Computers, Materials & Continua CMC, vol.65, no.3, pp.1905-1924, 2020

Analysis and Dynamics of Illicit Drug Use Described by Fractional Derivative with Mittag-Leffler Kernel

Berat Karaagac[1, 2], Kolade Matthew Owolabi[1, 3, *] and Kottakkaran Sooppy Nisar[4]

Abstract: Illicit drug use is a significant problem that causes great material and moral losses and threatens the future of the society. For this reason, illicit drug use and related crimes are the most significant criminal cases examined by scientists. This paper aims at modeling the illegal drug use using the Atangana-Baleanu fractional derivative with Mittag-Leffler kernel. Also, in this work, the existence and uniqueness of solutions of the fractional-order Illicit drug use model are discussed via Picard-Lindelöf theorem which provides successive approximations using a convergent sequence. Then the stability analysis for both disease-free and endemic equilibrium states is conducted. A numerical scheme based on the known Adams-Bashforth method is designed in fractional form to approximate the novel Atangana-Baleanu fractional operator of order $0 < \alpha \leq 1$. Finally, numerical simulation results based on different values of fractional order, which also serve as control parameter, are presented to justify the theoretical findings.

Keywords: Atangana-Baleanu fractional operator, illicit drug use, existence and uniqueness of solutions, stability analysis.

1 Introduction

Illicit drug use is both a serious threat to human health and a common problem for the nations and the world due to material and moral losses, the decrease in national welfare, loss and misery of individual lives, serious infectious diseases, deaths, and the increase in crimes associated with addiction. Certain drugs are natural such as cannabis and cocaine or could be derived from natural substances such as heroin. Some may be synthetic such as amphetamines and various others. These drugs are classified into four main categories based on their effects: stimulants, depressants, opioids and hallucinogens. Stimulant substances improve attention, energy, blood pressure, pulse and respiratory rates. Depressant substances reduce pain and make the individuals feel sleepy. Furthermore,

[1] Faculty of Mathematics and Statistics, Ton Duc Thang University, Ho Chi Minh City, Vietnam.

[2] Department of Mathematics Education, Faculty of Education, Adıyaman University, Adıyaman, Turkey.

[3] Department of Mathematical Sciences, Federal University of Technology, Akure, Nigeria.

[4] Department of Mathematics, College of Arts and Sciences, Wadi Aldawaser, Prince Sattam Bin Abdulaziz University, Al Kharj, Saudi Arabia.

[*] Corresponding Author: Kolade Matthew Owolabi. Email: koladematthewowolabi@tdtu.edu.vn.

Received: 20 May 2020; Accepted: 03 August 2020.

these substances slow down pulse and respiration. Opioid substances slow down all physical functions and hallucinogens alter the consciousness leading to visions. The first impact of illicit drug use includes mental and nervous diseases such as insanity, early dementia, insomnia, paralysis and hallucination. It leads to nausea, vomiting, bleeding and injuries in the digestive tract. In the liver and kidneys, which play the most important role in the elimination of the harmful substances included in the drugs, failure and occlusions are observed. In the long term, Type B hepatitis, HIV/AIDS, and malaria could be observed. Some reasons for using illicit drugs might be to achieve the original feeling; eliminate the sense of loneliness or cope with peer and family problems. Another reason, especially in the case of adolescents and young adults, is peer pressure, as illicit drug use could be observed in individuals with a peer circle where drug use or drug use attempts are common. The presence of genetic predisposition could lead to drug abuse of between 30% and 70%.

Although drug abuse was a personal decision in western countries until the late 19th century. In 1961, the United Nations Single Convention on Narcotic Drugs prohibited the production and procurement of certain narcotic drugs unless licensed for special purposes such as medical treatments and scientific research (Single Convention on Narcotic Drugs, 1961, Convention on Psychotropic Substances, 1971, Convention against the Illicit Traffic in Narcotic Drugs and Psychotropic Substances, 1988). But, what about the history of illicit drugs use until about 58 years ago? Since the beginning of human history, individuals have tried to change their level of consciousness and sensual perceptions by ingesting herbs, alcohol and other substances. These substances have been used by clergies in religious ceremonies, for medical purposes and their use was socially approved by the majority of the society. More than 4000 years ago in Asia, the monks and shamans used Amanita muscaria, a red mushroom with white spots, often observed in fairy tales and famous cartoons, to trigger trance. For similar purposes, psilocybin mushrooms were used in Central America and the Aztecs in Pre-Columbian Mexico used Lophophora williamsii, a needleless cactus species with protrusions. The use of these plants slowly disappeared before the 1950s.

Certain drugs were used for therapeutic purposes in most human history. For instance, opium, the first opioid, was used by Egyptians and Persians around 3400 BC to treat warriors. After the 18th century, its use became prevalent in Europe, India and China in the treatment of cancer and labor pain. In the late 18th century, the addictive property of the drug was identified. In 1805, morphine and codeine were isolated from opium and used to treat opium dependency. The scientists aimed to produce heroin from morphine as a replacement, however, it was also found to be highly addictive. The social attitudes towards illicit drug use have changed dramatically over time. Although policies and penalties against drug use vary significantly between nations, there have been several attempts to legalize and prohibit illicit drug use. Past experiences, the damages encountered by the society and the individual due to drug abuse demonstrated that there are justified reasons for the prohibition of the use of such drugs today. According to the 2018 data published on the United Nations Office on Drugs and Crime (UNODC) official web site, more than 275 million people use drugs worldwide despite the knowledge that the adverse effects of drug abuse are more severe than previously claimed. Despite public research, education and social programs and legal practices against illicit drugs, permanent poverty, long-term unemployment and deterioration of living conditions, psychological problems, addiction-oriented factors have led to sudden increase in illicit drug use and trade.

Successful mathematical algorithms have observed real life problems and obtained exact solutions of the problems which define nature nature [Li and Zhu (2018); Chen, Xu, Zuo et al. (2019); Ghanbari, Osman and Baleanu (2019); Osman (2019)]. Similarly, mathematical models on illicit drug use assist us in understanding the phenomenon by assuming that real-life phenomena were accurately defined. Thus, it allows us to examine the behavior related to the phenomenon under different conditions using various parameters, and to observe the differences in the outcome. More importantly, with the help of constructed models, we can find answers to our 'if' scenarios. Research has shown that modeling of real-life phenomena with the concept of fractional calculus is most accurate and acceptable [Ghanbari, Kumar and Kumar (2020); Kheiri and Jafari (2019); Naik, Zu and Owolabi (2020); Owolabi and Atagana (2019b); Podlubny (1999); Yavuz and Bonyah (2019)]. This technique has been applied under various auspices with record of success [Karaagac (2018, 2019a, 2019b); Owolabi (2018a, 2018b, 2018c, 2018d, 2018e, 2019); Owolabi and Atangana (2018a, 2018b, 2018c, 2019a); Owolabi and Gómez-Aguilar (2018); Owolabi and Hammouch (2019)] and references therein. Standing on these various achievements of fractional derivatives, the illicit drug use model is defined with Atangana-Baleanu fractional derivative in this paper. The presence and uniqueness of the solution are demonstrated by the Picard-Lindelöf theorem which is significant for initial value problems in ordinary differential equations. The stability of the system is examined by considering the eigenvalues for the disease-free and endemic states. In the following sections, a numerical diagram is obtained for the solution of the system and numerical results are presented using tables and charts.

Let us consider the fractional illicit drug use model defined by Atangana–Baleanu fractional derivative given by an ordinary differential equation system:

$$\begin{aligned}
{}_a^{ABC}D_t^{\alpha} S(t) &= (1-S)\mu - \beta(E+\tau I)S \\
{}_a^{ABC}D_t^{\alpha} E(t) &= (\beta S - a - \gamma - \sigma - \mu - \nu)E + \beta\tau IS \\
{}_a^{ABC}D_t^{\alpha} I(t) &= aE - (\rho + \theta + \mu + d)I \\
{}_a^{ABC}D_t^{\alpha} R(t) &= \sigma E + \theta I - (\varepsilon + \mu + \delta)R \\
{}_a^{ABC}D_t^{\alpha} V(t) &= \gamma E + \rho I + \varepsilon R - (\mu + w)V
\end{aligned} \tag{1}$$

We let $N(t)$ depict the studied population, which is clustered into sub-classes

$N(t) = \{S(t), E(t), I(t), R(t), V(t)\}.$

Among these sub-classes, $S(t)$ depicts the individuals who do not abuse drugs, but are in contact with individuals who abuse drugs, $E(t)$ depicts the class of individuals with mild drug use or infrequent illicit drug use. $I(t)$ represents the group of individuals with severe illicit drug use. $R(t)$ represents the group of individuals who experienced psychological problems due to illicit drug use and $V(t)$ depicts the group of individuals who are identified as illicit drug users. The systemic parameters are described as follows:

μ is a constant number depicting the number of recruitment and non-illicit related death rate in the population within the time period t. β is the parameter that measures the strength of the interaction between sensitive individuals and illicit drug users. a is the rate of mild drug users who become heavy users in time. γ,ϵ,ρ are the detection and rehabilitation for the individuals in E,R,I. σ,θ are the rates of light and heavy users developing psychological (mental) illness. ν, d are the light and heavy users that exited the model due to quitting drug use or drug use-related death. δ includes mentally ill individuals who exited the model permanently by drug use-related death. w depicts the users who permanently exited the system due to rehabilitation recovery. τ is a modification factor; it explains a comparison of heavy drug users with light drug use affecting new drug users. The most important feature that distinguishes the present model from other models is the fact that individuals with psychological diseases are also included in the system [Mushayabasa and Tapedzesa (2015)]. The measurement of the impact of illicit drugs on psychological diseases, which are the most dangerous threats to society, is a significant point that needs to be noted here.

The remaining sections are arranged as follows: Some useful preliminaries based on the Atangana-Baleanu fractional operator are given in Section 2. The existence and uniqueness result of the main model is presented in Section 3. The main results involving stability analysis of the equilibria (drug-free equilibrium state and endemic equilibrium state) are reported in Section 4. Numerical experiments to justify the theoretical findings in some instances of fractional order are given to reveal the behavior of dynamics components. The conclusion is finally drawn in the last section

2 Preliminaries

In this section, some definitions and properties related to fractional calculus are reported. In the process, we are going to refer to the following given specific definitions and properties of the Atangana-Baleanu fractional derivatives of Caputo type [Atangana and Baleanu (2016)] that are peculiar to this study.

Definition 2.1 *The left- and right-sided Caputo fractional derivative for order* $\alpha > 0$ *is respectively defined as*

$$ {}_{a}^{C}D_{t}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{a}^{t} (t-\xi)^{n-\alpha-1} f^{(n)}(\xi)\, d\xi \quad (left), $$

$$ {}_{a}^{C}D_{t}^{\alpha} f(t) = \frac{(-1)}{\Gamma(n-\alpha)} \int_{t}^{b} (t-\xi)^{n-\alpha-1} f^{(n)}(\xi)\, d\xi \quad (right) $$

where $n-1 < \alpha \leq n, n \in N, f \in C^{n-1}[0,t]$.

Definition 2.2 *The left- and right-side of the Atangana-Baleanu fractional order derivative for a given function f(t) in the sense of Caputo is defined as*

$$ {}_{a}^{ABC}D_{t}^{\alpha} f(t) = \frac{B(\alpha)}{1-\alpha} \int_{a}^{t} \frac{df(\xi)}{d\xi} E_{\alpha}\left[-\frac{\alpha}{1-\alpha}(t-\xi)^{\alpha}\right] d\xi \quad (left), $$

$$ {}_{a}^{ABC}D_t^{\alpha} f(t) = -\frac{B(\alpha)}{1-\alpha}\int_t^b E_{\alpha}\left[-\frac{\alpha}{1-\alpha}(t-\xi)^{\alpha}\right] d\xi \quad (right) $$

where $B(\alpha)=(1-\alpha)+\frac{\alpha}{\Gamma(\alpha)}$ is a normalization function and $E_{\alpha}(\cdot)$ is the Mittag-Leffler function.

Definition 2.3 *Atangana-Baleanu fractional integral order* α *is defined as*

$$ {}^{AB}I_t^{\alpha}(f(t)) = \frac{1-\alpha}{B(\alpha)} f(t) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_a^t f(\xi)(t-\xi)^{\alpha-1} d\xi , $$

if $f(t)$ is a constant, integral will be zero.

3 Existence and uniqueness

In general, a first order ordinary differential equation given with the initial value problem is as follows:

$$ \frac{dy}{dt} = f(y,t), \quad y(t_0) = y_0 \tag{2} $$

There might be no solution for the differential equation given in Eq. (2), it might have an infinite solution, or it might have a single solution. Researchers who are interested in such problems look for answers to two questions: Under which conditions does the solution to problem (2) exist (question of existence)? Under which conditions is the solution unique (the question of uniqueness)? Hence, we provide answers to these two questions under this section. We study the illicit drug use model which is formulated with the Caputo fractional derivative.

In what follows, we construct the dynamic system given in Eq. (1) as

$$ \begin{aligned} {}_0^{ABC}D_t^{\alpha} S(t) &= f_1(t,S,E,I,R,V) = (1-S)\mu - \beta(E+\tau I)S \\ {}_0^{ABC}D_t^{\alpha} E(t) &= f_2(t,S,E,I,R,V) = (\beta S - a - \gamma - \sigma - \mu - \nu)E + \beta\tau IS \\ {}_0^{ABC}D_t^{\alpha} I(t) &= f_3(t,S,E,I,R,V) = aE - (\rho + \theta + \mu + d)I \\ {}_0^{ABC}D_t^{\alpha} R(t) &= f_4(t,S,E,I,R,V) = \sigma E + \theta I - (\varepsilon + \mu + \delta)R \\ {}_0^{ABC}D_t^{\alpha} V(t) &= f_5(t,S,E,I,R,V) = \gamma E + \rho I + \varepsilon R - (\mu + w)V \end{aligned} \tag{3} $$

and

$X(t) = (S,E,I,R,V), F(X(t),t) = (f_1(X(t)), f_2(X(t)), f_3(X(t)), f_4(X(t)), f_5(X(t)))$.

Considering the solutions of the system as continuous functions from a finite time-interval J to a bounded subset $U \in IR^5$, with the initial value $X(t_0) = X_0$ we set, $J = [t - t_0, t + t_0], U = \overline{R(x,r)}$ and

$C_{a,r_1} = J\times\left[x-r_1, x+r_1\right] = J\times R_1, \quad C_{a,r_2} = J\times\left[x-r_2, x+r_2\right] = J\times R_2,\ C_{a,r_3}$
$= J\times\left[x-r_3, x+r_3\right] = J\times R_3,$
$C_{a,r_4} = J\times\left[x-r_4, x+r_4\right] = J\times R_4, \quad C_{a,r_5} = J\times\left[x-r_5, x+r_5\right] = J\times R_5$

R_i are the closed balls in IR^m around x with radius r_i. We let $C_{a,r}\left(J,U\right)$ denote a set of continuous function from J to U equipped with the norms

$$M_1 = \sup_{C_{a,r_1}}\left\|f_1\left(t, X\left(t\right)\right)\right\|, \quad M_2 = \sup_{C_{a,r_2}}\left\|f_2\left(t, X\left(t\right)\right)\right\|,\ M_3 = \sup_{C_{a,r_3}}\left\|f_3\left(t, X\left(t\right)\right)\right\|,$$
$$M_4 = \sup_{C_{a,r_4}}\left\|f_4\left(t, X\left(t\right)\right)\right\|,\ M_5 = \sup_{C_{a,r_5}}\left\|f_5\left(t, X\left(t\right)\right)\right\|,\ M = \sup_{C_{a,r_5}}\left\|f_5\left(t, X\left(t\right)\right)\right\|$$

where supremum of all $M_1, M_2,\ M_3, M_4$ and M_5 values is M. Let us define a Picard operator T from $C_{a,r}\left(J,U\right)$ to $C_{a,r}\left(J,U\right)$ such as

$$T : C_{a,r}\left(J,U\right) \to C_{a,r}\left(J,U\right)$$
$$T\left(X\left(t\right)\right) = X\left(t_0\right) + \frac{1-\alpha}{B\left(\alpha\right)}F\left(X\left(t\right)\right) + \frac{\alpha}{B\left(\alpha\right)\Gamma\left(\alpha\right)}\int_{t_0}^{t}F\left(X\left(s\right)\right)\left(t-s\right)^{\alpha-1}ds,$$

Applying the Picard-Lindelöf Existence Uniqueness theorem to the illicit drug model with the initial value problem, results to the following theorem.

Theorem 3.1 *(Picard-Lindelöf) Suppose* $C_{a,r} : \left[t_0 - a, t_0 + a\right]\times U = \overline{R\left(x,r\right)} \to IR^m$ *is continuous and bounded by a constant* η. *Furthermore, that* $C_{a,r}$ *is Lipschitz continuous with Lipschitz constant* L *for every* $t\in\left[t_0 - a, t_0 + a\right]$. *Then initial value problem has a unique solution* $x\in C_{a,r}\left(J,U\right)$ *as long as the time-interval is chosen with a satisfying* $0 < a < \min\left(1/L, r/\eta\right)$.

Step1: The first aim is to show that the Picard operator T is a contraction on the space $C_{a,r}\left(J,U\right)$. We note that f_j functions are continuous because $C_{a,r}$ is Lipschitz. U is closed and bounded, so f_j take their min and max value on U. Therefore, $X(t)\in C_{a,r}\left(J,U\right)$

$$\left\|T\left(X(t)\right)-X\left(t_0\right)\right\|=\left\|\frac{1-\alpha}{B(\alpha)}F\left(X(t)\right)+\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_{t_0}^{t}F\left(X(s)\right)(t-s)^{\alpha-1}ds\right\|$$

$$\leqslant\left\|\frac{1-\alpha}{B(\alpha)}F\left(X(t)\right)\right\|+\left\|\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_{t_0}^{t}F\left(X(s)\right)(t-s)^{\alpha-1}ds\right\|$$

$$\leqslant\frac{1-\alpha}{B(\alpha)}\left\|F\left(X(t)\right)\right\|+\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_{t_0}^{t}\left\|F\left(X(s)\right)(t-s)^{\alpha-1}\right\|ds=\left(\frac{1-\alpha}{B(\alpha)}+\frac{\alpha T_{\max}^{\alpha}}{B(\alpha)\Gamma(\alpha)}\right)M$$

so, it remains that T maps $C_{a,r}(J,U)$ into itself.

Step 2: Consider two elements $X_1(t)$, $X_2(t)\in C_{a,r}(J,U)$, then

$$\left\|T\left(X_1(t)\right)-T\left(X_2(t)\right)\right\|=\left\|\frac{1-\alpha}{B(\alpha)}F\left(X_1(t)\right)+\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_{t_0}^{t}F\left(X_1(s)\right)(t-s)^{\alpha-1}ds\right.$$

$$\left.-\frac{1-\alpha}{B(\alpha)}F\left(X_2(t)\right)-\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_{t_0}^{t}F\left(X_2(s)\right)(t-s)^{\alpha-1}ds\right\|$$

$$\leqslant\frac{1-\alpha}{B(\alpha)}\left\|\left(F\left(X_1(t)\right)-F\left(X_2(t)\right)\right)\right\|$$

$$+\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_{t_0}^{t}\left\|\left(F\left(X_1(s)\right)-F\left(X_2(s)\right)\right)(t-s)^{\alpha-1}\right\|ds\leqslant\left(\frac{L}{\Gamma(n-\alpha)}+\frac{\alpha T_{\max}^{\alpha}}{B(\alpha)\Gamma(\alpha)}\right)L$$

4 Stability analysis

In this section, the analysis of equilibria of the illicit drug model is considered. The model given in Eq. (1) has two equilibrium points, one is the drug free equilibrium $\tilde{E}_0$ and the other is the endemic equilibrium $\tilde{E}_1$.

4.1 Drug-free equilibrium state

It is obvious that if drug abuse is absent in the community, the equilibrium for drug-free state requires the number of light or heavy drug users or the number of having mental disease people who suffer from mental illness due to illegal drug use and detected illicit drug users to be equal to zero. Drug free equilibrium is $\tilde{E}_0=\left(S_0,E_0,I_0,R_0,V_0\right)$. Thus, we get the basic reproductive number which is a matrix used to predict the potential of a disease spread in a population, following

$$\tilde{R}=\frac{\beta\left(d+\mu+\rho+a\tau+\theta\right)}{\left(a+\gamma+\mu+\nu+\sigma\right)\left(d+\mu+\rho+\theta\right)}.$$

When we solve the system given in Eq. (1) by setting each fractional differential equation equal to zero, drug-free equilibrium is obtained following
$\left(S_0, E_0, I_0, R_0, V_0\right) = (1,0,0,0,0)$.

The general Jacobian matrices for the illicit drug model is

$$J = \begin{bmatrix} -\mu-\beta(E+\tau I) & -\beta S & -\beta\tau S & 0 & 0 \\ \beta E+\beta\tau I & \mathfrak{M} & \beta\tau S & 0 & 0 \\ 0 & a & -(\rho+\theta+\mu+d) & 0 & 0 \\ 0 & \sigma & \theta & -(\varepsilon+\mu+\delta) & 0 \\ 0 & \gamma & \rho & \varepsilon & -(\mu+w) \end{bmatrix}$$

where $\mathfrak{M} = \beta S - a - \gamma - \sigma - \mu - \nu$ is the resulting Jacobian matrix in this case, and $\left(S_0, E_0, I_0, R_0, V_0\right) = (1,0,0,0,0)$

$$J\left(\tilde{E}_0\right) = \begin{bmatrix} -\mu & -\beta & -\beta\tau & 0 & 0 \\ 0 & \beta-a-\gamma-\sigma-\mu-\nu & \beta\tau & 0 & 0 \\ 0 & a & -(\rho+\theta+\mu+d) & 0 & 0 \\ 0 & \sigma & \theta & -(\varepsilon+\mu+\delta) & 0 \\ 0 & \gamma & \rho & \varepsilon & -(\mu+w) \end{bmatrix} \tag{4}$$

the characteristics polynomial of the matrices given in (4) is

$$\begin{aligned}&(-\lambda-\mu)(-w-\lambda-\mu)(-\delta-\varepsilon-\lambda-\mu)\\ &\times\left(-a\beta\tau+\left(-a+\beta-\gamma-\lambda-\mu-\nu-\sigma\left(-d-\lambda-\mu-\rho-\theta\right)\right)\right)=0.\end{aligned} \tag{5}$$

Solving this eigenvalue problem, we get the eigenvalues

$$\lambda_1 = -\mu, \lambda_2 = -w-\mu, \lambda_3 = -\delta-\varepsilon-\mu; \lambda_{4,5} = \beta-d-a-\gamma-2\mu-\nu-\rho-\sigma-\theta \mp \sqrt{\Delta}$$

where

$$\begin{aligned}\Delta &= d^2+a^2-2d\left(a+\gamma+\upsilon+\sigma-\beta-\rho-\theta\right)+2a\left(\gamma+\nu+\sigma+2\beta\tau-\beta-\rho-\theta\right)\\ &+\left(\beta+\rho+\theta-\gamma-\nu-\sigma\right)^2.\end{aligned}$$

Therefore, if $\beta < d+a+\gamma+2\mu+\nu+\rho+\sigma+\theta \mp \sqrt{\Delta}$, the drug-free equilibrium will be stable.

4.2 Endemic equilibria state

The endemic equilibrium state is the state where the drug use cannot be totally eradicated but remains in the population. In other words
$\tilde{E}_1 = \left(S^*, E^*, I^*, R^*, V^*\right) \neq (0,0,0,0,0)$. At the endemic equilibrium state, $\tilde{E}_1$, the Jacobian Matrix becomes

$$J(\tilde{E}_1)=\begin{bmatrix} -m_1 & m_1 & -m_2 & 0 & 0 \\ \dfrac{\mu\left(\beta-\gamma-\mu-v-\sigma+a\left(\dfrac{\beta\tau}{(d+\mu+\rho+\theta)}-1\right)\right)}{(a+\gamma+\mu+v+\sigma)} & \dfrac{a\tau(a+\gamma+\mu+v+\sigma)}{(d+\mu+\rho+a\tau+\theta)} & m_2 & 0 & 0 \\ 0 & a & -(\rho+\theta+\mu+d) & 0 & 0 \\ 0 & \sigma & \theta & -(\varepsilon+\mu+\delta) & 0 \\ 0 & \gamma & \rho & \varepsilon & -(\mu+w) \end{bmatrix}.$$

$$m_1=\frac{(a+\gamma+\mu+v+\sigma)(d+\mu+\rho+\theta)}{(d+\mu+\rho+a\tau+\theta)}, \text{ and}$$

$$m_2=\frac{\tau(a+\gamma+\mu+v+\sigma)(d+\mu+\rho+\theta)}{(d+\mu+\rho+a\tau+\theta)}.$$

The endemic equilibrium points of the illicit drug use model depend on the solutions of the characteristic equation of the Jacobian Matrix given in $J(\tilde{E}_1)$. If real parts of all eigenvalues are negative, then the equilibrium is stable. If at least one eigenvalue has a positive real part, then the equilibrium is unstable. The first two roots are $\lambda_1=-\mu-w,\ \lambda_2=-\delta-\varepsilon-\mu$. Computations to obtain the other eigenvalues from the characteristic equation are very complex. However, we can represent the eigenvalues of the matrices $J(\tilde{E}_1)$ graphically in Fig. 1 with the chosen value of parameters given in caption.

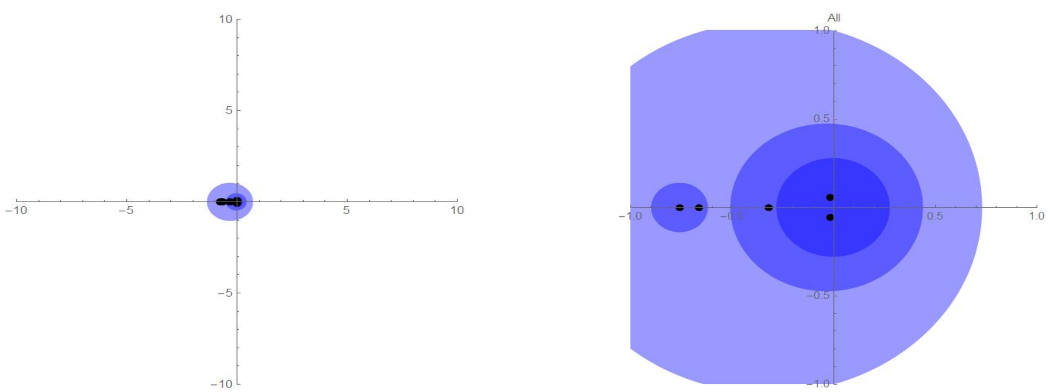

Figure 1: Eigenvalues of endemic state matrices $\tilde{E}_1$ for $w=0.3$; $\mu=0.02$; $\tau=1.25$; $\gamma=0.1$; $\rho=0.35$; $\varepsilon=0.6$; $a=0.01$ $v=0.035$; $\delta=0.14$; $d=0.2$; $\sigma=0.05$; $\theta=0.09$

5 Numerical method

In the present part of this paper, numerical results for fractional illicit drug use model are obtained. For this purpose, we will use the numerical scheme derived by Atangana et al. in [Atangana and Owolabi (2018)]. Considering the first equation of illicit drug use model, applying the fundamental theorem of calculus, one obtains

$$S(t)-S(t_0)=\frac{1-\alpha}{B(\alpha)}f_1(t,X)+\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_{t_0}^{t}f_1(t,X)(t-\xi)^{\alpha-1}d\xi,$$

where $t_0=0$. Thus at $t=t_{n+1}$ for $n=0,1,2,...$

$$S(t_{n+1})-S(0)=\frac{1-\alpha}{B(\alpha)}f_1(t_n,X_n)+\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^{t_{n+1}}(t_{n+1}-\xi)^{\alpha-1}f_1(t,X)dt, \tag{6}$$

if we rename the given integral such as

$$II_1=\int_0^{t_{n+1}}(t_{n+1}-\xi)^{\alpha-1}f_1(t,X)dt$$

where $n<\alpha<n+1$ and $\alpha\in(0,1]$. For calculating the integral given above, we need the Lagrange polynomials for polynomial interpolation;

$$P_k(t)=\frac{f_1(t_n,X_n)}{h}(t-t_{n-1})-\frac{f_1(t_{n-1},X_{n-1})}{h}(t-t_n).$$

Substituting polynomial $P_k(t)$ into II_1 and II_2

$$\begin{aligned}II_1&=\int_0^{t_{n+1}}(t_{n+1}-\xi)^{\alpha-1}\left(\frac{f_1(t_n,X_n)}{h}(t-t_{n-1})-\frac{f_1(t_{n-1},X_{n-1})}{h}(t-t_n)\right)dt\\&=\frac{f_1(t_n,X_n)}{h}\int_0^{t_{n+1}}(t_{n+1}-\xi)^{\alpha-1}(t-t_{n-1})dt-\frac{f_1(t_{n-1},X_{n-1})}{h}\int_0^{t_{n+1}}(t_{n+1}-\xi)^{\alpha-1}(t-t_n)dt\end{aligned}.$$

By using the transformation $y=t_{n+1}-\xi$, one obtains

$$\begin{aligned}II_1=\sum_{k=0}^{n}&\left\{\frac{h^\alpha f_1(t_n,X_n)}{\alpha(\alpha+1)}\Big(\big((n+1-k)^\alpha(n-k+2+\alpha)-(n-k)^\alpha(n-k+2+2\alpha)\big)\Big)\right.\\&\left.-\left(\frac{h^\alpha f_1(t_{n-1},X_{n-1})}{\alpha(\alpha+1)}\Big(\big((n+1-k)^{\alpha+1}-(n-k)^\alpha(n-k+1+\alpha)\big)\Big)\right)\right\}.\end{aligned} \tag{7}$$

where $t_0=0$. Putting Eq. (7) into Eq. (6) implies

$$S(t_{n+1}) = S(0) + \frac{1-\alpha}{B(\alpha)} f_1(t_n, X_n)$$
$$+\frac{\alpha}{B(\alpha)\Gamma(\alpha)} \sum_{k=0}^{n} \left\{ \frac{h^{\alpha} f_1(t_n, X_n)}{\alpha(\alpha+1)} \left(\left((n+1-k)^{\alpha} (n-k+2+\alpha) - (n-k)^{\alpha} (n-k+2+2\alpha) \right) \right) \right.$$
$$\left. -\left(\frac{h^{\alpha} f_1(t_{n-1}, X_{n-1})}{\alpha(\alpha+1)} \left(\left((n+1-k)^{\alpha+1} - (n-k)^{\alpha} (n-k+1+\alpha) \right) \right) \right) \right\}.$$

If we apply the same process to all equations of Illicit drug use model, we obtain the following numerical schemes

$$E(t_{n+1}) = E(0) + \frac{1-\alpha}{B(\alpha)} f_2(t_n, X_n)$$
$$+\frac{\alpha}{B(\alpha)\Gamma(\alpha)} \sum_{k=0}^{n} \left\{ \frac{h^{\alpha} f_2(t_n, X_n)}{\alpha(\alpha+1)} \left(\left((n+1-k)^{\alpha} (n-k+2+\alpha) - (n-k)^{\alpha} (n-k+2+2\alpha) \right) \right) \right.$$
$$\left. -\left(\frac{h^{\alpha} f_2(t_{n-1}, X_{n-1})}{\alpha(\alpha+1)} \left(\left((n+1-k)^{\alpha+1} - (n-k)^{\alpha} (n-k+1+\alpha) \right) \right) \right) \right\}$$

$$I(t_{n+1}) = I(0) + \frac{1-\alpha}{B(\alpha)} f_3(t_n, X_n)$$
$$+\frac{\alpha}{B(\alpha)\Gamma(\alpha)} \sum_{k=0}^{n} \left\{ \frac{h^{\alpha} f_3(t_n, X_n)}{\alpha(\alpha+1)} \left(\left((n+1-k)^{\alpha} (n-k+2+\alpha) - (n-k)^{\alpha} (n-k+2+2\alpha) \right) \right) \right.$$
$$\left. -\left(\frac{h^{\alpha} f_3(t_{n-1}, X_{n-1})}{\alpha(\alpha+1)} \left(\left((n+1-k)^{\alpha+1} - (n-k)^{\alpha} (n-k+1+\alpha) \right) \right) \right) \right\}$$

$$R(t_{n+1}) = R(0) + \frac{1-\alpha}{B(\alpha)} f_4(t_n, X_n)$$
$$+\frac{\alpha}{B(\alpha)\Gamma(\alpha)} \sum_{k=0}^{n} \left\{ \frac{h^{\alpha} f_4(t_n, X_n)}{\alpha(\alpha+1)} \left(\left((n+1-k)^{\alpha} (n-k+2+\alpha) - (n-k)^{\alpha} (n-k+2+2\alpha) \right) \right) \right.$$
$$\left. -\left(\frac{h^{\alpha} f_4(t_{n-1}, X_{n-1})}{\alpha(\alpha+1)} \left(\left((n+1-k)^{\alpha+1} - (n-k)^{\alpha} (n-k+1+\alpha) \right) \right) \right) \right\}$$

$$V\left(t_{n+1}\right)=V\left(0\right)+\frac{1-\alpha}{B\left(\alpha\right)}f_5\left(t_n,X_n\right)$$
$$+\frac{\alpha}{B\left(\alpha\right)\Gamma\left(\alpha\right)}\sum_{k=0}^{n}\left\{\frac{h^{\alpha}f_5\left(t_n,X_n\right)}{\alpha\left(\alpha+1\right)}\left(\left(\left(n+1-k\right)^{\alpha}\left(n-k+2+\alpha\right)-\left(n-k\right)^{\alpha}\left(n-k+2+2\alpha\right)\right)\right)\right.$$
$$\left.-\left(\frac{h^{\alpha}f_5\left(t_{n-1},X_{n-1}\right)}{\alpha\left(\alpha+1\right)}\left(\left(\left(n+1-k\right)^{\alpha+1}-\left(n-k\right)^{\alpha}\left(n-k+1+\alpha\right)\right)\right)\right)\right\}.$$

The above system is going to solve for a sequence of time steps. By using the initial condition for starting to guess for the iterative method, one obtains solutions in $\left(n+1\right)\_th$ step by the solutions in n_th step. This process goes until desired final time t_{final}.

6 Numerical experiments

In this section, the numerical experiment of the illicit drug system as discussed in the previous sections is carried out with the aid of the Matlab 2013a package. The aim of this section is two-fold. The first is to draw a comparison between the classical order derivative and the Atangana-Baleanu fractional operator. Secondly, we intend to examine the behavior of the dynamic system at different instances of fractional power $\alpha \in (0,1)$.

In the experiment, simulation is done with $h = 0.01, t = 10$ and parameters

$$\begin{aligned}&w = 0.3, \mu = 0.02, \tau = 1.25, \beta = 0.35, \gamma = 0.1, \rho = 0.35, \epsilon = 0.6,\\ &a = 0.01, \nu = 0.035, \delta = 0.14, d = 0.2, \sigma = 0.05, \theta = 0.09,\\ &S(0) = 0.85, E(0) = 0.02, I(0) = 0.01, R(0) = 0.00, V(0) = 0.00.\end{aligned} \tag{8}$$

In Figs. 2-5, we draw a comparison between the classical system obtained when $\alpha = 1.00$ with different instances of $\alpha \in (0,1)$ as shown in the figures captions. It is evident in Fig. 2 that both integer and non-integer order systems display almost similar distribution. But as the value of α is decreasing from 1 the effect of the fractional derivative power becomes pronounced. Figs. 6 and 7 correspond to the coexistence and individual response to the fractional index α. It is obvious as shown in the population profiles that the number of people with severe drug use I, those that are badly affected due to illicit drug use R and the group identified as drug users V decline in the population with respect to changes in α.

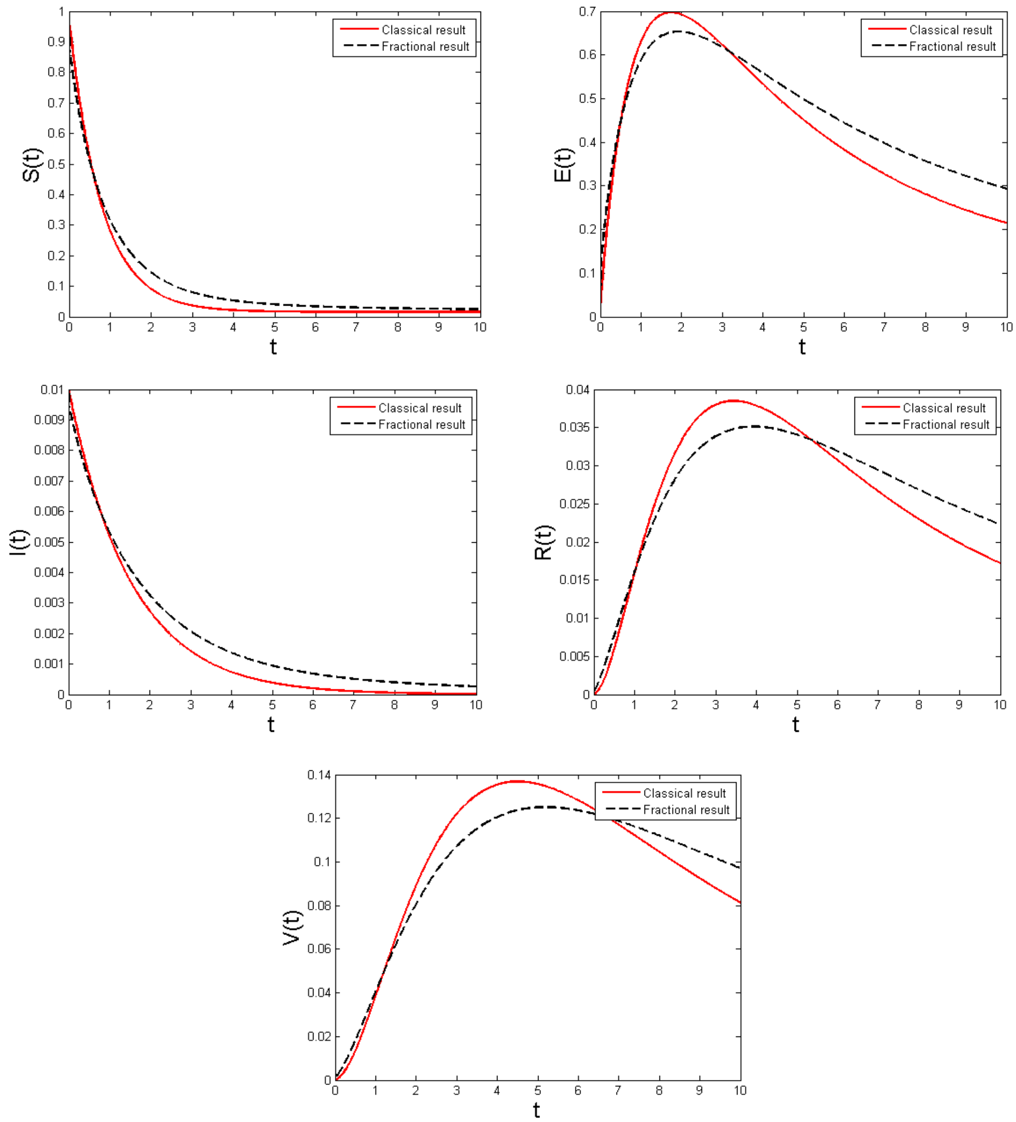

Figure 2: Numerical results for integer-order system with $\alpha = 1$ (thick-red) and fractional-order case $\alpha = 0.93$ (dash-black)

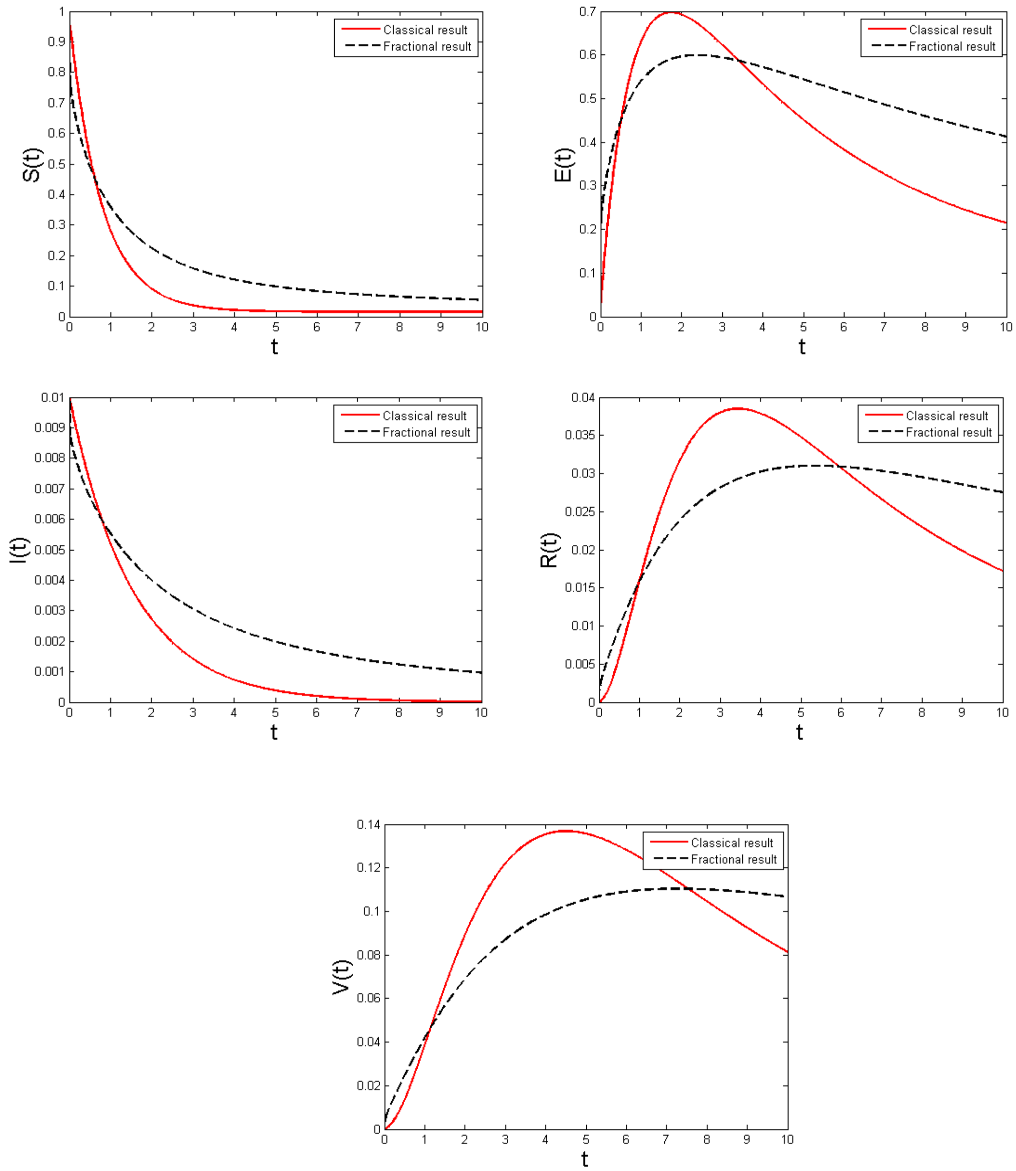

Figure 3: Numerical results for integer-order system with $\alpha = 1$ (thick-red) and fractional-order case $\alpha = 0.81$ (dash-black)

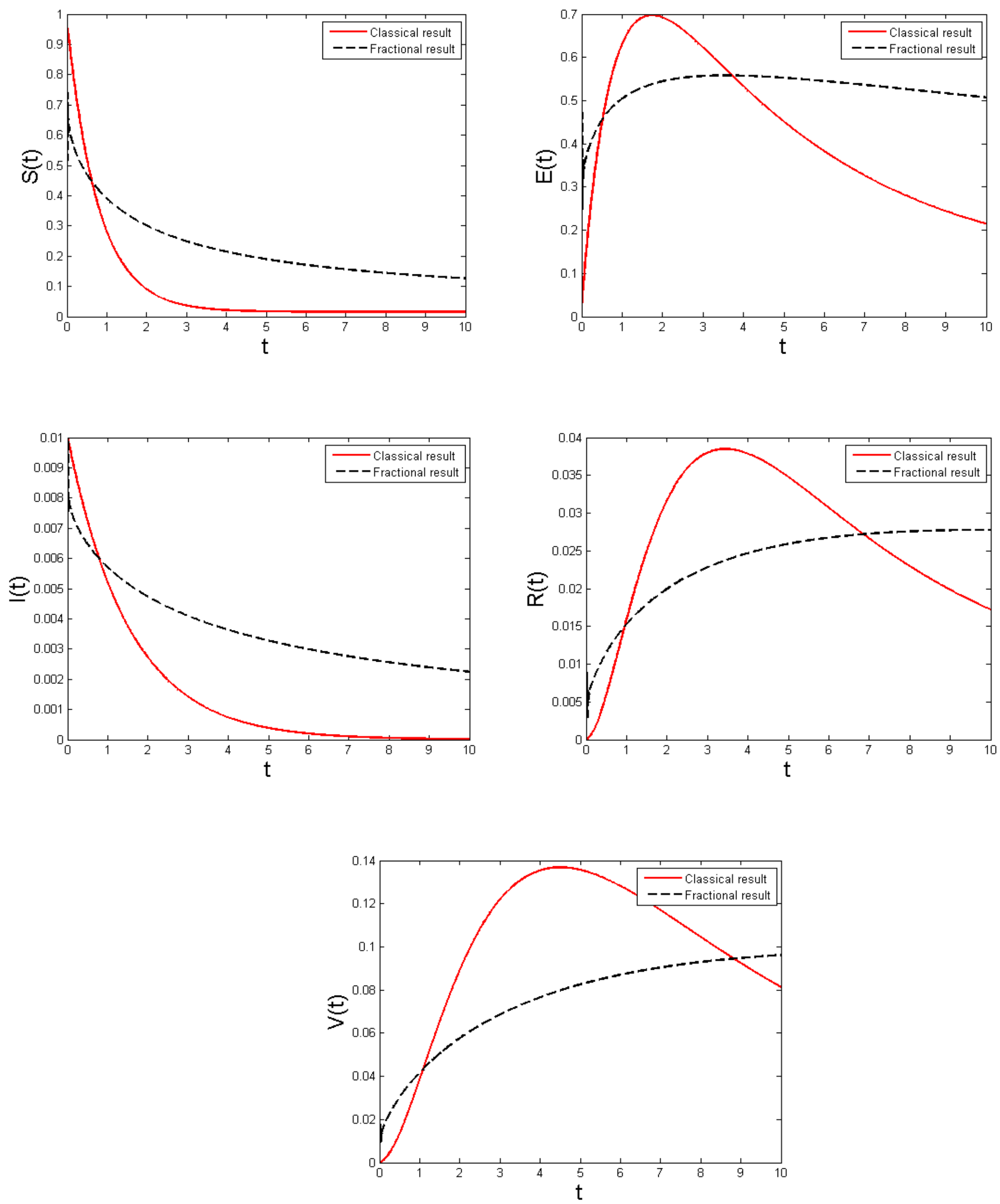

Figure 4: Numerical results for integer-order system with $\alpha = 1$ (thick-red) and fractional-order case $\alpha = 0.66$ (dash-black)

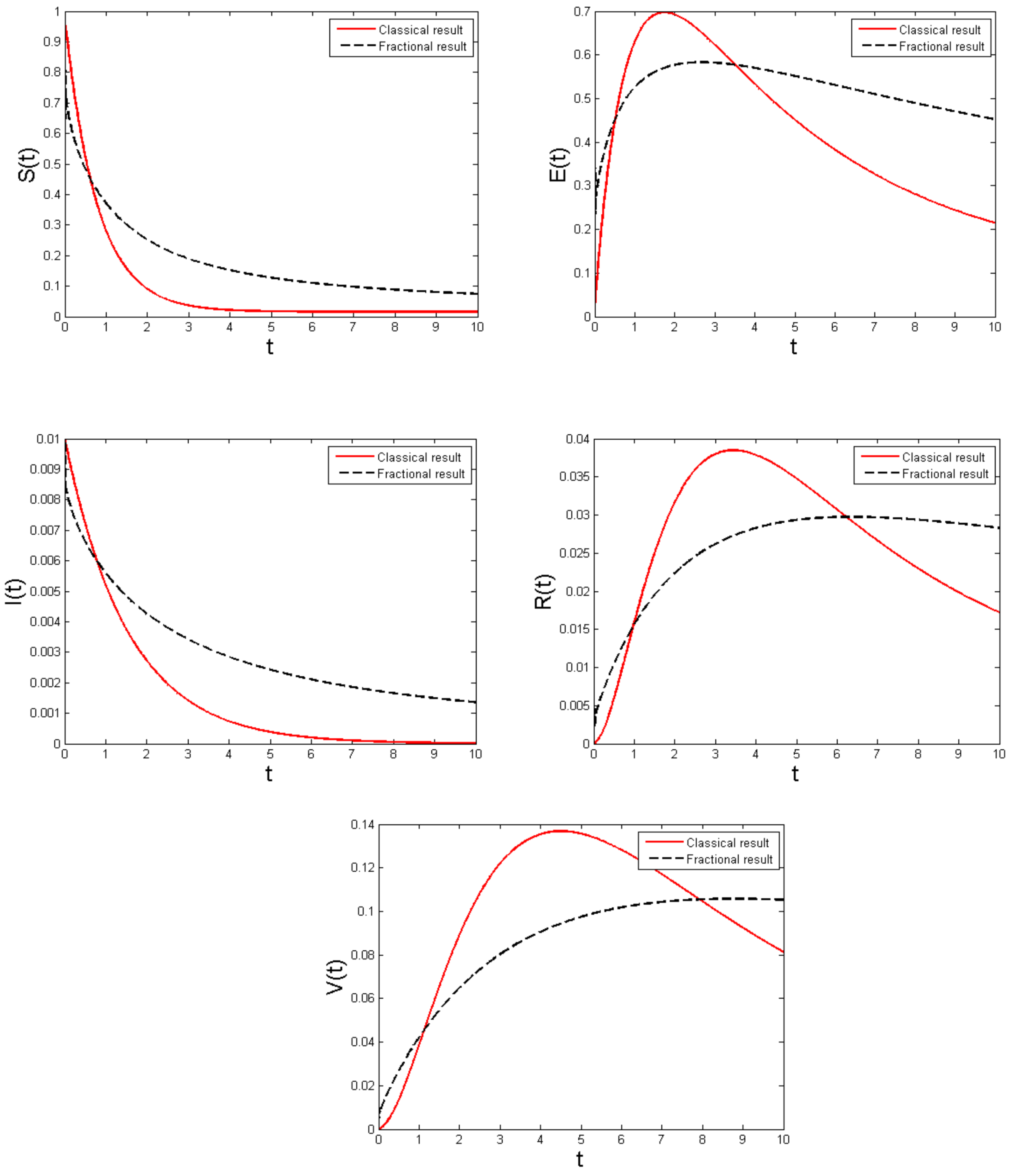

Figure 5: Numerical results for integer-order system with $\alpha = 1$ (thick-red) and fractional-order case $\alpha = 0.77$ (dash-black)

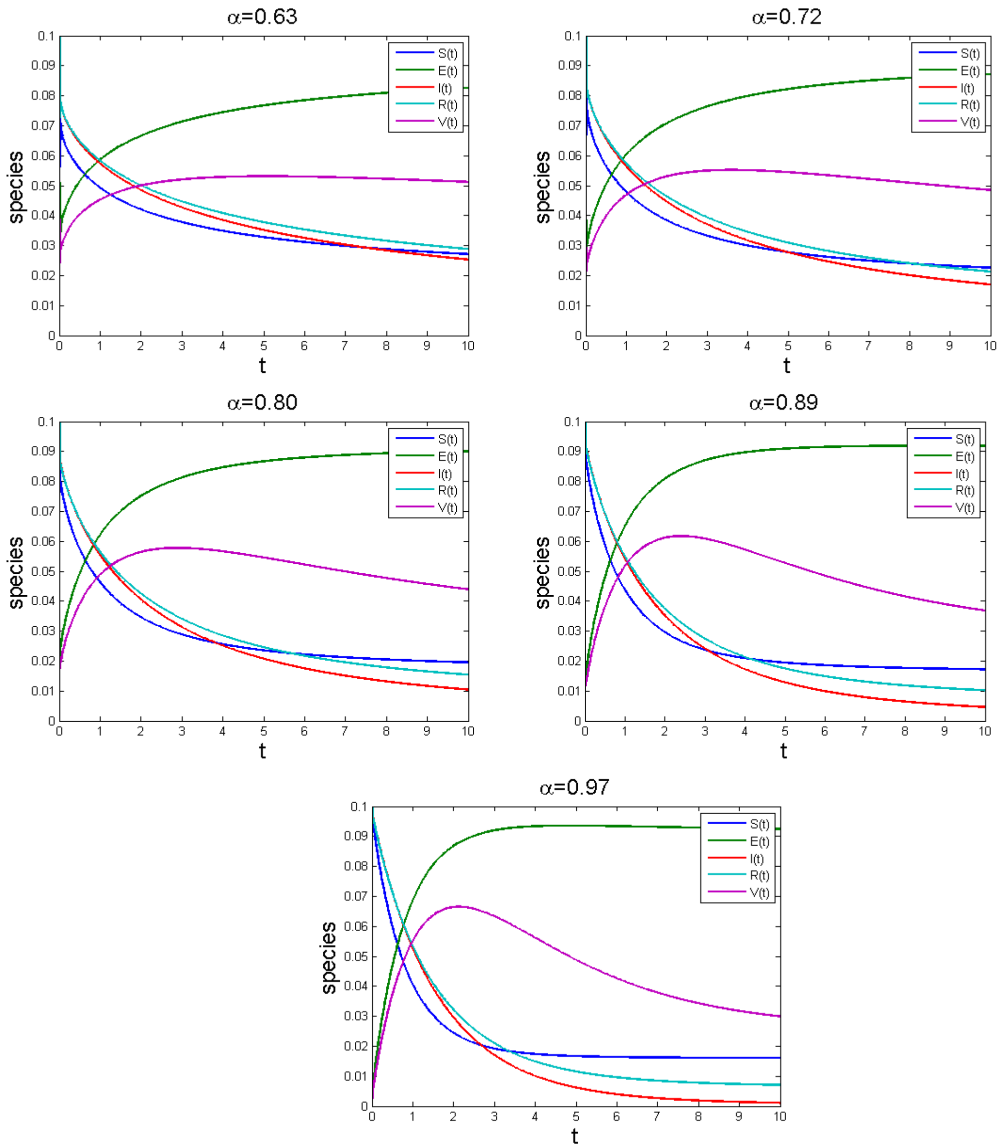

Figure 6: Time series distribution of fractional system (1) showing the coexistence of the species for different instances of α

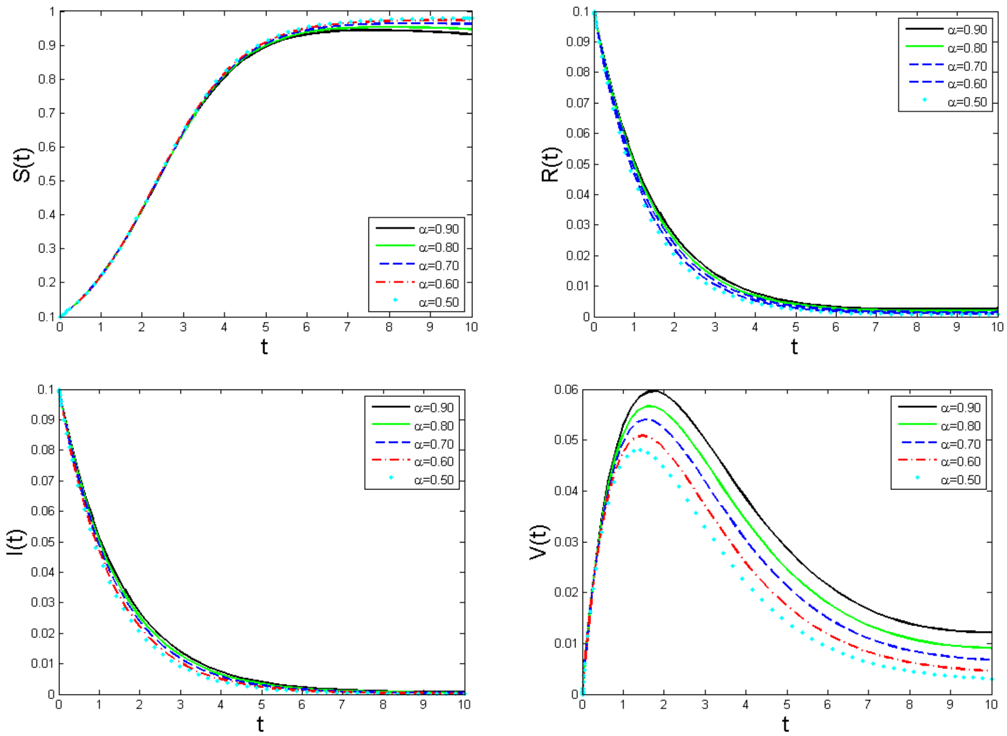

Figure 7: Individual species responses to some fractional α values

7 Conclusion

Recently, mathematical modeling of real-life phenomena in physical sciences using the concept of fractional-order operators for the system of differential equations has been a subject of investigations. The nonlocal and non-singular derivatives have some important properties called the memory effect, and such a rare characteristic is completely missing in the classical derivatives. Hence, the effect of illicit drug use with the new fractional derivative whose formulation is based on the nonlocal and non-singular kernel is investigated in this paper. The integer-order derivative in time is replaced with this new fractional operator called the Atangana-Baleanu fractional derivative. We adopt the novel numerical scheme based on the fractional version of the Adams-Bashforth method for approximating the dynamical system. Stability analysis for both the endemic and epidemic dynamics, with emphasis on the threshold dynamical system is characterized by the basic reproduction number. Existence and uniqueness of solutions are also conducted. Numerical results for some instances of fractional power are given to justify the theoretical findings. Extension to other derivatives such as the Caputo, Caputo-Fabrizio, among several other existing operators to model complex models arising in mathematical epidemiology, is left for future research.

Funding Statement: The authors received no specific funding for this study.

Conflicts of Interest: The authors declare that they have no conflicts of interest to report regarding the present study.

References

Atangana, A.; Baleanu, D. (2016): New fractional derivatives with non-local and non-singular kernel: theory and application to heat transfer model. *Thermal Science*, vol. 20, no. 2, pp. 763-769.

Atangana, A.; Owolabi, K. M. (2018): New numerical approach for fractional differential equations. *Mathematical Modelling of Natural Phenomena*, vol. 13, no. 1, pp. 1-21.

Chen, Y. T.; Xu, W. H.; Zuo, J. W.; Yang, K. (2019): The fire recognition algorithm using dynamic feature fusion and IV-SVM classifier. *Cluster Computing*, vol. 22, no. 3, pp. 7665-7675.

Ghanbari, B.; Kumar, S.; Kumar, R. (2020): A study of behaviour for immune and tumor cells in immunogenic tumour model with non-singular fractional derivative. *Chaos, Solitons and Fractals*, vol. 133, pp. 1-11.

Ghanbari, B.; Osman, M. S.; Baleanu, D. (2019): Generalized exponential rational function method for extended Zakharov-Kuzetsov equation with conformable derivative. *Modern Physics Letters A*, vol. 34, no. 20, 1950155.

Karaagac, B. (2019a): A study on fractional Klein Gordon equation with non-local and non-singular kernel. *Chaos, Solitons and Fractals*, vol. 126, pp. 218-229.

Karaagac, B. (2019b): New exact solutions for some fractional order differential equations via improved sub-equation method. *Discrete and Continuous Dynamical Systems Series S*, vol. 12, no. 3, pp. 447-454.

Karaagac, B. (2018): Analysis of the cable equation with non-local and non-singular kernel fractional derivative. *The European Physical Journal Plus*, vol. 133, no. 54, pp. 1-9.

Kheiri, H.; Jafari, M. (2019): Stability analysis of a fractional order model for the HIV/AIDS epidemic in a patchy environment. *Journal of Computational and Applied Mathematics*, vol. 346, pp. 323-339.

Li, W. J.; Zhu, B. H. (2018): A 2k-vertex kernel for vertex cover based on crown decomposition. *Theoretical Computer Science*, vol. 739, pp. 80-85.

Mushayabasa, S.; Tapedzesa, G. (2015): Modeling illicit drug use dynamics and its optimal control analysis. *Computational and Mathematical Methods in Medicine*, pp. 1-11.

Naik, P. A.; Zu, J.; Owolabi, K. M. (2020): Modeling the mechanics of viral kinetics under immune control during primary infection of HIV-1 with treatment in fractional order. *Physica A*, vol. 545, no. 123816, pp. 1-19.

Osman, M. S. (2019): New analytical study of water waves described by coupled fractional variant Boussinesq equation in fluid dynamics. *Pramana*, vol. 93, no. 2, pp. 26.

Owolabi, K. M. (2018a): Numerical approach to fractional blow-up equations with Atangana-Baleanu derivative in Riemann-Liouville sense. *Mathematical Modelling of Natural Phenomena*, vol. 13, no. 7, pp. 1-17.

Owolabi, K. M. (2018d): Analysis and numerical simulation of multicomponent system with Atangana-Baleanu fractional derivative. *Chaos, Solitons and Fractals*, vol. 115, pp. 127-134.

Owolabi, K. M. (2018e): Numerical patterns in system of integer and non-integer order derivatives. *Chaos, Solitons and Fractals*, vol. 115, pp. 143-153.

Owolabi, K. M. (2019): Computational study of noninteger order system of predation. *Chaos*, vol. 29, no. 1, 013120.

Owolabi, K. M. (2018b): Modelling and simulation of a dynamical system with the Atangana-Baleanu fractional derivative. *The European Physical Journal Plus*, vol. 133, no. 15, pp. 1-13

Owolabi, K. M. (2018c): Efficient numerical simulation of non-integer-order space-fractional reaction-diffusion equation via the Riemann-Liouville operator. *The European Physical Journal Plus*, vol. 133, no. 98, pp. 1-16

Owolabi, K. M.; Atangana, A. (2019b): *Numerical Method for Fractional Differentiation*. Springer, Singapore.

Owolabi, K. M.; Atangana, A. (2019a): On the formulation of Adams-Bashforth scheme with Atangana-Baleanu-Caputo fractional derivative to model chaotic problems. *Chaos*, vol. 29, no. 2, 29023111.

Owolabi, K. M.; Atangana, A. (2018a): Modelling and formation of spatiotemporal patterns of fractional predation system in subdiffusion and superdiffusion scenarios. *The European physical Journal Plus*, vol. 133, no. 43, pp. 1-13

Owolabi, K. M.; Atangana, A. (2018b): Robustness of fractional difference schemes via the Caputo subdiffusion-reaction equations. *Chaos, Solitons and Fractals*, vol. 111, pp. 119-127.

Owolabi, K. M.; Atangana, A. (2018c): Chaotic behaviour in system of noninteger-order ordinary differential equations. *Chaos, Solitons and Fractals*, vol. 115, pp. 362-370.

Owolabi, K. M.; Gómez-Aguilar, J. F. (2018): Numerical simulations of multilingual competition dynamics with nonlocal derivative. *Chaos, Solitons and Fractals*, vol. 117, pp. 175-182.

Owolabi, K. M.; Hammouch, Z. (2019): Mathematical modeling and analysis of two-variable system with noninteger-order derivative. *Chaos*, vol. 29, no. 1, 013145.

Podlubny, I. (1999): *Fractional Differential Equations*. Academic Press, New York.

Yavuz, M.; Bonyah, E. (2019): New approaches to the fractional dynamics of schistosomiasis disease model. *Physica A*, vol. 525, pp. 373-393.