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Article

Qualitative Analysis of a Fractional Pandemic Spread Model of the Novel Coronavirus (COVID-19)

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Abstract: In this study, we classify the genera of COVID-19 and provide brief information about the root of the spread and the transmission from animal (natural host) to humans. We establish a model of fractional-order differential equations to discuss the spread of the infection from the natural host to the intermediate one, and from the intermediate one to the human host. At the same time, we focus on the potential spillover of bat-borne coronaviruses. We consider the local stability of the co-existing critical point of the model by using the Routh–Hurwitz Criteria. Moreover, we analyze the existence and uniqueness of the constructed initial value problem. We focus on the control parameters to decrease the outbreak from pandemic form to the epidemic by using both strong and weak Allee Effect at time t. Furthermore, the discretization process shows that the system undergoes Neimark–Sacker Bifurcation under specific conditions. Finally, we conduct a series of numerical simulations to enhance the theoretical findings.

Keywords: Allee Effect; coronavirus; fractional-order differential equations; local stability; Neimark–Sacker bifurcation

1 Introduction

In the last few months, nature has showed its laws in establishing the environment of the 21st century. It is out of our primary objective whether the coronavirus (COVID-19) is used as a biological weapon or not. The main point is now that humans are fighting against something to survive that has a genome size of 27 to 34 kilobases. Coronaviruses are members of the sub-family coronavirinae in the family coronaviridae and the order Nidovirales [1,2]. They show four genera, which are given in Tab. 1.

The natural host of SARS-CoV, MERS-CoV, HCoV-NL63, and HCoV-229e are bats, while HCoV-OC43 and HKU1 have originated from rodents [3,4]. In the spread of transmission, domestic animals have only intermediate host role from the natural host to the human one. Covid-19 was not considered as highly pathogenic, until the outbreak of SARS-CoV in 2002 and MERS-CoV in 2012. The spread of

SARS-CoV in China (Guangdong) showed a COVID-19 that was transmitted from bats to an intermediate host, like market civets from which the transmission spreads to the human host. At the same time, the outbreak of MERS-CoV in the Middle East Countries also came from bats to dromedary camels as an intermediate host, and from the dromedary camels to humans [5–8]. These viruses cause respiratory and intestinal infections, with symptoms including fever, dizziness, and cough. In December 2019, a novel Coronaviridae was reported in China (Wuhan). The outbreak was associated again with intermediate hosts like reptilians, while the natural host was assumed as bats. This virus was designated later as Covid-19 by the WHO.

Table 1: Genera of COVID-19 and the pathogenic class

Coronavirinae genera	α-CoV	β-CoV	γ-CoV	δ-CoV
Pathogenic class	Mammals	Mammals	Both non-mammal and mammals	Both non-mammal and mammals

Covid-19 was characterized by two members of β-coronavirus; the human-origin coronavirus (SARS-CoV Tor2) and bat-origin coronavirus (bat-SL-CoVZC45). Intensive studies show that it was most closely related to the bat-origin coronavirus [9]. Thus, the primary assumption formed was that the natural host of Covid-19 spreads by infected bats of genus Rhinolophus that are mainly in the area of Shatan River Valley.

Domestic animals, like snakes in that area, were hunted for the food market in Wuhan, which played an intermediate host role in the transmission. Finally, this virus spillover from the intermediate hosts to cause several diseases in human. A virus that started with an endemic pathogenic behavior in China (Wuhan) reaches somehow to a pandemic point worldwide with the infection from human-to-human.

2 The Model Description

It has been realized that the dynamics of many biological and medical phenomena can be characterized via mathematical models. Over the years, many models are formulated mathematically to analyze events in biological and medicine such as infections, treatments, or environmental phenomena [10–13]. The study of these phenomena has been restricted to models of integer-order differential equations (IDEs). However, it is seen that many problems in biology, as well as in other fields like engineering, finance, and economics, can be successfully formulated by the so-called fractional-order differential equations (FDEs); see, for instance, the papers [14–20]. The nonlocal property of models of FDEs is not only depending on the current state but also provides an adequate description for the historical ones. It is evidenced that FDEs can model certain phenomena that cannot be modeled by IDEs. Thus, FDEs are mainly used on biological models since they are relevant to systems with memory and hereditary [21–27].

In this paper, we establish a model that describes the pandemic infection, which occurs when the virus is transmitted from the human body to the intermediate host and continues to spread from human-to-human. The model consists of five fractional differential equations. The first three equations show an SI (susceptible-infected) model to explain the transmission from human-to-human, where S is the susceptible class, C_1 is the infected type that does not know they are infected because of the late occurred symptoms of COVID-19 and C_2 shows the infected class that knows they are infected. The spillover from the intermediate infected class M to the human host S denotes a predator-prey mathematical model, while for the transmission from the natural host N, which is the bat population, to intermediate host M is a host-parasite model of Holling Type II.

Indeed, the mathematical model of this biological phenomena has the form:

$$\begin{cases} D^{\alpha}S(t) = r_1S(t)(p - \mu_1S(t)) - \beta_1S(t)C_1(t) - \beta_2M(t)S(t) + \sigma_1M(t)S(t) \\ D^{\alpha}C_1(t) = r_2C_1(t)(1 - \mu_2C_1(t)) + \beta_1(1 - \varepsilon_1)S(t)C_1(t) - \theta C_1(t) + \beta_2(1 - \varepsilon_2)M(t)S(t) \\ D^{\alpha}C_2(t) = C_2(t)(1 - \mu_3C_2(t)) + \theta C_1(t)C_2(t) + \beta_1\varepsilon_1S(t)C_1(t) + \beta_2\varepsilon_2M(t)S(t) \\ D^{\alpha}M(t) = M(t)r_3(1 - \mu_4M(t)) - \sigma_2M(t) - \gamma f(t)N(t) \\ D^{\alpha}N(t) = N(t)r_4(1 - \mu_5N(t)) + \delta f(t)N(t) \end{cases} \tag{1}$$

where

$$f(t) = \frac{M(t)}{1 + he\omega M(t)} \tag{2}$$

represents the Holling type II function and all the parameters of the model (1) belong to $\mathbb{R}^+$ and $t \in [0, \infty)$.

The susceptible S is composed of individuals that have not contacted the infection but can get infected through contacts from the human that does not know they are infected and from the intermediate hosts. The parameter r_1 is the population growth rate of the susceptible population and μ_1 denotes the logistic rate. p is a rate of the susceptible population per year. The susceptible lost their class following contacts with infectives C_1 and the intermediate host M at a rate β_1 and β_2, respectively. The parameter σ_1 links the parameter of the interaction between the hunted M class and the predator S population.

The C_1 class does not know that they have COVID-19. In this equation, r_2 is the population growth rate of the class, while μ_2 is the logistic rate. The population of this class decreases after screening at a rate θ and be aware of the infection. Another possibility is that after the S-C_1 contact, the symptoms occur in early stages so that both classes noticed that they are infected, which is given with the rate ε_1. The intermediate host infected group could also show early symptoms to be aware of the infection, which is provided by a rate of ε_2. The logistic rate of C_2 is denoted as μ_3.

M is the domestic animal as an intermediate class in the corona transmission spread. r_3 is the intrinsic growth rate of the population, while μ_4 is the logistic rate. σ_2 shows the effect on the hunted M during the interaction between the intermediate host and susceptible class. γ denotes the predation rate in the host-parasite scheme.

N represents the natural host (bat population) of COVID-19 in this dynamic system. r_4 is the intrinsic growth rate and μ_5 is the logistic rate of the population. δ shows the conversion factor of the natural host. e is the attack rate of the bat population to infect the M, while ω $(0 < \omega \leq 1)$ represents the fraction of the potential infectivity of the natural host. h is the rate of average time spend on infecting the domestic intermediate class, which is also known as the handling time.

Tab. 2 shows description of the parameters that are given in system (1).

Table 2: Description of the parameters

Parameter	Symbol	rate
The growth rate of $\boldsymbol{S(t)}$	r_1	0.012
The growth rate of $\boldsymbol{C_1(t)}$	r_2	0.009
The growth rate of $\mathbf{M(t)}$	r_3	0.014
The growth rate of $\mathbf{N(t)}$	r_4	0.01
Logistic rate of $\boldsymbol{S(t)}$	μ_1	0.05
Logistic rate of $\boldsymbol{C_1(t)}$	μ_2	0.1

(Continued)

Table 2 (continued).

Parameter	Symbol	rate
Logistic rate of $\mathbf{C}_2(\mathbf{t})$	μ_3	0.15
Logistic rate of $\mathbf{M}(\mathbf{t})$	μ_4	0.01
Logistic rate of $\mathbf{N}(\mathbf{t})$	μ_5	0.01
Rate of the $\boldsymbol{S}(\boldsymbol{t})$ population per year	p	1.6
Parametric lost from class $\boldsymbol{S}(\boldsymbol{t})$ to $\boldsymbol{C}_1(\boldsymbol{t})$	β_1, β_2	0.00134, 0.00044
Rate of interaction between $\boldsymbol{S}(\boldsymbol{t}) - \mathbf{M}(\mathbf{t})$	σ_1, σ_2	0.0001
Predation rate	γ	0.0044
Rate of screening	θ	[0.01, 0.05]
Recognition of infection	$\varepsilon_1, \varepsilon_2$	[0.1, 0.4]
A conversion factor of $\mathbf{N}(\mathbf{t})$	δ	0.0045
The attack rate of $\mathbf{N}(\mathbf{t})$ to $\mathbf{M}(\mathbf{t})$	e	0.15
Rate of average time on infecting $\mathbf{M}(\mathbf{t})$	h	0.15
Potential infectivity of $\mathbf{N}(\mathbf{t})$	ω	$\omega \in (0, 1]$

Definition 2.1 Podlubny [25] The fractional integral of order $\alpha > 0$ of a function $f : R^+ \rightarrow R$ is given by

$$I_0^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) dt, \tag{3}$$

defined on R^+.

Definition 2.2. Podlubny [25] Let $f : R^+ \rightarrow R$ be a continuous function. The Caputo fractional derivative of order $\alpha \in (n-1,\ n)$ is given by

$$D^\alpha f(x) = I_0^{n-\alpha} D^n f(x), \quad D = \frac{d}{dx}. \tag{4}$$

Definition 2.3. Podlubny [25] The function

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+\alpha k)}, \ \alpha \in \mathbb{C},\ \mathbb{R}(\alpha) > 0 \text{ and } z \in \mathbb{C} \tag{5}$$

with $\mathbb{C}$ being the set of complex numbers is called the Mittag–Leffler function of one parameter.

3 Stability Analysis of the Co-Existing Critical Point

Consider the model

$$\begin{cases} D^{\alpha}S(t) = f(S(t), C_1(t), C_2(t), M(t), N(t)) \\ \qquad = r_1S(t)(p - \mu_1S(t)) - \beta_1S(t)C_1(t) - \beta_2M(t)S(t) + \sigma_1M(t)S(t) \\ D^{\alpha}C_1(t) = g(S(t), C_1(t), C_2(t), M(t), N(t)) \\ \qquad = r_2C_1(t)(1 - \mu_2C_1(t)) + \beta_1(1 - \varepsilon_1)S(t)C_1(t) - \theta C_1(t) + \beta_2(1 - \varepsilon_2)M(t)S(t) \\ D^{\alpha}C_2(t) = h(S(t), C_1(t), C_2(t), M(t), N(t)) \\ \qquad = C_2(t)(1 - \mu_3C_2(t)) + \theta C_1(t)C_2(t) + \beta_1\varepsilon_1S(t)C_1(t) + \beta_2\varepsilon_2M(t)S(t) \\ D^{\alpha}M(t) = j(S(t), C_1(t), C_2(t), M(t), N(t)) \\ \qquad = M(t)r_3(1 - \mu_4M(t)) - \sigma_2M(t) - \gamma f(t)N(t) \\ D^{\alpha}N(t) = k(S(t), C_1(t), C_2(t), M(t), N(t)) \\ \qquad = N(t)r_4(1 - \mu_5N(t)) + \delta f(t)N(t). \end{cases} \tag{6}$$

To analyze the stability of model (6), we perturb the equilibrium point by adding $\varepsilon_i(t) > 0, i = 1,\ 2,\ 3,\ 4,\ 5$, that is,

$$S(t) - \bar{S} = \varepsilon_1(t),\ C_1(\mathrm{t}) - \overline{C_1} = \varepsilon_2(t), C_2(\mathrm{t}) - \overline{C_2} = \varepsilon_3(t),\ M(t) - \bar{M} = \varepsilon_4(t) \text{ and } N(t) - \bar{N} = \varepsilon_5(t) \tag{7}$$

Thus, we have

$$D^{\alpha}(\varepsilon_1(t)) \simeq f(\bar{S}, \overline{C_1}, \overline{C_2}, \bar{M}, \bar{N}) + \frac{\partial f(\bar{S}, \overline{C_1}, \overline{C_2}, \bar{M}, \bar{N})}{\partial S}\varepsilon_1(t) + \frac{\partial f(\bar{S}, \overline{C_1}, \overline{C_2}, \bar{M}, \bar{N})}{\partial C_1}\varepsilon_2(t) + \frac{\partial f(\bar{S}, \overline{C_1}, \overline{C_2}, \bar{M}, \bar{N})}{\partial C_2}\varepsilon_3(t) + \frac{\partial f(\bar{S}, \overline{C_1}, \overline{C_2}, \bar{M}, \bar{N})}{\partial M}\varepsilon_4(t) + \frac{\partial f(\bar{S}, \overline{C_1}, \overline{C_2}, \bar{M}, \bar{N})}{\partial N}\varepsilon_5(t),$$

$$D^{\alpha}(\varepsilon_2(t)) \simeq g(\bar{S}, \overline{C_1}, \overline{C_2}, \bar{M}, \bar{N}) + \frac{\partial g(\bar{S}, \overline{C_1}, \overline{C_2}, \bar{M}, \bar{N})}{\partial S}\varepsilon_1(t) + \frac{\partial g(\bar{S}, \overline{C_1}, \overline{C_2}, \bar{M}, \bar{N})}{\partial C_1}\varepsilon_2(t) + \frac{\partial g(\bar{S}, \overline{C_1}, \overline{C_2}, \bar{M}, \bar{N})}{\partial C_2}\varepsilon_3(t) + \frac{\partial g(\bar{S}, \overline{C_1}, \overline{C_2}, \bar{M}, \bar{N})}{\partial M}\varepsilon_4(t) + \frac{\partial g(\bar{S}, \overline{C_1}, \overline{C_2}, \bar{M}, \bar{N})}{\partial N}\varepsilon_5(t),$$

$$D^{\alpha}(\varepsilon_3(t)) \simeq h(\bar{S}, \overline{C_1}, \overline{C_2}, \bar{M}, \bar{N}) + \frac{\partial h(\bar{S}, \overline{C_1}, \overline{C_2}, \bar{M}, \bar{N})}{\partial S}\varepsilon_1(t) + \frac{\partial h(\bar{S}, \overline{C_1}, \overline{C_2}, \bar{M}, \bar{N})}{\partial C_1}\varepsilon_2(t) + \frac{\partial h(\bar{S}, \overline{C_1}, \overline{C_2}, \bar{M}, \bar{N})}{\partial C_2}\varepsilon_3(t) + \frac{\partial h(\bar{S}, \overline{C_1}, \overline{C_2}, \bar{M}, \bar{N})}{\partial M}\varepsilon_4(t) + \frac{\partial h(\bar{S}, \overline{C_1}, \overline{C_2}, \bar{M}, \bar{N})}{\partial N}\varepsilon_5(t),$$

$$D^{\alpha}(\varepsilon_4(t)) \simeq j(\bar{S}, \overline{C_1}, \overline{C_2}, \bar{M}, \bar{N}) + \frac{\partial j(\bar{S}, \overline{C_1}, \overline{C_2}, \bar{M}, \bar{N})}{\partial S}\varepsilon_1(t) + \frac{\partial j(\bar{S}, \overline{C_1}, \overline{C_2}, \bar{M}, \bar{N})}{\partial C_1}\varepsilon_2(t) + \frac{\partial j(\bar{S}, \overline{C_1}, \overline{C_2}, \bar{M}, \bar{N})}{\partial C_2}\varepsilon_3(t) + \frac{\partial j(\bar{S}, \overline{C_1}, \overline{C_2}, \bar{M}, \bar{N})}{\partial M}\varepsilon_4(t) + \frac{\partial j(\bar{S}, \overline{C_1}, \overline{C_2}, \bar{M}, \bar{N})}{\partial N}\varepsilon_5(t),$$

and

$$D^{\alpha}(\varepsilon_5(t)) \simeq k(\bar{S}, \overline{C_1}, \overline{C_2}, \bar{M}, \bar{N}) + \frac{\partial k(\bar{S}, \overline{C_1}, \overline{C_2}, \bar{M}, \bar{N})}{\partial S}\varepsilon_1(t) + \frac{\partial k(\bar{S}, \overline{C_1}, \overline{C_2}, \bar{M}, \bar{N})}{\partial C_1}\varepsilon_2(t) + \frac{\partial k(\bar{S}, \overline{C_1}, \overline{C_2}, \bar{M}, \bar{N})}{\partial C_2}\varepsilon_3(t) + \frac{\partial k(\bar{S}, \overline{C_1}, \overline{C_2}, \bar{M}, \bar{N})}{\partial M}\varepsilon_4(t) + \frac{\partial k(\bar{S}, \overline{C_1}, \overline{C_2}, \bar{M}, \bar{N})}{\partial N}\varepsilon_5(t).$$

Thus, we obtain a linearized system about the equilibrium point of the form

$$D^{\alpha}Z = JZ, \tag{8}$$

where $Z = (\varepsilon_1(t), \varepsilon_2(t), \varepsilon_3(t), \varepsilon_4(t), \varepsilon_5(t))$. Moreover, J is the Jacobian matrix at the equilibrium:

$$J(\Lambda) = \begin{pmatrix} \frac{\partial f(\bar{S},\overline{C_1},\overline{C_2},\bar{M},\bar{N})}{\partial S} & \frac{\partial f(\bar{S},\overline{C_1},\overline{C_2},\bar{M},\bar{N})}{\partial C_1} & \frac{\partial f(\bar{S},\overline{C_1},\overline{C_2},\bar{M},\bar{N})}{\partial C_2} & \frac{\partial f(\bar{S},\overline{C_1},\overline{C_2},\bar{M},\bar{N})}{\partial M} & \frac{\partial f(\bar{S},\overline{C_1},\overline{C_2},\bar{M},\bar{N})}{\partial N} \\ \frac{\partial g(\bar{S},\overline{C_1},\overline{C_2},\bar{M},\bar{N})}{\partial S} & \frac{\partial g(\bar{S},\overline{C_1},\overline{C_2},\bar{M},\bar{N})}{\partial C_1} & \frac{\partial g(\bar{S},\overline{C_1},\overline{C_2},\bar{M},\bar{N})}{\partial C_2} & \frac{\partial g(\bar{S},\overline{C_1},\overline{C_2},\bar{M},\bar{N})}{\partial M} & \frac{\partial g(\bar{S},\overline{C_1},\overline{C_2},\bar{M},\bar{N})}{\partial N} \\ \frac{\partial h(\bar{S},\overline{C_1},\overline{C_2},\bar{M},\bar{N})}{\partial S} & \frac{\partial h(\bar{S},\overline{C_1},\overline{C_2},\bar{M},\bar{N})}{\partial C_1} & \frac{\partial h(\bar{S},\overline{C_1},\overline{C_2},\bar{M},\bar{N})}{\partial C_2} & \frac{\partial h(\bar{S},\overline{C_1},\overline{C_2},\bar{M},\bar{N})}{\partial M} & \frac{\partial h(\bar{S},\overline{C_1},\overline{C_2},\bar{M},\bar{N})}{\partial N} \\ \frac{\partial j(\bar{S},\overline{C_1},\overline{C_2},\bar{M},\bar{N})}{\partial S} & \frac{\partial j(\bar{S},\overline{C_1},\overline{C_2},\bar{M},\bar{N})}{\partial C_1} & \frac{\partial j(\bar{S},\overline{C_1},\overline{C_2},\bar{M},\bar{N})}{\partial C_2} & \frac{\partial j(\bar{S},\overline{C_1},\overline{C_2},\bar{M},\bar{N})}{\partial M} & \frac{\partial j(\bar{S},\overline{C_1},\overline{C_2},\bar{M},\bar{N})}{\partial N} \\ \frac{\partial k(\bar{S},\overline{C_1},\overline{C_2},\bar{M},\bar{N})}{\partial S} & \frac{\partial k(\bar{S},\overline{C_1},\overline{C_2},\bar{M},\bar{N})}{\partial C_1} & \frac{\partial k(\bar{S},\overline{C_1},\overline{C_2},\bar{M},\bar{N})}{\partial C_2} & \frac{\partial k(\bar{S},\overline{C_1},\overline{C_2},\bar{M},\bar{N})}{\partial M} & \frac{\partial k(\bar{S},\overline{C_1},\overline{C_2},\bar{M},\bar{N})}{\partial N} \end{pmatrix} \tag{9}$$

where the co-existing equilibrium point is $\Lambda = \left(\bar{S},\overline{C_1},\overline{C_2},\bar{M},\bar{N}\right)$. Then, we have $B^{-1}JB = C$, where C is given by

$$C = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 & 0 \\ 0 & 0 & 0 & \lambda_4 & 0 \\ 0 & 0 & 0 & 0 & \lambda_5 \end{pmatrix}, \tag{10}$$

and $\lambda_i (i = 1,\ 2,\ 3,\ 4,\ 5)$ are the eigenvalues and B the eigenvectors of J. Therefore, we get

$$\begin{cases} D_*^\alpha \eta_1 = \lambda_1 \eta_1 \\ D_*^\alpha \eta_2 = \lambda_2 \eta_2 \\ D_*^\alpha \eta_3 = \lambda_3 \eta_3 \\ D_*^\alpha \eta_4 = \lambda_4 \eta_4 \\ D_*^\alpha \eta_5 = \lambda_5 \eta_5 \end{cases}, \text{where } \eta = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \\ \eta_5 \end{pmatrix}, \text{ and } \eta = B^{-1} Z, \tag{11}$$

whose solutions are given by Mittag–Leffler functions

$$\eta_1(t) = \sum_{n=0}^{\infty} \frac{(\lambda_1)^n t^{n\alpha}}{\Gamma(n\alpha + 1)} \eta_1(0) = E_\alpha(\lambda_1 t^\alpha)\eta_1(0), \tag{12}$$

$$\eta_2(t) = \sum_{n=0}^{\infty} \frac{(\lambda_2)^n t^{n\alpha}}{\Gamma(n\alpha + 1)} \eta_2(0) = E_\alpha(\lambda_2 t^\alpha)\eta_2(0), \tag{13}$$

$$\eta_3(t) = \sum_{n=0}^{\infty} \frac{(\lambda_3)^n t^{n\alpha}}{\Gamma(n\alpha + 1)} \eta_3(0) = E_\alpha(\lambda_3 t^\alpha)\eta_3(0), \tag{14}$$

$$\eta_4(t) = \sum_{n=0}^{\infty} \frac{(\lambda_4)^n t^{n\alpha}}{\Gamma(n\alpha + 1)} \eta_4(0) = E_\alpha(\lambda_4 t^\alpha)\eta_4(0) \tag{15}$$

and

$$\eta_5(t) = \sum_{n=0}^{\infty} \frac{(\lambda_5)^n t^{n\alpha}}{\Gamma(n\alpha + 1)} \eta_5(0) = E_\alpha(\lambda_5 t^\alpha)\eta_5(0). \tag{16}$$

By using the result of [28], if $|arg(\lambda_i)| > \frac{\alpha\pi}{2} (i = 1,\ 2,\ 3,\ 4,\ 5)$, then $\eta_i(t)\ (i = 1,\ 2,\ 3,\ 4,\ 5)$ are decreasing and therefore we conclude that $\varepsilon_i(t)\ (i = 1,\ 2,\ 3,\ 4,\ 5)$ are decreasing. Let $(\varepsilon_1(t), \varepsilon_2(t), \varepsilon_3(t), \varepsilon_4(t), \varepsilon_5(t))$ be the solution of Eq. (8). If the solution of Eq. (8) is increasing, then Λ is unstable and if $(\varepsilon_1(t), \varepsilon_2(t), \varepsilon_3(t), \varepsilon_4(t), \varepsilon_5(t))$ is decreasing, then Λ is locally asymptotically stable.

Evaluating the Jacobian matrix (9) for the co-existing equilibrium point Λ, we obtain

$$J(\Lambda) = \begin{pmatrix} a_{11} & a_{12} & 0 & a_{14} & 0 \\ a_{21} & a_{22} & 0 & a_{23} & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} & 0 \\ 0 & 0 & 0 & a_{44} & a_{45} \\ 0 & 0 & 0 & a_{54} & a_{55} \end{pmatrix} \tag{17}$$

where

$a_{11} = r_1 p - 2\mu_1 r_1 \bar{S} - \beta_1 \overline{C_1} - (\beta_2 - \sigma_1)\bar{M}$, $a_{12} = -\beta_1 \bar{S}$, $a_{14} = -(\beta_2 - \sigma_1)\bar{S}$,
$a_{21} = \beta_1(1 - \varepsilon_1)\overline{C_1} + \beta_2(1 - \varepsilon_2)\bar{M}$,
$a_{22} = r_2 - 2\mu_2 r_2 \overline{C_1} + \beta_1(1 - \varepsilon_1)\bar{S} - \theta$, $a_{23} = \beta_2(1 - \varepsilon_2)\bar{S}$,
$a_{31} = \beta_1\varepsilon_1\overline{C_1} + \beta_2\varepsilon_2\bar{M}$, $a_{32} = \theta\overline{C_2} + \beta_1\varepsilon_1\bar{S}$, $a_{33} = 1 - 2\mu_3\overline{C_2} + \theta\overline{C_1}$, $a_{34} = \beta_2\varepsilon_2$,
$a_{44} = r_3 - 2\mu_4 r_3 \bar{M} - \sigma_2 - \dfrac{\gamma\bar{N}}{(1 + he\omega\bar{M})^2}$, $a_{45} = -\dfrac{\gamma\bar{M}}{1 + he\omega\bar{M}}$,
and
$a_{54} = \dfrac{\delta\bar{N}}{(1 + he\omega\bar{M})^2}$, $a_{55} = r_4 - 2r_4\mu_5\bar{N} + \dfrac{\delta\bar{M}}{1 + he\omega\bar{M}}$.

The characteristic equation of the matrix (17) is given as

$$\{(a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21}\}\{(a_{44} - \lambda)(a_{55} - \lambda) - a_{45}a_{54}\} = 0 \tag{18}$$

and

$$\lambda = a_{33} < 0, \tag{19}$$

if

$$1 - 2\mu_3\overline{C_2} + \theta\overline{C_1} < 0 \Rightarrow \overline{C_2} > \frac{\theta\overline{C_1} + 1}{2\alpha_3}. \tag{20}$$

From Eq. (18), we have two quadratic equations, which are

$$\lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22}\left(1 - \frac{a_{12}a_{21}}{a_{11}a_{22}}\right) = 0$$

or

$$\lambda^2 + (-a_{11} - a_{22})\lambda + a_{11}a_{22}(1 - R_{01}) = 0 \tag{21}$$

and

$$\lambda^2 - (a_{44} + a_{55})\lambda + a_{44}a_{55}\left(1 - \frac{a_{45}a_{54}}{a_{44}a_{55}}\right) = 0$$

or

$$\lambda^2 + (-a_{44} - a_{55})\lambda + a_{44}a_{55}(1 - R_{02}) = 0, \tag{22}$$

where $R_{01} = \dfrac{a_{12}a_{21}}{a_{11}a_{22}}$ and $R_{02} = \dfrac{a_{45}a_{54}}{a_{44}a_{55}}$. R_{01} is the basic reproduction number, which represents the transmission potential of $S - C_1$ class, while R_{02} shows the transmission potential of the intermediate-natural host classes $M - N$.

For the following theorems in this section, we consider the case, where both $R_{01} < 1$ and $R_{02} < 1$, which hold for the following statements:

(i) $\bar{S} > \dfrac{\theta - r_2}{\beta_1(2-\varepsilon_1)}$,

(ii) $p > \dfrac{\beta_1(2-\varepsilon_1)\overline{C_1} + 2\mu_1 r_1 \bar{S} + (\beta_2(2-\varepsilon_2) - \sigma_1)\bar{M}}{r_1}$,

(iii) $\overline{C_1} > \dfrac{\beta_1(2-\varepsilon_1)\bar{S} - (\theta - r_2)}{2\mu_2 r_2}$,

(iv) $\beta_2 > \sigma_1,\ \theta > r_2, \delta > \ \gamma, \varepsilon_1 < 1$ and $\varepsilon_2 < 1$,

(v) $\dfrac{\gamma}{\delta} < \bar{N} < 0.5{\mu_5}^{-1}$ and $\bar{M} > 0.5{\mu_4}^{-1}$.

Theorem 3.1. Let Λ be the co-existing critical point of system (6) and assume that (i)–(iv) hold such that $R_{01} < 1$ and $R_{02} < 1$. Moreover, let $r_1 \in \left(\dfrac{\beta_1(2-\varepsilon_1)}{2\mu_1}, \infty\right), r_2 \in \left(\dfrac{\beta_1(1-\varepsilon_1)}{2\mu_2}, \theta\right),\ r_3 \in \left(0, \dfrac{\delta}{2\mu_4 + he\omega}\right)$ and $r_3 + r_4 < \dfrac{2r_4\mu_5\gamma + \sigma_2\delta}{\delta}$. If

$$\overline{C_1} \in \left(\frac{\beta_2(1-\varepsilon_2)\bar{M} + 2\mu_1 r_1 \bar{S}}{2\mu_2 r_2 - \beta_1(1-\varepsilon_1)}, \infty\right) \text{ and } \bar{M} \in \left(\frac{\delta - 2\mu_4 r_3}{2\mu_4 r_3 he\omega}, \infty\right),$$

where

$$p \in \left(\frac{\beta_1(2-\varepsilon_1)\overline{C_1} + 2\mu_1 r_1 \bar{S} + (\beta_2(2-\varepsilon_2) - \sigma_1)\bar{M}}{r_1}, \frac{(\beta_1 + 2\mu_2 r_2)\overline{C_1} + (\beta_2 - \sigma_1)\bar{M}}{r_1}\right),$$

then all roots of Eq. (18) are real or complex conjugates with negative real parts and $|arg(\lambda_i)| > \dfrac{\alpha\pi}{2}$, $(i = 1,\ 2,\ 3,\ 4)$, is equivalent to the Routh–Hurwitz criteria. This implies that Λ is locally asymptotically stable.

Proof. Let us consider the case for $a_{11} + a_{22} < 0$ to have eigenvalues with negative real parts. Thus, we have

$$r_1 > \frac{\beta_1(1-\varepsilon_1)}{2\mu_1} \tag{23}$$

and

$$p < \frac{(\beta_1 + 2\mu_2 r_2)\overline{C_1} + (\beta_2 - \sigma_1)\bar{M}}{r_1}. \tag{24}$$

From (ii) and Eq. (24), we obtain

$$p \in \left(\frac{\beta_1(2-\varepsilon_1)\overline{C_1} + 2\mu_1 r_1 \bar{S} + (\beta_2(2-\varepsilon_2) - \sigma_1)\bar{M}}{r_1}, \frac{(\beta_1 + 2\mu_2 r_2)\overline{C_1} + (\beta_2 - \sigma_1)\bar{M}}{r_1}\right), \tag{25}$$

if

$$\overline{C_1} > \frac{\beta_2(1-\varepsilon_2)\bar{M} + 2\mu_1 r_1 \bar{S}}{2\mu_2 r_2 - \beta_1(1-\varepsilon_1)}, \tag{26}$$

where $r_2 > \frac{\beta_1(1-\varepsilon_1)}{2\mu_2}$.

In considering both (iii) and Eq. (26), we get

$$\overline{C_1} > \frac{\beta_2(1-\varepsilon_2)\bar{M} + 2\mu_1 r_1 \bar{S}}{2\mu_2 r_2 - \beta_1(1-\varepsilon_1)} > \frac{\beta_1(2-\varepsilon_1)\bar{S} - (\theta - r_2)}{2\mu_2 r_2}, \tag{27}$$

where $r_1 > \frac{\beta_1(2-\varepsilon_1)}{2\mu_1}$. Moreover, the discriminant of Eq. (21) is, in this case, positive.

Let us consider now the case for $a_{44} + a_{55} < 0$ to have eigenvalues with negative real parts. Thus, from

$$\left(\frac{\delta}{1+he\omega\bar{M}} - 2\mu_4 r_3\right)\bar{M} + \left\{(r_3 + r_4 - \sigma_2) - \left(\frac{\gamma}{(1+he\omega\bar{M})^2} + 2r_4\mu_5\right)\bar{N}\right\} < 0 \tag{28}$$

we obtain

$$\bar{M} > \frac{\delta - 2\mu_4 r_3}{2\mu_4 r_3 he\omega} \text{ for } r_3 < \frac{\delta}{2\mu_4} \tag{29}$$

and

$$\bar{N} > \frac{(r_3 + r_4 - \sigma_2)}{2r_4\mu_5} \text{ for } r_4 > \sigma_2 > r_3. \tag{30}$$

From (v) and Eqs. (28)–(29), we obtain

$$\bar{M} > \frac{\delta - 2\mu_4 r_3}{2\mu_4 r_3 he\omega} > 0.5{\alpha_4}^{-1} \text{ for } r_3 < \frac{\delta}{2\mu_4 + he\omega} < \frac{\delta}{2\mu_4}$$

and

$$0.5{\mu_5}^{-1} > \bar{N} > \frac{\gamma}{\delta} > \frac{(r_3 + r_4 - \sigma_2)}{2r_4\mu_5} \text{ for } r_3 + r_4 < \frac{2r_4\mu_5\gamma + \sigma_2\delta}{\delta}.$$

Since the discriminant of Eq. (22) is positive, the proof is complete.

Remark 3.1. Theorem 3.1. shows that among the human hosts, those who do not know they are infected, are the control class in the spread. In contrast, between the animal hosts, the intermediate class plays a dominant role, since that one has the essential role in transmitting from animal to human. The transmission potential for both $S - C_1$ and $M - N$ are $R_{01} < 1$ and $R_{02} < 1$. Moreover, the susceptible class and the C_1 class is stable based on two parameters, which are the awareness of the symptoms and the screening rate.

Theorem 3.2. Let Λ be the co-existing critical point of system (6) and assume that (i)–(iv) hold such that $R_{01} < 1$ and $R_{02} < 1$. Furthermore, let $r_1 \left\langle \frac{\beta_1(1-\varepsilon_1)}{2\mu_1},\ r_2 \right\rangle \frac{\beta_1(1-\varepsilon_1)}{2\mu_2}, \sigma_2 < r_4 < r_3 < \frac{\delta}{2\mu_4 + he\omega}$ and $p \in \left(\frac{(\beta_1 + 2\mu_2 r_2)\overline{C_1} + (\beta_2 - \sigma_1)\bar{M}}{r_1}, \infty\right)$. If

$\overline{C_1} > \dfrac{\beta_2(1-\varepsilon_2)\bar{M} + 2\mu_1 r_1 \bar{S}}{2\mu_2 r_2 - \beta_1(1-\varepsilon_1)}$ and $\dfrac{\delta - 2\mu_4 r_3}{2\mu_4 r_3 he\omega} > \bar{M} > 0.5{\mu_4}^{-1}$,

and the ratio between the susceptible and intermediate host is given by $\dfrac{\bar{M}}{\bar{S}} > \dfrac{\beta_1(2-\varepsilon_1)}{\beta_2(1-\varepsilon_2)}$, where

$$\left| tan^{-1}\left(-\left(4\frac{\left(r_1 p - 2\mu_1 r_1 \bar{S} - \beta_1 \overline{C_1} - (\beta_2 - \sigma_1)\bar{M}\right)\left(r_2 - 2\mu_2 r_2 \overline{C_1} + \beta_1(1-\varepsilon_1)\bar{S} - \theta\right)(1 - R_{01})}{\left(r_1 p + r_2 - \theta - (\beta_1 + 2\mu_2 r_2)\overline{C_1} - (\beta_2 - \sigma_1)\bar{M} + (\beta_1(1-\varepsilon_1) - 2\mu_1 r_1)\bar{S}\right)^2} - 1 \right)^{\frac{1}{2}} \right) \right| > \frac{\alpha\pi}{2}$$

and

$$\left| tan^{-1}\left(-\left(4\frac{\left(r_3 - 2\mu_4 r_3 \bar{M} - \sigma_2 - \dfrac{\gamma \bar{N}}{(1 + he\omega \bar{M})^2}\right)\left(r_4 - 2r_4\mu_5 \bar{N} + \dfrac{\delta \bar{M}}{1 + he\omega \bar{M}}\right)(1 - R_{02})}{\left(\left(\dfrac{\delta}{1 + he\omega \bar{M}} - 2\mu_4 r_3\right)\bar{M} + \left\{(r_3 + r_4 - \sigma_2) - \left(\dfrac{\gamma}{(1 + he\omega \bar{M})^2} + 2r_4\mu_5\right)\bar{N}\right\}\right)^2} - 1 \right)^{\frac{1}{2}} \right) \right| > \frac{\alpha\pi}{2}.$$

Then all roots of Eq. (18) are complex conjugates with positive real parts, which implies that Λ is locally asymptotically stable.

Proof. Let us consider the case for $a_{11} + a_{22} > 0$ to have eigenvalues with positive real parts. This holds if

$$r_1 < \frac{\beta_1(1-\varepsilon_1)}{2\mu_1} \tag{31}$$

and

$$p > \frac{(\beta_1 + 2\mu_2 r_2)\overline{C_1} + (\beta_2 - \sigma_1)\bar{M}}{r_1}. \tag{32}$$

From (ii) and Eq. (32) we obtain

$$p \in \left(\frac{(\beta_1 + 2\mu_2 r_2)\overline{C_1} + (\beta_2 - \sigma_1)\bar{M}}{r_1}, \infty \right) \tag{33}$$

if

$$\overline{C_1} > \frac{\beta_2(1-\varepsilon_2)\bar{M} + 2\mu_1 r_1 \bar{S}}{2\mu_2 r_2 - \beta_1(1-\varepsilon_1)}, \tag{34}$$

where $r_2 > \dfrac{\beta_1(1-\varepsilon_1)}{2\mu_2}$. In considering both (iii) and Eq. (34), we obtain

$$\overline{C_1} > \frac{\beta_2(1-\varepsilon_2)\bar{M} + 2\mu_1 r_1 \bar{S}}{2\mu_2 r_2 - \beta_1(1-\varepsilon_1)} > \frac{\beta_1(2-\varepsilon_1)\bar{S} - (\theta - r_2)}{2\mu_2 r_2},$$

where

$$\frac{\bar{M}}{\bar{S}} > \frac{\beta_1(2-\varepsilon_1)}{\beta_2(1-\varepsilon_2)}. \tag{35}$$

Additionally, we get $\sqrt{4a_{11}a_{22}(1-R_{01})-(a_{11}+a_{22})^2} > 0$, since $R_{01} < 1$, where

$$\left|tan^{-1}\left(-\left(4\frac{\left(r_1p-2\mu_1r_1\bar{S}-\beta_1\overline{C_1}-(\beta_2-\sigma_1)\bar{M}\right)\left(r_2-2\mu_2r_2\overline{C_1}+\beta_1(1-\varepsilon_1)\bar{S}-\theta\right)(1-R_{01})}{\left(r_1p+r_2-\theta-(\beta_1+2\mu_2r_2)\overline{C_1}-(\beta_2-\sigma_1)\bar{M}+(\beta_1(1-\varepsilon_1)-2\mu_1r_1)\bar{S}\right)^2}-1\right)^{\frac{1}{2}}\right)\right| > \frac{\alpha\pi}{2}.$$

Similarly, let us consider the case for $a_{44}+a_{55}>0$ to have eigenvalues with positive real parts. From

$$\left(\frac{\delta}{1+he\omega\bar{M}}-2\mu_4r_3\right)\bar{M}+\left\{(r_3+r_4-\sigma_2)-\left(\frac{\gamma}{(1+he\omega\bar{M})^2}+2r_4\mu_5\right)\bar{N}\right\}>0 \tag{36}$$

we obtain

$$\bar{M} < \frac{\delta-2\mu_4r_3}{2\mu_4r_3he\omega} \text{ for } r_3 < \frac{\delta}{2\mu_4} \tag{37}$$

and

$$\bar{N} < \frac{(r_3+r_4-\sigma_2)(1+he\omega\bar{M})^2}{\gamma+2r_4\mu_5(1+he\omega\bar{M})^2} \text{ for } r_3 > r_4 > \sigma_2. \tag{38}$$

From (v) and Eqs. (37)–(38) we have

$$\frac{\delta-2\mu_4r_3}{2\mu_4r_3he\omega} > \bar{M} > 0.5{\mu_4}^{-1} \text{ for } r_3 < \frac{\delta}{2\mu_4+he\omega} < \frac{\delta}{2\mu_4}$$

and

$$\frac{(r_3+r_4-\sigma_2)}{2r_4\mu_5} > 0.5{\mu_5}^{-1} > \bar{N} > \frac{\gamma}{\delta} \text{ for } r_3 > \sigma_2.$$

Moreover, we get $\sqrt{4a_{44}a_{55}(1-R_{02})-(a_{44}+a_{55})^2} > 0$, since $R_{02} < 1$, where

$$\left|tan^{-1}\left(-\left(4\frac{\left(r_3-2\mu_4r_3\bar{M}-\sigma_2-\frac{\gamma\bar{N}}{(1+he\omega\bar{M})^2}\right)\left(r_4-2r_4\mu_5\bar{N}+\frac{\delta\bar{M}}{1+he\omega\bar{M}}\right)(1-R_{02})}{\left(\left(\frac{\delta}{1+he\omega\bar{M}}-2\mu_4r_3\right)\bar{M}+\left\{(r_3+r_4-\sigma_2)-\left(\frac{\gamma}{(1+he\omega\bar{M})^2}+2r_4\mu_5\right)\bar{N}\right\}\right)^2}-1\right)^{\frac{1}{2}}\right)\right| > \frac{\alpha\pi}{2}.$$

This completes the proof.

Remark 3.2. In Theorem 3.2., we emphasize that class C_1 should be more aware of the symptoms that might become from the susceptible class as well as from the intermediate class, than the S class to stop the outbreak. For the susceptible class, it is more important to keep the population rate per year non-infected. The transmission of the virus to the offspring would reach an uncontrollable phenomenon worldwide.

Theorem 3.3. Let Λ be the co-existing critical point of system (6) and assume that (i)–(iv) hold such that $R_{01} < 1$ and $R_{02} < 1$.

(i) Let $r_1 \in \left(\frac{\beta_1(2-\varepsilon_1)}{2\mu_1}, \infty\right)$, $r_2 \in \left(\frac{\beta_1(1-\varepsilon_1)}{2\mu_2}, \theta\right)$, $\sigma_2 < r_4 < r_3 < \frac{\delta}{2\mu_4 + he\omega}$ and

$$p \in \left(\frac{\beta_1(2-\varepsilon_1)\overline{C_1} + 2\mu_1 r_1 \bar{S} + (\beta_2(2-\varepsilon_2) - \sigma_1)\bar{M}}{r_1}, \frac{(\beta_1 + 2\mu_2 r_2)\overline{C_1} + (\beta_2 - \sigma_1)\bar{M}}{r_1}\right).$$

If

$$\overline{C_1} \in \left(\frac{\beta_2(1-\varepsilon_2)\bar{M} + 2\mu_1 r_1 \bar{S}}{2\mu_2 r_2 - \beta_1(1-\varepsilon_1)}, \infty\right) \text{and } \frac{\delta - 2\mu_4 r_3}{2\mu_4 r_3 he\omega} > \bar{M} > 0.5\mu_4^{-1}$$

where

$$\left| tan^{-1}\left(-\left(4\frac{\left(r_3 - 2\mu_4 r_3 \bar{M} - \sigma_2 - \frac{\gamma\bar{N}}{(1+he\omega\bar{M})^2}\right)\left(r_4 - 2r_4\mu_5\bar{N} + \frac{\delta\bar{M}}{1+he\omega\bar{M}}\right)(1-R_{02})}{\left(\left(\frac{\delta}{1+he\omega\bar{M}} - 2\mu_4 r_3\right)\bar{M} + \left\{(r_3 + r_4 - \sigma_2) - \left(\frac{\gamma}{(1+he\omega\bar{M})^2} + 2r_4\mu_5\right)\bar{N}\right\}\right)^2} - 1\right)^{\frac{1}{2}}\right)\right| > \frac{\alpha\pi}{2}.$$

then the $S - C_1$ class represents real or complex conjugates with negative real parts, while the $M - N$ class shows complex conjugates with positive real parts.

(ii) Let $r_1 \left\langle \frac{\beta_1(1-\varepsilon_1)}{2\mu_1}, r_2 \right\rangle \frac{\beta_1(1-\varepsilon_1)}{2\mu_2}$, $r_3 \in \left(0, \frac{\delta}{2\mu_4 + he\omega}\right)$, $r_3 + r_4 < \frac{2r_4\mu_5\gamma + \sigma_2\delta}{\delta}$ and

$$p \in \left(\frac{(\beta_1 + 2\mu_2 r_2)\overline{C_1} + (\beta_2 - \sigma_1)\bar{M}}{r_1}, \infty\right).$$

If

$$\overline{C_1} > \frac{\beta_2(1-\varepsilon_2)\bar{M} + 2\mu_1 r_1 \bar{S}}{2\mu_2 r_2 - \beta_1(1-\varepsilon_1)}, \quad \bar{M} \in \left(\frac{\delta - 2\mu_4 r_3}{2\mu_4 r_3 he\omega}, \infty\right)$$

and the ratio between the susceptible and intermediate host is given by $\frac{\bar{M}}{\bar{S}} > \frac{\beta_1(2-\varepsilon_1)}{\beta_2(1-\varepsilon_2)}$, where

$$\left| tan^{-1}\left(-\left(4\frac{(r_1 p - 2\mu_1 r_1 \bar{S} - \beta_1\bar{C}_1 - (\beta_2 - \sigma_1)\bar{M}(r_2 - 2\mu_2 r_2 \bar{C}_1 + \beta_1(1-\varepsilon_1)\bar{S} - \theta)(1-R_{01})}{\left(r_1 p + r_2 - \theta - (\beta_1 + 2\mu_2 r_2)\overline{C_1} - (\beta_2 - \sigma_1)\bar{M} + (\beta_1(1-\varepsilon) - 2\mu_1 r_1 \bar{S}\right)^2} - 1\right)^{\frac{1}{2}}\right)\right| > \frac{\alpha\pi}{2},$$

then the $S - C_1$ class represents complex conjugates with positive real parts, while the $M - N$ class shows real or complex conjugates with negative real parts.

Example. In this part, we present numerical simulations that are in good agreement with our theoretical results. We assume the initial conditions of the system (1) as $S(0) = 1000,\ C_1(0) = 80, C_2(0) = 40,\ M(0) = 30$ and $N(0) = 10$.

In Fig. 1 the blue graph denotes the susceptible class S and the red graph shows C_1 who does not know they are infected. Fig. 1 represents the transmission of the infection that occurs as an epidemic case in some areas, but it spreads intensively to a pandemic case and covers almost the susceptible class. Here we want to emphasize the point of screening, where we assume that about $\%1$ do testing in the hospitals ***before*** the symptoms appear. Additionally, we consider that the symptoms appear late, and thus the awareness of the infection is also at $\%1$. This changes the endemic spread from epidemic to an uncontrolled pandemic form.

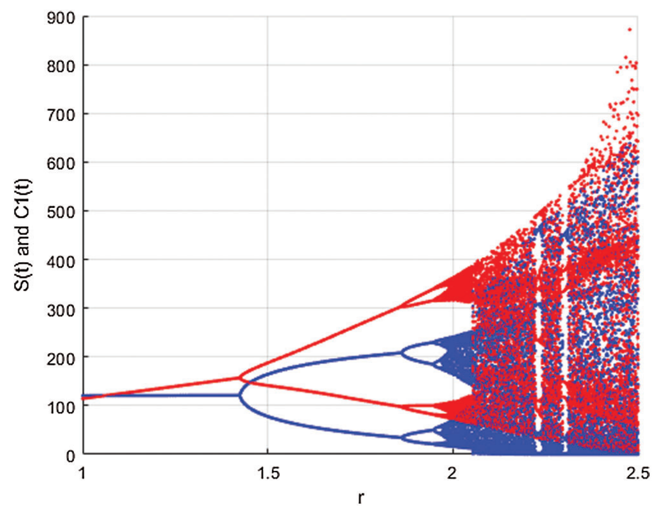

Figure 1: Spread of the C_1 class and effect on the susceptible S class, where $\theta = 0.01$ and $\varepsilon_1 = \varepsilon_2 = 0.1$

In Fig. 2, we keep the screening parameter as $\theta = 0.01$, while we consider the case that the people become aware of the virus and the symptoms of it through media and health organizations. An organized and constant information flood from media might increase the awareness up to $\varepsilon_1 = \varepsilon_2 = 0.4$.

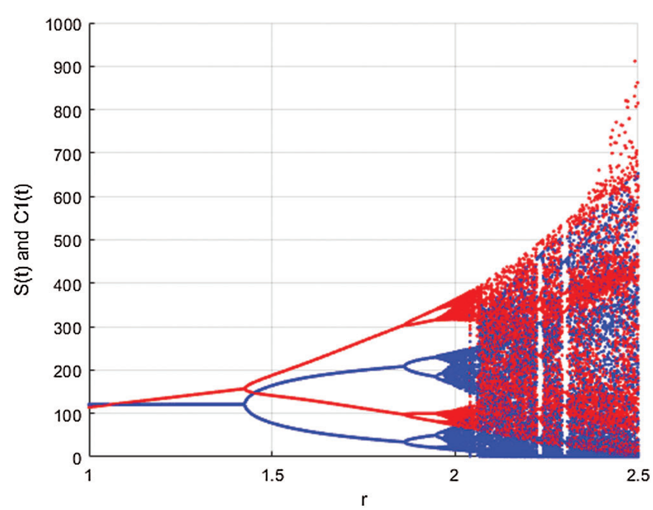

Figure 2: Spread of the C_1 class and effect on the susceptible S class, where $\theta = 0.01$ and $\varepsilon_1 = \varepsilon_2 = 0.4$

This awareness of the people through media and health organizations let them go to hospitals for screening so that the class who does not know they are infected decreases. Fig. 3 shows the effect of the testing when it reaches to %5. The spread is under control and returns to an epidemic form.

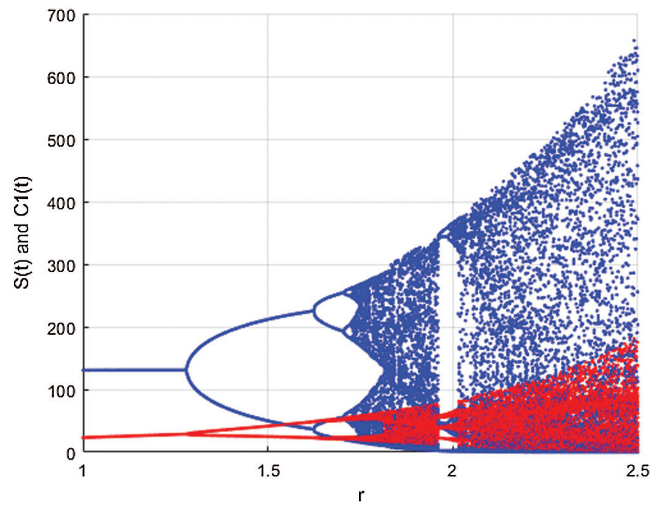

Figure 3: Spread of the C_1 class and effect on the susceptible S class, where $\theta = 0.05$ and $\varepsilon_1 = \varepsilon_2 = 0.4$

We considered in these examples the infection from human-to-human since the pandemic case reaches from the human transmission. We want to emphasize the strong coordination between health organizations and the media which is an essential tool for two critical parameters, which are θ and ε_i $(i = 1,\ 2)$

The design of nature keeps the natural host and intermediate host in a stable dynamical system in the habitat. The intermediate host had only a transmission role from animal to human, while the main spread happens through human to human from the C_1 class who does not know they are infected.

4 Existence and Uniqueness of the Initial Value Fractional-Order Problem

Considering system (6) with the initial conditions $S(0) > 0, C_1(0) > 0,\ C_2(0) > 0, M(0) > 0$ and $N(0) > 0$, the initial value problem can be written in matrix form as

$$\begin{cases} D^{\alpha}U(t) = AU(t) + S(t)BU(t) + C_1(t)CU(t) + C_2(t)DU(t) + M(t)EU(t) + N(t)FU(t), \\ U(0) = U_0 \end{cases} \tag{39}$$

for $t \in (0,T]$, where $U(t) = \begin{bmatrix} S(t) \\ C_1(t) \\ C_2(t) \\ M(t) \\ N(t) \end{bmatrix}$ and $U(0) = \begin{bmatrix} S(0) \\ C_1(0) \\ C_2(0) \\ M(0) \\ N(0) \end{bmatrix}$.

Let us assume that $0 < M(0) \leq \vartheta$, and $S(0) > 0,\ \ C_1(0) > 0, C_2(0) > 0,\ N(0) > 0,$

when $t > \sigma \geq 0$. In this case, the following definitions can be adopted to the main theorems in this section.

Definition 4.1. Let $C^*[0,\ T]$ be the class of continuous column vector $U(t)$ whose components $S(t),\ C_1(t), C_2(t),\ M(t),\ N(t) \in C[0,\ T]$ are the class of continuous functions on the interval $[0,\ T]$. The norm of $U \in C^*[0,\ T]$ is given by

$$\| U \| = \sup_t |e^{-Wt}S(t)| + \sup_t |e^{-Wt}C_1(t)| + \sup_t |e^{-Wt}C_2(t)| + \sup_t |e^{-Wt}M(t)| + \sup_t |e^{-Wt}N(t)|$$

when $t > \sigma \geq 0$, we write $C^*_\sigma[0,\ T]$ and $C_\sigma[0,\ T]$.

Definition 4.2. Let the initial value problem Eq. (39) has a solution given by $U \in C^*[0,\ T]$. If

(i) $(t,\ U(t)) \in D, t \in [0,\ T]$ where $D = [0, T] \times K$ and

$K = \{(S(t), C_1(t), C_2(t), M(t),\ N(t))\colon |\ S(t)| \leq \upsilon_1 |C_1(t)| \leq \upsilon_2, |C_2(t)| \leq \upsilon_3,\ |M(t)| \leq \vartheta, |\ N(t)| \leq \upsilon_4\}$.

(ii) $U(t)$ satisfies Eq. (39).

Theorem 4.1. The initial value problem Eq. (39) has a unique solution $U \in C^*[0,\ T]$.

Proof. Because of Eq. (39), we have

$$I^{1-\alpha} \frac{d}{dt} U(t) = AU(t) + S(t)BU(t) + C_1(t)CU(t) + C_2(t)DU(t) + M(t)EU(t) + N(t)FU(t). \tag{40}$$

Operating I^α on Eq. (40), we obtain

$$U(t) = U(0) + I^\alpha (AU(t) + S(t)BU(t) + C_1(t)CU(t) + C_2(t)DU(t) + M(t)EU(t) + N(t)FU(t)). \tag{41}$$

Define the operator $F : C^*[0,\ T] \to C^*[0,\ T]$ by

$$FU(t) = U(0) + I^\alpha (AU(t) + S(t)BU(t) + C_1(t)CU(t) + \ C_2(t)DU(t) + M(t)EU(t) + N(t)FU(t)). \tag{42}$$

It follows that

$$e^{-Wt} \parallel FU - FV \parallel = e^{-Wt} I^\alpha (A(U(t) - V(t)) + S(t)B(U(t) - V(t)) + C_1(t)C(U(t) - V(t)) + C_1(t)D(U(t) - V(t)) + M(t)E(U(t) - V(t)) + M(t)F(U(t) - V(t)))$$

$$\leq \frac{1}{\Gamma(\alpha)} {}_0^t (t-s)^{\alpha-1} e^{-W(t-s)} (U(s) - V(s)) e^{-Ws} (A + \upsilon_1 B + \upsilon_2 C + \upsilon_3 D + \vartheta E + \upsilon_4 F) ds$$

$$\leq \frac{(A + \upsilon_1 B + \upsilon_2 C + \upsilon_3 D + \vartheta E + \upsilon_4 F)}{W^\alpha} \parallel U - V \parallel \int_0^t \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds.$$

This implies that $\parallel FU - FV \parallel \leq \frac{(A + \upsilon_1 B + \upsilon_2 C + \upsilon_3 D + \vartheta E + \upsilon_4 F)}{W^\alpha} \parallel U - V \parallel$. If we choose W such that $W^\alpha > A + \upsilon_1 B + \upsilon_2 C + \upsilon_3 D + \vartheta E + \upsilon_4 F$, then we obtain $\parallel FU - FV \parallel \leq k \parallel U - V \parallel,\ \ 0 < k < 1$. Therefore, using the Banach fixed point theorem, we conclude that the operator F given by Eq. (42) has a unique fixed point. Consequently, Eq. (41) has a unique solution $U \in C^*[0,\ T]$. From Eq. (41), we have

$$\begin{aligned} U(t) = U(0) + &\left(\frac{t^\alpha}{\Gamma(\alpha+1)} (AU(t) + S(t)BU(t) + C_1(t)CU(t) + C_2(t)DU(t) + M(t)EU(t) + N(t)FU(t)) \right) \\ &+ I^{\alpha+1} (AU'(t) + S'(t)BU(t) + S(t)BU'(t) + {C_1}'(t)CU(t) + C_1(t)CU'(t) \\ &+ {C_2}'(t)DU(t) + C_2(t)DU'(t) + M'(t)EU(t) + M(t)EU'(t) + N'(t)FU(t) + N(t)FU'(t)) \end{aligned}$$

and

$$\begin{aligned} \frac{U(t)}{dt} = &\frac{t^{\alpha-1}}{\Gamma(\alpha)} (AU(0) + S(0)BU(0) + C_1(0)CU(0) + C_2(0)DU(0) + M(0)EU(0) + N(0)FU(0)) \\ &+ I^\alpha (AU'(t) + S'(t)BU(t) + S(t)BU'(t) + C_1^{'(t)} CU(t) + C_1(t)CU'(t) + C_2^{'(t)} DU(t) + C_2(t)DU'(t) \\ &+ M'(t)EU(t) + M(t)EU'(t) + N'(t)FU(t) + N(t)FU'(t)), \end{aligned}$$

which implies

$$e^{-Nt}\left(\frac{U(t)}{dt}\right) = e^{-Nt}\left(\frac{t^{\alpha-1}}{\gamma(\alpha)}(AU(0) + S(0)BU(0) + C_1(0)CU(0) + C_2(0)DU(0) + M(0)EU(0) + N(0)FU(0))\right.$$
$$+ I^{\alpha}(U'(t) + S'(t)BU(t) + S(t)BU'(t) + C_1{}'(t)CU(t) + C_1(t)CU'(t) + C_2{}'(t)DU(t)$$
$$\left. + C_2(t)DU'(t) + M'(t)EU(t) + M(t)EU'(t) + N'(t)FU(t) + N(t)FU'(t))\right)$$

from which we can deduce that $U' \in C_{\sigma}^{*}[0,\ T]$. Thus, we have

$$\frac{dU(t)}{dt} = \frac{d}{dt}I^{\alpha}(AU(t) + S(t)BU(t) + C_1(t)CU(t) + C_2(t)DU(t) + M(t)EU(t) + N(t)FU(t)).$$

It follows that

$$I^{1-\alpha}\frac{dU(t)}{dt} = I^{1-\alpha}\frac{d}{dt}I^{\alpha}(AU(t) + S(t)BU(t) + C_1(t)CU(t) + C_2(t)DU(t) + M(t)EU(t) + N(t)FU(t)),$$

which implies

$$D^{\alpha}U(t) = AU(t) + S(t)BU(t) + C_1(t)CU(t) + C_2(t)DU(t) + M(t)EU(t) + N(t)FU(t)$$

and thus

$$U(0) = U_0 + I^{\alpha}(AU(0) + S(0)BU(0) + C_1(0)CU(0) + C_2(0)DU(0) + M(0)EU(0) + N(0)FU(0))$$
$$= U_0$$

Therefore, this IVP is equivalent to Eq. (39), which completes the proof.

5 The Case of Extinction via Strong Allee Effect

In 1838, Pierre Verhulst [29] considered the *logistic growth* function to explain mono-species growth. Later on, it is demonstrated that the logistic equation needs modifications to explain the growth of the population in low density-size, which is known as *the Allee effect*.

The Allee effect can be divided into two main types:

(i) strong Allee effect and

(ii) weak Allee effect.

A population with a strong Allee effect will have a critical population size, which is the threshold of the population, and any size that is less than the threshold will go to extinction without any further aid. However, a population with a weak Allee effect will reduce the *per capita* growth rate at lower population density or size [30–34].

Let us incorporate an Allee function to the $C_1(t)$ class at time t such as

$$\begin{cases} D^{\alpha}\mathrm{S(t)} = \mathrm{r_1S(t)(p} - \mu_1\mathrm{S(t))} - \beta_1\mathrm{S(t)}C_1\mathrm{(t)} - \beta_2\mathrm{M(t)S(t)} + \sigma_1 M(t)\mathrm{S(t)} \\ D^{\alpha}C_1\mathrm{(t)} = \mathcal{H}(C_1(t))\{\mathrm{r_2}C_1(t)(1 - \mu_2 C_1(t)) + \beta_1(1-\varepsilon_1)S(\mathrm{t})C_1(t) - \theta C_1(t) + \beta_2(1-\varepsilon_2)\mathrm{M}(t)S(t)\} \\ D^{\alpha}C_2\mathrm{(t)} = \mathrm{C_2(t)}(1 - \mu_3 C_2\mathrm{(t)}) + \theta\mathrm{C_1(t)C_2(t)} + \beta_1\varepsilon_1\mathrm{S}(t)\mathrm{C_1(t)} + \beta_2\varepsilon_2\mathrm{M(t)S(t)} \\ D^{\alpha}M(t) = M(t)\mathrm{r_3}(1 - \mu_4 M(t)) - \sigma_2 M(t) - \gamma f(t)N(t) \\ D^{\alpha}N(t) = N(t)\mathrm{r_4}(1 - \mu_5 N(t)) + \delta f(t)N(t) \end{cases} \tag{43}$$

where

$$f(t) = \frac{M(t)}{1 + he\omega M(t)} \tag{44}$$

is a function of Holling Type II and $\mathcal{H}(C_1(t))$ is an Allee function at time t.

Let

$$\mathcal{M}(t)=\frac{D^{\alpha}C_1(\mathrm{t})}{\mathrm{C}_1(t)}=\mathcal{H}(C_1(t))\left\{\mathrm{r}_2(1-\mu_2C_1(t))+\beta_1(1-\varepsilon_1)S(\mathrm{t})-\theta+\beta_2(1-\varepsilon_2)\frac{\mathrm{M}(t)S(t)}{\mathrm{C}_1(t)}\right\},$$

where we obtain $\frac{\partial\mathcal{M}(t)}{\partial \mathrm{C}_1(\mathrm{t})}<0$, if

$$\frac{\beta_1(1-\varepsilon_1)S(\mathrm{t})-\mathcal{H}(C_1(t))\mathrm{r}_2\mu_2-\frac{d\mathcal{H}(C_1(t))}{dC_1(t)}(\theta-\mathrm{r}_2)}{\mathrm{r}_2\mu_2}<C_1(t)<\frac{\mathcal{H}(C_1(t))}{\frac{d\mathcal{H}(C_1(t))}{dC_1(t)}} \tag{45}$$

and

$$\frac{\mathcal{H}(C_1(t))\mathrm{r}_2\mu_2\left(\frac{d\mathcal{H}(C_1(t))}{dC_1(t)}+1\right)+\left(\frac{d\mathcal{H}(C_1(t))}{dC_1(t)}\right)^2(\theta-\mathrm{r}_2)}{\beta_1(1-\varepsilon_1)\mathcal{H}(C_1(t))}>S(\mathrm{t})>\frac{\mathcal{H}(C_1(t))\mathrm{r}_2\mu_2+\frac{d\mathcal{H}(C_1(t))}{dC_1(t)}(\theta-\mathrm{r}_2)}{\beta_1(1-\varepsilon_1)}, \tag{46}$$

where

$$\frac{\mathcal{H}(C_1(t))}{\frac{d\mathcal{H}(C_1(t))}{dC_1(t)}}>1. \tag{47}$$

Remark 5.1 The susceptible class and the classes who do not know they are infected are the main populations that affect the Allee function in stabilizing the spread of transmission. While it is essential to keep human non-infected, the other essential aim is to detect the infected class before the symptoms occur.

The characteristic equation of system (43) is given by

$$\left\{(a_{11}-\lambda)\left(\breve{a}_{22}-\lambda\right)-a_{12}\,\breve{a}_{21}\right\}\{(a_{44}-\lambda)(a_{55}-\lambda)-a_{45}a_{54}\}=0 \tag{48}$$

and

$$\lambda=a_{33}<0\Rightarrow 1-2\mu_3\overline{C_2}+\theta\overline{C_1}<0\Rightarrow\overline{C_2}>\frac{\theta\overline{C_1}+1}{2\mu_3}, \tag{49}$$

where

$\breve{a}_{21}=\mathcal{H}\left(\overline{C_1}\right)\left(\beta_1(1-\varepsilon_1)\overline{C_1}+\beta_2(1-\varepsilon_2)\bar{M}\right)$, $\breve{a}_{22}=\mathcal{H}\left(\overline{C_1}\right)\left(\mathrm{r}_2-2\mu_2\mathrm{r}_2\overline{C_1}+\beta_1(1-\varepsilon_1)\bar{S}-\theta\right)$.

From Eq. (48), we have two quadratic equations, which are

$$\lambda^2+\left(-a_{11}-\breve{a}_{22}\right)\lambda+a_{11}\,\breve{a}_{22}\,(1-R_{01})=0 \tag{50}$$

and

$$\lambda^2+(-a_{44}-a_{55})\lambda+a_{44}a_{55}(1-R_{02})=0. \tag{51}$$

where $\breve{R}_{01}=R_{01}=\frac{a_{12}\,\breve{a}_{21}}{a_{11}\,\breve{a}_{22}}$ and $R_{02}=\frac{a_{45}a_{54}}{a_{44}a_{55}}$. $\breve{R}_{01}$ is the basic reproduction number, which represents the transmission potential of the $S-C_1$ class in the case of early detection, while R_{02} shows the transmission potential of the intermediate-natural host classes. This indicates that the reproduction numbers are not dependent on the Allee function.

For a strong Allee effect, let us assume that the Allee function is given by

$$\mathcal{H}(C_1(t)) = \left(\frac{C_1(t)}{K_0} - 1\right), \tag{52}$$

where K_0 represents the Allee threshold of the infected class, that do not know they are infected.

The following Theorem is given without proof since it is similar to the stability analysis of Section 3.

Theorem 5.1. Let Λ be the co-existing critical point of system (43) and assume that (i)–(iv) hold with Eqs. (45)–(47) such that $R_{01} < 1$ and $R_{02} < 1$.

(i) Let $r_1 \in \left(\frac{\beta_1 \mathcal{H}(\overline{C_1})(2-\varepsilon_1)}{2\mu_1}, \infty\right)$, $r_2 \in \left(\frac{\beta_1(2-\mathcal{H}(\overline{C_1})-\varepsilon_1)}{2\mu_2\mathcal{H}(\overline{C_1})}, \theta\right)$, $r_3 \in \left(0, \frac{\delta}{2\mu_4 + he\omega}\right)$, $r_3 + r_4 < \frac{2r_4\mu_5\gamma + \sigma_2\delta}{\delta}$ and $\mathcal{H}(\overline{C_1}) + \varepsilon_1 < 2$. If

$$\overline{C_1} \in \left(\frac{\beta_2(1-\varepsilon_2)\bar{M} + 2\mu_1 r_1 \bar{S}}{2\mu_2 r_2 \mathcal{H}(\overline{C_1}) - \beta_1(2 - \mathcal{H}(\overline{C_1}) - \varepsilon_1)}, \infty\right) \text{ and } \bar{M} \in \left(\frac{\delta - 2\mu_4 r_3}{2\mu_4 r_3 he\omega}, \infty\right),$$

where

$$p \in \left(\frac{\beta_1(2-\varepsilon_1)\overline{C_1} + 2\mu_1 r_1 \bar{S} + (\beta_2(2-\varepsilon_2) - \sigma_1)\bar{M}}{r_1}, \frac{(\beta_1 + 2\mu_2 r_2)\mathcal{H}(\overline{C_1})\overline{C_1} + (\beta_2 - \sigma_1)\bar{M}}{r_1}\right),$$

then all the roots of the system are real or complex conjugates with negative real parts.

(ii) Let $r_1 < \frac{\beta_1(1-\varepsilon_1)}{2\mu_1}$, $r_2 \in \left(\frac{\beta_1(2-\mathcal{H}(\overline{C_1})-\varepsilon_1)}{2\mu_2\mathcal{H}(\overline{C_1})}, \theta\right)$, $\sigma_2 < r_4 < r_3 < \frac{\delta}{2\mu_4 + he\omega}$, $\mathcal{H}(\overline{C_1}) + \varepsilon_1 < 2$ and

$$p \in \left(\frac{(\beta_1 + 2\mu_2 r_2)\mathcal{H}(\overline{C_1})\overline{C_1} + (\beta_2 - \sigma_1)\bar{M}}{r_1}, \infty\right).$$

If

$$\overline{C_1} \in \left(\frac{\beta_2(1-\varepsilon_2)\bar{M} + 2\mu_1 r_1 \bar{S}}{2\mu_2 r_2 \mathcal{H}(\overline{C_1}) - \beta_1(2 - \mathcal{H}(\overline{C_1}) - \varepsilon_1)}, \infty\right) \text{ and } \bar{M} \in \left(0.5\mu_4^{-1}, \frac{\delta - 2\mu_4 r_3}{2\mu_4 r_3 he\omega}\right),$$

and the ratio between the susceptible and intermediate host is given by $\frac{\bar{M}}{\bar{S}} > \frac{\beta_1(2-\varepsilon_1)}{\beta_2(1-\varepsilon_2)}$, where

$$\left| tan^{-1}\left(-\left(4\frac{(r_1 p - 2\mu_1 r_1 \bar{S} - \beta_1 \overline{C_1} - (\beta_2 - \sigma_1)\bar{M})(r_2 - 2\mu_2 r_2 \overline{C_1} + \beta_1(1-\varepsilon_1)\bar{S} - \theta)\mathcal{H}(\overline{C_1})(1 - R_{01})}{(r_1 p + \mathcal{H}(\overline{C_1})(r_2 - \theta) - (\beta_1 + 2\mu_2 r_2 \mathcal{H}(\overline{C_1}))\overline{C_1} - (\beta_2 - \sigma_1)\bar{M} + (\beta_1(1-\varepsilon_1)\mathcal{H}(\overline{C_1}) - 2\mu_1 r_1)\bar{S})^2} - 1\right)^{\frac{1}{2}}\right)\right| > \frac{\alpha\pi}{2}$$

and

$$\left|tan^{-1}\left(-\left(4\frac{\left(r_3-2\mu_4r_3\bar{M}-\sigma_2-\frac{\gamma\bar{N}}{(1+he\omega\bar{M})^2}\right)\left(r_4-2r_4\mu_5\bar{N}+\frac{\delta\bar{M}}{1+he\omega\bar{M}}\right)(1-R_{02})}{\left(\left(\frac{\delta}{1+he\omega\bar{M}}-2\mu_4r_3\right)\bar{M}+\left\{(r_3+r_4-\sigma_2)-\left(\frac{\gamma}{(1+he\omega\bar{M})^2}+2r_4\mu_5\right)\bar{N}\right\}\right)^2}-1\right)^{\frac{1}{2}}\right)\right|>\frac{\alpha\pi}{2}.$$

Thus, all roots of the system are complex conjugates with positive real parts.

(iii) Let $r_1\in\left(\frac{\beta_1\mathcal{H}(\overline{C_1})(2-\varepsilon_1)}{2\mu_1},\infty\right)$, $r_2\in\left(\frac{\beta_1(2-\mathcal{H}(\overline{C_1})-\varepsilon_1)}{2\mu_2\mathcal{H}(\overline{C_1})},\theta\right)$, $\sigma_2<r_4<r_3<\frac{\delta}{2\mu_4+he\omega}$, $\mathcal{H}(\overline{C_1})+\varepsilon_1<2$ and

$$p\in\left(\frac{\beta_1(2-\varepsilon_1)\overline{C_1}+2\mu_1r_1\bar{S}+(\beta_2(2-\varepsilon_2)-\sigma_1)\bar{M}}{r_1},\frac{(\beta_1+2\mu_2r_2)\mathcal{H}(\overline{C_1})\overline{C_1}+(\beta_2-\sigma_1)\bar{M}}{r_1}\right).$$

If

$$\overline{C_1}\in\left(\frac{\beta_2(1-\varepsilon_2)\bar{M}+2\mu_1r_1\bar{S}}{2\mu_2r_2\mathcal{H}(\overline{C_1})-\beta_1(2-\mathcal{H}(\overline{C_1})-\varepsilon_1)},\infty\right)\text{ and }\bar{M}\in\left(0.5\alpha_4{}^{-1},\frac{\delta-2\mu_4r_3}{2\mu_4r_3he\omega}\right),$$

where

$$\left|tan^{-1}\left(-\left(4\frac{\left(r_3-2\mu_4r_3\bar{M}-\sigma_2-\frac{\gamma\bar{N}}{(1+he\omega\bar{M})^2}\right)\left(r_4-2r_4\mu_5\bar{N}+\frac{\delta\bar{M}}{1+he\omega\bar{M}}\right)(1-R_{02})}{\left(\left(\frac{\delta}{1+he\omega\bar{M}}-2\mu_4r_3\right)\bar{M}+\left\{(r_3+r_4-\sigma_2)-\left(\frac{\gamma}{(1+he\omega\bar{M})^2}+2r_4\mu_5\right)\bar{N}\right\}\right)^2}-1\right)^{\frac{1}{2}}\right)\right|>\frac{\alpha\pi}{2},$$

then the $S-C_1$ class represents real or complex conjugates with negative real parts, while the $M-N$ class shows complex conjugates with positive real parts.

(iv) Let $r_1<\frac{\beta_1(1-\varepsilon_1)}{2\mu_1}$, $r_2\in\left(\frac{\beta_1(2-\mathcal{H}(\overline{C_1})-\varepsilon_1)}{2\mu_2\mathcal{H}(\overline{C_1})},\theta\right)$, $r_3\in\left(0,\frac{\delta}{2\mu_4+he\omega}\right)$, $r_3+r_4<\frac{2r_4\mu_5\gamma+\sigma_2\delta}{\delta}$, $\mathcal{H}(\overline{C_1})+\varepsilon_1<2$ and

$$p\in\left(\frac{(\beta_1+2\mu_2r_2)\mathcal{H}(\overline{C_1})\overline{C_1}+(\beta_2-\sigma_1)\bar{M}}{r_1},\infty\right).$$

If

$$\overline{C_1}\in\left(\frac{\beta_2(1-\varepsilon_2)\bar{M}+2\mu_1r_1\bar{S}}{2\mu_2r_2\mathcal{H}(\overline{C_1})-\beta_1(2-\mathcal{H}(\overline{C_1})-\varepsilon_1)},\infty\right),\ \bar{M}\in\left(\frac{\delta-2\mu_4r_3}{2\mu_4r_3he\omega},\infty\right)$$

and the ratio between the susceptible and intermediate host is given by $\frac{\bar{M}}{\bar{S}}>\frac{\beta_1(2-\varepsilon_1)}{\beta_2(1-\varepsilon_2)}$, where

$$\left| tan^{-1}\left(-\left(4\frac{(\mathrm{r}_1 p - 2\mu_1 \mathrm{r}_1 \bar{S} - \beta_1 \overline{C_1} - (\beta_2 - \sigma_1)\bar{M})(\mathrm{r}_2 - 2\mu_2 \mathrm{r}_2 \overline{C_1} + \beta_1(1-\varepsilon_1)\bar{S} - \theta)\mathcal{H}(\overline{C_1})(1 - R_{01})}{\left(\mathrm{r}_1 p + \mathcal{H}(\overline{C_1})(\mathrm{r}_2 - \theta) - (\beta_1 + 2\mu_2 \mathrm{r}_2 a(\overline{C_1}))\overline{C_1} - (\beta_2 - \sigma_1)\bar{M} + (\beta_1(1-\varepsilon_1)\mathcal{H}(\overline{C_1}) - 2\mu_1 \mathrm{r}_1)\bar{S}\right)^2} - 1 \right)^{\frac{1}{2}} \right) \right| > \frac{\alpha\pi}{2},$$

then the $S - C_1$ class represents complex conjugates with positive real parts, while the $M - N$ class shows real or complex conjugates with negative real parts.□

6 Neimark–Sacker Bifurcation of the Dynamical Behavior with Discretization

In this section, we consider the discretization process to analyze Neimark–Sacker bifurcation. We will modify our system in (1) in considering the discrete-time effect on the model. The discretization of system (1) is as follows:

$$\begin{cases} D^\alpha \mathrm{S}(\mathrm{t}) = \mathrm{r}_1 \mathrm{S}\left(\left[\frac{t}{x}\right]x\right)\left(\mathrm{p} - \mu_1 \mathrm{S}\left(\left[\frac{t}{x}\right]x\right)\right) - \beta_1 \mathrm{S}\left(\left[\frac{t}{x}\right]x\right) C_1\left(\left[\frac{t}{x}\right]x\right) - \beta_2 \mathrm{M}\left(\left[\frac{t}{x}\right]x\right)\mathrm{S}\left(\left[\frac{t}{x}\right]x\right) + \sigma_1 M\left(\left[\frac{t}{x}\right]x\right)\mathrm{S}\left(\left[\frac{t}{x}\right]x\right) \\ D^\alpha C_1(\mathrm{t}) = \mathrm{r}_2 C_1\left(\left[\frac{t}{x}\right]x\right)\left(1 - \mu_2 C_1\left(\left[\frac{t}{x}\right]x\right)\right) + \beta_1(1-\varepsilon_1) S\left(\left[\frac{t}{x}\right]x\right) C_1\left(\left[\frac{t}{x}\right]x\right) - \theta C_1\left(\left[\frac{t}{x}\right]x\right) + \beta_2(1-\varepsilon_2)\mathrm{M}\left(\left[\frac{t}{x}\right]x\right) S\left(\left[\frac{t}{x}\right]x\right) \\ D^\alpha C_2(\mathrm{t}) = \mathrm{C}_2\left(\left[\frac{t}{x}\right]x\right)\left(1 - \mu_3 C_2\left(\left[\frac{t}{x}\right]x\right)\right) + \theta \mathrm{C}_1\left(\left[\frac{t}{x}\right]x\right)\mathrm{C}_2\left(\left[\frac{t}{x}\right]x\right) + \beta_1\varepsilon_1 \mathrm{S}\left(\left[\frac{t}{x}\right]x\right)\mathrm{C}_1\left(\left[\frac{t}{x}\right]x\right) + \beta_2\varepsilon_2 \mathrm{M}\left(\left[\frac{t}{x}\right]x\right)\mathrm{S}\left(\left[\frac{t}{x}\right]x\right) \\ D^\alpha M(t) = M\left(\left[\frac{t}{x}\right]x\right)\mathrm{r}_3\left(1 - \mu_4 M\left(\left[\frac{t}{x}\right]x\right)\right) - \sigma_2 M\left(\left[\frac{t}{x}\right]x\right) - \gamma f\left(\left[\frac{t}{x}\right]x\right) N\left(\left[\frac{t}{x}\right]x\right) \\ D^\alpha N(t) = N\left(\left[\frac{t}{x}\right]x\right)\mathrm{r}_4\left(1 - \mu_5 N\left(\left[\frac{t}{x}\right]x\right)\right) + \delta f\left(\left[\frac{t}{x}\right]x\right) N\left(\left[\frac{t}{x}\right]x\right) \end{cases} \tag{53}$$

where

$$f\left(\left[\frac{t}{x}\right]x\right) = \frac{M\left(\left[\frac{t}{x}\right]x\right)}{1 + he\omega M\left(\left[\frac{t}{x}\right]x\right)}. \tag{54}$$

The solution of system (53) for $t \in [0, h)$, $\frac{t}{h} \in [0, 1)$ is given by

$$\begin{cases} \mathrm{S}(1) = \mathrm{S}(0) + \frac{t^\alpha}{\Gamma(\alpha+1)}\{\mathrm{r}_1 \mathrm{S}(0)(\mathrm{p} - \mu_1 \mathrm{S}(0)) - \beta_1 \mathrm{S}(0) C_1(0) - \beta_2 \mathrm{M}(0)\mathrm{S}(0) + \sigma_1 M(0)\mathrm{S}(0)\} \\ C_1(1) = C_1(0) + \frac{t^\alpha}{\Gamma(\alpha+1)}\{\mathrm{r}_2 C_1(0)(1 - \mu_2 C_1(0)) + \beta_1(1-\varepsilon_1) S(0) C_1(0) - \theta C_1(0) + \beta_2(1-\varepsilon_2)\mathrm{M}(0) S(0)\} \\ C_2(1) = C_2(0) + \frac{t^\alpha}{\Gamma(\alpha+1)}\{\mathrm{C}_2(0)(1 - \mu_3 C_2(0)) + \theta \mathrm{C}_1(0)\mathrm{C}_2(0) + \beta_1\varepsilon_1 \mathrm{S}(0)\mathrm{C}_1(0) + \beta_2\varepsilon_2 \mathrm{M}(0)\mathrm{S}(0)\} \\ M(1) = M(0) + \frac{t^\alpha}{\Gamma(\alpha+1)}\{M(0)\mathrm{r}_3(1 - \mu_4 M(0)) - \sigma_2 M(0) - \gamma f(0) N(0)\} \\ N(1) = N(0) + \frac{t^\alpha}{\Gamma(\alpha+1)}\{N(0)\mathrm{r}_4(1 - \mu_5 N(0)) + \delta f(0) N(0)\}. \end{cases}$$

If we repeat the discretization process n times, we get

$$\begin{cases} S(n+1) = S(n) + \frac{(t-nh)^{\alpha}}{\Gamma(\alpha+1)}\{r_1 S(n)(p-\mu_1 S(n)) - \beta_1 S(n)C_1(n) - \beta_2 M(n)S(n) + \sigma_1 M(n)S(n)\} \\ C_1(n+1) = C_1(n) + \frac{(t-nh)^{\alpha}}{\Gamma(\alpha+1)}\{r_2 C_1(n)(1-\mu_2 C_1(n)) + \beta_1(1-\varepsilon_1)S(n)C_1(n) - \theta C_1(n) + \beta_2(1-\varepsilon_2)M(n)S(n)\} \\ C_2(n+1) = C_2(n) + \frac{(t-nh)^{\alpha}}{\Gamma(\alpha+1)}\{C_2(n)(1-\mu_3 C_2(n)) + \theta C_1(n)C_2(n) + \beta_1\varepsilon_1 S(n)C_1(n) + \beta_2\varepsilon_2 M(n)S(n)\} \\ M(n+1) = M(n) + \frac{(t-nh)^{\alpha}}{\Gamma(\alpha+1)}\{M(n)r_3(1-\mu_4 M(n)) - \sigma_2 M(n) - \gamma f(n)N(n)\} \\ N(n+1) = N(n) + \frac{(t-nh)^{\alpha}}{\Gamma(\alpha+1)}\{N(n)r_4(1-\mu_5 N(n)) + \delta f(n)N(n)\}. \end{cases}$$

For $t \in [nh, (n+1)\cdot h)$ and $t \to (n+1)\cdot h$, while $\alpha \to 1$, we have

$$\begin{cases} S(n+1) = S(n) + \frac{h^{\alpha}}{\Gamma(\alpha+1)}\{r_1 S(n)(p-\mu_1 S(n)) - \beta_1 S(n)C_1(n) - \beta_2 M(n)S(n) + \sigma_1 M(n)S(n)\} \\ C_1(n+1) = C_1(n) + \frac{h^{\alpha\alpha}}{\Gamma(\alpha+1)}\{r_2 C_1(n)(1-\mu_2 C_1(n)) + \beta_1(1-\varepsilon_1)S(n)C_1(n) - \theta C_1(n) + \beta_2(1-\varepsilon_2)M(n)S(n)\} \\ C_2(n+1) = C_2(n) + \frac{h^{\alpha\alpha}}{\Gamma(\alpha+1)}\{C_2(n)(1-\mu_3 C_2(n)) + \theta C_1(n)C_2(n) + \beta_1\varepsilon_1 S(n)C_1(n) + \beta_2\varepsilon_2 M(n)S(n)\} \\ M(n+1) = M(n) + \frac{h^{\alpha}}{\Gamma(\alpha+1)}\left\{M(n)r_3(1-\mu_4 M(n)) - \sigma_2 M(n) - \frac{\gamma M(n)N(n)}{1+he\omega M(n)}\right\} \\ N(n+1) = N(n) + \frac{h^{\alpha}}{\Gamma(\alpha+1)}\left\{N(n)r_4(1-\mu_5 N(n)) + \frac{\delta M(n)N(n)}{1+he\omega M(n)}\right\}. \end{cases} \quad (55)$$

The Jacobian matrix of (55) around the co-existing equilibrium point Λ is

$$J(\Lambda) = \begin{pmatrix} b_{11} & b_{12} & 0 & b_{14} & 0 \\ b_{21} & b_{22} & b_{23} & 0 & 0 \\ b_{31} & b_{32} & b_{33} & 0 & 0 \\ 0 & 0 & 0 & b_{44} & b_{45} \\ 0 & 0 & 0 & b_{54} & b_{55} \end{pmatrix}, \quad (56)$$

where

$$b_{11} = 1 + \frac{h^{\alpha}}{\Gamma(\alpha+1)}\left(r_1 p - 2\mu_1 r_1 \bar{S} - \beta_1 \overline{C_1} - (\beta_2 - \sigma_1)\bar{M}\right),\ b_{12} = -\frac{\beta_1 \bar{S} h^{\alpha}}{\Gamma(\alpha+1)},\ b_{14} = \frac{-(\beta_2 - \sigma_1)\bar{S}}{\Gamma(\alpha+1)}$$

$$b_{21} = \frac{h^{\alpha}\left(\beta_1(1-\varepsilon_1)\overline{C_1} + \beta_2(1-\varepsilon_2)\bar{M}\right)}{\Gamma(\alpha+1)},\ b_{22} = 1 + \frac{h^{\alpha}}{\Gamma(\alpha+1)}\left(r_2 - 2\mu_2 r_2 \overline{C_1} + \beta_1(1-\varepsilon_1)\bar{S} - \theta\right),\ b_{23} = \frac{\beta_2(1-\varepsilon_2)\bar{S}h^{\alpha}}{\Gamma(\alpha+1)}$$

$$b_{31} = \frac{h^{\alpha}\left(\beta_1\varepsilon_1\overline{C_1} + \beta_2\varepsilon_2\bar{M}\right)}{\Gamma(\alpha+1)},\ b_{32} = \frac{h^{\alpha}\left(\theta\overline{C_2} + \beta_1\varepsilon_1\bar{S}\right)}{\Gamma(\alpha+1)},\ b_{33} = 1 + \frac{h^{\alpha}}{\Gamma(\alpha+1)}\left(1 - 2\mu_3\overline{C_2} + \theta\overline{C_1}\right),\ b_{34} = \frac{\beta_2\varepsilon_2\bar{S}h^{\alpha}}{\Gamma(\alpha+1)}$$

$$b_{44} = 1 + \frac{h^{\alpha}}{\Gamma(\alpha+1)}\left(r_3 - 2\mu_4 r_3 \bar{M} - \sigma_2 - \frac{\gamma\bar{N}}{(1+he\omega\bar{M})^2}\right), b_{45} = -\frac{\gamma\bar{M}h^{\alpha}}{\Gamma(\alpha+1)(1+he\omega\bar{M})},$$

$$b_{54} = \frac{\delta\bar{N}h^{\alpha}}{\Gamma(\alpha+1)(1+he\omega\bar{M})^2},\ b_{55} = 1 + \frac{h^{\alpha}}{\Gamma(\alpha+1)}\left(r_4 - 2r_4\mu_5\bar{N} + \frac{\delta\bar{M}}{1+he\omega\bar{M}}\right)$$

We obtain the characteristic equation of the matrix such as

$$\lambda^2 + (-b_{11} - b_{22})\lambda + b_{11}b_{22}(1 - R_{01}) = 0 \tag{57}$$

and

$$\lambda^2 + (-b_{44} - b_{55})\lambda + b_{44}b_{55}(1 - R_{02}) = 0, \tag{58}$$

where (i)-(v) hold and

$$\left|1 + \frac{h^\alpha}{\Gamma(\alpha+1)}\left(1 - 2\mu_3\overline{C_2} + \theta\overline{C_1}\right)\right|\langle 1 \Rightarrow \overline{C_2}\rangle \frac{\theta\overline{C_1} + 1}{2\mu_3}. \tag{59}$$

To analyze the conditions for Neimark-Sacker Bifurcation, we use the following Theorem.

Theorem 6.1. [35] For a quadratic polynomial $P(\lambda) = 0$ such as

$$\lambda^2 + \ell_1\lambda + \ell_0 = 0, \tag{60}$$

a pair of complex conjugate roots of (1) lie on the unit circle if and only if

(a) $P(1) = 1 + \ell_1 + \ell_0 > 0$

(b) $P(-1) = 1 - \ell_1 + \ell_0 > 0$

(c) $D_1^+ = 1 + \ell_0 > 0$

(d) $D_1^- = 1 - \ell_0 = 0,$

Theorem 6.2. Let Λ be the co-existing critical point of system (55) and assume that (i)–(v) hold. If

$$h_1 = \left(\Gamma(\alpha+1)\sqrt{\frac{(\mathrm{r}_1 p - (\theta - \mathrm{r}_2) - (2\mu_2\mathrm{r}_2 + \beta_1)\overline{C_1} + (\beta_1(1-\varepsilon_1) - 2\mu_1\mathrm{r}_1)\bar{S} - (\beta_2 - \sigma_1)\bar{M})(1 - R_{01}) + \sqrt{\Delta_1}}{2\left(\mathrm{r}_1 p - 2\mu_1\mathrm{r}_1\bar{S} - \beta_1\overline{C_1} - (\beta_2 - \sigma_1)\bar{M}\right)\left(-\mathrm{r}_2 + 2\mu_2\mathrm{r}_2\overline{C_1} - \beta_1(1-\varepsilon_1)\bar{S} + \theta\right)(1 - R_{01})}}\right)^{\frac{1}{\alpha}}$$

where $\mathrm{r}_1 < \dfrac{\beta_1(1-\varepsilon_1)}{2\mu_1}$, then the $S - C_1$ class undergoes a Neimark-Sacker bifurcation. Additionally, if

$$h_2 = \left(\Gamma(\alpha+1)\sqrt{\frac{\left(\left(\frac{\delta}{1 + he\omega\bar{M}} - 2\mu_4\mathrm{r}_3\right)\bar{M} + \mathrm{r}_3 + \mathrm{r}_4 - \sigma_2 - \left(\frac{\gamma}{(1 + he\omega\bar{M})^2} + 2\mathrm{r}_4\mu_5\right)\bar{N}\right) + \sqrt{\Delta_2}}{2\left(-\mathrm{r}_3 + 2\mu_4\mathrm{r}_3\bar{M} + \sigma_2 + \frac{\gamma\bar{N}}{(1 + he\omega\bar{M})^2}\right)\left(\mathrm{r}_4 - 2\mathrm{r}_4\mu_5\bar{N} + \frac{\delta\bar{M}}{1 + he\omega\bar{M}}\right)(1 - R_{02})}}\right)^{\frac{1}{\alpha}}$$

where $\bar{M} < \dfrac{\delta - 2\mu_4\mathrm{r}_3}{2\mu_4\mathrm{r}_3 he\omega}$ and $\bar{N} < \dfrac{(\mathrm{r}_3 + \mathrm{r}_4 - \sigma_2)(1 + he\omega\bar{M})^2}{\gamma + 2\mathrm{r}_4\mu_5(1 + he\omega\bar{M})^2}$ for $\mathrm{r}_3 < \dfrac{\delta}{2\mu_4}$ and $\mathrm{r}_4 > \sigma_2$, then the $M - N$ classes shows also a dynamical behavior of Neimark–Sacker bifurcation.

Proof. Let us first consider the statements in Theorem 6.1 for Eq. (57). Thus, from (a)-(c) together with (i) we have

$$2 - R_{01} + \frac{h^{\alpha}}{\Gamma(\alpha+1)}\big(r_1 p + (\beta_1(1-\varepsilon_1) - 2\mu_1 r_1)\bar{S} + (2\mu_2 r_2 + \beta_1)\overline{C_1} - (\beta_2 - \sigma_1)\bar{M} - (\theta - r_2)\big)(1 - R_{01})$$
$$< \frac{h^{2\alpha}}{\gamma^2(\alpha+1)}\big(r_1 p - 2\mu_1 r_1 \bar{S} - \beta_1 \overline{C_1} - (\beta_2 - \sigma_1)\bar{M}\big)\big(-r_2 + 2\mu_2 r_2 \overline{C_1} - \beta_1(1-\varepsilon_1)\bar{S} + \theta\big)(1 - R_{01}),$$

which holds for

$$h < \left(\Gamma(\alpha+1)\sqrt{\frac{2 - R_{01}}{\big(r_1 p - 2\mu_1 r_1 \bar{S} - \beta_1 \overline{C_1} - (\beta_2 - \sigma_1)\bar{M}\big)\big(-r_2 + 2\mu_2 r_2 \overline{C_1} - \beta_1(1-\varepsilon_1)\bar{S} + \theta\big)(1 - R_{01})}}\right)^{\frac{1}{\alpha}} \quad (61)$$

where $r_1 < \frac{\beta_1(1-\varepsilon_1)}{2\mu_1}$.

Finally, from (d) we obtain

$$R_{01} - \frac{h^{\alpha}}{\Gamma(\alpha+1)}\big(r_1 p - (\theta - r_2) - (2\mu_2 r_2 + \beta_1)\overline{C_1} + (\beta_1(1-\varepsilon_1) - 2\mu_1 r_1)\bar{S} - (\beta_2 - \sigma_1)\bar{M}\big)(1 - R_{01})$$
$$+ \frac{h^{2\alpha}}{\Gamma^2(\alpha+1)}\big(r_1 p - 2\mu_1 r_1 \bar{S} - \beta_1 \overline{C_1} - (\beta_2 - \sigma_1)\bar{M}\big)\big(-r_2 + 2\mu_2 r_2 \overline{C_1} - \beta_1(1-\varepsilon_1)\bar{S} + \theta\big)(1 - R_{01}) = 0,$$

which gives

$$h = \left(\Gamma(\alpha+1)\sqrt{\frac{\big(r_1 p - (\theta - r_2) - (2\mu_2 r_2 + \beta_1)\overline{C_1} + (\beta_1(1-\varepsilon_1) - 2\mu_1 r_1)\bar{S} - (\beta_2 - \sigma_1)\bar{M}\big)(1 - R_{01}) + \sqrt{\Delta}}{2\big(r_1 p - 2\mu_1 r_1 \bar{S} - \beta_1 \overline{C_1} - (\beta_2 - \sigma_1)\bar{M}\big)\big(-r_2 + 2\mu_2 r_2 \overline{C_1} - \beta_1(1-\varepsilon_1)\bar{S} + \theta\big)(1 - R_{01})}}\right)^{\frac{1}{\alpha}} \quad (62)$$

where

$$\Delta_1 = \big(r_1 p - (\theta - r_2) - (2\mu_2 r_2 + \beta_1)\overline{C_1} + (\beta_1(1-\varepsilon_1) - 2\mu_1 r_1)\bar{S} - (\beta_2 - \sigma_1)\bar{M}\big)^2(1 - R_{01})^2$$
$$- 4\big(r_1 p - 2\mu_1 r_1 \bar{S} - \beta_1 \overline{C_1} - (\beta_2 - \sigma_1)\bar{M}\big)\big(-r_2 + 2\mu_2 r_2 \overline{C_1} - \beta_1(1-\varepsilon_1)\bar{S} + \theta\big)(1 - R_{01})R_{01} > 0.$$

In considering both Eqs. (61) and (62), we get

$$h_1 = \left(\Gamma(\alpha+1)\sqrt{\frac{\big(r_1 p - (\theta - r_2) - (2\mu_2 r_2 + \beta_1)\overline{C_1} + (\beta_1(1-\varepsilon_1) - 2\mu_1 r_1)\bar{S} - (\beta_2 - \sigma_1)\bar{M}\big)(1 - R_{01}) + \sqrt{\Delta_1}}{2\big(r_1 p - 2\mu_1 r_1 \bar{S} - \beta_1 \overline{C_1} - (\beta_2 - \sigma_1)\bar{M}\big)\big(-r_2 + 2\mu_2 r_2 \overline{C_1} - \beta_1(1-\varepsilon_1)\bar{S} + \theta\big)(1 - R_{01})}}\right)^{\frac{1}{\alpha}}$$
$$< \left(\Gamma(\alpha+1)\sqrt{\frac{2 - R_{01}}{\big(r_1 p - 2\mu_1 r_1 \bar{S} - \beta_1 \overline{C_1} - (\beta_2 - \sigma_1)\bar{M}\big)\big(-r_2 + 2\mu_2 r_2 \overline{C_1} - \beta_1(1-\varepsilon_1)\bar{S} + \theta\big)(1 - R_{01})}}\right)^{\frac{1}{\alpha}},$$

which completes the proof of the $S - C_1$ class.

The characteristic equation Eq. (58) holds for Theorem 5.1./(a)–(c), if

$$2-R_{02}+\frac{h^{\alpha}}{\Gamma(\alpha+1)}\left(\left(\frac{\delta}{1+he\omega\bar{M}}-2\mu_4 r_3\right)\bar{M}+r_3+r_4-\sigma_2-\left(\frac{\gamma}{(1+he\omega\bar{M})^2}+2r_4\mu_5\right)\bar{N}\right)(1-R_{02})$$
$$+\frac{h^{2\alpha}}{\Gamma^2(\alpha+1)}\left(r_3-2\mu_4 r_3\bar{M}-\sigma_2-\frac{\gamma\bar{N}}{(1+he\omega\bar{M})^2}\right)\left(r_4-2r_4\mu_5\bar{N}+\frac{\delta\bar{M}}{1+he\omega\bar{M}}\right)(1-R_{02})>0,$$

then

$$\bar{M}<\frac{\delta-2\mu_4 r_3}{2\mu_4 r_3 he\omega}\ \text{for } r_3<\frac{\delta}{2\mu_4}, \tag{63}$$

$$\bar{N}<\frac{(r_3+r_4-\sigma_2)(1+he\omega\bar{M})^2}{\gamma+2r_4\mu_5(1+he\omega\bar{M})^2}\ \text{for } r_4>\sigma_2 \tag{64}$$

and

$$h<\left(\Gamma(\alpha+1)\sqrt{\frac{2-R_{02}}{\left(-r_3+2\mu_4 r_3\bar{M}+\sigma_2+\frac{\gamma\bar{N}}{(1+he\omega\bar{M})^2}\right)\left(r_4-2r_4\mu_5\bar{N}+\frac{\delta\bar{M}}{1+he\omega\bar{M}}\right)(1-R_{02})}}\right)^{1/\alpha} \tag{65}$$

Finally, from (d) we get

$$R_{02}-\frac{h^{\alpha}}{\Gamma(\alpha+1)}\left(\left(\frac{\delta}{1+he\omega\bar{M}}-2\mu_4 r_3\right)\bar{M}+r_3+r_4-\sigma_2-\left(\frac{\gamma}{(1+he\omega\bar{M})^2}+2r_4\mu_5\right)\bar{N}\right)(1-R_{02})$$
$$+\frac{h^{2\alpha}}{\Gamma^2(\alpha+1)}\left(-r_3+2\mu_4 r_3\bar{M}+\sigma_2+\frac{\gamma\bar{N}}{(1+he\omega\bar{M})^2}\right)\left(r_4-2r_4\mu_5\bar{N}+\frac{\delta\bar{M}}{1+he\omega\bar{M}}\right)(1-R_{02})=0 \tag{66}$$

which holds for

$$h_2=\left(\Gamma(\alpha+1)\sqrt{\frac{\left(\left(\frac{\delta}{1+he\omega\bar{M}}-2\mu_4 r_3\right)\bar{M}+r_3+r_4-\sigma_2-\left(\frac{\gamma}{(1+he\omega\bar{M})^2}+2r_4\mu_5\right)\bar{N}\right)+\sqrt{\Delta_2}}{2\left(-r_3+2\mu_4 r_3\bar{M}+\sigma_2+\frac{\gamma\bar{N}}{(1+he\omega\bar{M})^2}\right)\left(r_4-2r_4\mu_5\bar{N}+\frac{\delta\bar{M}}{1+he\omega\bar{M}}\right)(1-R_{02})}}\right)^{\frac{1}{\alpha}}$$
$$<\left(\Gamma(\alpha+1)\sqrt{\frac{2-R_{02}}{\left(-r_3+2\mu_4 r_3\bar{M}+\sigma_2+\frac{\gamma\bar{N}}{(1+he\omega\bar{M})^2}\right)\left(r_4-2r_4\mu_5\bar{N}+\frac{\delta\bar{M}}{1+he\omega\bar{M}}\right)(1-R_{02})}}\right)^{1/\alpha},$$

where

$$\Delta_2 = \left(\left(\frac{\delta}{1 + he\omega\bar{M}} - 2\mu_4 r_3 \right)\bar{M} + r_3 + r_4 - \sigma_2 - \left(\frac{\gamma}{(1 + he\omega\bar{M})^2} + 2r_4\mu_5 \right)\bar{N} \right)^2 (1 - R_{02})^2$$

$$-4\left(-r_3 + 2\mu_4 r_3\bar{M} + \sigma_2 + \frac{\gamma\bar{N}}{(1 + he\omega\bar{M})^2} \right)\left(r_4 - 2r_4\mu_5\bar{N} + \frac{\delta\bar{M}}{1 + he\omega\bar{M}} \right)(1 - R_{02})R_{02}.$$

This completes the proof.

7 Conclusion

In this paper, we classified the coronaviruses and their spread from the natural host to the human host. We proposed a model of the novel coronavirus, which is known as COVID-19, as a system of fractional-order differential equations. We divided the system into five sub-classes:

- the susceptible class S, the infected class C_1, that does not know they are infected since specific symptoms did not appear,
- the infected class C_2 that knows they are infected because of some symptoms such as respiratory and intestinal infections, including fever, dizziness, and cough, appeared.
- the intermediate domestic host M, that has a transmission role from the natural host to the human host
- the natural host N, that are bats of genus *Rhinolophus*.

We consider the pandemic infection case; animal to human and human to human. Therefore, the first three equations in the constructed model show human to human transmission. The spillover from the intermediate infected class to the human host denotes a predator-prey mathematical model, and the transmission from the natural host to intermediate host M is a host-parasite model of Holling Type II.

In Sections 3 and 4, we analyzed the local stability of the co-existing equilibrium point by using the Routh–Hurwitz Criteria. We proved the existence and the uniqueness of the initial value problem.

Theorem 3.1., shows that among the human hosts, those who do not know they are infected are the control class in the spread. While between the animal hosts, the intermediate class plays a dominant role in the spread since that class has an essential role in transmitting the virus from animal to human. The transmission potential for both $S - C_1$ and $M - N$ is $R_{01} < 1$ and $R_{02} < 1$, respectively. Also, the susceptible class and the C_1 class is stable based on two parameters, which is the awareness of the symptoms and the screening rate.

In Theorem 3.2., we emphasized that C_1 class should be more aware of the symptoms that might become from the susceptible class as well as from the intermediate class, than the S class to stop the outbreak. For the susceptible class, it is more important to keep the population rate per year non-infected. The transmission of the virus to the offspring would reach an uncontrollable phenomenon worldwide.

In Section 5, we incorporate the Allee function at time t. The strong Allee effect is analyzed so that the screening for possible inflectional cases is an essential control parameter to support the Allee function in stabilizing the effect of the spread.

In Section 6, we deduced that the system demonstrates a Neimark–Sacker bifurcation under specific conditions.

Availability of Data and Material: All data generated or analyzed during this study are included in this published article.

Authors' Contributions: Yousef and Bozkurt conceived the study and was in charge of overall direction and planning. Bozkurt and Yousef designed the mathematical model and set up the main parts of the study. They proved the theorems. Bozkurt, Yousef, and Abdeljawad collected the data and analyzed them. All authors interpreted the data and carried out this implementation. Bozkurt and Yousef conducted the simulation results using MATLAB 2019. All the authors are involved in writing and editing the manuscript. There is no Ghost-writing.

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