Computer Modeling in Engineering & Sciences
DOI:10.32604/cmes.2020.010688
Article

A Bayesian Updating Method for Non-Probabilistic Reliability Assessment of Structures with Performance Test Data

Jiaqi He[1] and Yangjun Luo[1,2,*]

[1]Province Key Laboratory of Advanced Technology for Aerospace Vehicles, School of Aeronautics and Astronautics, Dalian University of Technology, Dalian, 116024, China
[2]State Key Laboratory of Structural Analysis for Industrial Equipment, Dalian University of Technology, Dalian, 116024, China
[*]Corresponding Author: Yangjun Luo. Email: yangjunluo@dlut.edu.cn
Received: 20 March 2020; Accepted: 17 July 2020

Abstract: For structures that only the predicted bounds of uncertainties are available, this study proposes a Bayesian method to logically evaluate the non-probabilistic reliability of structures based on multi-ellipsoid convex model and performance test data. According to the given interval ranges of uncertainties, we determine the initial characteristic parameters of a multi-ellipsoid convex set. Moreover, to update the plausibility of characteristic parameters, a Bayesian network for the information fusion of prior uncertainty knowledge and subsequent performance test data is constructed. Then, an updated multi-ellipsoid set with the maximum likelihood of the performance test data can be achieved. The credible non-probabilistic reliability index is calculated based on the Kriging-based surrogate model of the performance function. Several numerical examples are presented to validate the proposed Bayesian updating method.

Keywords: Convex model; Bayesian method; non-probabilistic reliability; information fusion

1 Introduction

In the modern industries, the consideration of uncertainties in the structural analysis and design is very important since even a little fluctuation of uncertain variables may result in structural performance failure. Therefore, there is a growing emphasis on the study of a quantitative mathematical model that fully characterizes aleatory or epistemic uncertainties. In general, the approaches of quantifying uncertainties can be divided into two categories: the probabilistic and non-probabilistic methods [1].

The probabilistic model that describes aleatory uncertainties to be probability distributions, has been extensively utilized because of its complete theory system [2]. However, in practical engineering applications, the production of precise probability distributions requires sufficient samples of a suitable quality that may be frequently unavailable [3]. In addition, the probabilistic reliability is very sensitive to the probability distributions of uncertain variables [4]. Therefore, for the circumstances of the structural systems suffering from epistemic uncertainties, in which the sample information is limited, the non-probabilistic method that describes uncertainties by utilizing a convex model, such as interval set and multi-ellipsoid set, has been developed as an attractive complementary tool [5,6]. Note that the interval model can be considered as a specific instance of the multi-ellipsoid model if each ellipsoid set comprises only one uncertain parameter. Based on the convex set model, Ben–Haim first introduced the concept of non-probabilistic reliability [7,8]. Over the past decades, numerous studies have been conducted for quantifying uncertainties and assessing the reliability in the non-probabilistic framework for various engineering problems involving incomplete information [9–19]. Moreover, for structures exhibiting both aleatory and epistemic uncertainties, structural reliability assessment and optimization methods based on probabilistic and convex set mixed models have also been substantially discussed by numerous researchers [10–20,25].

In the convex set-based non-probabilistic reliability theory, the multi-ellipsoid model has become one of the most realistic, and general descriptions of epistemic uncertainties. Therefore, based on experimental data, how to accurately construct the multi-ellipsoid convex model is critically important. By employing a transformation matrix for a rotating n-dimensional coordinate system, Zhu et al. [26] typically determined the minimum-volume ellipsoid that contains all uncertain inputs. Jiang et al. [27] used the inverse correlation matrix as the characteristic matrix to construct the ellipsoid model. This correlation matrix comprised variances and co-variances calculated by utilizing each pair of uncertainties that fail to enclose all the sample points in the constructed ellipsoids in some cases. Based on measured uncertainties, Kang et al. [28] recently proposed a general framework for constructing an ellipsoid convex model. This method transforms the ellipsoidal convex model construction problem into a semi-definite programming (SDP) optimization formulation that can be solved by applying a standard SDP optimizer. It is noteworthy that the aforementioned methods are completely based on the measured data of uncertain variables.

However, in many practical engineering problems, the available information of uncertainties is rather limited. In these situations, it is not only the probabilistic distribution of uncertainties but also the constructed convex model, as well as their corresponding reliability assessment, may be inaccurate and unjustified. In some cases, the performance test data, such as displacements, and stresses of structures, may be much more easily obtained when compared with the direct measurement of uncertain variables. Therefore, studies on predicting the uncertainties from performance test data have been carried out by using the inverse problem analysis method [29–35]. Based on the derivation of forward formulas and prior uncertainty knowledge, the inverse problem can be constructed driven by performance test data and iterative regularization and solved by the optimization method to yield the final solution. However, a reasonable inverse problem may be hardly obtained for some practical engineering applications with limited cognition of their forward problems. As revealed by the literature survey, the systematical method that fuses prior uncertainty knowledge and subsequent performance test data for a reasonable assessment of non-probabilistic reliability has not been discussed to the best in previous related publications, and it is the main novelty of this paper.

To this end, we develop a new Bayesian updating method for the multi-ellipsoid convex model based on performance test data in this study. The Bayesian method has been extensively utilized to solve updating problems driven by data in numerous research fields, such as structural health monitoring system [36], model validation [37], structural identification [38], and machine learning [39]. In addition, since the Bayesian method does not require numerous information for the maximum likelihood estimation, it is also useful for reliability analysis with a small number of samples [40,41]. By employing the Bayesian network, we establish the inherent relationship between the plausibility of convex model parameters and performance evidence herein, and we then achieve an updated index of non-probabilistic reliability. Besides, to deal with the highly nonlinear performance function, the Kriging surrogate model is adopted.

As illustrated in Fig. 1, this study mainly evaluates the structural non-probabilistic reliability based on the multi-ellipsoid convex model and the information fusion method of the Bayesian network. By applying the prior knowledge information about uncertainties, the initial parameterized multi-ellipsoid model, and the initial ranges of characteristic parameters are first constructed. Then, the available performance test samples are inserted into a Bayesian network to update the plausibility of each characteristic parameter and further determine the multi-ellipsoid model with maximum likelihood. Based on the updated ellipsoidal convex model, the credible non-probabilistic reliability index can be obtained. Finally, several numerical examples are presented to validate the proposed Bayesian updating method.

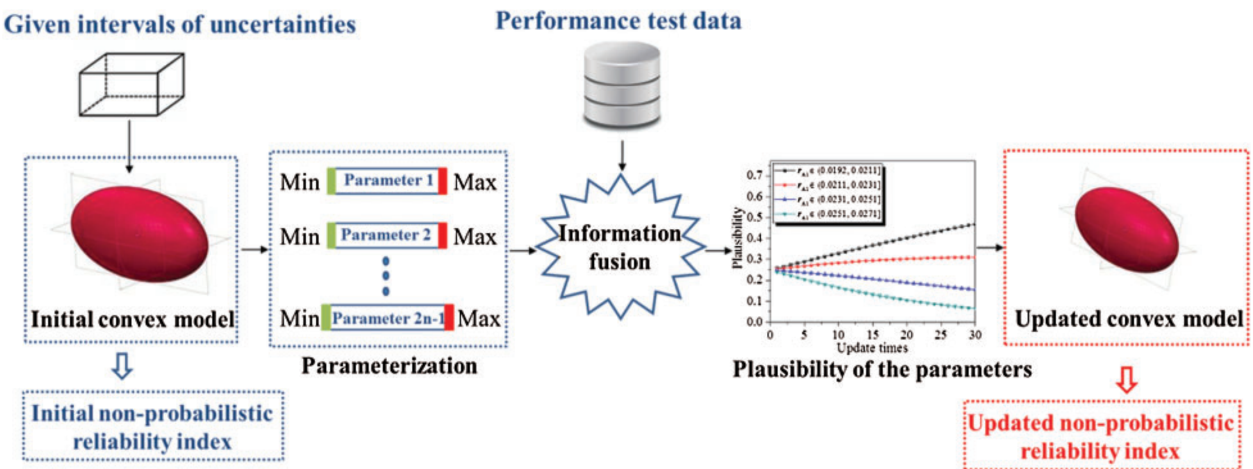


Figure 1: Information fusion of prior uncertainty knowledge and subsequent performance test data Jones et al.

2 The Multi-Ellipsoid Convex Model Used for Bayesian Updating

Assume there are a set of uncertain variables x_j $(j = 1, 2, \ldots, N)$ and their mean value $\hat{x}_j$. Then, the uncertain variables x_j can be transformed as dimensionless variables δ_j as follows:

$$\delta_j = \frac{x_j - \hat{x}_j}{\hat{x}_j}, \quad (j = 1, 2, \ldots, N) \tag{1}$$

The uncertainties may arise from multiple sources in practical applications. For such a situation, the group that comprises uncertainties from the same or similar sources can be described by one ellipsoid set; thus several ellipsoid sets are constructed based on various uncertain sources [18]. The so-called multi-ellipsoid convex model can be defined as follows:

$$\Omega = \left\{ \boldsymbol{\delta} \,\middle|\, \boldsymbol{\delta}_i^{\mathrm{T}} \mathbf{W}_i \boldsymbol{\delta}_i \le 1 \right\} \quad (i = 1, 2, \ldots, N_E) \tag{2}$$

where $\boldsymbol{\delta} = \{\delta_1, \delta_2, \ldots, \delta_N\}^{\mathrm{T}}$ denotes the vector of dimensionless uncertain variables, $\mathbf{W}_i$ denotes the characteristic matrix of the ith ellipsoid set, and N_E denotes the total number of ellipsoid sets.

2.1 Parameterization of Multi-Ellipsoid Convex Model

Characteristic matrix $\mathbf{W}_i$ contains the mixed information of direction and size of the ith ellipsoidal set. To update the convex model by employing a Bayesian method, it is convenient to express the multi-ellipsoid convex model by several characteristic parameters. An eigenvalue decomposition for matrix $\mathbf{W}_i$ is performed by applying the following equation:

$$\mathbf{W}_i = \mathbf{Q}_i{}^{\mathrm{T}} \Sigma_i \mathbf{Q}_i \quad (i = 1, 2, \ldots, N_E) \tag{3}$$

where Σ_i denotes a diagonal matrix, in which the diagonal elements are the eigenvalues of the matrix $\mathbf{W}_i$, and $\mathbf{Q}_i$ denotes the eigenvector matrix that indicates the direction of the ith ellipsoid set.

For the ith ellipsoid set that includes N_i uncertain parameters, the $N_i \times N_i$ characteristic matrix $\mathbf{W}_i$ can be directly related to the semi-axis lengths r_k $(k = 1, 2, \ldots, N_i)$ and the angles θ_k $(k = 1, 2, \ldots, N_i - 1)$ of the ith ellipsoid based on the Gramm–Schmidt orthogonalization procedure [21] as follows:

$$\begin{aligned} &\Sigma_i = diag\left(r_1, r_2, \ldots, r_{N_i}\right) \\ &\mathbf{Q}_i = \left[\mathbf{U}_1, \mathbf{U}_2, \ldots, \mathbf{U}_{N_i}\right] \end{aligned} \tag{4}$$

where

$$\mathbf{U}_1 = \begin{Bmatrix} \cos\theta_1 \\ \sin\theta_2 \cos\theta_2 \\ \vdots \\ \sin\theta_1 \sin\theta_2 \ldots \sin\theta_{N_i-2} \cos\theta_{N_i-1} \\ \sin\theta_1 \sin\theta_2 \ldots \sin\theta_{N_i-2} \sin\theta_{N_i-1} \end{Bmatrix}, \quad \mathbf{U}_k = \begin{Bmatrix} \mathbf{0}_{k-2} \\ -\sin\theta_{k-1} \\ \cos\theta_{k-1} \cos\theta_k \\ \vdots \\ \cos\theta_{k-1} \sin\theta_k \ldots \sin\theta_{N_i-2} \cos\theta_{N_i-1} \\ \cos\theta_{k-1} \sin\theta_k \ldots \sin\theta_{N_i-2} \sin\theta_{N_i-1} \end{Bmatrix}, \quad k = 2, 3, \ldots, N_i \tag{5}$$

and vector $\mathbf{0}_{k-2}$ comprises $k-2$ zero elements. Here, θ_k denotes the angle between the major axis of the ellipsoid and the kth Cartesian coordinate axis.

As depicted in Fig. 2, a two-dimensional ellipsoid model can be typically represented by the above-parameterized expressions with the following three characteristic parameters: (r_1, r_2, θ). After the parameterization in Eq. (4), the multi-ellipsoid convex model is determined by utilizing a total of $2N_i - 1$ characteristic parameters (r_k and θ_k) as follows:

$$\Omega = \left\{ \mathbf{x} \,\middle|\, \boldsymbol{\delta}_i{}^{\mathrm{T}} \mathbf{Q}_i{}^{\mathrm{T}} \Sigma_i \mathbf{Q}_i \delta_i \le 1 \right\} \quad (i = 1, 2, \ldots, N_E) \tag{6}$$

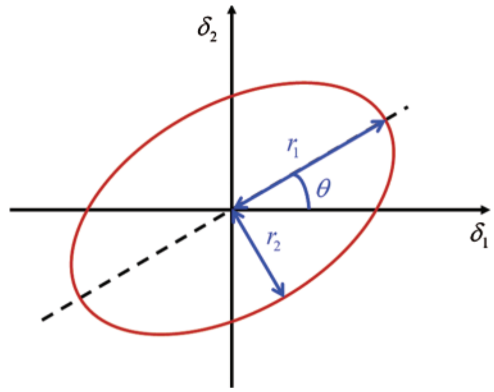


Figure 2: The parameterization of a two-dimensional ellipsoid

2.2 Construction of the Initial Ellipsoid Convex Model

It is assumed that the dimensionless uncertain variables $\delta_j (j = 1, 2, \ldots, N)$ are within the given intervals, namely $\delta_j \in \left[-\Delta_j, \Delta_j\right]$, where $-\Delta_j$ and Δ_j denote the lower bound and the upper bound of the uncertain variables δ_j, respectively. In this study, we selected the initial ith ellipsoid convex set to be the minimum circumscribed ellipsoid of the given intervals of the ith group of the uncertain variables. Then, the semi-axis lengths r_j^0 of the initial ith ellipsoid set can be calculated by employing the following equation:

$$r_k^0 = \sqrt{N_i}\Delta_k, \quad (k = 1, 2, \ldots, N_i), \tag{7}$$

while the corresponding characteristic angles are given by $\theta_1^0 = 0$, $\theta_k^0 = \pi/2, (k = 2, \ldots, N_i - 1)$.

During the updating process of the multi-ellipsoid convex model, the allowable ranges of the characteristic parameters r_k and θ_k are defined as follows:

$$\begin{aligned} & r_k \in \left[\boldsymbol{\delta}_k, r_k^0\right] \quad (k = 1, 2, \ldots, N_i) \\ & \theta_k \in \left[\theta_k^0 - \frac{\pi}{4}, \theta_k^0 + \frac{\pi}{4}\right] \quad (k = 1, 2, \ldots, N_i - 1) \end{aligned} \tag{8}$$

3 Assessment of Non-Probabilistic Reliability Based on Kriging Surrogate Model

We define the normalized vectors of uncertain variables as follows:

$$\mathbf{q}_i = \Sigma_i{}^{1/2}\mathbf{Q}_i{}^{\mathrm{T}}\boldsymbol{\delta}_i \quad i = 1, 2, \ldots, N_E, \tag{9}$$

The multi-ellipsoid domain Ω in Eq. (6) is mapped into the normalized $\mathbf{q}$-space and a multiple spheres of a unit radius in each sub-dimensional space spanned by $\mathbf{q}_i$ is defined as follows:

$$EM = \left\{\mathbf{q} \left| \sqrt{\mathbf{q}_i{}^{\mathrm{T}}\mathbf{q}_i} \leq 1 \right.\right\}, \quad i = 1, 2 \ldots, N_E \tag{10}$$

After the aforementioned normalization of uncertain variables, the normalized performance function can be denoted by $G(\mathbf{q})$ in $\mathbf{q}$-space. By considering that the performance function $G(\mathbf{q})$ may be highly nonlinear and multi-peak, the Kriging surrogate model associated with a training point strategy is applied to provide good approximates and error estimates. In addition, the Kriging surrogate model [42] is assumed as a Gaussian process, and the approximation performance function is constructed as follows:

$$G(\mathbf{q}) \approx \mathbf{h}(\mathbf{q})^{\mathrm{T}} \boldsymbol{\beta} + Z(\mathbf{q}) \tag{11}$$

where $\boldsymbol{\beta}$ denotes the vector of undetermined coefficients, $\mathbf{h}(\mathbf{q})^{\mathrm{T}}$ can be set as any function of $\mathbf{q}$, and the stochastic error $Z(\mathbf{q})$ is defined by a Gaussian process with the mean of zero and the variance of σ^2.

The correlation between two errors at the points of $\mathbf{q}^{(a)}$ and $\mathbf{q}^{(b)}$ can be defined as follows

$$\mathrm{Cov}\left[Z\left(\mathbf{q}^{(a)}\right), Z\left(\mathbf{q}^{(b)}\right)\right] = h^2 \exp\left[-\sum_{k=1}^{N} p_k \left(q_k^{(a)} - q_k^{(b)}\right)^2\right] \tag{12}$$

where N denotes the dimensionality of $\mathbf{q}$, while h and p_k are parameters to be determined by the maximum likelihood estimation. To obtain a relatively accurate surrogate model, we perform two steps in this study. First, an initial surrogate model is constructed based on samples generated by the orthogonal-maximin Latin hypercube sampling method [43]. Second, several training points are added into the model point by point to obtain a more accurate model, which is called the efficient global optimization method [44]. Here, two expected improvement functions are given as the rule for adding training points.

The first expected improvement function enables the surrogate model to be more accurate in the whole design domain. A new sample point $\mathbf{q}_\sigma{}^*$ is defined as the maximum departure of the model in the whole design domain. It can be obtained by solving the following optimization problem:

$$\begin{aligned} &\mathrm{Max} \quad \left[\sigma^2(\mathbf{q})\right] \\ &s.t. \quad \underline{\mathbf{q}} \le \mathbf{q} \le \bar{\mathbf{q}} \end{aligned} \tag{13}$$

where $\underline{\mathbf{q}}$ and $\bar{\mathbf{q}}$ denote the lower and upper bounds of variable $\mathbf{q}$, respectively.

The second expected improvement function enables the prediction to be more accurate near the failure surface where the performance function $G(\mathbf{q})$ is equal to zero inside the design domain. Therefore, the improvement function, F, can be defined as follows [45]:

$$F = \varepsilon - \left|\tilde{G}(\mathbf{q})\right|, \quad \tilde{G}(\mathbf{q}) \in [-\varepsilon, +\varepsilon] \tag{14}$$

where $\tilde{G}(\mathbf{q})$ denotes the predicted response of $\mathbf{q}$, and ε can be taken as 2σ based on [44]. Solving the following problem yields a new sample point $\mathbf{q}_{EF}{}^*$:

$$\begin{aligned} &\mathrm{Max} \quad E[F(\mathbf{q})] \\ &s.t. \quad \underline{\mathbf{q}} \le \mathbf{q} \le \bar{\mathbf{q}} \end{aligned} \tag{15}$$

When the cumulative variance (EISE) of M test samples is less than the given criterion, the process of adding sample points ends:

$$EISE = \frac{1}{M} \sum_{m=1}^{M} \left(G(\mathbf{q}_m) - \tilde{G}(\mathbf{q}_m)\right)^2 \tag{16}$$

where $\mathbf{q}_m$ denotes the mth test sample, while $G(\mathbf{q}_m)$ and $\tilde{G}(\mathbf{q}_m)$ denote the actual and predicted responses of $\mathbf{q}_m$, respectively.

Additionally, based on the multi-ellipsoidal convex model EM and the performance function $G(\mathbf{q})$, the safety margin of structures can be evaluated by utilizing both the non-probabilistic reliability index η [20] and the non-probabilistic reliability R_c [10]. As depicted in Fig. 3, the non-probabilistic reliability index is defined as the minimum distance from the origin of the coordinates to the boundary of the failure domain in $\mathbf{q}$-space. This index can be obtained by solving the following min–max optimization problem:

$$\begin{aligned} &\eta = sgn \cdot \min_{\mathbf{q}} \left\{ \max \left(\sqrt{\mathbf{q}_i{}^T \mathbf{q}_i} \right) \right\} \quad i = 1, 2, \ldots, N_E \\ &\text{s.t.} \quad G(\mathbf{q}) = 0 \\ &sgn = \begin{cases} 1 & \text{if} \quad G(\mathbf{0}) > 0 \\ 0 & \text{if} \quad G(\mathbf{0}) = 0 \\ -1 & \text{if} \quad G(\mathbf{0}) < 0 \end{cases} \end{aligned} \tag{17}$$

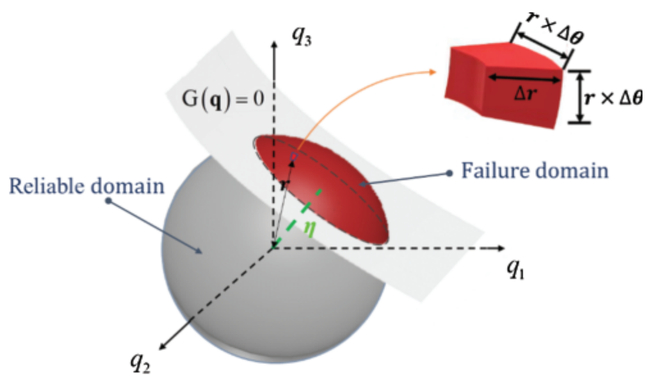


Figure 3: A schematic representation of the non-probabilistic reliability assessment

The non-probabilistic reliability R_c is defined as the multi-dimensional volume ratio of the reliable domain to the whole multi-dimensional spherical domain [9], while the corresponding dangerous degree can be defined as $F_c = 1 - R_c$. As shown in Fig. 3, by considering the case of a single three-dimensional ellipsoid convex model in $\mathbf{q}$-space, the convex set is equally divided into as follows: m_r small lengths (Δr) in the polar radius direction, m_1^θ small angles ($\Delta\theta$), and m_2^θ small angles ($\Delta\theta$) in the polar angle directions. The whole uncertainty domain is thus divided into $M = m_r \times m_1^\theta \times m_2^\theta$ element regions. The performance function responses inside the element region can be represented by the response of its center point. Thus, the non-probabilistic reliability can be calculated by utilizing the ratio of the element region within the reliable domain to the whole convex set region. Obviously, the non-probabilistic reliability can be easily extended to an n-dimensional convex model defined as follows:

$$R_c = 1 - \frac{\Omega_{G<0}}{\Omega} = 1 - \frac{\sum_{G<0} (r_m \cdot \Delta\theta)^{n-1} \cdot \Delta r}{\sum_{m=1}^{M} (r_m \cdot \Delta\theta)^{n-1} \cdot \Delta r} = 1 - \frac{\sum_{G<0} r_m{}^{n-1}}{\sum_{m=1}^{M} r_m{}^{n-1}} \tag{18}$$

where Ω denotes the whole convex set domain, $\Omega_{G<0}$ denotes the failure domain, and r_m denotes the polar radius of the mth element region in the polar coordinate system.

For the case of a multi-ellipsoid convex model, the non-probabilistic reliability can be easily evaluated by

$$R_c = 1 - \frac{\Omega_{G<0}}{\Omega} = 1 - \frac{\sum_{G<0}\left[\prod_{i=1}^{N_E}\left(r_{i,\mathrm{m}} \cdot \Delta\theta\right)^{N_i-1}\Delta r\right]}{\sum_{m=1}^{M}\left[\prod_{i=1}^{N_E}\left(r_{i,m} \cdot \Delta\theta\right)^{N_i-1}\Delta r\right]} = 1 - \frac{\sum_{G<0}\left[\prod_{i=1}^{N_E} r_{i,\mathrm{m}}{}^{N_i-1}\right]}{\sum_{m=1}^{M}\left[\prod_{i=1}^{N_E} r_{i,m}{}^{N_i-1}\right]} \tag{19}$$

where the subscript i denotes the ith ellipsoid model among the N_E ellipsoid models, and N_i is the number of uncertain parameters included in the ith ellipsoidal set.

4 Bayesian Updating Method for Multi-Ellipsoid Convex Model

4.1 Bayesian Network

In this study, we assume that the uncertain variables are uniformly distributed within the multi-ellipsoid convex model. In this case, the definition of non-probabilistic reliability R_c in Eq. (19) coincides with that of the reliability concept in the probabilistic framework. Moreover, the probability of an event or performance function $G(\mathbf{q})$ can be considered depending on the uncertainty of the characteristic parameters $\mathbf{X} = (r_1, r_2, \ldots, r_N, \theta_1, \theta_2, \ldots, \theta_{N-1})$, which denotes the variation of the multi-ellipsoid convex model. Thus, based on the Bayesian' rule, one can develop the inverse statistical inferences from the observation data of E to the prior knowledge of characteristic parameters $\mathbf{X}$ as follows:

$$p(\mathbf{X}|E) = \frac{p(\mathbf{X})\,p(E|\mathbf{X})}{p(E)} \tag{20}$$

where $p(\mathbf{X})$ denotes the prior distribution obtained from the prior knowledge, $p(\mathbf{X}|E)$ denotes the posteriori distribution that can be interpreted as the amended prior knowledge based on observed data, $p(E|\mathbf{X})$ denotes the data distribution in terms of the given $\mathbf{X}$ based on the observed data E, and $p(E) = \sum_{\mathbf{X}} p(\mathbf{X})\,p(E|\mathbf{X})$.

Herein, we adopt the concept of plausibility of the characteristic parameters, which is defined as the probability of the characteristic parameter X falling within the subinterval I as follows:

$$\mathrm{Pl}(X \in I) = \mathrm{Pr}(X|\, X \in I) \tag{21}$$

Based on the Bayesian theory, the updating process of the convex model can be considered as the plausibility inferences of the ellipsoidal characteristic parameters driven by performance test data. For the case of a multi-ellipsoid model with multiple characteristic parameters, the updating problem can be described as a multi-hierarchical Bayesian network, as demonstrated in Fig. 4. In the proposed Bayesian network, the leaf node E denotes the evidence variable that represents the test data information. Additionally, the second level node EM_i denotes the ith ellipsoid convex set. The root nodes $(X_{i,1}, X_{i,2}, \ldots, X_{i,2N_i-1})$ denote the characteristic parameters of the ith ellipsoid convex model. Therefore, the plausibility of the root nodes can be updated by applying the multi-hierarchical Bayesian calculation when the leaf node information is inputted.

4.2 Prior Probability and Conditional Probabilities

During the updating process, the allowable range of each root node variable is divided into m subintervals. For the cases with limited prior knowledge about the characteristic parameters, the prior probability of the jth characteristic variable $X_{i,j}$ of the ith ellipsoid convex set falling within the qth subinterval $I_q = \left(X_{i,j}^l, X_{i,j}^u\right]$ can be calculated as follows,

$$\Pr\left(X_{i,j} \middle| X_{i,j} \in I_q\right) = \frac{1}{m} \tag{22}$$

In addition, any other distribution of the plausibility of the characteristic parameters with available prior knowledge can be used.

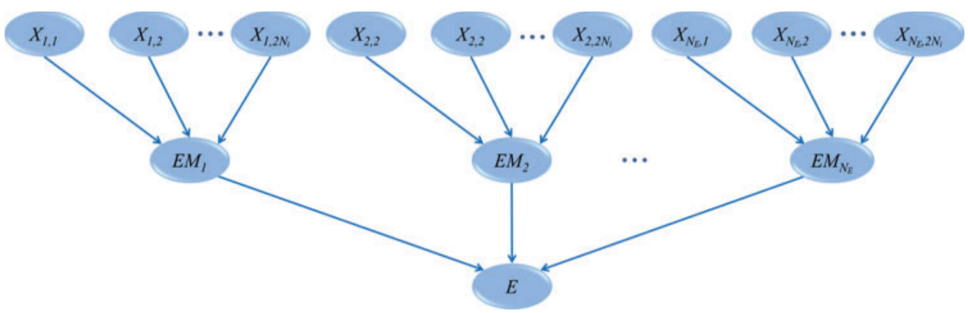


Figure 4: A schematic diagram of a Bayesian network

Moreover, if the root node variable $X_{i,j}$ is determined, the ellipsoid convex model EM_i can be uniquely determined, yielding the following conditional probability:

$$\Pr\left(EM_i \middle| X_{i,j}\right) = 1, \quad (i = 1, 2, \ldots, N_E;\ j = 1, 2, \ldots, N_i) \tag{23}$$

In addition, the evidence variable E can be defined as follows. If the structural performance test fails to satisfy the specified requirements, then $E = 0$; otherwise, $E = 1$, that is,

$$\begin{cases} E = 0 & if \quad G(\mathbf{q}_K) < 0 \\ E = 1 & if \quad G(\mathbf{q}_K) \geq 0 \end{cases} \tag{24}$$

where $\mathbf{q}_K$ denotes the Kth performance test sample.

Thus, the conditional probability expression can be defined as follows:

$$\Pr(E = 0 \,|EM) = F_c; \quad \Pr(E = 1 \,|EM) = R_c \tag{25}$$

4.3 Bayesian Updating Process

To achieve the initial plausibility of the characteristic parameters, the range of each characteristic parameter is divided into m subintervals before the Bayesian updating process. Then, based on the Bayesian network, the posteriori distribution of each root node variable $X_{i,j}$ can be calculated by the input evidence variable E. Moreover, the current prior distribution is considered as the posteriori distribution in the former step as follows:

$$\begin{aligned} \Pr^{\kappa+1}\left(X_{i,j} \in \left(X_{i,j}^l, X_{i,j}^u\right]\right) &= \Pr^{\kappa}\left(X_{i,j} \in \left(X_{i,j}^l, X_{i,j}^u\right] \middle| E = 0\right) \\ \Pr^{\kappa+1}\left(X_{i,j} \in \left(X_{i,j}^l, X_{i,j}^u\right]\right) &= \Pr^{\kappa}\left(X_{i,j} \in \left(X_{i,j}^l, X_{i,j}^u\right] \middle| E = 1\right) \end{aligned} \tag{26}$$

where the superscript κ denotes the κth iteration process.

When the updated plausibility of the characteristic parameters is obtained by applying Eq. (26), the allowable range of characteristic parameters can be reduced by ignoring the subintervals with small plausibility (e.g., lower than $\frac{1}{m}$). Thus, the processes of updating plausibility and reducing the range of characteristic parameters are repeated until the convergence condition is satisfied. The convergence criterion is given as follows: the characteristic parameter range $\left(X_{i,j}^{l}, X_{i,j}^{u}\right]$ for the next iteration step is actually small, or the difference between the maximum and minimum plausibilities of the subintervals within the present characteristic parameter range is actually small, that is,

$$\begin{cases} \zeta_1 = \left(X_{i,j}^{u} - X_{i,j}^{l}\right)/I^0 \leq \varepsilon \\ \zeta_2 = \max\left[\mathrm{Pl}\left(X_{i,j} \in I_q\right)\right] - \min\left[\mathrm{Pl}\left(X_{i,j} \in I_q\right)\right] \leq \varepsilon, \quad q = 1, 2, \ldots, m \end{cases} \tag{27}$$

where I^0 denotes the initial allowable range of the characteristic parameters, I_q denotes the qth subinterval within the present characteristic parameter range, m denotes the total number of the subintervals, and $\varepsilon = 0.01$ is a small number. Finally, by taking the characteristic parameters as the mean value of the subinterval with the maximum plausibility in the last updating step, an updated ellipsoidal convex model can be achieved.

The main computational cost of the updating procedure is the calculation of non-probabilistic reliabilities based on various multi-ellipsoid convex models. However, the calculation of R_c based on each multi-ellipsoid convex model is independent with others, so the parallel computing can be carried out to reduce the computational time tremendously.

It is noteworthy that the proposed Bayesian updating method is effective only when the initial non-probabilistic reliability index is less than 1. However, if the initial non-probabilistic reliability index is greater than 1, there is no unreliable point inside the ellipsoid domain. In general, whichever characteristic parameters are taken within the characteristic parameter ranges, the structural performance failure possibility is identically equal to zero, resulting in an invalid updating of the proposed Bayesian method. In this case, a new performance function $G_{new} = G - \Delta$ and its corresponding performance test data are used to update the convex model. Here, the parameter Δ is determined by setting the initial non-probabilistic reliability index of G_{new} to be equal to the given value of $\underline{\eta} = 0.99$. Furthermore, after the final updated convex set is obtained, the non-probabilistic reliability index of G can be determined by solving the optimization problem of Eq. (17).

5 Programming Procedure

The flow chart of the proposed method is shown in Fig. 5. Additionally, a detailed algorithm for the Bayesian updating method is the following.

Step 1: The initial parameterized multi-ellipsoidal model is constructed as the minimum circumscribed ellipsoid of the given intervals of uncertainties. Based on the given intervals of uncertainties, the bounds of the characteristic parameters that comprise semi-axis lengths and directions are calculated, as shown in Section 2.

Step 2: Based on the initial multi-ellipsoidal convex model obtained in Step 1, the initial non-probabilistic reliability index can be calculated by solving the min–max optimization problem.

Step 3: If the non-probabilistic reliability index η is greater than $\underline{\eta}$, the new performance function is determined by moving the failure surface. Then, return to Step 2; otherwise, move on to Step 4.

Step 4: The Kriging surrogate model used for approximating the performance function is constructed. First, based on the samples given by orthogonal-maximin Latin hypercube sampling, an initial Kriging model is constructed, and the subsequent samples are added to enable the surrogate model to be more accurate based on the two sampling criteria expressed in Eqs. (13) and (15).

Step 5: As demonstrated in Section 4, a Bayesian network is constructed to update the plausibility of the characteristic parameter driven by the performance test data. The final subintervals of characteristic parameters that satisfy the convergence condition defined by Eq. (26) are achieved. Then, the updated characteristic parameters and the updated convex model are produced.

Step 6: Based on the updated ellipsoidal convex model obtained in Step 5, the non-probabilistic reliability index can be calculated.

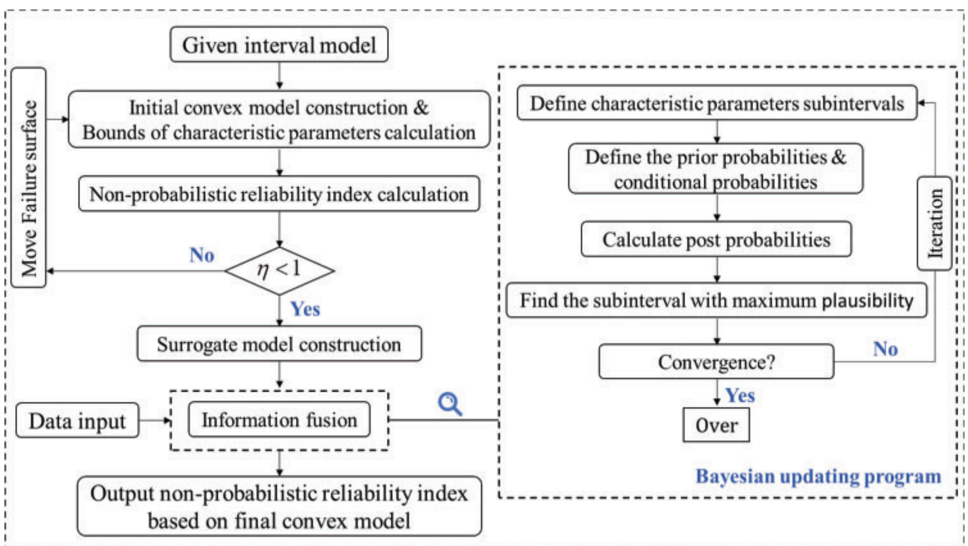


Figure 5: A flow chart of the proposed method

6 Numerical Examples

6.1 A Cantilever Beam with an Initial Non-Probabilistic Reliability Index Less Than 1

As demonstrated in Fig. 6, we consider a cantilever beam with the inertia moment of the cross section $I_m = 118.6 \times 10^6$ mm^4 under a uniformly distributed load. The uncertain variables are the uniformly distributed load and the Young's modulus of the material. The available information about the uncertain variables are their bounds, namely, $P \in [20, 30]$ KN/m and $E \in [195, 205]$ Gpa.

If the angle displacement α of node 1 exceeds 0.0015 Rad, the structure is deemed to have failed. Then, the performance function is expressed as follows: $g = 0.0015^2 - \alpha^2$. After generating the uncertain variables within their interval ranges respectively, the performance test samples (including 30 angular displacement samples of node1) can be obtained by numerical simulations which are presented in Tab. 1.

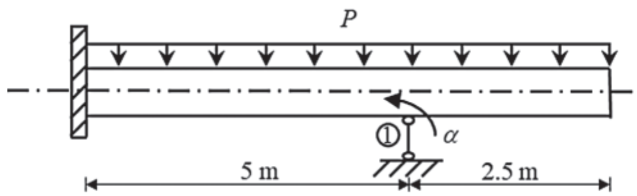


Figure 6: A cantilever beam

Table 1: Performance test samples $\left(\times 10^{-3}\ \text{Rad}\right)$

1.144	1.290	1.244	1.197	1.414	1.115	1.135	1.279	1.146	1.186
1.261	1.275	1.209	1.233	1.272	1.254	1.151	1.413	1.127	1.246
1.128	1.279	1.209	1.122	1.105	1.287	1.116	1.190	1.262	1.228

As demonstrated in Section 2, a two-dimensional ellipsoid convex model is constructed as follows:

$$\begin{pmatrix} E \\ P \end{pmatrix}^T \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \begin{bmatrix} r_E^{-2} & 0 \\ 0 & r_P^{-2} \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}^T \begin{pmatrix} E \\ P \end{pmatrix} \leq 1 \tag{28}$$

where r_E, r_P, and θ denote characteristic parameters to be determined by the Bayesian updating method. The initial non-probabilistic reliability index is calculated based on the initial ellipsoidal convex model, $\eta_{Initial} = 0.8589$, meaning that the structure design is unreliable with the considered uncertainties. To incorporate the performance test data into the inference, we performed the Bayesian network based on the method demonstrated in Section 4 by making the three parameters (r_E, r_P, and θ) as root variables and the performance test data as evidence variables. The range of the root variables is first divided into four subintervals, and the prior distribution is defined based on the method illustrated in Subsection 4.1. Moreover, the calculation result of the first iteration step is demonstrated in Fig. 7.

According to Eq. (27), we can observe that the characteristic parameter r_E satisfies the convergence condition. Thus, it is unnecessary to take a further iteration step to obtain a more precise subinterval of r_E, and r_E can be set as the mean value of the subinterval with the maximum plausibility in the first update step. For the characteristic parameter r_P, the plausibilities of the subinterval (0.1341, 0.1456] and (0.1456, 0.1571] are both lower than 0.25, which can be ignored in further iteration steps. Therefore, a new range [0.1111, 0.1314] is used for the next step. Analogously, a new range $(-\pi/4,\ 0]$ of the characteristic parameter θ is used for the next iteration step. After dividing the new ranges into four subintervals, the second iteration step is performed; the calculation result is shown in Tab. 2. Both characteristic parameters r_P and θ satisfy the convergence condition in the second iteration step, and r_P and θ can be set as the mean value of the subintervals (0.1111, 0.1341] and $(-3\pi/16, -\pi/8]$, respectively. As depicted in Fig. 8, the updated ellipsoidal convex model is constructed by the final characteristic parameters.

Based on the updated ellipsoidal convex model, the calculated non-probabilistic reliability index is greater than 1. Thus, rather than the initial assessment of judging the structure performance as a failure, the actual structure performance is much more likely to be safe when the structure performance test information is involved in the inference. Therefore, to apply the proposed method, the variable uncertainties can be determined by fusing the performance function test data in the inference, thereby yielding a more credible structural reliability assessment.

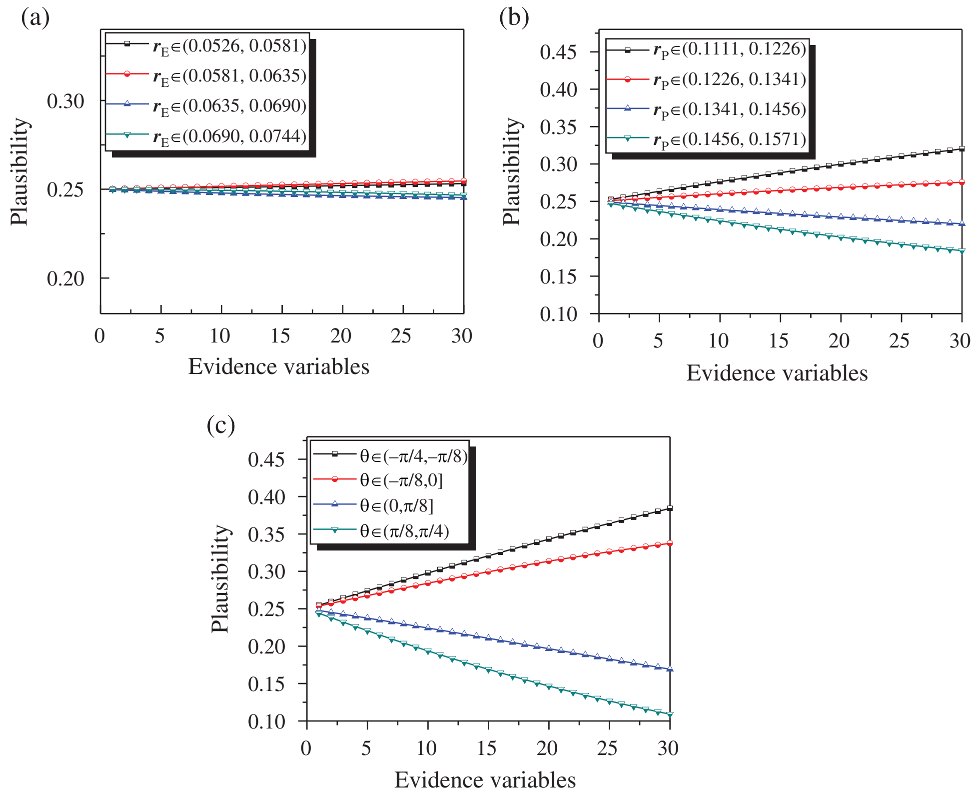


Figure 7: The plausibility calculation result of first iteration step. (a) r_E (b) r_P (c) θ

Table 2: The calculation result of the second iteration step

r_P	Subinterval	(0.1111, 0.1169]	(0.1169, 0.1226]	(0.1226, 0.1284]	(0.1284, 0.1341]	ζ_1	ζ_2
	Plausibility	0.2505	0.2505	0.2505	0.2485	0.3761	0.0020
θ	Subinterval	$(-\pi/4, -3\pi/16]$	$(-3\pi/16, -\pi/8]$	$(-\pi/8, -\pi/16]$	$(-\pi/16, 0]$	ζ_1	ζ_2
	Plausibility	0.2504	0.2508	0.2503	0.2485	0.5000	0.0020

6.2 A Cantilever Beam with an Initial Non-Probabilistic Reliability Index Greater Than 1

A mechanical model, which is defined in Subsection 6.1, is used in this numerical example with a reliable initial assessment. The bounded intervals of the uniformly distributed load and the Young's modulus are $P \in [18, 26]$ KN/m and $E \in [195, 225]$ Gpa, respectively. Although the test data are all reliable as presented in Tab. 3, a relatively more credible ellipsoidal model can be achieved by applying the proposed method.

Table 3: Performance test samples $(\times 10^{-3}\ \text{Rad})$

1.235	1.356	1.190	1.148	1.006	1.277	1.280	1.138	1.078	1.083
1.139	1.114	1.201	1.430	1.178	1.096	1.234	1.147	1.495	1.383
1.401	1.103	1.212	1.345	1.076	1.151	1.121	1.193	1.191	1.291
1.035	1.086	1.068	1.157	1.372	1.430	1.351	1.286	1.304	1.154
0.992	1.230	1.085	1.023	1.391	1.069	1.328	1.126	1.239	1.096

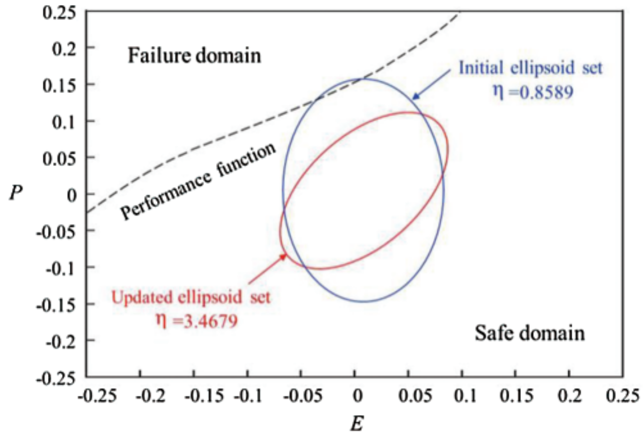


Figure 8: The updated ellipsoidal convex model

First, the performance function is moved to the boundary of the uncertain domain, achieving the initial non-probabilistic reliability index as 0.99 s, the original test data can be disposed as shown in Subsection 4.3, achieving one failure sample. Then, the Bayesian update programming can be performed to obtain the most credible characteristic parameters driven by the test data. The calculation result in the first iteration step is shown in Fig. 9.

By removing the insensitive characteristic parameters and reducing the range of the sensitive characteristic parameters according to Eq. (27), we performed the subsequent iteration steps. The results are presented in Tabs. 4–6. The characteristic parameters r_P and θ satisfy the convergence condition in the third and sixth steps, respectively. As shown in Fig. 9, when the 19th sample is inputted in the Bayesian updating program, the plausibility of each parameter changes significantly. It is because the 19th sample is a failure sample after moving the performance function, and then the occurrence of the failure sample makes the plausibility of the combination of the parameters which can make the failure occur in a relative higher probability increase, simultaneously, making the plausibility of other combinations of the parameters decrease suddenly.

Finally, by taking the characteristic parameters as the mean value of the subintervals with the maximum plausibility, we achieved the updated ellipsoidal convex model illustrated in Fig. 10.

6.3 The Reliability Analysis of a Frame Structure

We consider a frame structure in which the uncertain variables are the elasticity modulus of each beam and the uniformly distributed load applied on each beam, as shown in Fig. 11. The bounded intervals of the Young's modulus of each beam (E_1, E_2, E_3) are $[2.9, 3.1] \times 10^{11}$ Pa, while their uniformly distributed loads are $P_1 \in [3800, 4400]$ N/m, $P_2 \in [2500, 3000]$ N/m, and $P_3 \in [2500, 3000]$ N/m.

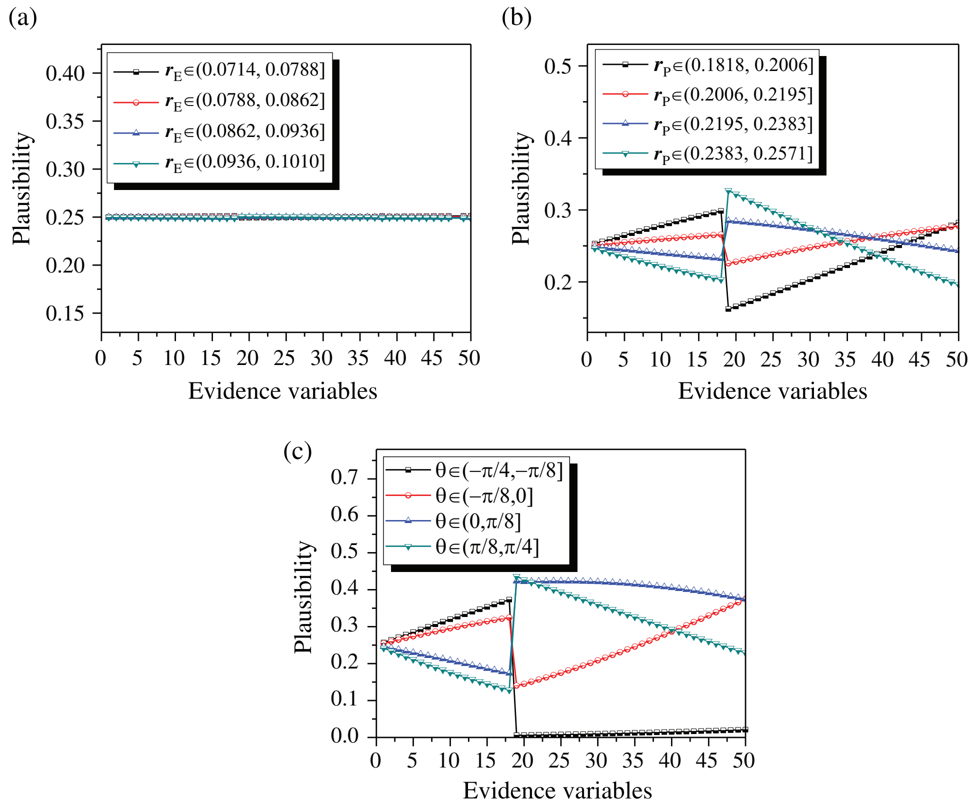


Figure 9: The plausibility calculation result of the first iteration step. (a) r_E, (b) r_P, (c) θ

Table 4: The calculation result of the second iteration step

r_P	Subinterval	(0.1818, 0.1912]	(0.1912, 0.2006]	(0.2006, 0.2101]	(0.2101, 0.2195]	ζ_1	ζ_2
	Plausibility	0.2517	0.2635	0.2531	0.2317	0.3758	0.0318
θ	Subinterval	$(-\pi/8, -\pi/16]$	$(-\pi/16, 0]$	$(0, \pi/16]$	$(\pi/16, \pi/8]$	ζ_1	ζ_2
	Plausibility	0	0.3412	0.3682	0.2907	0.5890	0.3682

Table 5: The calculation result of the third iteration step

r_P	Subinterval	(0.1818, 0.1889]	(0.1889, 0.1959]	(0.1959, 0.2030]	(0.2030, 0.2101]	ζ_1	ζ_2
	Plausibility	0.2515	0.2564	0.2528	0.2394	0.2815	0.0170
θ	Subinterval	$(-\pi/16, -\pi/64]$	$(-\pi/64, \pi/32]$	$(\pi/32, 5\pi/64]$	$(5\pi/64, \pi/8]$	ζ_1	ζ_2
	Plausibility	0.0960	0.3353	0.3027	0.2660	0.2188	0.3682

When the angle displacement α of node 1 exceeds 0.0015 Rad, the structure is regarded as a failure. A total of 30 performance test samples are presented in Tab. 7.

Table 6: The calculation result of the characteristic parameter θ in subsequent iteration steps

Step 4	Subinterval Plausibility	(−0.0491, 0.0614] 0.2620	(0.0614, 0.1718] 0.2737	(0.1718, 0.2823] 0.2432	(0.2823, 0.3927] 0.2211	ζ_1 0.1406	ζ_2 0.0526
Step 5	Subinterval Plausibility	(−0.0491, 0.0061] 0.1937	(0.0061, 0.0609] 0.2601	(0.0609, 0.1166] 0.2767	(0.1166, 0.1718] 0.2696	ζ_1 0.1055	ζ_2 0.0830
Step 6	Subinterval Plausibility	(0.0061, 0.0476] 0.2328	(0.0476, 0.0890] 0.2556	(0.0890, 0.1304] 0.2591	(0.1304, 0.1718] 0.2524	ζ_1 0.0790	ζ_2 0.0263

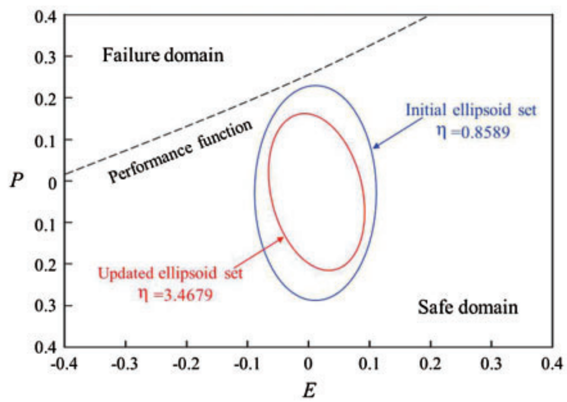


Figure 10: The updated ellipsoidal convex model

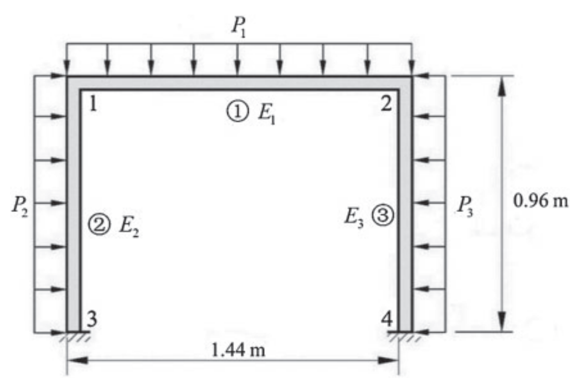
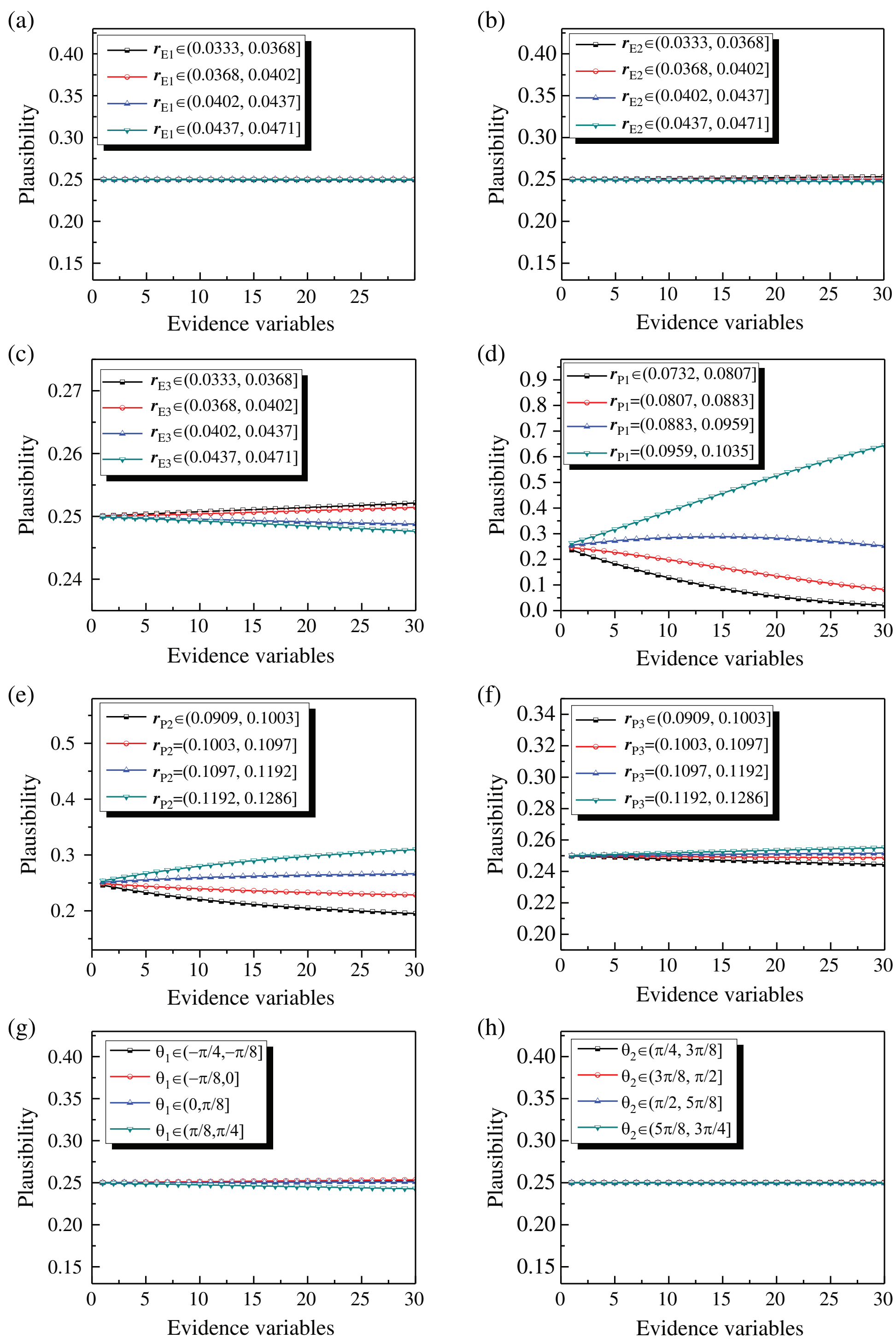


Figure 11: A frame structure

Table 7: Performance test samples $\left(\times 10^{-3}\ \text{Rad}\right)$

1.441	1.220	1.432	1.425	1.498	1.358	1.441	1.174	1.310	1.407
1.437	1.486	1.278	1.274	1.433	1.338	1.285	1.231	1.323	1.430
1.462	1.285	1.259	1.443	1.263	1.360	1.432	1.289	1.472	1.316


(a)
r_{E1}∈(0.0333, 0.0368]
r_{E1}∈(0.0368, 0.0402]
r_{E1}∈(0.0402, 0.0437]
r_{E1}∈(0.0437, 0.0471]
Plausibility
Evidence variables
(b)
r_{E2}∈(0.0333, 0.0368]
r_{E2}∈(0.0368, 0.0402]
r_{E2}∈(0.0402, 0.0437]
r_{E2}∈(0.0437, 0.0471]
Plausibility
Evidence variables
(c)
r_{E3}∈(0.0333, 0.0368]
r_{E3}∈(0.0368, 0.0402]
r_{E3}∈(0.0402, 0.0437]
r_{E3}∈(0.0437, 0.0471]
Plausibility
Evidence variables
(d)
r_{P1}∈(0.0732, 0.0807]
r_{P1}=(0.0807, 0.0883]
r_{P1}=(0.0883, 0.0959]
r_{P1}=(0.0959, 0.1035]
Plausibility
Evidence variables
(e)
r_{P2}∈(0.0909, 0.1003]
r_{P2}=(0.1003, 0.1097]
r_{P2}=(0.1097, 0.1192]
r_{P2}=(0.1192, 0.1286]
Plausibility
Evidence variables
(f)
r_{P3}∈(0.0909, 0.1003]
r_{P3}=(0.1003, 0.1097]
r_{P3}=(0.1097, 0.1192]
r_{P3}=(0.1192, 0.1286]
Plausibility
Evidence variables
(g)
θ_1∈(−π/4,−π/8]
θ_1∈(−π/8,0]
θ_1∈(0,π/8]
θ_1∈(π/8,π/4]
Plausibility
Evidence variables
(h)
θ_2∈(π/4, 3π/8]
θ_2∈(3π/8, π/2]
θ_2∈(π/2, 5π/8]
θ_2∈(5π/8, 3π/4]
Plausibility
Evidence variables

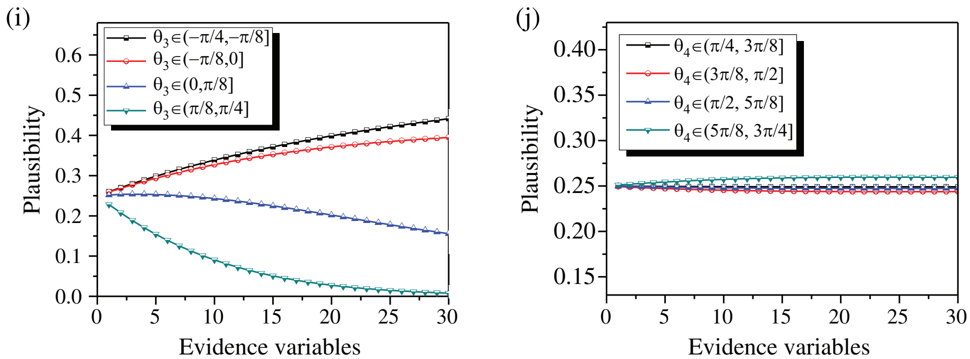


Figure 12: The plausibility calculation results of the first iteration step. (a) r_{E1}, (b) r_{E2}, (c) r_{E3}, (d) r_{P1}, (e) r_{P2}, (f) r_{P3}, (g) θ_1, (h) θ_2, (i) θ_3, (j) θ_4

In this example, the uncertain variables of Young's modulus and loads are classified into two ellipsoidal sets, and the allowable range of each characteristic parameters for the convex model is divided into four subintervals. The plausibility of the characteristic parameters in the first iteration step is depicted in Fig. 12. By removing the insensitive characteristic parameters and reducing the range of the sensitive characteristic parameters, the subsequent iteration steps are performed as demonstrated in Tab. 8, while the updated ellipsoidal convex model is illustrated in Fig. 13.

Table 8: The calculation result of the subsequent iteration steps

Iteration step	Mean of the subinterval with the maximum plausibility			Convergence criterion ζ_1			Convergence criterion ζ_2		
	r_{P1}	r_{P2}	θ_3	r_{P1}	r_{P2}	θ_3	r_{P1}	r_{P2}	θ_3
1	0.0997	0.1239	−0.3927	0.2508	0.2493	0.5000	0.6232	0.1147	0.4337
2	0.1026	0.1251	−0.4909	0.0627	0.1857	0.3750	0.2036	0.1807	0.2659
3	0.1019	0.1201	−0.5645	0.0165	0.0451	0.0937	0.1669	0.1525	0.1822
4	0.1017	0.1203	−0.5829	0.0165	0.0133	0.0234	0.1669	0.1168	0.2586

6.4 The Buckling Reliability Assessment of a Hull Stiffening Plate

The stiffened panel is the basic composite of the hull structure. In addition, the global deformation of the hull structure under wave load is mainly hogging or sagging. Decks above and below the neutral surfaces are suffered press loads undergoing sagging and hogging deformations, respectively. Decks comprise several stiffening plate elements. As shown in Fig. 14, the evaluation of the buckling strength of the stiffening plate is a significant component of the hull structure strength assessment.

For this numerical example, there are four uncertain variables (wave bending moment, still water bending moment, as well as the thickness of the plate and stiffeners) that exist from two sources, which are load moment and structural inherent uncertainties. Two ellipsoid convex models are constructed to contain two sources of uncertain variables. The first ellipsoid convex model contains the wave bending moment variable ($P_1 \in [1.8,\ 2.0] \times 10^5$ N/m) and the still water bending moment variable ($P_2 \in [7.5,\ 8.0] \times 10^5$ N/m). Additionally, the second ellipsoid convex model contains the thickness variables of plate ($T_1 \in [8, 12]$ mm) and stiffeners ($T_2 \in [11, 13]$ mm). The performance function is defined as that the critical buckling factor is greater than 1. Besides, the performance test sample set comprises 30 samples that are all reliable without the buckling phenomena. According to the method proposed in this study, three iteration steps are performed during the Bayesian updating process. The calculation result of the first iteration step is depicted in Fig. 15, while the calculation results of the second and third iteration steps are presented in Tabs. 9 and 10, respectively.

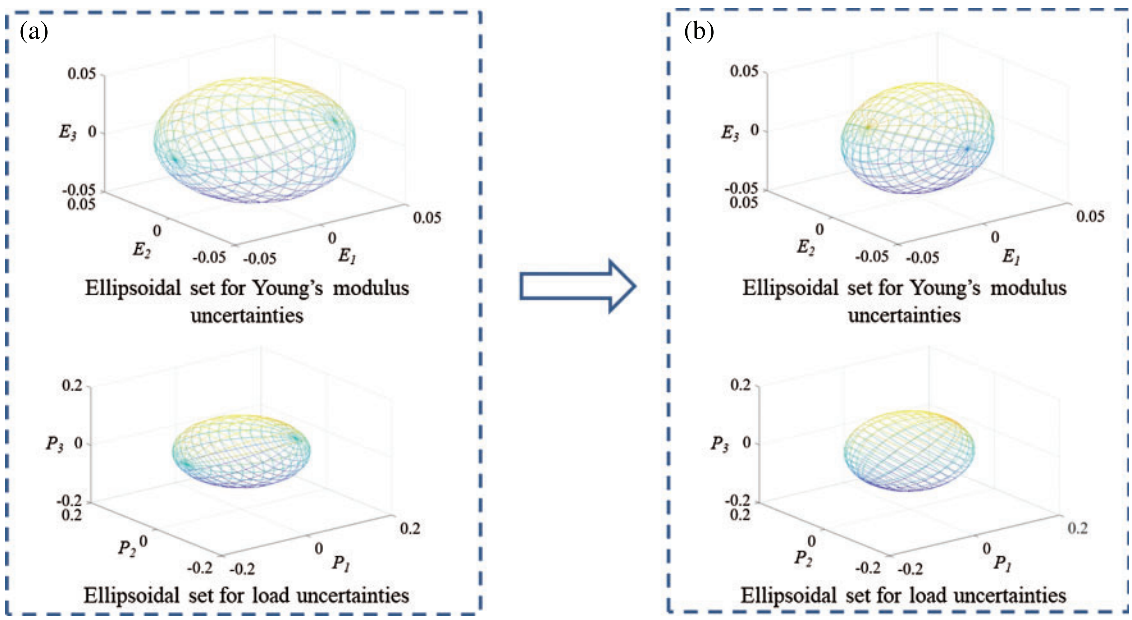


Figure 13: The updated multi-ellipsoidal convex model. (a) initial ellipsoid set, $\eta = 0.9883$, (b) updated ellipsoid set, $\eta = 1.1420$

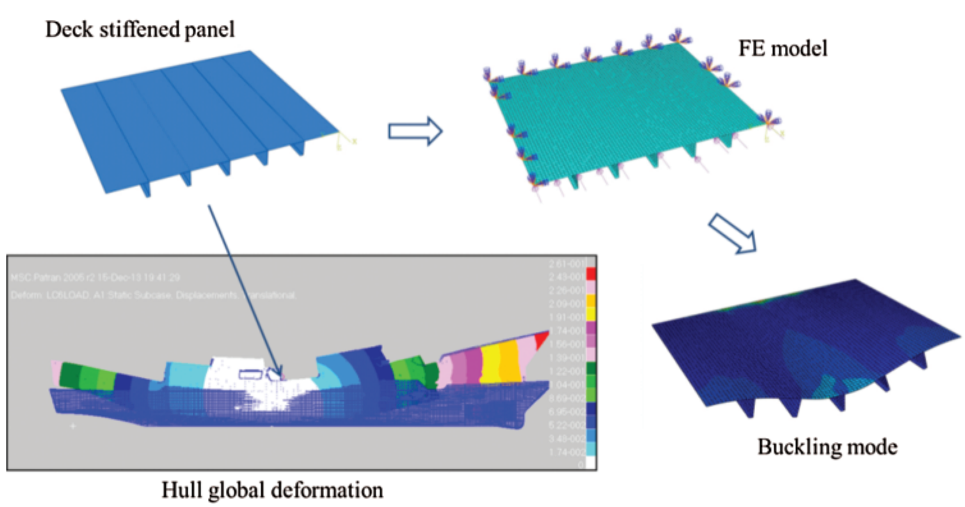


Figure 14: A hull stiffening plate

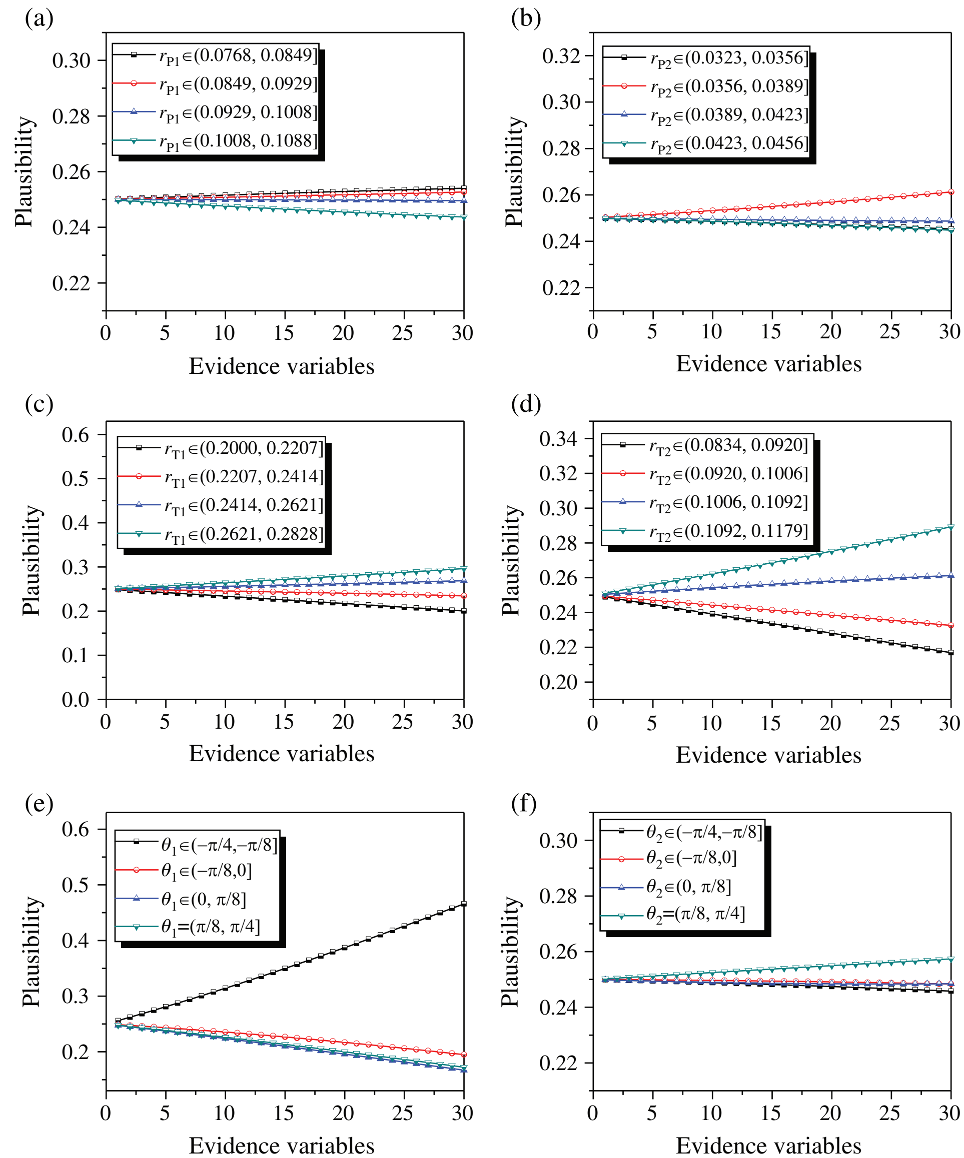


Figure 15: The plausibility calculation results of the first iteration step. (a) r_{P1} (b) r_{P2} (c) r_{T1} (d) r_{T2} (e) θ_1 (f) θ_2

Table 9: The calculation result of the second iteration step

r_{T1}	Subinterval	(0.2414, 0.2518]	(0.2518, 0.2621]	(0.2621, 0.2725]	(0.2725, 0.2828]	ζ_1	ζ_2
	Plausibility	0.2423	0.2415	0.2570	0.2592	0.2500	0.0177
r_{T2}	Subinterval	(0.1006, 0.1049]	(0.1049, 0.1093]	(0.1093, 0.1136]	(0.1136, 0.1179]	ζ_1	ζ_2
	Plausibility	2537	0.2469	0.2497	0.2497	0.1250	0.0068
θ_1	Subinterval	$(-\pi/4, -7\pi/32]$	$(-7\pi/32, -3\pi/16]$	$(-3\pi/16, -5\pi/32]$	$(-5\pi/32, -\pi/8]$	ζ_1	ζ_2
	Plausibility	0.5919	0.2229	0.1091	0.0761	0.0625	0.5158

Table 10: The calculation result of the third iteration step

θ_1	Subinterval	$(-0.7854, -0.7609]$	$(-0.7609, -0.7363]$	$(-0.7363, -0.7118]$	$(-0.7118, -0.6872]$	ζ_1	ζ_2
	Plausibility	0.2388	0.2586	0.2533	0.2493	0.0313	0.0208

By considering the characteristic parameters as the mean value of the subintervals with the maximum plausibility, the updated ellipsoidal convex model is achieved, as shown in Fig. 16. Based on the updated multi-ellipsoidal convex model, the non-probabilistic reliability index calculation is 1.1037. When compared with the initial non-probabilistic reliability index (0.9964), the actual structure may be more reliable than that of the initial assessment.

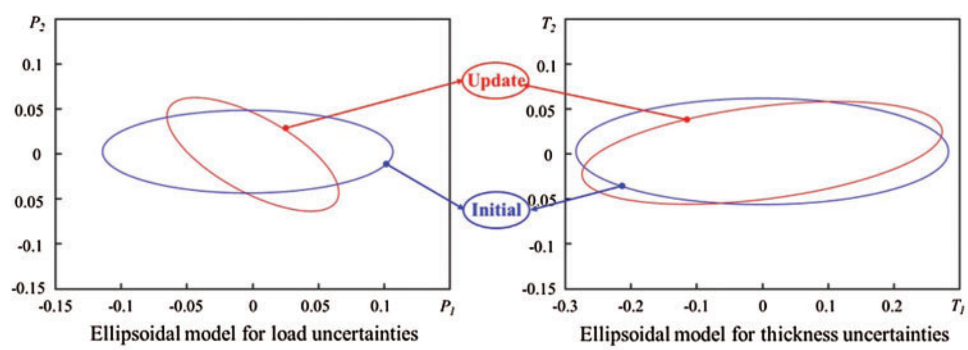


Figure 16: The updated multi-ellipsoidal convex model

7 Conclusions

In this study, we proposed a Bayesian updating method for making more credible non-probabilistic reliability assessments through the fusion of the information about the bounds of uncertainties and the performance test data. Based on the parameterization of multi-ellipsoidal convex models, a Bayesian network was established to update the plausibility of the characteristic parameters driven by evidence variables. This method can be applied to obtain the most credible convex model according to the performance test data. Based on the most credible convex model, a more credible non-probabilistic reliability index compared with the initial non-probabilistic reliability index can be achieved which is modified by the performance test samples. It would be very helpful in the safety assessment of practical engineering structures when considering the uncertain variables with less cognition. Future work will focus on extending the proposed Bayesian updating method to the topology optimization or novel design of some practical problems [46–48] considering uncertainties.

Replication of Results: Code and data will be made available on request.

Funding Statement: This work was supported financially by the National Key R&D Program of China (2017YFB0203604), the National Natural Science Foundation of China (11972104, 11772077) and the Liaoning Revitalization Talents Program (XLYC1807187).

Conflicts of Interest: The authors declare that they have no conflicts of interest to report regarding the present study.

References

1. Schuëller, G. I., Jensen, H. A. (2008). Computational methods in optimization considering uncertainties—An overview. *Computer Methods in Applied Mechanics and Engineering, 198(1),* 2–13. DOI 10.1016/j.cma.2008.05.004.
2. Dubois, D. (2010). Representation, propagation, and decision issues in risk analysis under incomplete probabilistic information. *Risk Analysis: An International Journal, 30(3),* 361–368. DOI 10.1111/j.1539-6924.2010.01359.x.
3. Beer, M., Ferson, S., Kreinovich, V. (2013). Imprecise probabilities in engineering analyses. *Mechanical Systems and Signal Processing, 37(1–2),* 4–29. DOI 10.1016/j.ymssp.2013.01.024.
4. Elishakoff, I. (1995). Essay on uncertainties in elastic and viscoelastic structures: from AM Freudenthal's criticisms to modern convex modeling. *Computers & Structures, 56(6),* 871–895. DOI 10.1016/0045-7949(94)00499-S.
5. Ben-Haim, Y., Elishakoff, I. (2013). *Convex models of uncertainty in applied mechanics*. Amsterdam, Netherlands: Elsevier.
6. Sofi, A., Muscolino, G., Elishakoff, I. (2015). Static response bounds of Timoshenko beams with spatially varying interval uncertainties. *Acta Mechanica, 226(11),* 3737–3748. DOI 10.1007/s00707-015-1400-9.
7. Ben-Haim, Y. (1994). A non-probabilistic concept of reliability. *Structural Safety, 14(4),* 227–245. DOI 10.1016/0167-4730(94)90013-2.
8. Ben-Haim, Y. (1995). A non-probabilistic measure of reliability of linear systems based on expansion of convex models. *Structural Safety, 17(2),* 91–109. DOI 10.1016/0167-4730(95)00004-N.
9. Au, F. T. K., Cheng, Y. S., Tham, L. G., Zeng, G. W. (2003). Robust design of structures using convex models. *Computers & Structures, 81(28–29),* 2611–2619. DOI 10.1016/S0045-7949(03)00322-5.
10. Jiang, C., Bi, R. G., Lu, G. Y., Han, X. (2013). Structural reliability analysis using non-probabilistic convex model. *Computer Methods in Applied Mechanics and Engineering, 254,* 83–98. DOI 10.1016/j.cma.2012.10.020.
11. Jiang, C., Zhang, Q. F., Han, X., Qian, Y. H. (2014). A non-probabilistic structural reliability analysis method based on a multidimensional parallelepiped convex model. *Acta Mechanica, 225(2),* 383–395. DOI 10.1007/s00707-013-0975-2.
12. Jiang, C., Ni, B. Y., Han, X., Tao, Y. R. (2014). Non-probabilistic convex model process: a new method of time-variant uncertainty analysis and its application to structural dynamic reliability problems. *Computer Methods in Applied Mechanics and Engineering, 268,* 656–676. DOI 10.1016/j.cma.2013.10.016.
13. Yang, L., Nie, S., Zhang, A. (2013). Non-probabilistic wear reliability analysis of swash-plate/slipper of water hydraulic piston motor based on convex model. *Proceedings of the Institution of Mechanical Engineers, Part C: Journal of Mechanical Engineering Science, 227(3),* 609–619. DOI 10.1177/0954406212463501.
14. Jiang, T., Chen, J. J., Xu, Y. L. (2007). A semi-analytic method for calculating non-probabilistic reliability index based on interval models. *Applied Mathematical Modelling, 31(7),* 1362–1370. DOI 10.1016/j.apm.2006.02.013.
15. Cao, H., Duan, B. (2005). An approach on the non-probabilistic reliability of structures based on uncertainty convex models. *Chinese Journal of Computational Mechanics, 22(5),* 546–549.
16. Zheng, J., Luo, Z., Jiang, C., Ni, B., Wu, J. (2018). Non-probabilistic reliability-based topology optimization with multidimensional parallelepiped convex model. *Structural and Multidisciplinary Optimization, 57(6),* 2205–2221. DOI 10.1007/s00158-017-1851-9.
17. Faes, M., Moens, D. (2019). Recent trends in the modeling and quantification of non-probabilistic uncertainty. *Archives of Computational Methods in Engineering, 27(3),* 1–39. DOI 10.1007/s11831-019-09327-x.
18. Luo, Y., Zhan, J., Xing, J., Kang, Z. (2019). Non-probabilistic uncertainty quantification and response analysis of structures with a bounded field model. *Computer Methods in Applied Mechanics and Engineering, 347,* 663–678. DOI 10.1016/j.cma.2018.12.043.

19. Meng, Z., Zhang, D., Li, G., Yu, B. (2019). An importance learning method for non-probabilistic reliability analysis and optimization. *Structural and Multidisciplinary Optimization, 59(4),* 1255–1271. DOI 10.1007/s00158-018-2128-7.
20. Luo, Y., Kang, Z., Li, A. (2009). Structural reliability assessment based on probability and convex set mixed model. *Computers & Structures, 87(21–22),* 1408–1415. DOI 10.1016/j.compstruc.2009.06.001.
21. Wang, L., Liang, J., Wu, D. (2018). A non-probabilistic reliability-based topology optimization (NRBTO) method of continuum structures with convex uncertainties. *Structural and Multidisciplinary Optimization, 58(6),* 2601–2620. DOI 10.1007/s00158-018-2040-1.
22. Luo, Y., Kang, Z., Luo, Z., Li, A. (2009). Continuum topology optimization with non-probabilistic reliability constraints based on multi-ellipsoid convex model. *Structural and Multidisciplinary Optimization, 39(3),* 297–310. DOI 10.1007/s00158-008-0329-1.
23. Kang, Z., Luo, Y. (2009). Non-probabilistic reliability-based topology optimization of geometrically nonlinear structures using convex models. *Computer Methods in Applied Mechanics and Engineering, 198(41–44),* 3228–3238. DOI 10.1016/j.cma.2009.06.001.
24. Rocchetta, R., Crespo, L. G., Kenny, S. P. (2020). A scenario optimization approach to reliability-based design. *Reliability Engineering & System Safety, 196,* 106755. DOI 10.1016/j.ress.2019.106755.
25. Eldred, M. S., Swiler, L. P., Tang, G. (2011). Mixed aleatory-epistemic uncertainty quantification with stochastic expansions and optimization-based interval estimation. *Reliability Engineering & System Safety, 96(9),* 1092–1113. DOI 10.1016/j.ress.2010.11.010.
26. Zhu, L. P., Elishakoff, I., Starnes, J. H. (1996). Derivation of multi-dimensional ellipsoidal convex model for experimental data. *Mathematical and Computer Modelling, 24(2),* 103–114. DOI 10.1016/0895-7177(96)00094-5.
27. Jiang, C., Han, X., Lu, G. Y., Liu, J., Zhang, Z. et al. (2011). Correlation analysis of non-probabilistic convex model and corresponding structural reliability technique. *Computer Methods in Applied Mechanics and Engineering, 200(33–36),* 2528–2546. DOI 10.1016/j.cma.2011.04.007.
28. Kang, Z., Zhang, W. (2016). Construction and application of an ellipsoidal convex model using a semi-definite programming formulation from measured data. *Computer Methods in Applied Mechanics and Engineering, 300,* 461–489. DOI 10.1016/j.cma.2015.11.025.
29. Bellis, C., Moulinec, H. (2016). A full-field image conversion method for the inverse conductivity problem with internal measurements. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences, 472(2187),* 20150488. DOI 10.1098/rspa.2015.0488.
30. Klibanov, M. V., Kolesov, A. E., Nguyen, L., Sullivan, A. (2017). Globally strictly convex cost functional for a 1-D inverse medium scattering problem with experimental data. *SIAM Journal on Applied Mathematics, 77(5),* 1733–1755. DOI 10.1137/17M1122487.
31. Fournier, A. E., Clerget, C. H., Bharadwaj, P., Merciu, A., Skar, G. (2019). A seismoelectric inverse problem with well-log data and borehole-confined acquisition. *SEG Technical Program Expanded Abstracts 2019,* pp. 4014–4018. Oklahoma, America: Society of Exploration Geophysicists.
32. Zhuang, Q., Yu, B., Jiang, X. (2015). An inverse problem of parameter estimation for time-fractional heat conduction in a composite medium using carbon-carbon experimental data. *Physica B: Condensed Matter, 456,* 9–15. DOI 10.1016/j.physb.2014.08.011.
33. Nguyen, D. L., Klibanov, M. V., Nguyen, L. H., Kolesov, A. E., Fiddy, M. A. et al. (2017). Numerical solution of a coefficient inverse problem with multi-frequency experimental raw data by a globally convergent algorithm. *Journal of Computational Physics, 345,* 17–32. DOI 10.1016/j.jcp.2017.05.015.
34. Alessandrini, G., Maarten, V., Faucher, F., Gaburro, R., Sincich, E. (2019). Inverse problem for the Helmholtz equation with Cauchy data: reconstruction with conditional well-posedness driven iterative regularization. *ESAIM: Mathematical Modelling and Numerical Analysis, 53(3),* 1005–1030. DOI 10.1051/m2an/2019009.
35. Nguyen, H. C., Zecchina, R., Berg, J. (2017). Inverse statistical problems: from the inverse Ising problem to data science. *Advances in Physics, 66(3),* 197–261. DOI 10.1080/00018732.2017.1341604.

36. Rocchetta, R., Broggi, M., Huchet, Q., Patelli, E. (2018). On-line bayesian model updating for structural health monitoring. *Mechanical Systems and Signal Processing, 103,* 174–195. DOI 10.1016/j.ymssp.2017.10.015.
37. Li, Y., Wang, X., Wang, C., Xu, M., Wang, L. (2018). Non-probabilistic Bayesian update method for model validation. *Applied Mathematical Modelling, 58,* 388–403. DOI 10.1016/j.apm.2018.02.011.
38. Behmanesh, I., Moaveni, B., Lombaert, G., Papadimitriou, C. (2015). Hierarchical Bayesian model updating for structural identification. *Mechanical Systems and Signal Processing, 64,* 360–376. DOI 10.1016/j.ymssp.2015.03.026.
39. Snoek, J., Larochelle, H., Adams, R. P. (2012). Practical bayesian optimization of machine learning algorithms. *Advances in Neural Information Processing Systems 25: 26th Annual Conference on Neural Information Processing Systems,* vol. 4, pp. 2951–2959. New York, America: Neural Information Processing Systems Foundation. DOI 10.1190/segam2019-3216905.1.
40. Van De Schoot, R., Broere, J. J., Perryck, K. H., Zondervan-Zwijnenburg, M., Van Loey, N. E. (2015). Analyzing small data sets using Bayesian estimation: the case of posttraumatic stress symptoms following mechanical ventilation in burn survivors. *European Journal of Psychotraumatology, 6(1),* 25216. DOI 10.3402/ejpt.v6.25216.
41. Fereshtehnejad, E., Banazadeh, M., Shafieezadeh, A. (2016). System reliability-based seismic collapse assessment of steel moment frames using incremental dynamic analysis and Bayesian probability network. *Engineering Structures, 118,* 274–286. DOI 10.1016/j.engstruct.2016.03.057.
42. Krige, D. G. (1951). A statistical approach to some basic mine valuation problems on the Witwatersrand. *Journal of the Southern African Institute of Mining and Metallurgy, 52(6),* 119–139.
43. Joseph, V. R., Hung, Y. (2008). Orthogonal-maximin Latin hypercube designs. *Statistica Sinica, 18(1),* 171–186. DOI 10.1080/03461230802487123.
44. Jones, D. R., Schonlau, M., Welch, W. J. (1998). Efficient global optimization of expensive black-box functions. *Journal of Global Optimization, 13(4),* 455–492. DOI 10.1023/A:1008306431147.
45. Bichon, B. J., Eldred, M. S., Swiler, L. P., Mahadevan, S., McFarland, J. M. (2008). Efficient global reliability analysis for nonlinear implicit performance functions. *AIAA Journal, 46(10),* 2459–2468. DOI 10.2514/1.34321.
46. Luo, Y., Bao, J. (2019). A material-field series-expansion method for topology optimization of continuum structures. *Computers & Structures, 225,* 106122. DOI 10.1016/j.compstruc.2019.106122.
47. Luo, Y., Xing, J., Kang, Z. (2020). Topology optimization based on Kriging surrogate model and material-field series expansion: an effective non-gradient method. *Computer Methods in Applied Mechanics and Engineering, 364,* 112966. DOI 10.1016/j.cma.2020.112966.
48. Li, M., Li, X., Che, L. X., Li, F. W., Kang, Z. (2020). Non-uniform global-buckling and local-folding in thin film of stretchable electronics. *International Journal of Mechanical Sciences, 175,* 105537. DOI 10.1016/j.ijmecsci.2020.105537.