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ARTICLE

New Computation of Unified Bounds via a More General Fractional Operator Using Generalized Mittag–Leffler Function in the Kernel

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ABSTRACT

In the present case, we propose the novel generalized fractional integral operator describing Mittag–Leffler function in their kernel with respect to another function Φ. The proposed technique is to use graceful amalgamations of the Riemann–Liouville (RL) fractional integral operator and several other fractional operators. Meanwhile, several generalizations are considered in order to demonstrate the novel variants involving a family of positive functions $n\,(n \in \mathbb{N})$ for the proposed fractional operator. In order to confirm and demonstrate the proficiency of the characterized strategy, we analyze existing fractional integral operators in terms of classical fractional order. Meanwhile, some special cases are apprehended and the new outcomes are also illustrated. The obtained consequences illuminate that future research is easy to implement, profoundly efficient, viable, and exceptionally precise in its investigation of the behavior of non-linear differential equations of fractional order that emerge in the associated areas of science and engineering.

KEYWORDS

Integral inequality; generalized fractional integral with respect to another function; increasing and decreasing functions; Mittag–Leffler function

1 Introduction

A study was initiated in Newton's time, but, lately, it has captivated the consideration of numerous researchers due to its intriguing nature, known as the fractional calculus. For the previous three decades, the most charming bounds in modelling and simulation have been discovered in the frame of fractional calculus. The idea of the fractional operators has been mechanized because of the complexities related to a heterogeneous phenomenon. The fractional differential operators are equipped for catching the conduct of multidimensional media as they have diffusion processes. It was considered to be an essential device, and numerous issues can be demonstrated more appropriately and more precisely with differential equations having an arbitrary order. Calculus with fractional order is related to concrete adventures and is widely utilized in nanotechnology, optics, diseases, chaos theory and different other fields [1–8]. The fractional derivatives for various sorts of equations, describe that these models have a significant job in depicting the idea of mathematical problems that are linked with science and technology, see the references [9–11].

In the present scenario, numerous significant fractional derivative and integral operators are systematically and successfully analyzed with the assistance of fractional calculus, see [10,11]. Several assorted operators that have been recommended by numerous senior researchers like, Riemann, Hadamard, Antagana–Baleanu, Caputo and Fabrizio. Nevertheless, these operators have their own repressions. Recently, Abdeljawad [1] reported fractional operators are known as fractional conformable derivatives and integrals. The exponential and Mittag–Leffler functions are used as kernels by several researchers for developing new fractional techniques that provide assistance to many researchers consult the derivative to model and integrals in order to analyze the nonlocal dynamics due to the presence of nonsingular kernels and helps to find the solution for diverse classes of non-linear complex problems. Kilbas et al. [11] and Almeida [9] introduced the generalized RL and Caputo derivative operators in the sense of another function. Rashid et al. [12] proposed another novel fractional approach which comes into existence in the theory of fractional calculus, which is known as generalized proportional fractional operators with respect to another function Ψ.

Adopting the aforementioned trend, we delineate the significance of the novel fractional operator and future plan, we, in the present the framework, consider the Mittag–Leffler function, which assumes a dynamic job in portraying the idea of porous media just as establishing the several generalizations by employing more generalized fractional operator with Mittag–Leffler in their kernels.

Mathematical inequalities considered to be a significant tool in diverse areas of science and technology, among others; especially we point out the initial value problem, the stability of linear transformation, integral differential equations, and impulse equations [13–15]. Variants regarding fractional integral operators are the use of noteworthy significant strategies and are also highly implemented in natural and social sciences to portray real-world problems.

Briefly, inequalities involve solid and rich communication among analysis, geometry and technology. In [16], Agarwal explored some new inequalities for Hadamard fractional integral operators. Also, Rashid et al. [17] established Hermite-Hadamard inequalities for exponentially m-convex functions via extended Mittag Leffler function. In [18], researchers have been focused their attention in order to find the distinguished version of the reverse Minkowski inequality for quantum Hahn fractional integral operators. Set et al. [19] derived Chebyshev type inequalities by employing fractional integral operator having Mittag–Leffler function in the kernel. For more details, see [20–38] and the reference cited therein. The diverse utilities of fractional integral operators compelled us to show the speculations by using a family of n positive functions involving generalized fractional integrals operators with respect to another function Ψ as well-known special function in their kernel.

The principal objective of this article is that we demonstrate the notations of our newly introduced operator generalized fractional integral with respect to another function Ψ by introducing Mittag–Leffler function in their kernel. Also, we present the results concerning for a class of family of n $(n \in \mathbb{N})$ continuous positive decreasing functions on $[\upsilon_1, \upsilon_2]$ by employing generalized fractional integral operator proposed by Andric [39], Salim et al. [40], Rahman et al. [41], Srivastava et al. [42], Prabhakar [43] and Riemann-Liouville fractional integral operators [44]. The idea is quite new and seems to have opened new doors of investigation towards various scientific fields of research including engineering, fluid dynamics, bio-sciences, chaos, meteorology, vibration analysis, bio-chemistry, aerodynamics and many more. The authors argued that the generalized fractional integral in Hilfer sense can capture a limited number of complex problems on one hand and on the other hands it can also capture different types of complexities, thus putting these two concepts together can help us to understand the complexities of existing nature in a much better way.

2 Prelude

Here, we define the basic notion of the more generalized fractional integral operator as Mittag–Leffler function introduced in their kernel.

We begin with fractional integral operators defined by Salim and Faraj in [40] containing generalized Mittag–Leffler function in their kernels as follows:

Definition 2.1. ([40]) Let $\nu,\tau,j,\vartheta,z,\kappa$ be positive real numbers and $\omega\in\mathbb{R}$. Then the generalized fractional integral operators containing Mittag–Leffler function for a real-valued continuous function Φ are defined by

$$\left(\mathscr{T}^{\vartheta,z,\kappa}_{\nu,\tau,j,\omega,\eta_1^+}\Phi\right)(s)=\int_{\eta_1}^{s}(s-\beta)^{\tau-1}E^{\vartheta,z,\kappa}_{\nu,\tau,j}\left(\omega(s-\beta)^{\sigma}\right)\Phi(\beta)d\beta, \tag{1}$$

$$\left(\mathscr{T}^{\vartheta,z,\kappa}_{\nu,\tau,j,\omega,\eta_2^-}\Phi\right)(s)=\int_{s}^{\eta_2}(\beta-s)^{\tau-1}E^{\vartheta,z,\kappa}_{\nu,\tau,j}\left(\omega(\beta-s)^{\sigma}\right)\Phi(\beta)d\beta, \tag{2}$$

where $E^{\vartheta,z,\kappa}_{\nu,\tau,j}(\beta)$ is the generalized Mittag–Leffler function defined as

$$E^{\vartheta,z,\kappa}_{\nu,\tau,j}(\beta)=\sum_{n=0}^{\infty}\frac{(\vartheta)_{\kappa n}\beta^n}{\Gamma(\nu n+\tau)(z)_{\delta n}}. \tag{3}$$

Andric et al. [39], defined the following fractional integral operators containing an extended generalized Mittag–Leffler function in their kernels:

Definition 2.2. ([39]) Let $\nu,\tau,j,\vartheta,\varepsilon\in\mathbb{C},\Re(\tau),\Re(j)>0$, $\Re(\varepsilon)>\Re(\vartheta)>0$, with $q\geq 0$, ν, $z>0$ and $0<\kappa\leq z+\nu$. Let $\Phi\in L_1([\eta_1,\eta_2])$ and $z\in[\eta_1,\eta_2]$. Then the generalized fractional integral operators $\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+}\Phi$ and $\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_2^-}\Phi$ are defined by

$$\left(\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+}\Phi\right)(s;q)=\int_{\eta_1}^{s}(s-\beta)^{\tau-1}E^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j}\left(\omega(s-\beta)^{\nu};q\right)\Phi(\beta)d\beta, \tag{4}$$

$$\left(\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_2^-}\Phi\right)(s;q)=\int_{s}^{\eta_2}(\beta-s)^{\tau-1}E^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j}\left(\omega(\beta-s)^{\nu};q\right)\Phi(\beta)d\beta, \tag{5}$$

where $E^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j}(\beta)$ is the generalized Mittag–Leffler function defined as

$$E^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j}(\beta;q)=\sum_{n=0}^{\infty}\frac{\mathbb{B}_q\left(\vartheta+n\kappa,\epsilon-\vartheta\right)_{n\kappa}\beta^n}{\mathbb{B}(\vartheta,\epsilon-\vartheta)\Gamma(\sigma n+\tau)(j)_{zn}}. \tag{6}$$

is the extended generalized Mittag–Leffler function.

Now we present the concept of the generalized fractional integral operator in the Hilfer sense having Mittag–Leffler function in the kernel as follows:

Definition 2.3. Let Φ: $[\eta_1, \eta_2] \to \mathbb{R}$, $0 < \eta_1 < \eta_2$ be a function such that Φ be a positive and integrable and Ψ be a differentiable and strictly increasing. Also, let $\nu, \tau, j, \omega, \vartheta, \epsilon \in \mathbb{C}, \Re(\tau), \Re(j) > 0, \Re(\epsilon) > \Re(\vartheta) > 0$ with $q \geq 0, \nu, z > 0$ and $0 < \kappa \leq z + \nu$. Then for $s \in [\eta_1, \eta_2]$, the integral operators ${}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+}\Phi$ and ${}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_2^-}\Phi$ are stated as:

$$\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+}\Phi\right)(s;q) = \int_{\eta_1}^{s} (\Psi(s) - \Psi(\beta))^{\tau-1} E^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j}\left(\omega(\Psi(s) - \Psi(\beta))^{\nu}; q\right) \Psi'(\beta)\Phi(\beta)d\beta \tag{7}$$

and

$$\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_2^-}\Phi\right)(s;q) = \int_{s}^{\eta_2} (\Psi(\beta) - \Psi(s))^{\tau-1} E^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j}\left(\omega(\Psi(\beta) - \Psi(s))^{\nu}; q\right) \Psi'(\beta)\Phi(\beta)d\beta. \tag{8}$$

Remark 2.1. Definition 2.3 is the generalizations of the several noteworthy exisiting integrals are stated as follows:

(1) Letting $\Psi(s) = s$ and $q = 0$, then the fractional integral operator (7) and (8) leads to the operators defined by Salim et al. [40].
(2) Letting $\Psi(s) = s$ and $j = z = 1$, then the fractional integral operator (7) and (8) leads to the operators defined by Rahman et al. [41].
(3) Letting $\Psi(s) = s$, $q = 0$ and $j = z = 1$, then the fractional integral operator (7) and (8) leads to the operators defined by Srivastava et al. [42].
(4) Letting $\Psi(s) = s$, $q = 0$ and $j = z = \kappa = 1$, then the fractional integral operator (7) and (8) leads to the operators defined by Prabhakar [43].
(5) Letting $\Psi(s) = s$ and $\omega = q = 0$ then the fractional integral operator (7) and (8) leads to the Riemann-Liouville fractional integral operators [44].

3 Main Results

In this section, we present the novel generalizations pertaining to the more generalized fractional integral operator using the Mittag–Leffler function in the kernel.

Theorem 3.1. Suppose Φ be a continuous positive decreasing function on $[\eta_1, \eta_2]$ with $\eta_1 < s \leq \eta_2$, $\sigma > 0$ and $\varsigma \geq \alpha > 0$. Suppose Ψ be a differentiable and strictly increasing. Then the more generalized fractional integral operator defined in (7), we have

$$\frac{\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+}\Phi^{\varsigma}\right)(s;q)}{\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+}\Phi^{\alpha}\right)(s;q)} \geq \frac{\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+}(s-\eta_1)^{\sigma}\Phi^{\varsigma}\right)(s;q)}{\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+}(s-\eta_1)^{\sigma}\Phi^{\alpha}\right)(s;q)}. \tag{9}$$

Proof. Using the hypothesis given in Theorem 3.1, we have

$$\left((\delta - \eta_1)^{\sigma} - (\beta - \eta_1)^{\sigma}\right)\left(\Phi^{\varsigma-\alpha}(\beta) - \Phi^{\varsigma-\alpha}(\delta)\right) \geq 0, \tag{10}$$

where $\sigma > 0$, $\varsigma \geq \alpha > 0$, $\eta_1 < s \leq \eta_2$ and $\beta, \delta \in [\eta_1, s]$.

By (10), we have

$$(\delta-\eta_1)^{\sigma}\Phi^{\varsigma-\alpha}(\beta)-(\beta-\eta_1)^{\sigma}\Phi^{\varsigma-\alpha}(\delta)-(\delta-\eta_1)^{\sigma}\Phi^{\varsigma-\alpha}(\delta)+(\beta-\eta_1)^{\sigma}\Phi^{\varsigma-\alpha}(\beta)\geq 0. \tag{11}$$

Let us define a function

$$\mathfrak{J}(s,\beta)=(\Psi(s)-\Psi(\beta))^{\tau-1}E^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j}\left(\omega\left(\Psi(s)-\Psi(\beta)\right)^{\nu};q\right)\Psi'(\beta). \tag{12}$$

Accordingly, the function $\mathfrak{J}(s,\beta)$ is positive for all $\beta\in(\eta_1,s)$, $\eta_1<s\leq\eta_2$, as every term of the supposed function is positive and explained in Theorem 3.1. Therefore, multiplying both sides of (11) with

$$\mathfrak{J}(s,\beta)\Phi^{\alpha}(\beta)=(\Psi(s)-\Psi(\beta))^{\tau-1}E^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j}\left(\omega\left(\Psi(s)-\Psi(\beta)\right)^{\nu};q\right)\Psi'(\beta)\Phi^{\alpha}(\beta),$$

we have

$$\begin{aligned}
&\mathfrak{J}(s,\beta)\left[(\delta-\eta_1)^{\sigma}\Phi^{\varsigma-\alpha}(\beta)-(\beta-\eta_1)^{\sigma}\Phi^{\varsigma-\alpha}(\delta)\right.\\
&\quad\left.-(\delta-\eta_1)^{\sigma}\Phi^{\varsigma-\alpha}(\delta)+(\beta-\eta_1)^{\sigma}\Phi^{\varsigma-\alpha}(\beta)\right]\Phi^{\alpha}(\beta)\\
&=(\delta-\eta_1)^{\sigma}\left(\Psi(s)-\Psi(\beta)\right)^{\tau-1}E^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j}\left(\omega\left(\Psi(s)-\Psi(\beta)\right)^{\nu};q\right)\Psi'(\beta)\Phi^{\alpha}(\beta)\Phi^{\varsigma-\alpha}(\beta)\\
&\quad-(\beta-\eta_1)^{\sigma}\left(\Psi(s)-\Psi(\beta)\right)^{\tau-1}E^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j}\left(\omega\left(\Psi(s)-\Psi(\beta)\right)^{\nu};q\right)\Psi'(\beta)\Phi^{\alpha}(\beta)\Phi^{\varsigma-\alpha}(\delta)\\
&\quad-(\delta-\eta_1)^{\sigma}\left(\Psi(s)-\Psi(\beta)\right)^{\tau-1}E^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j}\left(\omega\left(\Psi(s)-\Psi(\beta)\right)^{\nu};q\right)\Psi'(\beta)\Phi^{\alpha}(\beta)\Phi^{\varsigma-\alpha}(\delta)\\
&\quad+(\beta-\eta_1)^{\sigma}\left(\Psi(s)-\Psi(\beta)\right)^{\tau-1}E^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j}\left(\omega\left(\Psi(s)-\Psi(\beta)\right)^{\nu};q\right)\Psi'(\beta)\Phi^{\alpha}(\beta)\Phi^{\varsigma-\alpha}(\beta)\geq 0.
\end{aligned} \tag{13}$$

Integrating on both sides with respect to β from η_1 to s, we have

$$\begin{aligned}
&(\delta-\eta_1)^{\sigma}\int_{\eta_1}^{s}\left(\Psi(s)-\Psi(\beta)\right)^{\tau-1}E^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j}\left(\omega\left(\Psi(s)-\Psi(\beta)\right)^{\nu};q\right)\Psi'(\beta)\Phi^{\alpha}(\beta)\Phi^{\varsigma-\alpha}(\beta)d\beta\\
&\quad-(\beta-\eta_1)^{\sigma}\int_{\eta_1}^{s}\left(\Psi(s)-\Psi(\beta)\right)^{\tau-1}E^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j}\left(\omega\left(\Psi(s)-\Psi(\beta)\right)^{\nu};q\right)\Psi'(\beta)\Phi^{\alpha}(\beta)\Phi^{\varsigma-\alpha}(\delta)d\beta\\
&\quad-(\delta-\eta_1)^{\sigma}\int_{\eta_1}^{s}\left(\Psi(s)-\Psi(\beta)\right)^{\tau-1}E^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j}\left(\omega\left(\Psi(s)-\Psi(\beta)\right)^{\nu};q\right)\Psi'(\beta)\Phi^{\alpha}(\beta)\Phi^{\varsigma-\alpha}(\delta)d\beta\\
&\quad+(\beta-\eta_1)^{\sigma}\int_{\eta_1}^{s}\left(\Psi(s)-\Psi(\beta)\right)^{\tau-1}E^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j}\left(\omega\left(\Psi(s)-\Psi(\beta)\right)^{\nu};q\right)\Psi'(\beta)\Phi^{\alpha}(\beta)\Phi^{\varsigma-\alpha}(\beta)d\beta\geq 0.
\end{aligned}$$

It follows that

$$\begin{aligned}
&(\delta-\eta_1)^{\sigma}\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+}\Phi^{\varsigma}\right)(s;q)+\Phi^{\varsigma-\alpha}(\delta)\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+}(s-\eta_1)^{\sigma}\Phi^{\alpha}\right)(s;q)\\
&\quad-(\delta-\eta_1)^{\sigma}\Phi^{\varsigma-\alpha}(\delta)\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+}\Phi^{\alpha}\right)(s;q)-\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+}(s-\eta_1)^{\sigma}\Phi^{\varsigma}(s)\right)(s;q)\geq 0.
\end{aligned} \tag{14}$$

Multiplying (14) by

$$\mathfrak{J}(s,\delta)\Phi^{\alpha}(\delta)=(\Psi(s)-\Psi(\delta))^{\tau-1}E^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j}\left(\omega\left(\Psi(s)-\Psi(\delta)\right)^{\nu};q\right)\Psi'(\delta)\Phi^{\alpha}(\delta),$$

for $\delta \in (\eta_1, s)$, $\eta_1 < s \leq \eta_2$, and integrating the subsequent identity w.r.t δ from η_1 to s shows

$$\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+}\Phi^{\varsigma}\right)(s;q)\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+}(s-\eta_1)^{\sigma}\Phi^{\alpha}\right)(s;q)$$
$$-\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+}(s-\eta_1)^{\sigma}\Phi^{\varsigma}\right)(s;q)\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+}\Phi^{\alpha}\right)(s;q) \geq 0.$$

Dividing the above inequality by $\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+}(s-\eta_1)^{\sigma}\Phi^{\alpha}\right)(s;q)\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+}\Phi^{\alpha}\right)(s;q)$, we get the desired inequality (9).

Some special remarkable cases of Theorem 3.1 are stated as follows:

Remark 3.1. In Theorem 3.1:

I. Take $\Psi(s) = s$. Then we get a new result for generalized fractional integral operator having Mittag–Leffler in the kernel as follows:

$$\frac{\left(\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+}\Phi^{\varsigma}\right)(s;q)}{\left(\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+}\Phi^{\alpha}\right)(s;q)} \geq \frac{\left(\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+}(s-\eta_1)^{\sigma}\Phi^{\varsigma}\right)(s;q)}{\left(\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+}(s-\eta_1)^{\sigma}\Phi^{\alpha}\right)(s;q)}.$$

II. Take $\Psi(s) = s$ along with $q = 0$ Then we have a new result

$$\frac{\left(\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+}\Phi^{\varsigma}\right)(s)}{\left(\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+}\Phi^{\alpha}\right)(s)} \geq \frac{\left(\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+}(s-\eta_1)^{\sigma}\Phi^{\varsigma}\right)(s)}{\left(\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+}(s-\eta_1)^{\sigma}\Phi^{\alpha}\right)(s)}.$$

III. Take $\Psi(s) = s$ along with $j = z = 1$ Then we have a new result

$$\frac{\left(\mathscr{T}^{\vartheta,\kappa,\epsilon}_{\nu,\tau,\omega,\eta_1^+}\Phi^{\varsigma}\right)(s;q)}{\left(\mathscr{T}^{\vartheta,\kappa,\epsilon}_{\nu,\tau,\omega,\eta_1^+}\Phi^{\alpha}\right)(s;q)} \geq \frac{\left(\mathscr{T}^{\vartheta,\kappa,\epsilon}_{\nu,\tau,\omega,\eta_1^+}(s-\eta_1)^{\sigma}\Phi^{\varsigma}\right)(s;q)}{\left(\mathscr{T}^{\vartheta,\kappa,\epsilon}_{\nu,\tau,\omega,\eta_1^+}(s-\eta_1)^{\sigma}\Phi^{\alpha}\right)(s;q)}.$$

IV. Take $\Psi(s) = s, q = 0$ along with $j = z = 1$ Then we have a new result

$$\frac{\left(\mathscr{T}^{\vartheta,\kappa,\epsilon}_{\nu,\tau,\omega,\eta_1^+}\Phi^{\varsigma}\right)(s)}{\left(\mathscr{T}^{\vartheta,\kappa,\epsilon}_{\nu,\tau,\omega,\eta_1^+}\Phi^{\alpha}\right)(s)} \geq \frac{\left(\mathscr{T}^{\vartheta,\kappa,\epsilon}_{\nu,\tau,\omega,\eta_1^+}(s-\eta_1)^{\sigma}\Phi^{\varsigma}\right)(s)}{\left(\mathscr{T}^{\vartheta,\kappa,\epsilon}_{\nu,\tau,\omega,\eta_1^+}(s-\eta_1)^{\sigma}\Phi^{\alpha}\right)(s)}.$$

V. Take $\Psi(s) = s, q = 0$ along with $j = z = \kappa = 1$ Then we have a new result

$$\frac{\left(\mathscr{T}^{\vartheta,\epsilon}_{\nu,\tau,\omega,\eta_1^+}\Phi^{\varsigma}\right)(s)}{\left(\mathscr{T}^{\vartheta,\epsilon}_{\nu,\tau,\omega,\eta_1^+}\Phi^{\alpha}\right)(s)} \geq \frac{\left(\mathscr{T}^{\vartheta,\epsilon}_{\nu,\tau,\omega,\eta_1^+}(s-\eta_1)^{\sigma}\Phi^{\varsigma}\right)(s)}{\left(\mathscr{T}^{\vartheta,\epsilon}_{\nu,\tau,\omega,\eta_1^+}(s-\eta_1)^{\sigma}\Phi^{\alpha}\right)(s)}.$$

Remark 3.2. If $\Psi(s)=s$, $\tau=1$ and $\omega=q=0$, then Theorem 3.1 reduces to Theorem 3 of [45]. Moreover, the inequality (15) will reverse if Φ is an increasing function on $[\eta_1,\eta_2]$.

Theorem 3.2. Suppose Φ be a continuous positive decreasing function on $[\eta_1,\eta_2]$ with $\eta_1 < s\le\eta_2$, $\sigma>0$ and $\varsigma\ge\alpha>0$. Suppose Ψ be a differentiable and strictly increasing. Then the more generalized fractional integral operator defined in (7), we have

$$\begin{aligned}
&\Big(\Big({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+}\Phi^{\varsigma}\Big)(s;q)\Big({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\lambda,j,\omega,\eta_1^+}(s-\eta_1)^{\sigma}\Phi^{\alpha}\Big)(s;q)\\
&\quad+\Big({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\lambda,j,\omega,\eta_1^+}\Phi^{\varsigma}\Big)(s;q)\Big({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+}(s-\eta_1)^{\sigma}\Phi^{\alpha}\Big)(s;q)\Big)\\
&\quad/\Big(\Big({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+}\Phi^{\alpha}\Big)(s;q)\Big({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+}(s-\eta_1)^{\sigma}\Phi^{\varsigma}\Big)(s;q)\\
&\quad+\Big({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+}\Phi^{\alpha}\Big)(s;q)\Big({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+}(s-\eta_1)^{\sigma}\Phi^{\varsigma}\Big)(s;q)\Big)\ge 1. \qquad (15)
\end{aligned}$$

Proof. Taking product on both sides of (14) by

$$\mathfrak{J}(s,\delta)\Phi^{\alpha}(\delta)=(\Psi(s)-\Psi(\delta))^{\lambda-1}E^{\vartheta,z,\kappa,\varepsilon}_{\nu,\lambda,j}\left(\omega\left(\Psi(s)-\Psi(\delta)\right)^{\nu};q\right)\Psi'(\delta)\Phi^{\alpha}(\delta),$$

for $\delta\in(\eta_1,r)$, $\eta_1<s\le\eta_2$ and integrating the subsequent identity w.r.t δ from η_1 to s shows

$$\begin{aligned}
&\Big({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+}\Phi^{\varsigma}\Big)(s;q)\Big({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\lambda,j,\omega,\eta_1^+}(s-\eta_1)^{\sigma}\Phi^{\alpha}\Big)(s;q)\\
&\quad+\Big({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\lambda,j,\omega,\eta_1^+}\Phi^{\varsigma}\Big)(s;q)\Big({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+}(s-\eta_1)^{\sigma}\Phi^{\alpha}(s)\Big)(s;q)\\
&\quad-\Big({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\lambda,j,\omega,\eta_1^+}\Phi^{\alpha}\Big)(s;q)\Big({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+}(s-\eta_1)^{\sigma}\Phi^{\varsigma}\Big)(s;q)\\
&\quad-\Big({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+}\Phi^{\alpha}\Big)(s;q)\Big({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\lambda,j,\omega,\eta_1^+}(s-\eta_1)^{\sigma}\Phi^{\varsigma}\Big)(s;q)\ge 0. \qquad (16)
\end{aligned}$$

Hence, dividing (16) by

$$\begin{aligned}
&\Big({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\lambda,j,\omega,\eta_1^+}\Phi^{\alpha}\Big)(s;q)\Big({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+}(s-\eta_1)^{\sigma}\Phi^{\varsigma}\Big)(s;q)\\
&\quad-\Big({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+}\Phi^{\alpha}\Big)(s;q)\Big({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\lambda,j,\omega,\eta_1^+}(s-\eta_1)^{\sigma}\Phi^{\varsigma}\Big)(s;q),
\end{aligned}$$

the proof of (15) is complete.

Remark 3.3. Letting $\tau=\lambda$ in inequality (15), then we attain the inequality (9).

Theorem 3.3 Let Φ be a continuous positive decreasing function on $[\eta_1,\eta_2]$ and $\mathscr{H}$ be a continuous positive increasing function on $[\eta_1,\eta_2]$ with $\eta_1<s\le\eta_2$, $\sigma>0$, and $\varsigma\ge\alpha>0$. Suppose

Ψ be a differentiable and strictly increasing. Then the more generalized fractional integral operator defined in (7), we have

$$\frac{\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+}\Phi^{\varsigma}\right)(s;q)\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+}\Phi^{\alpha}\mathscr{H}^{\sigma}\right)(s;q)}{\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+}\Phi^{\alpha}\right)(s;q)\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+}\Phi^{\varsigma}\mathscr{H}^{\sigma}\right)(s;q)}\geq 1. \tag{17}$$

Proof. Using the hypothesis given in Theorem 3.3, we have

$$\left(\mathscr{H}^{\sigma}(\delta)-\mathscr{H}^{\sigma}(\beta)\right)\left(\Phi^{\varsigma-\alpha}(\beta)-\Phi^{\varsigma-\alpha}(\delta)\right)\geq 0, \tag{18}$$

where $\sigma>0$, $\varsigma\geq\alpha>0$ and $\beta,\delta\in[\eta_1,s]$. From (18), we have

$$\mathscr{H}^{\sigma}(\delta)\Phi^{\varsigma-\alpha}(\beta)+\mathscr{H}^{\sigma}(\beta)\Phi^{\varsigma-\alpha}(\delta)-\mathscr{H}^{\sigma}(\delta)\Phi^{\varsigma-\alpha}(\delta)-\mathscr{H}^{\sigma}(\beta)\Phi^{\varsigma-\alpha}(\beta)\geq 0. \tag{19}$$

Taking product of (19) by

$$\mathfrak{J}(s,\beta)\Phi^{\alpha}(\beta)=(\Psi(s)-\Psi(\beta))^{\tau-1}E^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j}\left(\omega(\Psi(s)-\Psi(\beta))^{\nu};q\right)\Psi'(\beta)\Phi^{\alpha}(\beta),$$

for $\beta\in(\eta_1,r)$, $\eta_1<s\leq\eta_2$, where $\mathfrak{J}(s,\beta)$ is defined by (12), we have

$$\begin{aligned}
&\mathfrak{J}(s,\beta)\Phi^{\alpha}(\beta)\big[\mathscr{H}^{\sigma}(\delta)\Phi^{\varsigma-\alpha}(\beta)+\mathscr{H}^{\sigma}(\beta)\Phi^{\varsigma-\alpha}(\delta)-\mathscr{H}^{\sigma}(\delta)\Phi^{\varsigma-\alpha}(\delta)\\
&\quad-\mathscr{H}^{\sigma}(\beta)\Phi^{\varsigma-\alpha}(\beta)\big]\\
&=\mathscr{H}^{\sigma}(\delta)(\Psi(s)-\Psi(\beta))^{\tau-1}E^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j}\left(\omega(\Psi(s)-\Psi(\beta))^{\nu};q\right)\Psi'(\beta)\Phi^{\varsigma}(\beta)\\
&\quad+\mathscr{H}^{\sigma}(\beta)(\Psi(s)-\Psi(\beta))^{\tau-1}E^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j}\left(\omega(\Psi(s)-\Psi(\beta))^{\nu};q\right)\Psi'(\beta)\Phi^{\varsigma-\alpha}(\delta)\Phi^{\alpha}(\beta)\\
&\quad-\mathscr{H}^{\sigma}(\delta)(\Psi(s)-\Psi(\beta))^{\tau-1}E^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j}\left(\omega(\Psi(s)-\Psi(\beta))^{\nu};q\right)\Psi'(\beta)\Phi^{\varsigma-\alpha}(\delta)\Phi^{\alpha}(\beta)\\
&\quad-\mathscr{H}^{\sigma}(\beta)(\Psi(s)-\Psi(\beta))^{\tau-1}E^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j}\left(\omega(\Psi(s)-\Psi(\beta))^{\nu};q\right)\Psi'(\beta)\Phi^{\varsigma}(\beta)\geq 0.
\end{aligned} \tag{20}$$

Integrating (20) with respect to β from η_1 to s, we have

$$\begin{aligned}
&\mathscr{H}^{\sigma}(\delta)\int_{\eta_1}^{s}(\Psi(s)-\Psi(\beta))^{\tau-1}E^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j}\left(\omega(\Psi(s)-\Psi(\beta))^{\nu};q\right)\Psi'(\beta)\Phi^{\varsigma}(\beta)d\beta\\
&\quad+\Phi^{\varsigma-\alpha}(\delta)\int_{\eta_1}^{s}(\Psi(s)-\Psi(\beta))^{\tau-1}E^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j}\left(\omega(\Psi(s)-\Psi(\beta))^{\nu};q\right)\Psi'(\beta)\mathscr{H}^{\sigma}(\beta)\Phi^{\alpha}(\beta)d\beta\\
&\quad-\Phi^{\varsigma-\alpha}(\delta)\mathscr{H}^{\sigma}(\delta)\int_{\eta_1}^{s}(\Psi(s)-\Psi(\beta))^{\tau-1}E^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j}\left(\omega(\Psi(s)-\Psi(\beta))^{\nu};q\right)\Psi'(\beta)\Phi^{\alpha}(\beta)d\beta\\
&\quad-\int_{\eta_1}^{s}(\Psi(s)-\Psi(\beta))^{\tau-1}E^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j}\left(\omega(\Psi(s)-\Psi(\beta))^{\nu};q\right)\Psi'(\beta)\Phi^{\varsigma}(\beta)\mathscr{H}^{\sigma}(\beta)d\beta\geq 0.
\end{aligned} \tag{21}$$

From (21), it follows that

$$\mathscr{H}^{\sigma}(\delta)\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+}\Phi^{\varsigma}\right)(s;q)+\Phi^{\varsigma-\alpha}(\delta)\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+}\mathscr{H}^{\sigma}\Phi^{\alpha}\right)(s;q)$$
$$-\mathscr{H}^{\sigma}(\delta)\Phi^{\varsigma-\alpha}(\delta)\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+}\Phi^{\alpha}\right)(s;q)-\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+}\mathscr{H}^{\sigma}\Phi^{\alpha}\right)(s;q)\geq 0. \tag{22}$$

Again, taking product (14) by

$\mathfrak{J}(s,\delta)\Phi^{\alpha}(\delta)=(\Psi(s)-\Psi(\delta))^{\tau-1}E^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j}\left(\omega\left(\Psi(s)-\Psi(\delta)\right)^{\nu};q\right)\Psi'(\delta)\Phi^{\alpha}(\delta),$

for $\delta\in(\eta_1,s)$, $\eta_1<s\leq\eta_2$ and integrating the subsequent identity w.r.t δ from η_1 to s shows

$$\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+}\Phi^{\varsigma}\right)(s;q)\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+}\Phi^{\alpha}\mathscr{H}^{\sigma}\right)(s;q)$$
$$-\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+}\Phi^{\alpha}\right)(s;q)\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+}\Phi^{\varsigma}\mathscr{H}^{\sigma}\right)(s;q)\geq 0, \tag{23}$$

which completes the desired inequality (17) of Theorem 3.3.

Theorem 3.4. Let Φ be a continuous positive decreasing function on $[\eta_1,\eta_2]$ and $\mathscr{H}$ be a continuous positive increasing function on $[\eta_1,\eta_2]$ with $\eta_1<s\leq\eta_2$, $\sigma>0$ and $\varsigma\geq\alpha>0$. Suppose Ψ be a differentiable and strictly increasing. Then the more generalized fractional integral operator defined in (7), we have

$$\Bigg(\left(\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+}\Phi^{\varsigma}\right)(s;q)\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\lambda,j,\omega,\eta_1^+}\mathscr{H}^{\sigma}\Phi^{\alpha}\right)(s;q)\right.$$
$$\left.+\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\lambda,j,\omega,\eta_1^+}\Phi^{\varsigma}\right)(s;q)\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+}\mathscr{H}^{\theta}\Phi^{\alpha}\right)(s;q)\right)$$
$$/\left(\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+}\mathscr{H}^{\sigma}\Phi^{\varsigma}\right)(s;q)\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\lambda,j,\omega,\eta_1^+}\Phi^{\alpha}\right)(s;q)\right.$$
$$\left.+\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\lambda,j,\omega,\eta_1^+}\mathscr{H}^{\sigma}\Phi^{\varsigma}\right)(s;q)\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\lambda,j,\omega,\eta_1^+}\Phi^{\alpha}\right)(s;q)\right)\geq 1. \tag{24}$$

Proof. Taking product on both sides of (22) by

$\mathfrak{J}(s,\delta)\Phi^{\alpha}(\delta)=(\Psi(s)-\Psi(\delta))^{\lambda-1}E^{\vartheta,z,\kappa,\varepsilon}_{\nu,\lambda,j}\left(\omega\left(\Psi(s)-\Psi(\delta)\right)^{\nu};q\right)\Psi'(\delta)\Phi^{\alpha}(\delta),$

for $\delta\in(\eta_1,s),\eta_1<s\leq\eta_2$, and integrating the subsequent identity w.r.t δ from η_1 to s shows

$$\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+}\Phi^{\varsigma}\right)(s;q)\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\lambda,j,\omega,\eta_1^+}\mathscr{H}^{\sigma}\Phi^{\alpha}\right)(s;q)$$
$$+\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\lambda,j,\omega,\eta_1^+}\Phi^{\varsigma}\right)(s;q)\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+}\mathscr{H}^{\theta}\Phi^{\alpha}\right)(s;q)$$

$$-\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+}\mathscr{H}^{\sigma}(s)\Phi^{\varsigma}\right)(s;q)\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\lambda,j,\omega,\eta_1^+}\Phi^{\alpha}\right)(s;q)$$

$$-\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\lambda,j,\omega,\eta_1^+}\mathscr{H}^{\sigma}\Phi^{\varsigma}\right)(s;q)\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\lambda,j,\omega,\eta_1^+}\Phi^{\alpha}\right)(s;q)\geq 0.$$

It follows that

$$\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+}\Phi^{\varsigma}\right)(s;q)\left({}^{\Psi}{}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\lambda,j,\omega,\eta_1^+}\mathscr{H}^{\sigma}\Phi^{\alpha}\right)(s;q)$$

$$+\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\lambda,j,\omega,\eta_1^+}\Phi^{\varsigma}\right)(s;q)\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+}\mathscr{H}^{\theta}\Phi^{\alpha}\right)(s;q)$$

$$\geq\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+}\mathscr{H}^{\sigma}(s)\Phi^{\varsigma}\right)(s;q)\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\lambda,j,\omega,\eta_1^+}\Phi^{\alpha}\right)(s;q)$$

$$+\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\lambda,j,\omega,\eta_1^+}\mathscr{H}^{\sigma}\Phi^{\varsigma}\right)(s;q)\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\lambda,j,\omega,\eta_1^+}\Phi^{\alpha}\right)(s;q).$$

Dividing the above inequality by

$$\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+}\mathscr{H}^{\sigma}\Phi^{\varsigma}\right)(s;q)\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\lambda,j,\omega,\eta_1^+}\Phi^{\alpha}\right)(s;q)$$

$$+\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\lambda,j,\omega,\eta_1^+}\mathscr{H}^{\sigma}\Phi^{\varsigma}\right)(s;q)\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\lambda,j,\omega,\eta_1^+}\Phi^{\alpha}\right)(s;q),$$

acquires the desired inequality (24).

Remark 3.4. Letting $\tau=\lambda$ in inequality (24), then we attain the inequality (17).

Now, we demonstrate the following generalizations for more generalized fractional to derive some novel inequalities for a class of n-decreasing positive functions.

Theorem 3.5. Let $\{\Phi_i,\ i=1,2,3,\ldots,n\}$ be a sequence of continuous positive decreasing functions on $[\eta_1,\eta_2]$. Let $\eta_1<s\leq\eta_2$, $\sigma>0$, $\varsigma\geq\alpha_p>0$ for any fixed $p\in\{1,2,3,\ldots,n\}$. Suppose Ψ be a differentiable and strictly increasing. Then the more generalized fractional integral operator defined in (7), we have

$$\frac{\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+}\prod_{i\neq p}^{n}\Phi_i^{\alpha i}\Phi_p^{\varsigma}\right)(s;q)}{\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+}\prod_{i=1}^{n}\Phi_i^{\alpha i}\right)(s;q)}\geq\frac{\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+}(s-\eta_1)^{\sigma}\prod_{i\neq p}^{n}\Phi_i^{\alpha i}\Phi_p^{\varsigma}\right)(s;q)}{\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+}(s-\eta_1)^{\sigma}\prod_{i=1}^{n}\Phi_i^{\alpha i}\right)(s;q)}.\tag{25}$$

Proof. Let $\{\Phi_i,\ i=1,2,3,\ldots,n\}$ be a sequence of continuous positive decreasing functions on $[\eta_1,s]$, we have

$$\left((\delta-\eta_1)^{\sigma}-(\beta-\eta_1)^{\sigma}\right)\left(\Phi_p^{\varsigma-\alpha p}(\beta)-\Phi_p^{\varsigma-\alpha p}(\delta)\right)\geq 0\tag{26}$$

for any fixed $p\in\{1,2,3,\ldots,n\},\sigma>0,\ \varsigma\geq\alpha_p>0,$ and $\beta,\delta\in[\eta_1,s]$.

By (26), we have

$$(\delta-\eta_1)^{\sigma}\Phi_p^{\varsigma-\alpha p}(\beta)+(\beta-\eta_1)^{\sigma}\Phi_p^{\varsigma-\alpha p}(\delta)\geq(\delta-\eta_1)^{\sigma}\Phi_p^{\varsigma-\alpha p}(\delta)+(\beta-\eta_1)^{\sigma}\Phi_p^{\varsigma-\alpha p}(\beta). \tag{27}$$

Taking product of (27) by

$$\mathfrak{J}(s,\beta)\prod_{i=1}^{n}\Phi_i^{\alpha_i}(\beta)=(\Psi(s)-\Psi(\beta))^{\tau-1}E_{\nu,\tau,j}^{\vartheta,z,\kappa,\varepsilon}\left(\omega\left(\Psi(s)-\Psi(\beta)\right)^{\nu};q\right)\Psi'(\beta)\prod_{i=1}^{n}\Phi_i^{\alpha_i}(\beta),$$

for $\beta\in(\eta_1,s)$, $\eta_1<s\leq\eta_2$, where $\mathfrak{J}(s,\beta)$ is defined by (12), and integrating the subsequent identity w.r.t β from η_1 to s we have

$$\begin{aligned}
&\mathfrak{J}(s,\beta)\Big[(\delta-\eta_1)^{\sigma}\Phi_p^{\varsigma-\alpha p}(\beta)+(\beta-\eta_1)^{\sigma}\Phi_p^{\varsigma-\alpha p}(\delta)-(\delta-\eta_1)^{\sigma}\Phi_p^{\varsigma-\alpha p}(\delta)\\
&\quad-(\beta-\eta_1)^{\sigma}\Phi_p^{\varsigma-\alpha p}(\beta)\Big]\prod_{i=1}^{n}\Phi_i^{\alpha_i}(\beta)\\
&=(\delta-\eta_1)^{\sigma}(\Psi(s)-\Psi(\beta))^{\tau-1}E_{\nu,\tau,j}^{\vartheta,z,\kappa,\varepsilon}\left(\omega\left(\Psi(s)-\Psi(\beta)\right)^{\nu};q\right)\Psi'(\beta)\prod_{i=1}^{n}\Phi_i^{\alpha_i}(\beta)\Phi_p^{\varsigma-\alpha p}(\beta)\\
&\quad+(\beta-\eta_1)^{\sigma}(\Psi(s)-\Psi(\beta))^{\tau-1}E_{\nu,\tau,j}^{\vartheta,z,\kappa,\varepsilon}\left(\omega\left(\Psi(s)-\Psi(\beta)\right)^{\nu};q\right)\Psi'(\beta)\prod_{i=1}^{n}\Phi_i^{\alpha_i}(\beta)\Phi_p^{\varsigma-\alpha p}(\delta)\\
&\quad-(\delta-\eta_1)^{\sigma}(\Psi(s)-\Psi(\beta))^{\tau-1}E_{\nu,\tau,j}^{\vartheta,z,\kappa,\varepsilon}\left(\omega\left(\Psi(s)-\Psi(\beta)\right)^{\nu};q\right)\Psi'(\beta)\prod_{i=1}^{n}\Phi_i^{\alpha_i}(\beta)\Phi_p^{\varsigma-\alpha p}(\delta)\\
&\quad-(\beta-\eta_1)^{\sigma}(\Psi(s)-\Psi(\beta))^{\tau-1}E_{\nu,\tau,j}^{\vartheta,z,\kappa,\varepsilon}\left(\omega\left(\Psi(s)-\Psi(\beta)\right)^{\nu};q\right)\Psi'(\beta)\prod_{i=1}^{n}\Phi_i^{\alpha_i}(\beta)\Phi_p^{\varsigma-\alpha p}(\beta)\geq 0.
\end{aligned}\tag{28}$$

Integrating (28) with respect to β from η_1 to s, we have

$$\begin{aligned}
&(\delta-\eta_1)^{\sigma}\int_{\eta_1}^{s}(\Psi(s)-\Psi(\beta))^{\tau-1}E_{\nu,\tau,j}^{\vartheta,z,\kappa,\varepsilon}\left(\omega\left(\Psi(s)-\Psi(\beta)\right)^{\nu};q\right)\Psi'(\beta)\prod_{i=1}^{n}\Phi_i^{\alpha_i}(\beta)\Phi_p^{\varsigma-\alpha p}(\beta)d\beta\\
&\quad+\Phi_p^{\varsigma-\alpha p}(\delta)\int_{\eta_1}^{s}(\beta-\eta_1)^{\sigma}(\Psi(s)-\Psi(\beta))^{\tau-1}E_{\nu,\tau,j}^{\vartheta,z,\kappa,\varepsilon}\left(\omega\left(\Psi(s)-\Psi(\beta)\right)^{\nu};q\right)\Psi'(\beta)\prod_{i=1}^{n}\Phi_i^{\alpha_i}(\beta)d\beta\\
&\quad-(\delta-\eta_1)^{\sigma}\int_{\eta_1}^{s}(\Psi(s)-\Psi(\beta))^{\tau-1}E_{\nu,\tau,j}^{\vartheta,z,\kappa,\varepsilon}\left(\omega\left(\Psi(s)-\Psi(\beta)\right)^{\nu};q\right)\Psi'(\beta)\prod_{i=1}^{n}\Phi_i^{\alpha_i}(\beta)\Phi_p^{\varsigma-\alpha p}(\delta)d\beta\\
&\quad-\int_{\eta_1}^{s}(\beta-\eta_1)^{\sigma}(\Psi(s)-\Psi(\beta))^{\tau-1}E_{\nu,\tau,j}^{\vartheta,z,\kappa,\varepsilon}\left(\omega\left(\Psi(s)-\Psi(\beta)\right)^{\nu};q\right)\Psi'(\beta)\prod_{i=1}^{n}\Phi_i^{\alpha_i}(\beta)\Phi_p^{\varsigma-\alpha p}(\beta)d\beta\geq 0.
\end{aligned}\tag{29}$$

From (29), it follows that

$$(\delta-\eta_1)^{\sigma}\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+}\prod_{i\neq p}^{n}\Phi_i^{\alpha i}\Phi_p^{\varsigma}\right)(s;q)$$

$$+\Phi_p^{\varsigma-\alpha p}(\delta)\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+}(s-\eta_1)^{\sigma}\prod_{i=1}^{n}\Phi_i^{\alpha i}\right)(s;q)$$

$$\geq(\delta-\eta_1)^{\sigma}\Phi_p^{\varsigma-\alpha p}(\delta)\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+}\prod_{i=1}^{n}\Phi_i^{\alpha i}\right)(s;q)$$

$$+\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+}(s-\eta_1)^{\sigma}\prod_{i\neq p}^{n}\Phi_i^{\alpha i}\Phi_p^{\varsigma}\right)(s;q). \tag{30}$$

Again, taking product of (30) by

$$\mathfrak{J}(s,\delta)\prod_{i=1}^{n}\Phi_i^{\alpha_i}(\delta)=(\Psi(s)-\Psi(\delta))^{\lambda-1}E^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j}\left(\omega\left(\Psi(s)-\Psi(\delta)\right)^{\nu};q\right)\Phi'(\delta)\prod_{i=1}^{n}\Phi_i^{\alpha_i}(\delta),$$

for $\delta\in(\eta_1,s),\eta_1<s\leq\eta_2$, where $\mathfrak{J}(r,\delta)$ is defined by (12), and integrating the subsequent identity w.r.t δ from η_1 to s, we have

$$\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+}\prod_{i\neq p}^{n}\Phi_i^{\alpha i}\Phi_p^{\varsigma}\right)(s;q)\left((s-\eta_1)^{\sigma}\prod_{i=1}^{n}\Phi_i^{\alpha i}\right)(s;q)$$

$$\geq\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+}(s-\eta_1)^{\sigma}\prod_{i\neq p}^{n}\Phi_i^{\alpha i}\Phi_p^{\varsigma}\right)(s;q)\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+}\prod_{i=1}^{n}\Phi_i^{\alpha i}\right)(s;q), \tag{31}$$

which gives the desired inequality (25).

Theorem 3.6. Let $\{\Phi_i,\ i=1,2,3,\ldots,n\}$ be a sequence of continuous positive decreasing functions on $[\eta_1,\eta_2]$. Let $\eta_1<s\leq\eta_2$, $\sigma>0$, $\varsigma\geq\alpha_p>0$ for any fixed $p\in\{1,2,3,\ldots,n\}$. Suppose Ψ be a differentiable and strictly increasing. Then the more generalized fractional integral operator defined in (7), we have

$$\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+}\prod_{i\neq p}^{n}\Phi_i^{\alpha i}\Phi_p^{\varsigma}\right)(s;q)\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\lambda,j,\omega,\eta_1^+}(s-\eta_1)^{\sigma}\prod_{i=1}^{n}\Phi_i^{\alpha i}\right)(s;q)$$

$$+\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\lambda,j,\omega,\eta_1^+}\prod_{i\neq p}^{n}\Phi_i^{\alpha i}\Phi_p^{\varsigma}\right)(s;q)\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+}(s-\eta_1)^{\sigma}\prod_{i=1}^{n}\Phi_i^{\alpha i}\right)(s;q)\Bigg)$$

$$/\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+}(s-\eta_1)^{\sigma}\prod_{i\neq p}^{n}\Phi_i^{\alpha i}\Phi_p^{\varsigma}\right)(s;q)\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\lambda,j,\omega,\eta_1^+}\prod_{i=1}^{n}\Phi_i^{\alpha i}\right)(s;q)$$

$$+\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\lambda,j,\omega,\eta_1^+}(s-\eta_1)^{\sigma}\prod_{i\neq p}^{n}\Phi_i^{\alpha i}\Phi_p^{\varsigma}\right)(s;q)\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+}\prod_{i=1}^{n}\Phi_i^{\alpha i}\right)(s;q)\Bigg)\geq 1. \tag{32}$$

Proof. Taking product on both sides of (30) by

$$\mathfrak{J}(s,\delta)\prod_{i=1}^{n}\Phi_i^{\alpha_i}(\delta)=(\Psi(s)-\Psi(\delta))^{\lambda-1}E^{\vartheta,z,\kappa,\varepsilon}_{\nu,\lambda,j}\left(\omega\left(\Psi(s)-\Psi(\delta)\right)^{\nu};q\right)\Phi'(\delta)\prod_{i=1}^{n}\Phi_i^{\alpha_i}(\delta),$$

for $\delta\in(\eta_1,s)$, $\eta_1<s\leq\eta_2$, where $\mathfrak{J}(s,\delta)$ is defined by (12) and integrating the subsequent identity w.r.t δ from η_1 to s, we have

$$\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+}\prod_{i\neq p}^{n}\Phi_i^{\alpha i}\Phi_p^{\varsigma}\right)(s;q)\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\lambda,j,\omega,\eta_1^+}(s-\eta_1)^{\sigma}\prod_{i=1}^{n}\Phi_i^{\alpha i}\right)(s;q)$$

$$+\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\lambda,j,\omega,\eta_1^+}\prod_{i\neq p}^{n}\Phi_i^{\alpha i}\Phi_p^{\varsigma}\right)(s;q)\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+}(s-\eta_1)^{\sigma}\prod_{i=1}^{n}\Phi_i^{\alpha i}\right)(s;q)$$

$$\geq\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+}(s-\eta_1)^{\sigma}\prod_{i\neq p}^{n}\Phi_i^{\alpha i}\Phi_p^{\varsigma}\right)(s;q)\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\lambda,j,\omega,\eta_1^+}\prod_{i=1}^{n}\Phi_i^{\alpha i}\right)(s;q)$$

$$+\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\lambda,j,\omega,\eta_1^+}(s-\eta_1)^{\sigma}\prod_{i\neq p}^{n}\Phi_i^{\alpha i}\Phi_p^{\varsigma}\right)(s;q)\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+}\prod_{i=1}^{n}\Phi_i^{\alpha i}\right)(s;q). \tag{33}$$

Dividing the above inequality by

$$\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+}(s-\eta_1)^{\sigma}\prod_{i\neq p}^{n}\Phi_i^{\alpha i}\Phi_p^{\varsigma}\right)(s;q)\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\lambda,j,\omega,\eta_1^+}\prod_{i=1}^{n}\Phi_i^{\alpha i}\right)(s;q)$$

$$+\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\lambda,j,\omega,\eta_1^+}(s-\eta_1)^{\sigma}\prod_{i\neq p}^{n}\Phi_i^{\alpha i}\Phi_p^{\varsigma}\right)(s;q)\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+}\prod_{i=1}^{n}\Phi_i^{\alpha i}\right)(s;q), \tag{34}$$

which gives the desired inequality (32).

Remark 3.5. Letting $\tau=\lambda$ in inequality (32), then we attain the inequality (25).

Theorem 3.7. Let $\mathscr{H}$ be a continuous positive increasing functions and $\{\Phi_i, i=1,2,3,\ldots,n\}$ be a sequence of continuous positive decreasing functions on $[\eta_1,\eta_2]$. Let

$\eta_1 < s \leq \eta_2$, $\sigma > 0$, $\varsigma \geq \alpha_p > 0$ for any fixed $p \in \{1, 2, 3, \ldots, n\}$. Suppose Ψ be a differentiable and strictly increasing. Then the more generalized fractional integral operator defined in (7), we have

$$\frac{\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+}\prod_{i\neq p}^{n}\Phi_i^{\alpha i}\Phi_p^{\varsigma}\right)(s;q)\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+}\mathscr{H}^{\sigma}\prod_{i=1}^{n}\Phi_i^{\alpha i}\right)(s;q)}{\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+}\mathscr{H}^{\sigma}\prod_{i\neq p}^{n}\Phi_i^{\alpha i}\Phi_p^{\varsigma}\right)(s;q)\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+}\prod_{i=1}^{n}\Phi_i^{\alpha i}\right)(s;q)} \geq 1. \tag{35}$$

Proof. Under the given hypothesis, we have

$$\left(\mathscr{H}^{\sigma}(\delta)-\mathscr{H}^{\sigma}(\beta)\right)\left(\Phi_p^{\varsigma-\alpha p}(\beta)-\Phi_p^{\varsigma-\alpha p}(\delta)\right)\geq 0, \tag{36}$$

for any fixed $p \in \{1, 2, 3, \ldots, n\}, \sigma > 0$, $\varsigma \geq \alpha_p > 0$, and $\beta, \delta \in [\eta_1, s]$.

From (41), we have

$$\mathscr{H}^{\sigma}(\delta)\Phi_p^{\varsigma-\alpha p}(\beta)+\mathscr{H}^{\sigma}(\beta)\Phi_p^{\varsigma-\alpha p}(\delta)-\mathscr{H}^{\sigma}(\delta)\Phi_p^{\varsigma-\alpha p}(\delta)-\mathscr{H}^{\sigma}(\beta)\Phi_p^{\varsigma-\alpha p}(\beta)\geq 0. \tag{37}$$

Taking product on both sides of (39) by

$$\mathfrak{J}(s,\beta)\prod_{i=1}^{n}\Phi_i^{\alpha_i}(\beta)=(\Psi(s)-\Psi(\beta))^{\tau-1}E^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j}\left(\omega\left(\Psi(s)-\Psi(\beta)\right)^{\nu};q\right)\Psi'(\beta)\prod_{i=1}^{n}\Phi_i^{\alpha_i}(\beta),$$

for $\beta \in (\eta_1, s), \eta_1 < s \leq \eta_2$, where $\mathfrak{J}(s,\beta)$ is defined by (12), and integrating the subsequent identity w.r.t β from η_1 to s, we have

$$\begin{aligned}
&\mathscr{H}^{\sigma}(\delta)\left(\Psi(s)-\Psi(\beta)\right)^{\tau-1}E^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j}\left(\omega\left(\Psi(s)-\Psi(\beta)\right)^{\nu};q\right)\Psi'(\beta)\prod_{i=1}^{n}\Phi_i^{\alpha_i}(\beta)\Phi_p^{\varsigma-\alpha p}(\beta)\\
&\quad+\mathscr{H}^{\sigma}(\beta)\left(\Psi(s)-\Psi(\beta)\right)^{\tau-1}E^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j}\left(\omega\left(\Psi(s)-\Psi(\beta)\right)^{\nu};q\right)\Psi'(\beta)\prod_{i=1}^{n}\Phi_i^{\alpha_i}(\beta)\Phi_p^{\varsigma-\alpha p}(\delta)\\
&\quad-\mathscr{H}^{\sigma}(\delta)\left(\Psi(s)-\Psi(\beta)\right)^{\tau-1}E^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j}\left(\omega\left(\Psi(s)-\Psi(\beta)\right)^{\nu};q\right)\Psi'(\beta)\prod_{i=1}^{n}\Phi_i^{\alpha_i}(\beta)\Phi_p^{\varsigma-\alpha p}(\delta)\\
&\quad-\mathscr{H}^{\sigma}(\beta)\left(\Psi(s)-\Psi(\beta)\right)^{\tau-1}E^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j}\left(\omega\left(\Psi(s)-\Psi(\beta)\right)^{\nu};q\right)\Psi'(\beta)\prod_{i=1}^{n}\Phi_i^{\alpha_i}(\beta)\Phi_p^{\varsigma-\alpha p}(\beta)\geq 0.
\end{aligned}\tag{38}$$

Integrating (41) with respect to β from η_1 to s, we have

$$\begin{aligned}&\mathscr{H}^{\sigma}(\delta)\int_{\eta_1}^{s}(\Psi(s)-\Psi(\beta))^{\tau-1}E_{\nu,\tau,j}^{\vartheta,z,\kappa,\varepsilon}\left(\omega(\Psi(s)-\Psi(\beta))^{\nu};q\right)\Psi'(\beta)\prod_{i=1}^{n}\Phi_i^{\alpha_i}(\beta)\Phi_p^{\varsigma-\alpha p}(\beta)d\beta\\&+\mathscr{H}^{\sigma}(\beta)\int_{\eta_1}^{s}(\Psi(s)-\Psi(\beta))^{\tau-1}E_{\nu,\tau,j}^{\vartheta,z,\kappa,\varepsilon}\left(\omega(\Psi(s)-\Psi(\beta))^{\nu};q\right)\Psi'(\beta)\prod_{i=1}^{n}\Phi_i^{\alpha_i}(\beta)\Phi_p^{\varsigma-\alpha p}(\delta)d\beta\\&-\mathscr{H}^{\sigma}(\delta)\int_{\eta_1}^{s}(\Psi(s)-\Psi(\beta))^{\tau-1}E_{\nu,\tau,j}^{\vartheta,z,\kappa,\varepsilon}\left(\omega(\Psi(s)-\Psi(\beta))^{\nu};q\right)\Psi'(\beta)\prod_{i=1}^{n}\Phi_i^{\alpha_i}(\beta)\Phi_p^{\varsigma-\alpha p}(\delta)d\beta\\&-\mathscr{H}^{\sigma}(\beta)\int_{\eta_1}^{s}(\Psi(s)-\Psi(\beta))^{\tau-1}E_{\nu,\tau,j}^{\vartheta,z,\kappa,\varepsilon}\left(\omega(\Psi(s)-\Psi(\beta))^{\nu};q\right)\Psi'(\beta)\prod_{i=1}^{n}\Phi_i^{\alpha_i}(\beta)\Phi_p^{\varsigma-\alpha p}(\beta)d\beta\geq 0.\end{aligned} \tag{39}$$

From (39), it follows that

$$\begin{aligned}&\mathscr{H}^{\sigma}(\delta)\left({}^{\Psi}\mathscr{T}_{\nu,\tau,j,\omega,\eta_1^+}^{\vartheta,z,\kappa,\varepsilon}\prod_{i\neq p}^{n}\Phi_i^{\alpha i}\Phi_p^{\varsigma}\right)(s;q)+\Phi_p^{\varsigma-\alpha p}(s)\left({}^{\Psi}\mathscr{T}_{\nu,\tau,j,\omega,\eta_1^+}^{\vartheta,z,\kappa,\varepsilon}\mathscr{H}^{\sigma}(s)\prod_{i=1}^{n}\Phi_i^{\alpha i}\right)(s;q)\\&-\mathscr{H}^{\sigma}(\delta)\Phi_p^{\varsigma-\alpha p}(\delta)\left({}^{\Psi}\mathscr{T}_{\nu,\tau,j,\omega,\eta_1^+}^{\vartheta,z,\kappa,\varepsilon}\prod_{i=1}^{n}\Phi_i^{\alpha i}\right)(s;q)-\left({}^{\Psi}\mathscr{T}_{\nu,\tau,j,\omega,\eta_1^+}^{\vartheta,z,\kappa,\varepsilon}\mathscr{H}^{\sigma}(s)\prod_{i\neq p}^{n}\Phi_i^{\alpha i}\Phi_p^{\varsigma}\right)(s;q)\geq 0.\end{aligned} \tag{40}$$

Again, taking product on both sides of (40) by

$$\mathfrak{J}(s,\delta)\prod_{i=1}^{n}\Phi_i^{\alpha_i}(\delta)=(\Psi(s)-\Psi(\delta))^{\tau-1}E_{\nu,\tau,j}^{\vartheta,z,\kappa,\varepsilon}\left(\omega(\Psi(s)-\Psi(\delta))^{\nu};q\right)\Psi'(\delta)\prod_{i=1}^{n}\Phi_i^{\alpha_i}(\delta),$$

for $\delta\in(\eta_1,s),\eta_1<s\leq\eta_2$, where $\mathfrak{J}(s,\delta)$ is defined by (12), and integrating the subsequent identity w.r.t δ from η_1 to s, we have

$$\begin{aligned}&\left({}^{\Psi}\mathscr{T}_{\nu,\tau,j,\omega,\eta_1^+}^{\vartheta,z,\kappa,\varepsilon}\prod_{i\neq p}^{n}\Phi_i^{\alpha i}\Phi_p^{\varsigma}\right)(s;q)\left({}^{\Psi}\mathscr{T}_{\nu,\tau,j,\omega,\eta_1^+}^{\vartheta,z,\kappa,\varepsilon}\mathscr{H}^{\sigma}\prod_{i=1}^{n}\Phi_i^{\alpha i}\right)(s;q)\\&-\left({}^{\Psi}\mathscr{T}_{\nu,\tau,j,\omega,\eta_1^+}^{\vartheta,z,\kappa,\varepsilon}\mathscr{H}^{\sigma}\prod_{i\neq p}^{n}\Phi_i^{\alpha i}\Phi_p^{\varsigma}\right)(s;q)\left({}^{\Psi}\mathscr{T}_{\nu,\tau,j,\omega,\eta_1^+}^{\vartheta,z,\kappa,\varepsilon}\prod_{i=1}^{n}\Phi_i^{\alpha i}\right)(s;q)\geq 0,\end{aligned} \tag{41}$$

which establishes the desired inequality (35).

Theorem 3.8. Let $\mathscr{H}$ be a continuous positive increasing functions and $\{\Phi_i, i=1,2,3,\ldots,n\}$ be a sequence of continuous positive decreasing functions on $[\eta_1,\eta_2]$. Let $\eta_1<s\leq\eta_2$, $\sigma>0$,

$\varsigma \geq \alpha_p > 0$ for any fixed $p \in \{1,2,3,\ldots,n\}$. Then the more generalized fractional integral operator defined in (7), we have

$$\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+} \prod_{i\neq p}^{n} \Phi_i^{\alpha i}\Phi_p^{\varsigma} \right)(s;q) \left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\lambda,j,\omega,\eta_1^+} \mathscr{H}^{\sigma}(s) \prod_{i=1}^{n} \Phi_i^{\alpha i} \right)(s;q)$$
$$+ \left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+} \mathscr{H}^{\sigma} \prod_{i=1}^{n} \Phi_i^{\alpha i} \right)(s;q) \left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\lambda,j,\omega,\eta_1^+} \prod_{i\neq p}^{n} \Phi_i^{\alpha i}\Phi_p^{\varsigma} \right)(s;q) \Bigg)$$
$$/ \left(\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\lambda,j,\omega,\eta_1^+} \prod_{i=1}^{n} \Phi_i^{\alpha i} \right)(s;q) \left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+} \mathscr{H}^{\sigma} \prod_{i\neq p}^{n} \Phi_i^{\alpha i}\Phi_p^{\varsigma} \right)(s;q) \right.$$
$$\left. + \left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\lambda,j,\omega,\eta_1^+} \mathscr{H}^{\sigma} \prod_{i\neq p}^{n} \Phi_i^{\alpha i}\Phi_p^{\varsigma} \right)(s;q) \left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+} \prod_{i=1}^{n} \Phi_i^{\alpha i} \right)(s;q) \right) \geq 1. \tag{42}$$

Proof. Taking product on both sides of (40) by

$$\mathfrak{J}(s,\delta) \prod_{i=1}^{n} \Phi_i^{\alpha_i}(\delta) = (\Psi(s) - \Psi(\delta))^{\lambda-1} E^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j} \left(\omega (\Psi(s) - \Psi(\delta))^{\nu}; q \right) \Psi'(\delta) \prod_{i=1}^{n} \Phi_i^{\alpha_i}(\delta)$$

for $\delta \in (\eta_1, s), \eta_1 < s \leq \eta_2$, where $\mathfrak{J}(s,\delta)$ is defined by (12) and integrating the subsequent identity w.r.t δ from η_1 to s, we have

$$\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+} \prod_{i\neq p}^{n} \Phi_i^{\alpha i}\Phi_p^{\varsigma} \right)(s;q) \left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\lambda,j,\omega,\eta_1^+} \mathscr{H}^{\sigma} \prod_{i=1}^{n} \Phi_i^{\alpha i} \right)(s;q)$$
$$+ \left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+} \mathscr{H}^{\sigma} \prod_{i=1}^{n} \Phi_i^{\alpha i} \right)(s;q) \left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\lambda,j,\omega,\eta_1^+} \prod_{i\neq p}^{n} \Phi_i^{\alpha i}\Phi_p^{\varsigma} \right)(s;q)$$
$$- \left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\lambda,j,\omega,\eta_1^+} \prod_{i=1}^{n} \Phi_i^{\alpha i} \right)(s;q) \left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+} \mathscr{H}^{\sigma} \prod_{i\neq p}^{n} \Phi_i^{\alpha i}\Phi_p^{\varsigma} \right)(s;q)$$
$$- \left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\lambda,j,\omega,\eta_1^+} \mathscr{H}^{\sigma} \prod_{i\neq p}^{n} \Phi_i^{\alpha i}\Phi_p^{\varsigma} \right)(s;q) \left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+} \prod_{i=1}^{n} \Phi_i^{\alpha i} \right)(s;q) \geq 0. \tag{43}$$

It follows that

$$\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+}\prod_{i\neq p}^{n}\Phi_i^{\alpha i}\Phi_p^{\varsigma}(s;q)\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\lambda,j,\omega,\eta_1^+}\mathscr{H}^{\sigma}\prod_{i=1}^{n}\Phi_i^{\alpha i}\right)(s;q)\right.$$

$$+\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+}\mathscr{H}^{\sigma}\prod_{i=1}^{n}\Phi_i^{\alpha i}\right)(s;q)\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\lambda,j,\omega,\eta_1^+}\prod_{i\neq p}^{n}\Phi_i^{\alpha i}\Phi_p^{\varsigma}\right)(s;q)$$

$$\geq\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\lambda,j,\omega,\eta_1^+}\prod_{i=1}^{n}\Phi_i^{\alpha i}\right)(s;q)\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+}\mathscr{H}^{\sigma}\prod_{i\neq p}^{n}\Phi_i^{\alpha i}\Phi_p^{\varsigma}\right)(s;q)$$

$$+\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\lambda,j,\omega,\eta_1^+}\mathscr{H}^{\sigma}\prod_{i\neq p}^{n}\Phi_i^{\alpha i}\Phi_p^{\varsigma}\right)(s;q)\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+}\prod_{i=1}^{n}\Phi_i^{\alpha i}\right)(s;q).$$

Dividing both sides by

$$\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\lambda,j,\omega,\eta_1^+}\prod_{i=1}^{n}\Phi_i^{\alpha i}\right)(s;q)\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+}\mathscr{H}^{\sigma}(s)\prod_{i\neq p}^{n}\Phi_i^{\alpha i}\Phi_p^{\varsigma}\right)(s;q)$$

$$+\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\lambda,j,\omega,\eta_1^+}\mathscr{H}^{\sigma}\prod_{i\neq p}^{n}\Phi_i^{\alpha i}\Phi_p^{\varsigma}\right)(s;q)\left({}^{\Psi}\mathscr{T}^{\vartheta,z,\kappa,\varepsilon}_{\nu,\tau,j,\omega,\eta_1^+}\prod_{i=1}^{n}\Phi_i^{\alpha i}\right)(s;q),$$

acquire the desired inequality (42).

Remark 3.6. Letting $\tau=\lambda$ in inequality (42), then we attain the inequality (35).

4 Conclusion

A new concept of integration is introduced in this paper is known as the generalized fractional integral operator in the Hilfer sense having Mittag–Leffler function in the kernel. This new formulation admits as particular cases of the well-known fractional integrals, introduced by Andric et al. [39], Salim et al. [40], Rahman et al. [41], Srivastava et al. [42], Prabhakar [43] and RL-fractional integral operators. The new integration is combining of Mittag–Leffler function and fractional integration. New features are presented and some new theorems established for a class of n positive continuous and decreasing functions on the interval $[\eta_1,\eta_2]$. The derived consequences are the generalizations of the results presented in [45,46]. In addition to this, the established outcomes for the new fractional integral concedes as specific cases the notable fractional integrals of Hilfer, Hilfer–Hadamard, Riemann–Liouville, Hadamard, Weyl and Liouville. The newly introduced scheme will be used to solve a couple of equations at the Darcy scale describing flow in a dual medium, the solution of the non-linear problem without considering any discretization, perturbation or transformations. Finally, we can obtain the analytical solutions, using the method of successive approximations, to some fractional differential equations by the proposed study. This new scheme will be opening new doors of investigation toward fractional and fractal differentiations.

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