ARTICLE

Nonlinear Problems via a Convergence Accelerated Decomposition Method of Adomian

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ABSTRACT

The present paper is devoted to the convergence control and accelerating the traditional Decomposition Method of Adomian (ADM). By means of perturbing the initial or early terms of the Adomian iterates by adding a parameterized term, containing an embedded parameter, new modified ADM is constructed. The optimal value of this parameter is later determined via squared residual minimizing the error. The failure of the classical ADM is also prevented by a suitable value of the embedded parameter, particularly beneficial for the Duan–Rach modification of the ADM incorporating all the boundaries into the formulation. With the presented squared residual error analysis, there is no need to check out the results against the numerical ones, as usually has to be done in the traditional ADM studies to convince the readers that the results are indeed converged to the realistic solutions. Physical examples selected from the recent application of ADM demonstrate the validity, accuracy and power of the presented novel approach in this paper. Hence, the highly nonlinear equations arising from engineering applications can be safely treated by the outlined method for which the classical ADM may fail or be slow to converge.

KEYWORDS

Nonlinear equations; Adomian decomposition method; modification; convergence acceleration

1 Introduction

Researchers prefer an easily accessible and user friendly method requiring less computational labor while accurately approximating highly nonlinear equations resulting from mathematical modeling of real-life phenomena. The Adomian decomposition method (ADM) is one such popular technique capable of dealing with the prevailing nonlinearities by means of Adomian polynomials [1,2]. A modification of the classical ADM is proposed within the current study based on the recent publications [3,4] successfully generating fast convergent ADM series solutions with as small Adomian polynomials as possible in the solution series.

A quick literature survey exhibits that ADM has been applied to many nonlinear equations [5]. To classify some of the recent bibliography, algebraic equations were contained within the references [6,7]. The ordinary differential equations were dealt within the citations [8–14].

The articles [15–18] covered the efforts to partial differential equations. Mathematical analysis of the convergence of ADM to certain nonlinear equations was fulfilled in the publications [19–21]. It can be successfully used to gain correct physical parameters domain [22]. A traffic model was also very recently treated in [23] via the Adomian method. The publications by [24,25] present investigation of some nonlinear problems via different numerical approaches.

It is now well-known that an inadequate arrangement of the classical ADM series may lead to non-convergent solutions or solutions with a poor convergence rate. To avoid these shortcomings, a parameter is generally inserted at the leading term of the Adomian series and later it is subtracted at the first order term not to break down the equation structure. This procedure was pursued by the recent publications [11,17,18]. However, how a proper value of the inserted parameter will be determined was not mentioned in these references. Instead, a randomly chosen value was assigned to it. A variety of modifications were also offered in the articles [26–31]. A successful formulation of the ADM was made in the recent work of [3] which was named as the optimal ADM. Further applications of the homotopy analytic approximate method may be found in the literature [32–35].

The motivation of the current work is, benefiting from the idea in [11], to devise a method that greatly improves the mathematical property of classical ADM. Within this aim, a reorganization of the ADM series is proposed by altering the early terms so that they incorporate extra controllable terms. The reason of such a treatment is to get a rapidly converging ADM solutions with the least Adomian polynomials. In place of randomly selecting, an optimum value of the introduced parameter is later determined through error on the grounds of total residual. With this value at our disposal, there is no doubt that the ADM method is convergent to the true solution in a most rapid way, not demanding a verification of the ADM solutions by numerical ones. The failure of the classical ADM in the usual form or in the Duan–Rach formulation is also prevented by a suitable value of the embedded parameter. The present approach can also extend the region of convergence of the traditional method. Examples of physical value are provided to justify and validate the given procedure.

2 Traditional Decomposition Method of Adomian

The usual steps of traditional ADM can be inferred from the aforementioned citations. The methodology in brief is such that under an invertible linear operator L and a forcing function f, it is desired to approximate the function u having the nonlinearity $N(u)$ and satisfying the general nonlinear equation

$$L(u) = N(u) + f, \tag{1}$$

with the initial and/or boundary restrictions

$$B(u) = b. \tag{2}$$

Having inverted (1) under the restrictions (2) generally leads to

$$u = g + L^{-1}[N(u)] + L^{-1}[f], \tag{3}$$

where g is due to the conditions in (2). If u is a single scalar parameter like (1) representing an algebraic equation, then there is no such g in (3), whereas, in the case of a variable u, L^{-1} denotes an integral operator giving rise to g in (3). Then (3) is a mixed Volterra-Fredholm type equation so-called as the Duan–Rach formulation in the recent literature, see for instance [13,14].

The subsequent series decompositions of u and $N(u)$

$$u = \sum_{n=0}^{\infty} u_n, \quad N(u) = \sum_{n=0}^{\infty} A_n, \tag{4}$$

in which A_n's are the classical Adomian polynomials, are later substituted into (3). The solution u of (1) is finally generated from the recurrence relation

$$u_0 = g + L^{-1}[f],$$

$$u_{n+1} = L^{-1}[A_n], \quad n \geq 0. \tag{5}$$

As a result, by means of the relations from (5), an approximate series solution of order M is obtained as

$$u = \sum_{n=0}^{M} u_n, \tag{6}$$

which serves for practical purposes.

In general, the procedure in (5) yields convergent ADM series solutions, see for instance [19–21]. If not, to achieve convergent solutions or for computational conveniences some modifications in the terms u_i in (5) are implemented as in the articles [11,12], without a proper mathematical evidence and support.

3 A Modified Decomposition Method of Adomian

To overcome the divergence of classical ADM or to speed up the convergence rate of the ADM series, the leading order term u_0 in (5) (which is in compliance with the previous implementations, in for instance [11]) or some of the early terms, call u_e, $0 \leq e \leq M_e$, in (5) will be modified so that the modification will account for the change in the ADM by incorporating parameterized terms $\tilde{u}_e(h)$ with h an embedded unknown parameter [3]. To exemplify, consider the modified version of (5) in the manner

$$u_0 = u_0 + \tilde{u}_0(h),$$

$$u_1 = u_1 + \tilde{u}_1(h),$$

$$\vdots,$$

$$u_e = u_e + \tilde{u}_e(h),$$

$$\vdots,$$

$$u_{n+1} = L^{-1}[A_n], \quad n \geq e. \tag{7}$$

The following conditions for parameterized terms in the new algorithm (7) should be added

$$\tilde{u}_0(0) = \tilde{u}_1(0) = \cdots = \tilde{u}_e(0) = 0, \tag{8}$$

so that it can be reduced to the traditional ADM for $h = 0$.

It is remarked that there is no a unique way of selecting the $\tilde{u}_e(h)$ terms, but they must be as simple as possible in line with the simplicity inspiration of the ADM method itself. For

instance, constants depending on h or simple power functions (or simple integrable functions) in combination with h can be used, refer to the illustrations belove for more details. We should remark that in the absence of tilde terms (or h) (7) dully conforms with (5). Having inserted such tilde terms in the modified ADM (7), the question now arises, how to determine the correct or proper value of h? So far, in ADM applications of this kind, only random or trial values are preassigned, see for instance [11] amongst others. On the other hand, we may either observe the least change in the plot of some physical quantities, the method is so-called as the constant h-level curves [36] giving rise to rough estimates for h, or we outline the following rigorous algorithm to determine the best or optimal h.

Algorithm. Consider the squared residual error corresponding to (1) defined by

$$Res(h) = \|L(u) - N(u) - f\|^2, \tag{9}$$

where either $L^2(\Omega)$ or $L(\Omega)$ norms are employed. Moreover, u in (8) is owing to the modified ADM from (7). The optimum value of h is such that it minimizes the squared residual error (8) at the approximation level M.

As a consequence, the above Algorithm will generate the best value of h which will ensure the convergence of ADM series solution (7) in a fastest rate of convergence. The minimization task of (8) may be fulfilled by means of contemporary softwares, such as MAPPLE or MATHEMATICA.

4 Applications

Potential applications of the introduced ADM in (7) are given here. To control the error, we use the norm

$$err = \|u_e - u(t)\| \tag{10}$$

with the exact u_e and ADM solution u.

4.1 An Algebraic Equation

As stated by Adomian [1] the classical ADM method (5) fails to result in a convergent solution of

$$u^2 - 2u - 2 = 0, \tag{11}$$

for the solution $u = -0.73205080757$. On the other hand, when the new ADM is built via

$$u_0 = -1 - h,$$

$$u_1 = h + \frac{1}{2}u_0^2,$$

$$u_{n+1} = \frac{1}{2}\sum_{k=0}^{n} u_k u_{n-k}, \quad n \geq 1, \tag{12}$$

Fig. 1 displays h-level curves at selected truncation orders M. The interval $h \in (-0.35, -0.25)$ is observed to yield convergent ADM solutions, which excludes the traditional ADM with $h = 0$ [1].

Through the residual minimization

$$Res(h) = u^2 - 2u - 2 \tag{13}$$

at the approximation level $M = 8$, $h = -0.2679492$ is obtained as the optimum. The history and why this value is the best for the convergence control, as compared to the failure of classical ADM can be visualized in Tab. 1.

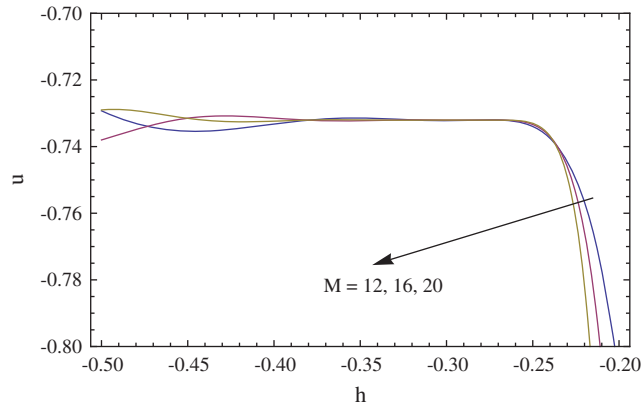**Figure 1:** Convergence control parameter h regarding (11)

Table 1: Convergence of modified ADM (12) for Eq. (11) with different h. Paranthesis is for the absolute error (10)

M	$h = 0.0$	$h = -0.20$	$h = -0.2679492$	$h = -0.40$
2	-1.000000000	-0.7760000000	-0.7320508035 (4.06×10^{-9})	-0.6880000000
4	-1.250000000	-0.7707200000	-0.7320508054 (2.17×10^{-9})	-0.7390400000
8	-4.234375000	-0.7794992000	-0.7320508069 (6.24×10^{-10})	-0.7295606400
12	-33.22070313	-0.8094297223	-0.7320508074 (1.79×10^{-10})	-0.7332243215
16	-349.4215698	-0.8769006132	-0.7320508075 (5.15×10^{-11})	-0.7314507918
20	-4071.603386	-1.0260654160	-0.7320508076 (1.48×10^{-11})	-0.7323668046

The convergent solution of (11) with the new modified ADM (12) at the approximation level $M = 8$ is found to be

$$u = \frac{1}{128}\left(-542 - 6435h - 37752h^2 - 117348h^3 - 210420h^4 - 223594h^5 \right.$$
$$\left. -140280h^6 - 50292h^7 - 9438h^8 - 715h^9\right), \tag{14}$$

for which the optimum h is tabulated in Tab. 1.

4.2 Equation Involving Integral

Consider the equation given in [30]

$$u(\eta) = \frac{1}{20}\left(300 + 315\eta^2 + 5\eta^4 + \eta^6\right) - \frac{1}{150}\int_0^\eta (\eta - \tau)u(\tau)^2 d\tau, \quad 0 \le \eta \le 1. \tag{15}$$

The modified ADM method (7) for the current integral problem is adopted as

$$u_0(\eta) = \frac{1}{20}\left(300 + 315\eta^2 + 5\eta^4 + \eta^6\right) - h\eta^2,$$

$$u_1(\eta) = h\eta^2 - \frac{1}{150}\int_0^\eta (\eta - \tau)u_0(\tau)^2 d\tau,$$

$$u_{n+1}(\eta) = -\frac{1}{150}\int_0^\eta (\eta - \tau)\sum_{k=0}^n u_k(\tau)u_{n-k}(\tau)d\tau, \quad n \geq 1. \tag{16}$$

We find $h \in (-1500, 1500)$ for the convergence interval in Fig. 2, refer also to [3]. A comparison and convergence accelerating feature of the present ADM can be visualized from the Tab. 2. It is noticed from Tab. 2 that even the 4th-order modified ADM (16) is able to produce the solution accurate of order 10^{-10}, which is adequate in practical purposes.
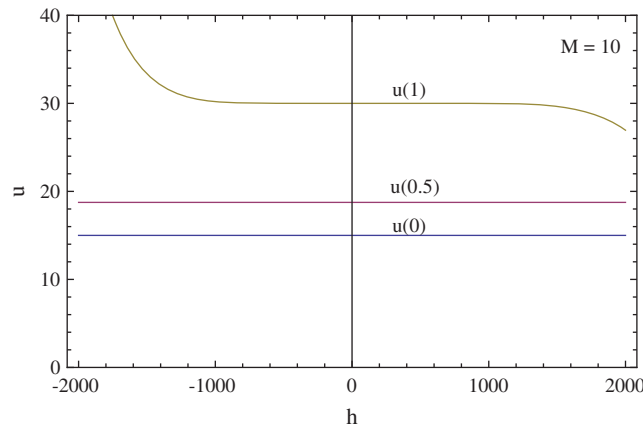

Figure 2: Convergence control parameter h regarding (15)

Table 2: Errors in (15) regarding (14). Parenthesis denotes the optimum values of h

M	$|ue(1) - u(1)|$ in [30]	$|ue(1) - u(1)|$ present
2	$3.568238353 \times 10^{-4}$	$2.585085279 \times 10^{-6}$ (0.8164225489)
4	$8.462701192 \times 10^{-8}$	$4.525954643 \times 10^{-11}$ (0.7993559975)
5	$1.233325816 \times 10^{-9}$	$1.783106924 \times 10^{-13}$ (0.7958113139)
10	$6.276712343 \times 10^{-19}$	$1.264455451 \times 10^{-25}$ (0.7889813873)

4.3 A Fin with Porosity Feature

As taken from [14], a porous fin can be modelled via

$$u'' = su^2 + \beta^2 u, \quad 0 < \eta < 1,$$
$$u'(0) = 0, \quad u(1) = 1. \tag{17}$$

In (17), temperature along the fin is u, and s and β are physical parameters.

The modified ADM algorithm (7) here is

$$u_0(\eta) = \theta_0 - h\eta^2,$$

$$u_1(\eta) = h\eta^2 + \int_0^\eta \int_0^\eta \left(su_0(\eta)^2 + \beta^2 u_0(\eta) \right) d\eta d\eta,$$

$$u_{n+1}(\eta) = \int_0^\eta \int_0^\eta \left(\beta^2 u_n(\eta) + s \sum_{k=0}^n u_k(\eta) u_{n-k}(\eta) \right) d\eta d\eta, \quad n \geq 1. \tag{18}$$

At the selected values $s = 5$ and $\beta = 1$, the history of convergence is tabulated in Tab. 3. It is seen how useful the modified ADM over the classical one by more than doubling the convergence rate at the same number of iterations. The CPU times evaluated by MATHEMATICA through the residual are shown in Tab. 3. The list proves the improved efficiency of the current method over the classical ADM.

Table 3: Residual errors of classical ADM and modified ADM (18) for Eq. (17) with $s = 5$ and $\beta = 1$. Parenthesis denote the optimum values of h

M	$\sqrt{Res(h)}$ ADM $(h = 0)$	CPU	$\sqrt{Res(h)}$ modified ADM	CPU
2	$9.275340168 \times 10^{-1}$	0.031	$2.9362768064 \times 10^{-2}$ (-0.5306610965)	0.031
4	$1.570940630 \times 10^{-1}$	0.156	$4.4652370216 \times 10^{-4}$ (-0.4863076732)	0.178
8	$2.113503722 \times 10^{-3}$	0.625	$5.6679833941 \times 10^{-8}$ (-0.4699160217)	0.703
12	$1.711030679 \times 10^{-5}$	2.026	$4.3828152149 \times 10^{-12}$ (-0.4655830387)	2.203

The values of $u(0)$ and $u'(1)$ are listed in Tab. 4 with $M = 12$ from the ADM and also with the optimums h_{opt}. The ADM (18) excellently compares with the full numerical solutions and hence, they can be used without consulting a numerical verification.

Table 4: Numerical and modified ADM results regarding (17) [3]

s	θ_0	$u'(1)$	θ_0 (OADM)	$u'(1)$ (OADM)	h_{opt}
1	0.5227380936	1.1393789158	0.5227380936	1.1393789158	-0.4071445322
2	0.4487571832	1.4182574617	0.4487571832	1.4182574617	-0.4410040002
3	0.3977684724	1.6480019963	0.3977684724	1.6480019963	-0.4567006260
4	0.3596753295	1.8474903623	0.3596753295	1.8474903623	-0.4635604524
5	0.3297469637	2.0261010880	0.3297469637	2.0261010880	-0.4655830387

Instead of the modification of ADM in (18), we may use the Duan–Rach formulation involving no unknown parameters within it except the embedded parameter h

$$u_0(\eta) = 1 - h,$$

$$u_1(\eta) = h + \int_1^\eta \int_0^\eta \left(su_0(\eta)^2 + \beta^2 u_0(\eta) \right) d\eta d\eta,$$

$$u_{n+1}(\eta) = \int_1^\eta \int_0^\eta \left(\beta^2 u_n(\eta) + s \sum_{k=0}^n u_k(\eta) u_{n-k}(\eta) \right) d\eta d\eta, \quad n \geq 1. \tag{19}$$

Choosing $s = \beta = 1$, Fig. 3 shows h−level curves, indicating that the prescription of h must be made within the range $(0.39, 0.49)$ to ensure the convergence of the modified ADM 19. The exact values of $u(0)$ and $u'(1)$ are respectively, 0.522738093570 and 1.13937891581.
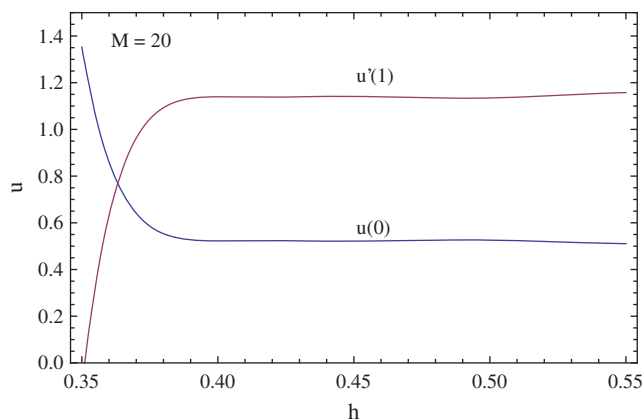

Figure 3: Convergence control parameter h regarding (17)

To demonstrate the power of the modified ADM (19), Tab. 5 shows the squared residual error (9) from both the novel and classical ADM. It is unfortunate to observe that the Duan–Rach formulation (19) with $h = 0$ fails to converge, however, the optimum embedded parameter h insures that the modified ADM is convergent for the present physical problem, even if the convergence is not as fast as the modified formulation in (18).

Table 5: Values of $\sqrt{Res(h)}$ for (17) from the classical and modified ADM methods

M	$\sqrt{Res(0)}$	$\sqrt{Res(h)}$	h
4	0.91×10^1	$9.379592752 \times 10^{-3}$	0.5717083134
8	5.90×10^2	$8.485113566 \times 10^{-5}$	0.4466738418
12	1.90×10^5	$1.712138085 \times 10^{-5}$	0.4205569783
16	9.50×10^7	$8.888688579 \times 10^{-6}$	0.4108495377
20	5.68×10^{10}	$3.872408211 \times 10^{-6}$	0.4061899784

4.4 Gelfand Equation

The Gelfand equation [5] involves exponential nonlinearity [8]

$$u'' = e^u, \quad u(\eta = 0) = u(\eta = 1) = 0, \tag{20}$$

with $u'(0) = -0.4636325917$ [3].

In line with the publications [5,8] when $h = 0$, the modified ADM is

$$u_0(\eta) = 0,$$

$$u_1(\eta) = -h - \frac{\eta}{2} + \frac{\eta^2}{2},$$

$$u_2(\eta) = h - \left(-\frac{1}{24} - \frac{h}{2}\right)\eta - \frac{h\eta^2}{2} - \frac{\eta^3}{12} + \frac{\eta^4}{24},$$

$$u_{n+1}(\eta) = -\eta \int_0^1 \int_0^\eta A_n d\eta d\eta + \int_0^\eta \int_0^\eta A_n d\eta d\eta, \quad n \geq 2, \tag{21}$$

where $A_n(h, \eta)$ are the polynomials of Adomian for $e^{u(\eta)}$, encompassing all boundary conditions in (20) within the Duan–Rach approach, and we have

$$u'(0) = -\int_0^1 \int_0^\eta A_n d\eta d\eta.$$

Fig. 4 shows the predicted convergence control parameters. With $M = 12$, an optimum value for the embedding parameter h is found to be -0.01274 from the Algorithm in (9). We find that the residual error is $1.6835117258 \times 10^{-15}$ with $u'(0) = -0.4636325917$ from the present approach. On the other hand, and error of $4.813740934 \times 10^{-9}$ with $u'(0) = -0.4636325899$ is calculated from the traditional ADM. Our value is given by

$$u'(0) = -\frac{767093598322389372691}{165452907128863850496} + \frac{15169262547373h}{37157048852152320000} + \frac{7945655577131h^2}{243290200817664000}$$

$$+ \frac{9908796519071h^3}{8536498274304000} + \frac{17536069217h^4}{980755776000} + \frac{38157057179h^5}{373621248000} + \frac{10097225749h^6}{57480192000}$$

$$+ \frac{93183637h^7}{1045094400} + \frac{1216277h^8}{81285120} + \frac{147689h^9}{174182400} + \frac{283h^{10}}{21772800} + \frac{h^{11}}{79833600}. \tag{22}$$

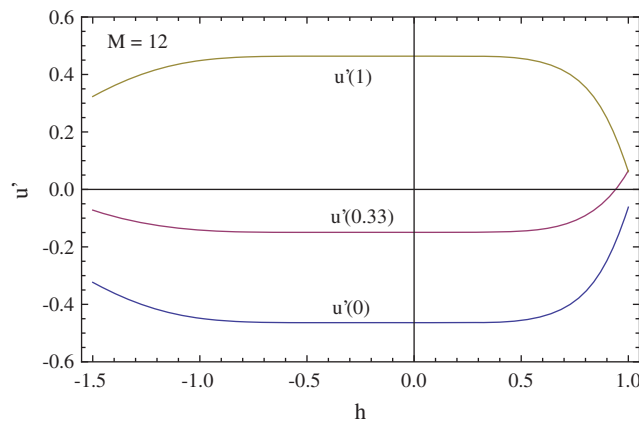The success of the present modified ADM (21) is thus obvious.

Figure 4: Convergence control parameter h regarding (20)

4.5 Electrostatic Cantilever Micro-Electromechanical System

The beam-type electrostatic actuators for the nonlinear cantilever micro-electro mechanical systems are modelled by the fourth-order boundary value problem from [11]

$$u'''' = -\frac{\alpha_K}{u^K} - \frac{\beta}{u^2} - \frac{\gamma}{u}, \quad 0 < \eta < 1, \quad K = 3, 4,$$

$$u(0) = 1, \quad u'(0) = u''(1) = u'''(1) = 0. \tag{23}$$

To comply with the Duan–Rach Adomian decomposition method in [11], the present modified ADM is

$$u_0(\eta) = 1 - h,$$

$$u_1(\eta) = h + \int_0^\eta \int_0^\eta \int_1^\eta \int_1^\eta A_0 \, d\eta d\eta d\eta d\eta$$

$$= h + \frac{\left(\alpha_K + \beta - h\beta + (-1+h)^2\gamma\right)\eta^2(6 + (-4 + \eta)\eta)}{24(-1+h)^3},$$

$$u_{n+1}(\eta) = \int_0^\eta \int_0^\eta \int_1^\eta \int_1^\eta A_n \, d\eta d\eta d\eta d\eta \quad n \geq 1, \tag{24}$$

where the Adomian polynomials $A_n(h, \eta)$ are due to the negative-power nonlinearities in (23), see [11] for more details.

For the fixed parameters $K = 3$, $\alpha_K = 0.2$, $\beta = 0.5$ and $\gamma = 0.25$, Fig. 5 produced from (24) helps us guess the proper values of embedded parameter h. Small values of h are seen to be adequate to get convergent ADM solutions.
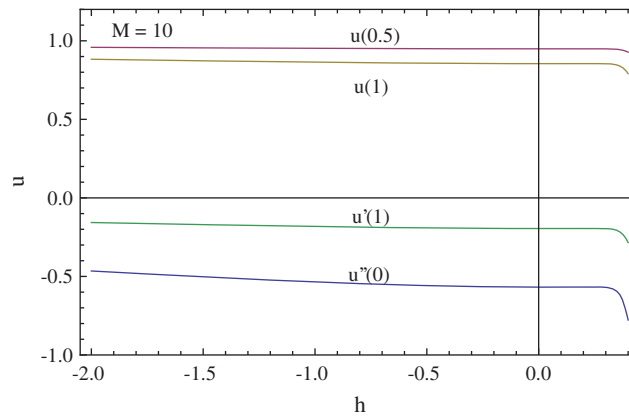

Figure 5: Convergence control parameter h regarding (23)

Fig. 6 demonstrates different approximation levels M, and it signifies to $h = 0.1045421730$ as the optimum h when $M = 10$. With this optimum value of the embedding parameter, the squared residual error for the current problem is $\sqrt{Res} = 3.2727623195 \times 10^{-9}$, whereas the classical residual error with $h = 0$ is $\sqrt{Res} = 8.7520447833 \times 10^{-5}$. This implies that much more Adomian series terms are required for the classical ADM to reach the accuracy of the modified ADM here. Moreover, from [11] the physical values are $u''(0) = -0.56764$ and $u'''(0) = 1.07585$

evaluated with 12 Adomian polynomials. The present values are $u''(0) = -0.5676684138$ and $u'''(0) = 1.0758898885$ evaluated with only 10 Adomian polynomials, correct to 9 decimal places. Hence, the advantage of the present modification is clear for the present highly nonlinear physical problem. We should remark that with the present Algorithm, the validation of the ADM results against the numerical ones is no longer a prerequisite. Also, there is no need to evaluate error remainder functions, nor to consult to Pade-approximates to increase the accuracy, both of which as implemented in [11].
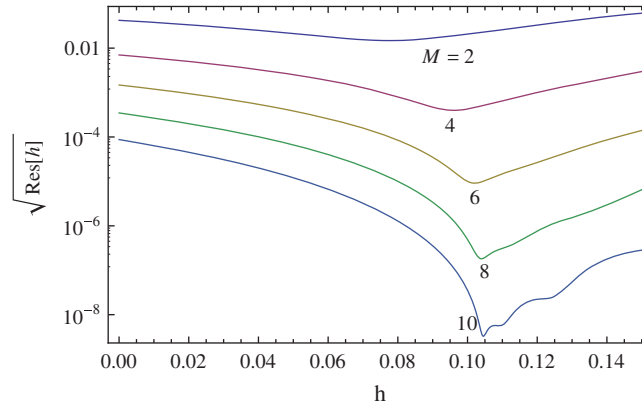

Figure 6: Error regarding (23) for various M

The convergence accelerating feature of the present modified ADM (24) as compared to the classical ADM is better visualized from the Tab. 6. Table also shows the comparable CPU times.

Table 6: Convergence history of modified ADM *vs.* classical ADM for Eq. (23)

M	$\sqrt{Res(0)}$	CPU	$\sqrt{Res(h)}$ (h_{opt})	CPU
2	$4.2251049880 \times 10^{-2}$	0.023	$1.4791163690 \times 10^{-2}$ (0.0777275068)	0.034
4	$7.0270444089 \times 10^{-3}$	0.041	$3.9824140521 \times 10^{-4}$ (0.0961306006)	0.058
6	$1.4789254421 \times 10^{-3}$	0.070	$9.1320572997 \times 10^{-6}$ (0.1020430905)	0.089
8	$3.4800777077 \times 10^{-4}$	0.105	$1.8150755783 \times 10^{-7}$ (0.1040090218)	0.131
10	$8.7520447833 \times 10^{-5}$	0.136	$3.2727623195 \times 10^{-9}$ (0.1045421730)	0.173

4.6 Electrostatic Cantilever Nano-Electromechanical System

Nonlinear model for the electrostatic double cantilever nano-electromechanical system in the case of Casimir force ($K = 4$) is given by [11]

$$u'''' = -\frac{\alpha_K}{u^K} - \frac{\beta}{u^2} - \frac{\gamma}{u}, \quad 0 < \eta < 1,$$

$$u(0) = u(1) = 1, \quad u'(0) = u'(1) = 0. \tag{25}$$

We adopt the subsequent modified ADM, that conforms to the classical ADM ($h = 0$) given in [15]

$$u_0(\eta) = 1 - h,$$

$$u_1(\eta) = h + \left(2\eta^3 - 3\eta^2\right) \int_0^1 \int_0^\eta \int_0^\eta \int_0^\eta A_0 d\eta d\eta d\eta d\eta$$

$$+ \left(\eta^2 - \eta^3\right) \int_0^1 \int_0^\eta \int_0^\eta A_0 d\eta d\eta d\eta + \int_0^\eta \int_0^\eta \int_0^\eta \int_0^\eta A_0 d\eta d\eta d\eta d\eta$$

$$= h - \frac{\left(\alpha_K + (-1+h)^2(\beta + \gamma - h\gamma)\right)(-1+\eta)^2 \eta^2}{24(-1+h)^4},$$

$$u_{n+1}(\eta) = \left(2\eta^3 - 3\eta^2\right) \int_0^1 \int_0^\eta \int_0^\eta \int_0^\eta A_n d\eta d\eta d\eta d\eta \tag{26}$$

$$+ \left(\eta^2 - \eta^3\right) \int_0^1 \int_0^\eta \int_0^\eta A_n d\eta d\eta d\eta + \int_0^\eta \int_0^\eta \int_0^\eta \int_0^\eta A_n d\eta d\eta d\eta d\eta, \quad n \geq 1,$$

where the Adomian polynomials $A_n(h, \eta)$ are due to the negative-power nonlinearities in (25), see [11] for more details.

For the specific parameters $\alpha_K = 1$, $\beta = 1.5$ and $\gamma = 0.5$, Fig. 7 displays the constant h-level curves drawn at the approximation level $M = 8$. Similar to the previous example, very small values of h are seen to be adequate to get convergent ADM.
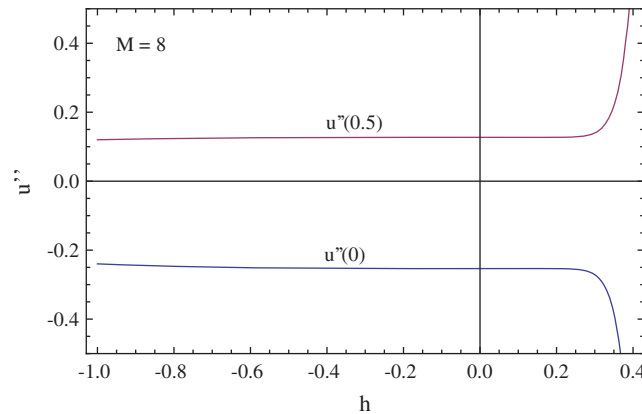

Figure 7: Convergence control parameter h regarding (25)

In order to evaluate the performance of modified ADM over the classical one, Tab. 7 shows the unknown physical quantities $u''(0) = u''(1)$ and $u'''(1)$ at several truncation orders M. The faster convergence rate of the present ADM is apparent.

To illustrate, the analytical formula computed via the present algorithm (26) at $M = 4$ for the value of $u''(0)$ is given by

$$u''(0) = \frac{1}{33476463820800(-1+h)^{19}} \left(8484398647110 - 161202053045835h\right.$$

$$+ 1450626144581925h^2 - 8210561431139865h^3 + 32657066457192189h^4$$

$$- 96557260487752602h^5 + 219137184603173574h^6 - 389199830265020934h^7$$

$$+ 547311661233315516h^8 - 613303909999850619h^9 + 548763779900139189h^{10}$$

$$- 391219305253946049h^{11} + 220658166826273313h^{12} - 97137704049404928h^{13}$$

$$+ 32601652101188352h^{14} - 8015172092928000h^{15} + 1344650417510400h^{16}$$

$$\left. - 133905855283200h^{17} + 5579410636800h^{18}\right), \tag{27}$$

which is of almost nine degree of accuracy as seen from Tab. 7.

Table 7: Values of $u''(0)$ and $u'''(1)$ for (25) from the classical and modified ADM methods

M	$u''(0)$ $(h=0)$	$u'''(1)$ $(h=0)$	$u''(0)$	$u'''(1)$	h
2	-0.2533482143	-1.5156250000	-0.2534302817	-1.5159601707	0.0053058312
4	-0.2534436938	-1.5160577858	-0.2534438205	-1.5160583465	0.0061391414
6	-0.2534438213	-1.5160583545	-0.2534438215	-1.5160583555	0.0063873039
8	-0.2534438215	-1.5160583555	-0.2534438215	-1.5160583555	0.0064510177

4.7 Lane–Emden Equation

We consider the Lane–Emden type boundary value problem from [9]

$$u'' + \frac{k}{x}u' - \frac{ru}{s+u} = 0, \quad 0 < x < 1,$$

$$u'(0) = 0, \quad 5u(1) + u'(1) = 5, \tag{28}$$

that models the oxygen diffusion in a spherical cell with Michaelis–Menten uptake kinetics. We take into account the subsequent constants to comply with the literature [9]

$$k = 4, \quad r = 0.76129, \quad s = 0.03119.$$

The modified ADM that is offered for the present problem is then

$$u_0(\eta) = a - h,$$

$$u_1(\eta) = h + \frac{76129(a-h)\eta^2}{10(3119 + 100000a - 100000h)},$$

$$u_{n+1}(\eta) = \frac{1}{3}\int_0^\eta t\left(1 - \left(\frac{t}{\eta}\right)^3\right)A_n dt, \quad n \geq 1, \tag{29}$$

which conforms with the classical ADM of [9] in the limit $h \to 0$ and the Adomian polynomials $A_n(h, \eta)$ are to account for the nonlinearity $\frac{ru}{s+u}$. The interest is to determine the physical value of $u(0) = a$.

We present Tab. 8 to demonstrate the performance of the modified ADM (29) versus the classical ADM. The expected practical accuracy is met at lower Adomian series approximations via the modified method.

Table 8: Values of $u(0) = a$ and $\sqrt{Res(h)}$ for (28) from the classical and modified ADM methods

M	$\sqrt{Res(0)}$	$\sqrt{Res(h)}$	$a \ (h = 0)$	a	h
2	$5.05920434 \times 10^{-5}$	$1.82258949 \times 10^{-5}$	0.896870943665	0.896877048532	-0.0289267765
4	$2.24898539 \times 10^{-7}$	$2.18219055 \times 10^{-8}$	0.896877026028	0.896877046709	-0.0321993310
6	$1.12229262 \times 10^{-9}$	$2.83931728 \times 10^{-11}$	0.896877046624	0.896877046709	-0.0335012655

4.8 The Fluid Flow of Jeffery–Hamel

The Jeffery–Hamel fluid flow problem is modelled via [22]

$$u''' + 2\alpha Re u u' + (4 - Ha)\alpha^2 u' = 0, \quad 0 < \eta < 1,$$
$$u(0) = 1, \quad u'(0) = u(1) = 0, \tag{30}$$

with α, Re and Ha are the physical parameters [13].

Following the successful Duan–Rach ADM formulation of the problem (30) in [17], we propose the following modified version

$$u_0(\eta) = 1 - \eta^2 - h,$$

$$u_1(\eta) = -\frac{1}{60}\alpha\eta^2\left(-1 + \eta^2\right)$$

$$\left(5(-4 + Ha)\alpha + 2Re\left(-4 + \eta^2\right)\right) + \frac{1}{6}h\left(6 - Re\alpha\eta^2\left(-1 + \eta^2\right)\right),$$

$$u_{n+1}(\eta) = -\eta^2 \int_0^1 \int_0^\eta \int_0^\eta A_n d\eta d\eta d\eta + \int_0^\eta \int_0^\eta \int_0^\eta A_n d\eta d\eta d\eta, \quad n \geq 1, \tag{31}$$

where $A_n(h, \eta) = -(4 - Ha)\alpha^2 u_n'(\eta) - 2\alpha Re \sum_{k=0}^n u_k(\eta)u_{n-k}'(\eta)$ are the Adomian polynomials.

For the diverging channel, considering the specific parameters $\alpha = 5^0$, $Re = 50$ and $Ha = 1000$ to be in line with [13], the constant h-level curves are depicted in Fig. 8 at the approximation order $M = 8$. Accordingly, we expect the convergency of the modified ADM (31) in the vicinity of zero.

The performance of modified ADM (31) is next measured by computing the centerline velocity $u(0.5)$ (numerical value is 0.764064240111) at different approximation levels M as shown in Tab. 9. It is observed that 10 digits of accuracy is quickly reached by the present ADM, whereas the classical ADM falls behind. Hence, even though it was not clearly mentioned in [13] (see Tab. 1 therein), the accuracy of order 10^{-8} as obtained via the classical ADM demands at least 15–20 Adomian polynomials, whereas only 6 Adomian polynomials are sufficient to gain the same accuracy with the present modification.
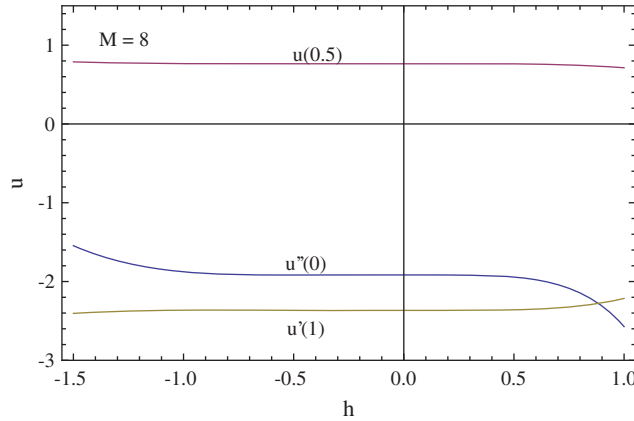
Figure 8: Convergence control parameter h regarding (30)

Table 9: Values of $u(0.5)$ and $\sqrt{Res(h)}$ for (30) from the classical and modified ADM methods

M	$\sqrt{Res(0)}$	$\sqrt{Res(h)}$	$u(0.5)$ $(h=0)$	$u(0.5)$	h
2	$6.92050730 \times 10^{-2}$	$1.13520536 \times 10^{-2}$	0.763699786377	0.764097626541	-0.0180187189
4	$1.94819917 \times 10^{-3}$	$1.87302552 \times 10^{-5}$	0.764051346961	0.764064279681	-0.0179816107
6	$7.53651632 \times 10^{-5}$	$3.95453415 \times 10^{-8}$	0.764063696627	0.764064239975	-0.0178807933
8	$3.34939742 \times 10^{-6}$	$9.72265613 \times 10^{-11}$	0.764064214850	0.764064240111	-0.0178880579

4.9 Squeezing Two Parallel Plates

The flow squeezed between two parallel plates are modelled by the nonlinear equations [22]

$$u^{(4)} - SA(1-\phi)^{2.5}(\eta u''' + 3u'' + u'u'' - uu''') = 0, \quad 0 < \eta < 1,$$
$$u(0) = u''(0) = 0, \quad u(1) = 1, \quad u'(1) = 0, \tag{32}$$

see [14] for the flow parameters.

In accordance with the Duan–Rach ADM formulation of the physical problem (32) in [14], we set the modified ADM in the form

$$u_0(\eta) = \frac{1}{2}\left(3\eta - \eta^3\right) - h\eta^3,$$

$$u_1(\eta) = \frac{1}{560}\left(560h\eta^3 + A(-27+2h)(1+2h)S\eta\left(-3+\eta^2\right)(1-\phi)^{2.5}\right.$$

$$\left. + 2A(1+2h)S\eta^5\left(-28+(1+2h)\eta^2\right)(1-\phi)^{5/2}\right.$$

$$\left. + 14A\left(-\frac{19}{2}+h\right)(1+2h)S\left(\eta-\eta^3\right)(1-\phi)^{2.5}\right),$$

$$u_{n+1}(\eta) = \frac{1}{2}\left(\eta - \eta^3\right)\int_0^1\int_0^\eta\int_0^\eta A_n d\eta d\eta d\eta + \frac{1}{2}\left(\eta^3 - 3\eta\right)\int_0^1\int_0^\eta\int_0^\eta\int_0^\eta A_n d\eta d\eta d\eta d\eta$$

$$+ \int_0^\eta\int_0^\eta\int_0^\eta\int_0^\eta A_n d\eta d\eta d\eta d\eta, \tag{33}$$

where

$$A_n(h,\eta) = SA(1-\phi)^{2.5}\left(\eta u_n'''(\eta) + 3u_n''(\eta) + \sum_{k=0}^n (u_k'(\eta)u_{n-k}''(\eta) - u_k(\eta)u_{n-k}'''(\eta))\right)$$

are the Adomian polynomials.

To make a comparison with the classical ADM in [14], we set the parameters $S = 1$, $\phi = 0.02$ and concentrate on Cu-Water nanofluid with $\rho_s = 8933$ and $\rho_f = 997.1$. The corresponding constant h-level curves are plotted in Fig. 9. It appears that the convergence of the modified ADM (32) is guaranteed if h is selected in the overlapping interval $(-1, 1)$.
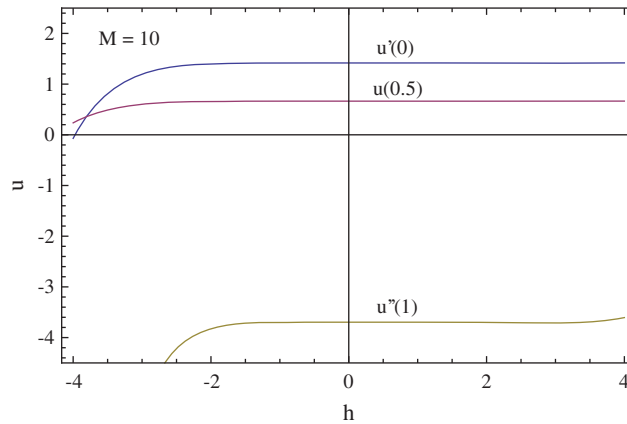

Figure 9: Convergence control parameter h regarding (32)

The effects of iterative number M on the skin friction $u''(1)$ are next demonstrated in Tab. 10. The advantage of the modified ADM (33) with optimum values of embedding parameter h is clearly observed against the classical ADM with $h = 0$. It appears that the accuracy of classical ADM of $O(10^{-8})$ as displayed in Tab. 2 of [18] clearly requires more Adomian series terms (see $M = 8$ in Tab. 10) as compared to the less terms needed in the modified ADM here.

Table 10: Values of $u''(1)$ and $\sqrt{Res(h)}$ for (32) from the classical and modified ADM methods

M	$\sqrt{Res(0)}$	$\sqrt{Res(h)}$	$u''(1)$ $(h=0)$	$u''(1)$	h
2	$2.67740258 \times 10^{-1}$	$6.68835495 \times 10^{-2}$	-3.6764267174	-3.6917310584	-0.0991340307
4	$1.61057990 \times 10^{-2}$	$7.92481375 \times 10^{-4}$	-3.6952425904	-3.6965216730	-0.1097576248
6	$1.09939458 \times 10^{-3}$	$6.61056361 \times 10^{-6}$	-3.6964553942	-3.6965506329	-0.1111585030
8	$8.10446170 \times 10^{-5}$	$7.71479681 \times 10^{-8}$	-3.6965431988	-3.6965505168	-0.1112323962
10	$6.28790630 \times 10^{-6}$	$1.09803715 \times 10^{-9}$	-3.6965499276	-3.6965505117	-0.1112270087

The following fourth-order modified ADM series solution for the skin friction may serve good to the purpose of engineering applications if not high accuracy is required

$$u''(1) = \frac{1}{2940824761875}\left(-8822474285625 - 2100589115625AS(1-\phi)^{5/2}\right.$$

$$-53295A^3\left(1036330 + h^2(1966073 + 36h(-77288 + 16563h))\right)S^3(1-\phi)^{15/2}$$

$$-72747675A^2\left(3687 + 1468h^3\right)S^2(-1+\phi)^5 + A^4(1+2h)\,(12606590933$$

$$+h\left(-17941167514 + h\left(12710854094 - 4619738103h + 509582226h^2\right)\right)\Big)$$

$$\left.S^4(-1+\phi)^{10}\right). \tag{34}$$

4.10 Nonlinear Oscillator Problem

Let us consider the nonlinear oscillator Duffing problem (see [36] (Chapter 5) and [4])

$$u'' + u + u^3 = 0, \quad u(0) = 1, \quad u'(0) = 0; \quad 0 \le t \le 3, \tag{35}$$

which involves a cubic nonlinearity.

The improved ADM can be given via

$$u_0(t) = 1 - ht,$$

$$u_1(t) = ht - \int_0^t \int_0^t (u_{n-1}(t) + A_0(t))dtdt,$$

$$u_{n+1}(t) = -\int_0^t \int_0^t A_n(t)dtdt, \quad n \ge 1, \tag{36}$$

with the Adomian terms $A_n(t)$ in (35).

The classical Adomian method with $h = 0$ in (36) is not convergent, whereas with $h = 0.68981924$, the residual error becomes $\sqrt{Res(h)} = 0.02589439$, and the convergency is satisfied as revealed in Fig. 10 for the domain of definition.
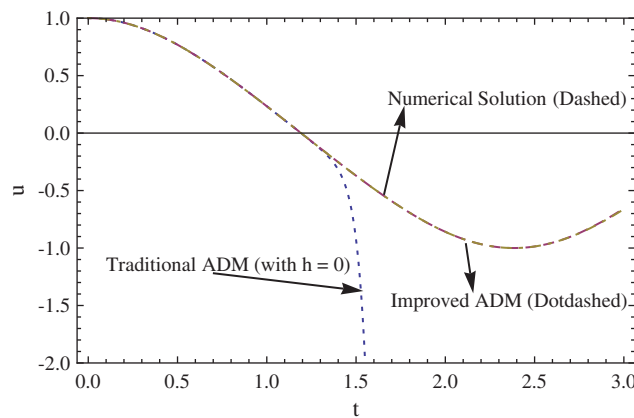

Figure 10: Convergence control parameter h regarding (35)

4.11 Diffusion Equation

Let us consider the nonlinear diffusion equation, see [18] and [3]

$$u_t + u + u^3 = (u^2 u_x)_x, \quad u(x,0) = e^{\frac{x}{\sqrt{3}}},$$
$$0 \le x \le 1, \quad 0 \le t \le 1, \tag{37}$$

for which [18] presents an exact solution.

The form of modified ADM for the partial differential equation (37) is

$$u_0(x,t) = e^{\frac{x}{\sqrt{3}}}(1 - ht),$$

$$u_1(x,t) = \frac{1}{2}e^{\frac{x}{\sqrt{3}}}t(-2 + h(2+t)),$$

$$u_{n+1}(x,t) = \int_0^t A_n(h,x,t)dt, \quad n \ge 1, \tag{38}$$

where

$$A_n(h,x,t)$$

are the Adomian polynomials corresponding to the function $-u - u^3 + (u^2 u_x)_x$.

Fig. 11 shows the constant h-level curves at the approximation level $M = 10$, indicating a very large range of embedding parameter h.
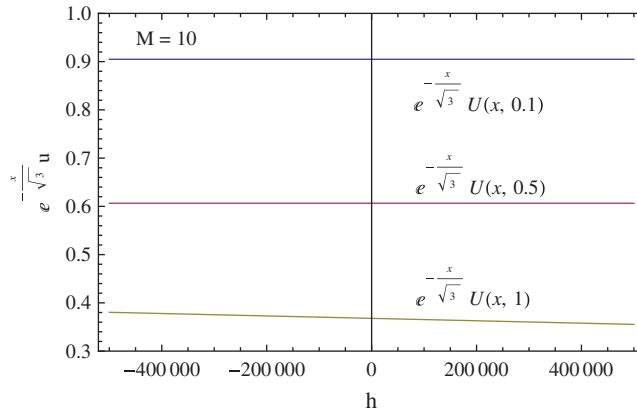

Figure 11: Convergence control parameter h regarding (37)

Actually at this truncation of the modified ADM series, it is obtained

$$u(x,t) = e^{\frac{x}{\sqrt{3}}}\left(-h\frac{t^{11}}{11!} + \sum_{n=0}^{10}\frac{(-1)^n t^n}{n!}\right). \tag{39}$$

Defining the squared residual error for (37) as

$$Res(h) = ||u_t + u + u^3 - (u^2 u_x)_x||^2 = \int_0^1 \int_0^1 (u_t + u + u^3 - (u^2 u_x)_x)^2 dxdt, \tag{40}$$

Tab. 11 tabulates how the modified ADM has smaller residual errors.

Table 11: The residual errors $\sqrt{Res(h)}$ for (37) from the classical and modified ADM methods

M	$\sqrt{Res(0)}$	$\sqrt{Res(h)}$	h
2	$3.067520559 \times 10^{-1}$	$1.126425884 \times 10^{-2}$	0.78155339805
4	$1.905329025 \times 10^{-2}$	$2.920732957 \times 10^{-4}$	0.84725848564
6	$5.284431922 \times 10^{-4}$	$4.431860700 \times 10^{-6}$	0.88282078470
8	$8.251967252 \times 10^{-6}$	$4.360598101 \times 10^{-8}$	0.90500266099
10	$8.249541273 \times 10^{-8}$	$2.997208214 \times 10^{-10}$	0.92013994524

4.12 Burger's Equation

The final example is known as Burger's equation [3]

$$u_t + uu_x = u_{xx}, \quad u(x,0) = 2x,$$
$$0 \le x \le 1, \quad 0 \le t \le 1, \tag{41}$$

with an exact solution $u(x,t) = \frac{2x}{1+2t}$.
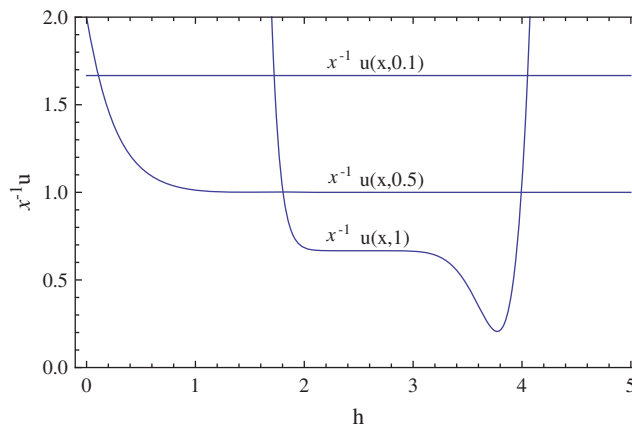

Figure 12: Convergence control parameter h regarding (41)

The form of modified ADM for the partial differential equation (40) is

$$u_0(x,t) = x(2 - ht),$$

$$u_1(x,t) = -\frac{1}{3}t(12 + h(-3 + t(-6 + ht)))x,$$

$$u_{n+1}(x,t) = \int_0^t A_n(h,x,t)dt, \quad n \ge 1, \tag{42}$$

where

$$A_n(h,x,t)$$

are the Adomian polynomials corresponding to the function $u_{xx} - uu_x$.

The traditional ADM with $h = 0$ in the domain $0 \leq x \leq 1$, $0 \leq t \leq 1$ is noticed to fail to produce a convergent series as also clear in Fig. 12, because $h = 0$ is not lying in the overlapping region.

With the definition

$$Res(h) = ||u_t + uu_x - u_{xx}||^2 = \int_0^1 \int_0^1 (u_t + uu_x - u_{xx})^2 dxdt, \tag{43}$$

Tab. 12 justifies the success of the present modified ADM over the classical divergent one, both in terms of accuracy and computational cost.

Table 12: The residual errors $\sqrt{Res(h)}$ for (41) from the classical and modified ADM methods

M	$\sqrt{Res(0)}$	CPU	$\sqrt{Res(h)}$ (h_{opt})	CPU
5	1.75×10^2	0.012	$4.674547981 \times 10^{-2}$ (2.2068296567)	0.039
10	1.67×10^5	0.053	$4.082713247 \times 10^{-3}$ (2.3147042279)	0.078
15	1.39×10^8	0.101	$6.755704561 \times 10^{-4}$ (2.3159080855)	0.145
20	1.24×10^{11}	0.242	$2.741368603 \times 10^{-5}$ (2.3820449852)	0.289

5 Concluding Remarks

The aim of the present work is to present superiority over the well-known Adomian decomposition method (ADM) often employed in the recent literature to analytically approximate solutions to highly nonlinear algebraic and differential equations of some real physical motions. Within this aim, a reformulation of the ADM is targeted to prevent first the failure and then convergence acceleration of the classical Adomian polynomials.

To accomplish the objective, the classical ADM is modified by inserting some simple parameterized terms into the early iterates involving an embedded parameter to control and pacing the convergence of the generated ADM series. In order to determine the best suitable value or the optimum value of this parameter, squared residual minimizing of the governing equation is proposed. This enables us to overcome the divergence of the classical ADM, and more importantly, there is no need to check out the results against the numerical ones, as usually has to be done in traditional ADM studies, since the optimum embedded parameter obtained is an insurance for ADM series convergence in a most rapid manner.

Physical examples selected from the recent application of ADM demonstrate the validity, accuracy and power of the present approach in terms of generating the convergent solution within the least number of iterations. In particular, the Duan-Rach modification of the ADM incorporating all the boundaries mostly used in the recent ADM applications takes great benefit of the present proposal, otherwise there is always the inevitable danger that it may lead to non physical solutions. The present approach successfully extends the convergence interval of the studied problem. In conclusion, the present formulation of ADM offers a promising tool to treat more strongly nonlinear equations/systems of real life phenomena.

References

1. Adomian, G., Rach, R. (1985). On the solution of algebraic equations by the decomposition method. *Journal of Mathematical Analysis and Applications, 105(1),* 141–166. DOI 10.1016/0022-247X(85)90102-7.
2. Adomian, G. (1988). A review of the decomposition method in applied mathematics. *Journal of Mathematical Analysis and Applications, 135(2),* 501–544. DOI 10.1016/0022-247X(88)90170-9.
3. Turkyilmazoglu, M. (2017). Parametrized Adomian decomposition method with optimum convergence. *ACM Transactions on Modeling and Computer Simulation, 27(4),* 1–22. DOI 10.1145/3106373.
4. Turkyilmazoglu, M. (2019). Accelerating the convergence of decomposition method of Adomian. *Journal of Computational Science, 31,* 54–59. DOI 10.1016/j.jocs.2018.12.014.
5. Duan, J. J., Rach, R., Baleanu, D., Wazwaz, A. M. (2012). A review of the Adomian decomposition method and its applications to fractional differential equations. *Commun in Fractional Calculus, 3(2),* 73–99.
6. Abbasbandy, S. (2003). Improving newton-raphson method for nonlinear equations by modified Adomian decomposition method. *Applied Mathematics and Computation, 145(2–3),* 887–893. DOI 10.1016/S0096-3003(03)00282-0.
7. Chun, C. (2006). A new iterative method for solving nonlinear equations. *Applied Mathematics and Computation, 178(2),* 415–422. DOI 10.1016/j.amc.2005.11.055.
8. Wazwaz, A. M. (2012). A reliable study for extensions of the bratu problem with boundary conditions. *Mathematical Methods in Applied Sciences, 35(7),* 845–856. DOI 10.1002/mma.1616.
9. Wazwaz, A. M., Rach, R., Duan, J. S. (2013). Adomian decomposition method for solving the volterra integral form of the lane-emden equations with initial values and boundary conditions. *Applied Mathematics and Computation, 219(10),* 5004–5019. DOI 10.1016/j.amc.2012.11.012.
10. Bhanja, D., Kundu, B., Mandal, P. K. (2013). Thermal analysis of porous pin fin used for electronic cooling. *Procedia Engineering, 64,* 956–965. DOI 10.1016/j.proeng.2013.09.172.
11. Duan, J. S., Rach, R., Wazwaz, A. M. (2013). Solution of the model of beam-type micro-and nano-scale electrostatic actuators by a new modified Adomian decomposition method for nonlinear boundary value problems. *International Journal of Non-Linear Mechanics, 49,* 159–169. DOI 10.1016/j.ijnonlinmec.2012.10.003.
12. Duan, J. S., Chaolu, T., Rach, R., Lu, L. (2013). The Adomian decomposition method with convergence acceleration techniques for nonlinear fractional differential equations. *Computers and Mathematics with Applications, 66(5),* 728–736. DOI 10.1016/j.camwa.2013.01.019.
13. Dib, A., Haiahem, A., Said, B. B. (2014). An analytical solution of the MHD jeffery-hamel flow by the modified Adomian decomposition method. *Computers & Fluids, 102,* 111–115. DOI 10.1016/j.compfluid.2014.06.026.
14. Dib, A., Haiahem, A., Said, B. B. (2015). Approximate analytical solution of squeezing unsteady nanofluid flow. *Powder Technology, 269,* 193–199. DOI 10.1016/j.powtec.2014.08.074.
15. Wazwaz, A. M. (2007). A comparison between the variational iteration method and Adomian decomposition method. *Journal of Computational and Applied Mathematics, 207(1),* 129–136. DOI 10.1016/j.cam.2006.07.018.
16. Olivares, A. G. (2003). Analytic solution of partial differential equations with Adomian's decomposition. *Kybernetes, 32(3–4),* 354–368. DOI 10.1108/03684920310458584.
17. Song, L., Wang, W. (2010). Approximate rational jacobi elliptic function solutions of the fractional differential equations via the enhanced Adomian decomposition method. *Physics Letters A, 374(31–32),* 3190–3196. DOI 10.1016/j.physleta.2010.05.057.
18. Song, L., Wang, W. (2013). A new improved Adomian decomposition method and its application to fractional differential equations. *Applied Mathematical Modelling, 37(3),* 1590–1598. DOI 10.1016/j.apm.2012.03.016.

19. Babolian, E., Biazar, J. (2002). On the order of convergence of Adomian method. *Applied Mathematics and Computation, 130(2),* 383–387. DOI 10.1016/S0096-3003(01)00103-5.

20. El-Kalla, I. L. (2008). Convergence of the Adomian method applied to a class of nonlinear integral equations. *Applied Mathematics Letters, 21(4),* 372–376. DOI 10.1016/j.aml.2007.05.008.

21. Abdelrazec, A., Pelinovsky, D. (2009). Convergence of the Adomian decomposition method for initial-value problems. *Numerical Methods for Partial Differential Equations, 27(4),* 749–766. DOI 10.1002/num.20549.

22. Turkyilmazoglu, M. (2016). Determination of the correct range of physical parameters in the approximate analytical solutions of nonlinear equations using the Adomian decomposition method. *Mediterranean Journal of Mathematics, 13(6),* 4019–4037. DOI 10.1007/s00009-016-0730-8.

23. Senturk, E., Coskun, S. B., Atay, M. T. (2018). Solution of jamming transition problem using Adomian decomposition method. *Engineering Computations, 35(5),* 1950–1964. DOI 10.1108/EC-12-2016-0437.

24. Zare, F., Heydari, M., Loghmani, G. B., Wazwaz, A. M. (2017). Numerical investigation of the beam-type nano-electrostatic actuator model by using the birkhoff interpolation method. *International Journal of Applied and Computational Mathematics, 3(4),* 129–146. DOI 10.1007/s40819-017-0417-2.

25. Heydari, M., Loghmani, G. B., Wazwaz, A. M. (2017). A numerical approach for a class of astrophysics equations using piecewise spectral-variational iteration method. *International Journal of Numerical Methods for Heat and Fluid Flow, 27(2),* 358–378. DOI 10.1108/HFF-09-2015-0379.

26. Wazwaz, A. M. (1999). A reliable modification of Adomian decomposition method. *Applied Mathematics and Computation, 102(1),* 77–86. DOI 10.1016/S0096-3003(98)10024-3.

27. Wazwaz, A. M., El-Sayed, S. M. (2001). A new modification of the Adomian decomposition method for linear and nonlinear operators. *Applied Mathematics and Computation, 122(3),* 393–405. DOI 10.1016/S0096-3003(00)00060-6.

28. Bakodah, H. O., Banaja, M. A., Alrigi, B. A., Ebaid, A., Rach, R. (2019). An efficient modification of the decomposition method with aconvergence parameter for solving korteweg de vries equations. *Journal of King Saud University of Science, 31(4),* 1424–1430. DOI 10.1016/j.jksus.2018.11.010.

29. Nuruddeen, R. I., Muhammad, L., Nass, A. M., Sulaiman, T. A. (2018). A review of the integral transforms-based decomposition methods and their applications in solving nonlinear pdes. *Palestine Journal of Mathematics, 7(1),* 262–280.

30. Hamoud, A. A., Ghadle, K. P. (2017). The combined modified laplace with Adomian decomposition method for solving the nonlinear volterra-fredholm integro-differential equations. *Journal of the Korean Society for Industrial and Applied Mathematics, 21,* 17–28. DOI 10.12941/jksiam.2017.21.017.

31. Hamoud, A. A., Ghadle, K. P. (2018). Modified Adomian decomposition method for solving fuzzy volterra-fredholm integral equations. *Journal of the Indian Mathematical Society, 85(1–2),* 52–69. DOI 10.18311/jims/2018/16260.

32. Rashidi, M. M., Domairry, G., Dinarvand, S. (2009). Approximate solutions for the burger and regularized long wave equations by means of the homotopy analysis method. *Communications in Nonlinear Science and Numerical Simulation, 14(3),* 708–717. DOI 10.1016/j.cnsns.2007.09.015.

33. Rashidi, M. M., Shahmohamadi, H. (2009). Analytical solution of three-dimensional navier stokes equations for the flow near an infinite rotating disk. *Communications in Nonlinear Science and Numerical Simulation, 14(7),* 2999–3006. DOI 10.1016/j.cnsns.2008.10.030.

34. Rashidi, M. M., Mohimanian, P. S. A. (2010). Analytic approximate solutions for unsteady boundary-layer flow and heat transfer due to a stretching sheet by homotopy analysis method. *Nonlinear Analysis: Modelling and Control, 15(1),* 83–95. DOI 10.15388/NA.2010.15.1.14366.

35. Rashidi, M. M., Freidoonimehr, N., Hosseini, A., Beg, O. A., Hung, T. K. (2014). Homotopy simulation of nanofluid dynamicsfrom a non-linearly stretching isothermal permeable sheet with transpiration. *Meccanica, 49(2),* 469–482. DOI 10.1007/s11012-013-9805-9.

36. Liao, S. J. (2014). *Advances in the homotopy analysis method.* Singapore: World Scientific. DOI 10.1142/8939.