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Exact Analysis of Non-Linear Fractionalized Jeffrey Fluid. A Novel Approach of Atangana-Baleanu Fractional Model

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Abstract: It is a very difficult task for the researchers to find the exact solutions to mathematical problems that contain non-linear terms in the equation. Therefore, this article aims to investigate the viscous dissipation (VD) effect on the fractional model of Jeffrey fluid over a heated vertical flat plate that suddenly moves in its own plane. Based on the Atangana-Baleanu operator, the fractional model is developed from the fractional constitutive equations. VD is responsible for the non-linear behavior in the problem. Upon taking the Laplace and Fourier sine transforms, exact expressions have been obtained for momentum and energy equations. The influence of relative parameters on fluid flow and temperature distribution is shown graphically. As special cases, and for the sake of correctness, the corresponding results for second-grade fluid and Newtonian viscous fluid are also obtained. It is interesting to note that fractional parameter α provides more than one line as compared to the classical model. This effect represents the memory effect in the fluid which is not possible to elaborate by the classical model. It is also worth noting that the temperature profile of the generalized Jeffrey fluid rises for higher values of Eckert number which is due to the enthalpy difference of the boundary layer.

Keywords: Viscous dissipation, atangana-baleanu fractional derivative, laplace transform, fourier sine transform.

1 Introduction

Fluids like castor oil, engine oil, different types of polymers, etc., having the properties of viscosity and elasticity are termed as viscoelastic fluid. These types of fluid have great importance and significance in various fields such as mechanical engineering, industrial engineering, polymerization, and automobile industry [Chhabra and Richardson (2011)].

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The viscoelastic fluid shows non-linear relation regarding shear stress, which is why it comes in the sub-class of non-Newtonian fluid. For a better understanding of the mechanical features of the non-Newtonian fluids, the Navier-Stokes equation is no longer applicable. Due to this failure of Navier-Stokes equations, different mathematicians have formulated various models for better interpretations of non-Newtonian fluid behavior, such as Jeffrey's fluid model, which describes the properties of non-Newtonian fluids better than other models. The Jeffrey fluid model has two generalized parameters, namely the retardation parameter and the ratio of retardation over the relaxation parameter. From the Jeffrey fluid model, the analyst can deduce the second grade and Newtonian fluid model by neglecting the effects of their generalized parameters. Keeping in view its properties and abilities, different studies have been investigated by various researchers like Khan [Khan (2015)] who found the closed-form expression for the natural convectional motion of Jeffrey fluid. Rahman et al. [Rahman, Ellahi, Nadeem et al. (2016)] explained the blood flow of Jeffrey fluid. Bhatti et al. [Bhatti, Ellahi and Zeeshan (2016)] studied Jeffrey fluid in a duct. In their research article, they also deduce their work to the Newtonian fluid by taking $\lambda_1 \to 0$. Shehzad et al. [Shehzad, Alsaedi and Hayat (2013)] inspected the rheology of viscoelastic Jeffrey fluid in the existence presence of Joule heating. The method of Homotopy analysis is used to develop the series solution of the governing equations. The characteristics of Jeffrey fluid with the amalgamate transportation of thermal energy and mass diffusion is examined by Qasim [Qasim (2013)]. Shehzad et al. [Shehzad, Hayat and Alhuthali (2014)] examined the non-linear 3-D flow of viscoelastic Jeffrey fluid together with the impact of Newtonian heating. Under the impact of the power-law of thermal energy and concentration [Narayana and Babu (2016)] developed a non-linear model for the viscoelastic Jeffrey fluid.

Fractional calculus which has been growing nowadays is vastly attracting increased attention due to its versatile and unique properties. The non-integer order derivative is solved through fractional calculus tools. Fractional calculus is an important and fruitful tool for describing many systems including memory. In the last few years, fractional calculus is used for many purposes in various fields, such as electrochemistry, transportation of water in ground level, electromagnetism, elasticity, diffusion, and in conduction of heat process. Keeping in view the tremendous mentioned properties, many researchers work directly or indirectly in the field of non-integer order derivative [Khalil, Al Horani, Yousef et al. (2014); Ali, Saqib, Khan et al. (2016); Shah and Khan (2016)]. In 2015, Caputo and Fabrizio contributed their potentials in the mentioned field and gave a new definition for non-integer order derivative based on exponential function with non-singular kernel [Caputo and Fabrizio (2015)]. By employing the idea of Caputo-Fabrizio's derivative (CFD), different inspections have been carried out by many researchers. For example, Saqib et al. [Saqib, Ali, Khan et al. (2018)] developed an exact solution for fractionalized Jeffrey fluid by utilizing the approach of CFD. Khan et al. [Khan, Abro, Tassaddiq et al. (2017)] discussed the time-fractional analysis of second-grade fluid via the concept of CFD. Some other interesting results can be seen in Abro et al. [Abro and Solangi (2017)]. However, after analysis, drawbacks regarding CFD came into view to the researchers, which is that the kernel of the CFD is marked local. To circumvent this issue, Atangana et al. [Atangana and Baleanu (2016)] presented a new expression for the time-fractional derivative with the non-local kernel. The biggest advantage of this expression is that it elaborates on the heredity of the function. Also, one can recover the original function from it by setting its order equal to one. Utilizing the idea of AB, Ali et al. [Ali, Murtaza, Khan et al. (2019)] investigated the comparative analysis of AB derivative and CFD in the Jeffrey nanofluid model. In another study, Ali et al. [Ali, Murtaza, Khan et al. (2019)] checked the impact of diffusion-thermo on Jeffrey nanofluid flow. They transformed the classical model of governing equations into a time-fractional model, namely AB derivative. Ali et al. [Ali, Yousaf, Khan et al. (2019)] discussed the effect of magnetic particles in blood flow using the AB time-fractional model. Murtaza et al. [Murtaza, Iftekhar, Ali et al. (2020)] examined the effect of electro-osmosis in viscoelastic fluid flow. Some other interesting other results can be seen in [Xue and Nie (2009); Li, Liu, Wang et al. (2019); Li and Zhu (2018)].

Viscous dissipation is referred to as the work done by fluid flow against viscous stresses. It can also be elaborated as "the changing of K. E into the I.E due to viscosity of the fluid". In various fields like engineering, medicine, mechanical engineering, biochemistry and biophysics, the heat transfer phenomenon including viscous dissipation has a great impact on the value of the yields. Therefore, many scientists and engineers have sought to better understand the heat transfer phenomena. It plays an indispensable role in the establishment of the heat exchanger, steam pipe, polymerization, and heat conductor, etc. Despite its importance many researchers ignored the term viscous dissipation in their studies. The role of viscous dissipation in a natural convectional flow was first discussed by Raju et al. [Raju, Kumar, Rahimi et al. (2019)]. They modeled the equations of motion in the form of PDEs. They obtained the solutions for the governing equation that involved a viscous dissipation term via a perturbation method. Certell [Certell (2008)] examined the influence of viscous dissipation on fluid passing over a stretched leaf. The analytical solution in the presence of viscous dissipation for the flow shear thickening fluid is discussed by Husnain et al. [Husnain, Salleh, Khan et al. (2018)]. The impact of carbon nanotubes on the rotational flow of viscous fluid in the presence of viscous dissipation has been discussed by Jawad et al. [Jawad, Shah, Islam et al. (2019)]. Afridi et al. [Afridi, Qasim and Makinde (2019)] inspected the visco-elastic fluid under the influence of a dissipative electric field flowing in a porous medium. Furthermore, Sene [Sene (2018)] investigated Stokes' first problem with AB fractional derivative along with the effect of viscous dissipation.

In the light of existing literature, in the present work, a non-linear problem of generalized Jeffrey and generalized second grade fluid in terms of AB fractional derivative is modeled for the first time in literature. The fluid is taken over a heated flat plate in the presence of viscous dissipation. The governing equations for considered phenomena are modeled by using Boussinesq approximation. Analytical expressions for thermal energy distributions and fluid motion are developed employing two different types of integral transformation, namely the Fourier sine transform (FST) and the Laplace transform techniques (LTT). The numerical results are computed in terms of several plots. Both exact and numerical solutions satisfy the imposed physical conditions. Furthermore, the exact expressions are reduced to the corresponding solutions of fractional second-grade fluid and Newtonian viscous fluid for the Stokes' first problem, which are in good agreement with the published works.

2 Mathematical modeling and solution

Consider a unidirectional and incompressible flow of Jeffrey fluid. The plate is taken along $x-axis$. At $t=0$, both the plate and fluid are at rest. At time $t=0^{+}$, the plate is disturbed with a sudden jerk and transmits the motion into the fluid, while the temperature of the fluid at $t>0$ is T_w, which represents wall temperature. VD is also considered in the energy equation.

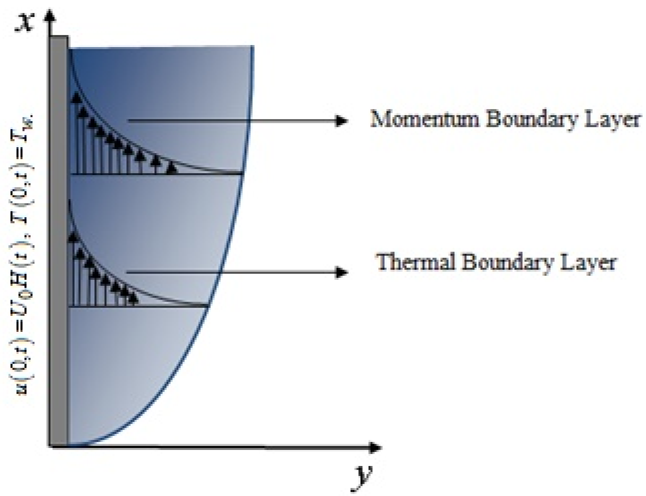

Figure 1: Schematic diagram of the flow regime

The velocity field according to the assumptions is,

$$\vec{V}=\{u(y,t),0,0\}, \tag{1}$$

and the constitutive equation is:

$$\rho\left\{\frac{d\vec{V}}{dt}\right\}=\nabla.\boldsymbol{T}, \tag{2}$$

where ρ represent density, $\frac{d\vec{V}}{dt}$ shows the material derivative of velocity field and the tensor $\boldsymbol{T}$ is the Cauchy stress tensor and defined as:

$$\boldsymbol{T}=-P\boldsymbol{I}+\mu\boldsymbol{S}, \tag{3}$$

where P, $\boldsymbol{I}$ and μ represent hydrostatic pressure, identity tensor and dynamic viscosity, respectively. The extra stress tensor $\boldsymbol{S}$ for Jeffrey fluid in Atangan-Baleanu sense can be written as:

$$\boldsymbol{S}=\frac{1}{1+\lambda_1}\left(1+\lambda_2\,{}^{AB}D_{\alpha}^{t}\right)\boldsymbol{A_1}, \tag{4}$$

where λ_1 and λ_2 are generalized Jeffrey fluid parameters, ${}^{AB}D_{\alpha}^{t}$ represents Atangana-Baleanu time-fractional derivative and its mathematical form is given as:

$$ {}^{AB}D_{\alpha}^{t} f(t) = \frac{N(\alpha)}{1-\alpha} \int_0^t f'(s) \mathbf{E}_{\alpha} \left(-\frac{\alpha}{1-\alpha} (t-s)^{\alpha} \right) ds; \qquad 0 < \alpha < 1, \tag{5} $$

where $N(\alpha)$ represents normalize function i.e. $N(0) = N(1) = 1$ and $\mathbf{E}_{\alpha}(s)$ represents generalized Mittage-Leffler function [(Atangana and Baleanu et al. (2016)].

The classical form of Eq. (4) is formulated by Khan [Khan (2015)], while $\boldsymbol{A}_1$ displays first Rivilin Erickson stress tensor and stated as:

$$ \boldsymbol{A}_1 = \boldsymbol{L} + \boldsymbol{L}^{\boldsymbol{T}}. \tag{6} $$

Here $\mathbf{L}$ shows the gradient of velocity.

Using the constitutive Eqs. (3), (4) and (6) along with Boussinesq's approximation [Gray and Giorgini (1976)] governing equations will take the form:

$$ \rho \frac{\partial u(y,t)}{\partial t} = \frac{\mu}{1+\lambda_1} \left(1 + \lambda_2 {}^{AB}D_{\alpha}^{t} \right) \frac{\partial^2 u(y,t)}{\partial y^2}, \tag{7} $$

and energy equation for temperature distribution along with the non-linear term of viscous dissipation is given as [Sene (2018)]:

$$ (\rho c_p)^{AB} D_{\alpha}^{t} T(y,t) = k \frac{\partial^2 T(y,t)}{\partial y^2} + \mu \left(\frac{\partial u(y,t)}{\partial y} \right)^2, \tag{8} $$

where ρ represent density while k and c_p show thermal conduction and capacity of specific heat, respectively. The last term of Eq. (8) represents viscous dissipation and refers to the changing of kinetic energy into the internal energy due to viscosity of the fluid. Although it has many important applications in different fields, many researchers ignore this term in their studies.

The relevant physical initial and boundary conditions (IBCs) are:

$$ \left. \begin{aligned} & u(y,0) = 0, \;\; u(0,t) = U_0 H(t), \;\; u(\infty,t) = 0. \\ & T(y,0) = T_{\infty}, \;\; T(0,t) = T_w, \qquad T(\infty,t) = T_{\infty}. \end{aligned} \right\} \tag{9} $$

2.1 Solution of velocity field

Introducing the following dimensionless variables:

$$ u^* = \frac{u}{U_0}, \quad y^* = \frac{yU_0}{\nu}, \quad t^* = \frac{tU_0^2}{\nu}, \quad \theta = \frac{T}{T_w}, \qquad \theta = \frac{T}{T_w} \tag{10} $$

in Eqs. (7)-(9), and neglecting (*) notation, the governing equation and conditions for velocity field become:

$$ \frac{\partial u(y,t)}{\partial t} = \left(a_1 + a_2 {}^{AB}D_{\alpha}^{t} \right) \frac{\partial^2 u(y,t)}{\partial y^2}, \tag{11} $$

$$u(y,0)=0,\ u(0,t)=1,\quad u(\infty,t)=0, \tag{12}$$

where

$$a_1=\frac{1}{1+\lambda_1},\qquad \lambda=\frac{\lambda_2 U_0^2}{\nu},\qquad a_2=a_1\lambda.$$

Taking Fourier sine transform of Eq. (11), we get:

$$F\left[\frac{\partial u(y,t)}{\partial t}\right]=F\left[\left(a_1+a_2{}^{AB}D_\alpha^t\right)\frac{\partial u^2(y,t)}{\partial y^2}\right], \tag{13}$$

where F denotes Fourier Sine transform and:

$$F_s\left[\frac{\partial u(y,t)}{\partial t}\right]=\frac{\partial}{\partial t}u_s(w,t),\quad F_s\left[\frac{\partial^2 u(y,t)}{\partial y^2}\right]=\sqrt{\frac{2}{\pi}}wu(0,t)-w^2u_s(w,t), \tag{14}$$

in which $u_s(w,t)$ shows Fourier sine transform of $u(y,t)$, incorporating Eq. (14) into Eq. (13) and using boundary condition defined in (12), we get:

$$\frac{\partial}{\partial t}u_s(w,t)=\left(a_1+a_2{}^{AB}D_\alpha^t\right)\left[\sqrt{\frac{2}{\pi}}w-w^2u_s(w,t)\right], \tag{15}$$

after rearranging, Eq. (15) becomes:

$$\frac{\partial}{\partial t}u_s(w,t)=-w^2\left(a_1+a_2{}^{AB}D_\alpha^t\right)u_s(w,t)+a_1\sqrt{\frac{2}{\pi}}w. \tag{16}$$

Now applying Laplace transform to Eq. (16) and using corresponding initial condition, we get:

$$p\overline{u_s}(w,p)=-w^2\left(a_1+a_2\frac{N(\alpha)}{1-\alpha}\cdot\frac{p^\alpha}{p^\alpha+\dfrac{\alpha}{1+\alpha}}\right)\overline{u_s}(w,p)+a_1\sqrt{\frac{2}{\pi}}\frac{w}{p}, \tag{17}$$

where p is Laplace transform variable. In abridged form, Eq. (17) can be written as:

$$\overline{u_s}(w,p)=a_1\sqrt{\frac{2}{\pi}}\left[\frac{w\left(p^\alpha(1-\alpha)+\alpha\right)}{p\left[p\left(p^\alpha(1-\alpha)+\alpha\right)+w^2a_1\left(p^\alpha(1-\alpha)+\alpha\right)+w^2a_2N(\alpha)p^\alpha\right]}\right]. \tag{18}$$

Assume $a_3=\dfrac{1-\alpha}{\alpha}$ and $a_4=\dfrac{a_2N(\alpha)+a_1(1-\alpha)}{\alpha}$, using this assumption Eq. (18) takes the form:

$$\overline{u_s}(w,p)=wa_1\sqrt{\frac{2}{\pi}}\left[\frac{p^\alpha a_3+1}{p\left(a_3p^{1+\alpha}+p+w^2a_1+p^\alpha w^2a_4\right)}\right], \tag{19}$$

for determining inverse Laplace transform, we introduce series:

$$\overline{u_s}(w,p)=a_1\frac{w^2}{w}\sqrt{\frac{2}{\pi}}\times\sum_{b=0}^{\infty}\frac{(-1)^b(a_3p+wa_4)p^{-b\alpha}}{(p+a_1w^2)}, \tag{20}$$

in more compact form, Eq. (20) can be displayed as:

$$\overline{u_s}(w,p)=\frac{a_1}{w}\sqrt{\frac{2}{\pi}}\bullet\bar{h}.(w,p), \tag{21}$$

where

$$\bar{h}(w,p)=\sum_{b=0}^{\infty}\sum_{c,d\geq 0}^{c+d=b}\frac{(-1)^b a_3^c w^{2(d+1)}(a_4)^4 b!}{c!d!}\times\left(\overline{\Re_1}(w,p)+\overline{\Re_2}(w,p)\right), \tag{22}$$

where

$$\overline{\Re_1}(w,p)=\frac{a_3p^{m-b\alpha+\alpha-1}}{(p+w^2a_1)^{b+1}}, \tag{23}$$

and

$$\overline{\Re_2}(w,p)=\frac{p^{c-b\alpha-1}}{(p+w^2a_1)^{b+1}}. \tag{24}$$

Applying the inverse transform operator of the Laplace on Eq. (21), we get:

$$u_s(w,t)=\frac{a_1}{w}\sqrt{\frac{2}{\pi}}\cdot h(w,t), \tag{25}$$

where $h(w,t)$ is defined as follows:

$$h(w,t)=\sum_{b=0}^{\infty}\sum_{c,d\geq 0}^{c+d=b}\frac{(-1)a_3^c w^{2(d+1)}(a_4)^4 b!}{c!d!}\bullet\left(\Re_1(w,t)+\Re_2(w,t)\right), \tag{26}$$

and

$$\left.\begin{aligned}\Re_1(w,t)&=a_3t^{b(\alpha+1)-m-\alpha+2}\boldsymbol{E}_{1,2-m+b\alpha-\alpha}^{(b)}\left(-w_1^2t\right)\\ \Re_2(w,t)&=-t^{b\alpha-m+2}\boldsymbol{E}_{1,2-m+b\alpha}^{(b)}\left(-w_1^2t\right)\end{aligned}\right\}, \tag{27}$$

Now, by applying the inverse transform operator of Fourier sine on Eq. (25), the closed form solution for momentum equation is obtained as:

$$u(y,t)=\frac{2}{\pi}\int_0^{\infty}\frac{\sin(wy)}{w}h(w,t)dw. \tag{28}$$

2.2 Solution of energy equation

Energy equation for temperature distribution in Atangana-Baleanu sense is given as:

$$\left(\rho c_p\right)^{AB}D_\alpha^t T\left(y,t\right)=k\frac{\partial^2 T\left(y,t\right)}{\partial y^2}+\mu\left(\frac{\partial u\left(y,t\right)}{\partial y}\right)^2, \tag{29}$$

subjected to the IBCs for temperature field is defined as:

$$T\left(y,0\right)=T_\infty,\quad T\left(0,t\right)=T_w,\quad T\left(\infty,t\right)=T_\infty. \tag{30}$$

Using the dimensionless variables defined in (10), energy Eq. (29) along with conditions (30) will be transformed to non-dimensional form:

$$^{AB}D_\alpha^t\theta\left(y,t\right)=\frac{1}{\Pr}\frac{\partial^2\theta\left(y,t\right)}{\partial y^2}+Ec\left(\frac{\partial u\left(y,t\right)}{\partial y}\right)^2, \tag{31}$$

with IBCs as:

$$\theta\left(y,0\right)=0,\quad \theta\left(0,t\right)=1,\quad \theta\left(\infty,t\right)=0, \tag{32}$$

where $Ec=\frac{U^2}{c_pT_w}$ shows Eckert number and $\Pr=\frac{\mu c_p}{k}$ represents Prandtl number.

Before solving the energy equation, and by substituting $f\left(y,t\right)=Ec\left(\frac{\partial u\left(y,t\right)}{\partial y}\right)^2$ in Eq. (31), we get:

$$^{AB}D_\alpha^t\theta\left(y,t\right)=\frac{1}{\Pr}\frac{\partial^2\theta\left(y,t\right)}{\partial y^2}+f\left(y,t\right). \tag{33}$$

For exact solution, first we apply Fourier sine transform on Eq. (33) along with boundary conditions, which yields:

$$^{AB}D_\alpha^t\theta\left(w,t\right)=\frac{1}{\Pr}\left(\sqrt{\frac{2}{\pi}}w-w^2\theta_s\left(w,t\right)\right)+f_s\left(w,t\right), \tag{34}$$

taking the Laplace transform of Eq. (34), we get:

$$\frac{N\left(\alpha\right)}{1-\alpha}\frac{p^\alpha\overline{\theta_s}\left(w,p\right)}{p^\alpha+\frac{\alpha}{1-\alpha}}=\frac{1}{\Pr}\left[\sqrt{\frac{2}{\pi}}\frac{w}{s}-w^2\overline{\theta_s}\left(w,p\right)\right]+\overline{f_s}\left(w,p\right). \tag{35}$$

After rearranging Eq. (35), we obtain:

$$\overline{\theta}_s(w,p)=\sqrt{\frac{2}{\pi}}\frac{w}{\Pr}\left[(1-\alpha)p^{\alpha}\left[p\left(N(\alpha)+\frac{w^2p^{\alpha}}{\Pr}(1-\alpha)+\frac{w^2\alpha}{\Pr}\right)\right]^{-1}\right]$$
$$+\sqrt{\frac{2}{\pi}}\frac{w}{\Pr}\left[\alpha p^{-1}\left(N(\alpha)+\frac{w^2p^{\alpha}}{\Pr}(1-\alpha)+\frac{w^2\alpha}{\Pr}\right)^{-1}\right]$$
$$+\overline{f_s}(w,p)(1-\alpha)p^{\alpha}\left[\left(N(\alpha)+\frac{w^2p^{\alpha}}{\Pr}(1-\alpha)+\frac{w^2\alpha}{\Pr}\right)^{-1}\right]$$
$$+\overline{f_s}(w,p)\,\alpha\left[\left(N(\alpha)+\frac{w^2p^{\alpha}}{\Pr}(1-\alpha)+\frac{w^2\alpha}{\Pr}\right)^{-1}\right], \tag{36}$$

Employing inverse transform operators of the Laplace and Fourier sine, the exact of Eq. (29) will be:

$$\theta(y,t)=\theta_1(y,t)+\theta_2(y,t)+\theta_3(y,t)+\theta_4(y,t), \tag{37}$$

where the function θ_1, θ_2, θ_3 and θ_4 are defined as:

$$\theta_1(y,t)=\frac{2}{\pi}\int_0^{\infty}(1-\alpha)\left(N(\alpha)+\frac{w^2}{\Pr}(1-\alpha)\right)^{-1}\frac{w}{\Pr}\sin(wy)\boldsymbol{E}_{\alpha,1}\left(-\frac{wt^{\alpha}}{\Pr}\left(N(\alpha)+\frac{w^2}{\Pr}(1-\alpha)\right)^{-1}\right)dw, \tag{38}$$

$$\theta_2(y,t)=\frac{2}{\pi}\int_0^{\infty}\alpha\left(N(\alpha)+\frac{w^2}{\Pr}(1-\alpha)\right)^{-1}\frac{wt^{\alpha}}{\Pr}\sin(wy)\boldsymbol{E}_{\alpha,\alpha+1}\left(-\frac{wt^{\alpha}}{\Pr}\left(N(\alpha)+\frac{w^2}{\Pr}(1-\alpha)\right)^{-1}\right)dw, \tag{39}$$

$$\theta_3(y,t)=\sqrt{\frac{2}{\pi}}\int_0^{\infty}(1-\alpha)\left(N(\alpha)+\frac{w^2}{\Pr}(1-\alpha)\right)^{-1}\frac{wt^{\alpha}}{\Pr}\sin(wy)f(w,t)$$
$$\boldsymbol{E}_{\alpha,\alpha+1}\left(-\frac{wt^{\alpha}}{\Pr}\left(N(\alpha)+\frac{w^2}{\Pr}(1-\alpha)\right)^{-1}\right)dw, \tag{40}$$

and

$$\theta_4(y,t)=\sqrt{\frac{2}{\pi}}\int_0^{\infty}\alpha\left(N(\alpha)+\frac{w^2}{\Pr}(1-\alpha)\right)^{-1}\frac{wt^{\alpha-1}}{\Pr}\sin(wy)f(w,t)$$
$$\boldsymbol{E}_{\alpha,\alpha+1}\left(-\frac{wt^{\alpha}}{\Pr}\left(N(\alpha)+\frac{w^2}{\Pr}(1-\alpha)\right)^{-1}\right)dw. \tag{41}$$

2.3 Limiting cases of momentum equation

In this section, some limiting cases are obtained for Jeffrey fluid.

2.3.1 Time fractional model of a second-grade fluid

For $\lambda_1 = 0$ the exact solution (28) will correspond to the exact solution obtained by Sene [Sene (2018)] for second-grade fluid.

Incorporating $\lambda_1 = 0$ in equation (19), we get:

$$\overline{u_s}(w,p) = w\sqrt{\frac{2}{\pi}} \times \frac{p^{\alpha} a_3 + 1}{p\left(a_3 p^{\alpha+1} + p + w^2 + p^{\alpha} w^2 a_5\right)}, \tag{42}$$

here $a_5 = \dfrac{a_2 N(\alpha) + (1-\alpha)}{\alpha}$.

Eq. (29) is almost identical to the Eq. (30) obtained by Sene [Sene (2018)] for second-grade fluid.

2.3.2 Classical model of a second-grade fluid

For $\alpha = 1$ and $\lambda_1 = 0$ the problem will correspond to the heated second-grade fluid [Shen, Tan and Zhao (2006)].

$$\overline{u_s}(w,p) = \sqrt{\frac{2}{\pi}} \times \frac{w}{p\left(p + w^2 \lambda p + w^2\right)} . \tag{43}$$

After decomposing the denominator, Eq. (50) will take the form:

$$\overline{u_s}(w,p) = \frac{1}{w}\sqrt{\frac{2}{\pi}} \times \left[\frac{1}{p} - \frac{1}{p + \dfrac{w^2}{1 + w^2 \lambda}}\right], \tag{44}$$

Taking inverse transform operator of the Fourier sine and Laplace, the solution is given as:

$$u(y,t) = \frac{2}{\pi}\int_0^{\infty} \frac{\sin(wy)}{w}\left[1 - e^{-\frac{w^2}{1+w^2\lambda}t}\right] dw. \tag{45}$$

This is quite in good agreement to the solution obtained by Shen et al. [Shen, Tan and Zhao (2006)].

2.3.3 Newtonian fluid

For $\alpha = 1$ and $\lambda = \lambda_1 = 0$, the Eq. (19) corresponds to Stokes' 1st problem for the classical Newtonian fluid [Shen, Tan and Zhao (2006), Xue and Nie (2009)].

$$\overline{u_s}(w,s)=\sqrt{\frac{2}{\pi}}\times\left\{\frac{w}{p\left(p+w^2\right)}\right\}, \tag{46}$$

After decomposition, Eq. (46) will take the form:

$$\overline{u_s}(w,s)=\sqrt{\frac{2}{\pi}}\times\left\{\frac{1}{p}-\frac{1}{p+w^2}\right\}\frac{1}{w}. \tag{47}$$

Applying inverse Laplace and inverse Fourier sine transformation, the exact solution for Newtonian fluid will be:

$$u(y,t)=erfc\left(\frac{y}{2\sqrt{t}}\right), \tag{48}$$

where $erfc(y)$ is complementary error function.

3 Results and discussion

In this section, graphical results are computed and discussed for embedded parameters. Physical model of the problem is shown in Fig. 1. To elucidate the effects of the relative parameters on velocity and temperature field, Figs. 2-5 are drawn. The influence of fractional parameters α on the velocity profile is highlighted in Fig. 2. It is observed that by increasing the magnitude of α, fluid motion accelerates. The impact of the material parameter λ on the velocity profile is shown in Fig. 3. It is worth noting that as the magnitude of λ rises, a drop in the velocity profile occurs due to enhancement in the viscous forces and elasticity of the flowing fluid.

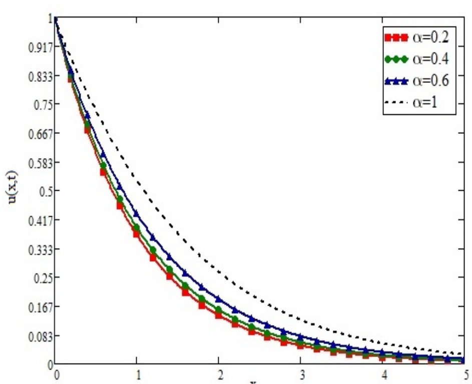

Figure 2: Velocity plot for α

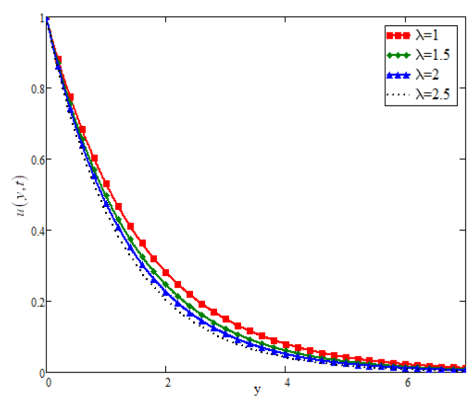

Figure 3: Velocity plot for λ

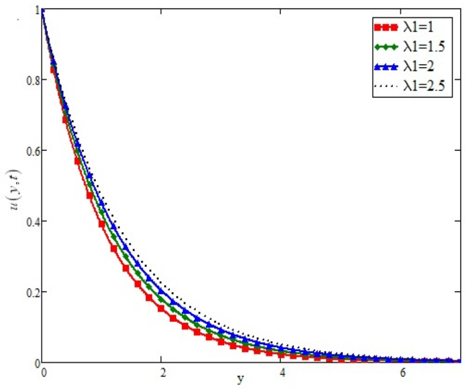

Figure 4: Velocity plot for λ_1

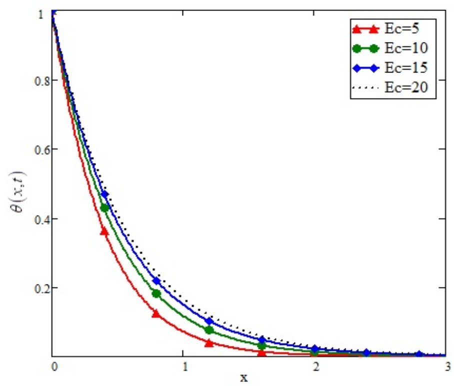

Figure 5: Temperature plot for Ec

Fig. 4 is drawn to check the rheology of fluid in the response of Jeffrey fluid parameter λ_1. As the domain of λ_1 increases, velocity of the fluid boost up also increases. This is due to the fact that an increase in the magnitude of λ_1 , a quick response to the tangential stress is gained. The effect of the Eckert number E_c on heat distribution is shown in Fig. 5. A rise in the temperature profile is reported when the magnitude of Ec increases. Physically it is true because Ec represents the ratio between the kinetic energy and an enthalpy difference of the boundary layer. The positive Ec shows the transfer of the heat to the fluid from the plate, hence greater dissipation of viscosity heat causes an enhancement in the temperature of the fluid.

4 Concluding remarks

For the analysis of Jeffrey fluid model along with the viscous dissipation, the phenomenon has been formulated by using the Atangana-Baleanu time-fractional derivative, where so far, no one has developed the exact solution in the presence of viscous dissipation. Exact expressions are acquired for momentum and energy equation via transformation techniques such as Fourier sine and Laplace. In order to elaborate the influence of implanted parameters on velocity and temperature profiles, the plots have been drawn. For the sake of verification, the present solutions are reduced to the published results in literature. The salient arguments concluded from this study are:

- An increase in fractional order parameter α reduces the velocity, which will be the benchmark for the experimental works.
- The velocity profile rises with the greater magnitude of Jeffrey fluid parameter λ_1, while the opposite trend is reported for the material parameter λ .
- Temperature profile rises for the higher values of Eckert number Ec,

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