the FMPC is smoother than one obtained by fractional-order $PI^{\lambda}$.

## 6. CONCLUSION

In this paper, two fractional order controllers has been introduced to fractional order systems. The first controller is the Fractional Model Predictive Control (FMPC), which is based on the

Grünwald - Letnikov's definition. Therefore, the output deviation approach is used to design the j-step ahead output predictor and the control law is obtained by solving a quadratic cost function. The second controller is the Fractional Order proportional integral, this controller is tuned by our analytical method based on Hermite–Biehler theorem gives a strong performance. The experimental results on a thermal system show that the FMPC and PI exhibits a good performance

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