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ARTICLE

Multi-Criteria Decision Making Based on Bipolar Picture Fuzzy Operators and New Distance Measures

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ABSTRACT

This paper aims to introduce the novel concept of the bipolar picture fuzzy set (BPFS) as a hybrid structure of bipolar fuzzy set (BFS) and picture fuzzy set (PFS). BPFS is a new kind of fuzzy sets to deal with bipolarity (both positive and negative aspects) to each membership degree (belonging-ness), neutral membership (not decided), and non-membership degree (refusal). In this article, some basic properties of bipolar picture fuzzy sets (BPFSs) and their fundamental operations are introduced. The score function, accuracy function and certainty function are suggested to discuss the comparability of bipolar picture fuzzy numbers (BPFNs). Additionally, the concept of new distance measures of BPFSs is presented to discuss geometrical properties of BPFSs. In the context of BPFSs, certain aggregation operators (AOs) named as "bipolar picture fuzzy weighted geometric (BPFWG) operator, bipolar picture fuzzy ordered weighted geometric (BPFOWG) operator and bipolar picture fuzzy hybrid geometric (BPFHG) operator" are defined for information aggregation of BPFNs. Based on the proposed AOs, a new multi-criteria decision-making (MCDM) approach is proposed to address uncertain real-life situations. Finally, a practical application of proposed methodology is also illustrated to discuss its feasibility and applicability.

KEYWORDS

Bipolar picture fuzzy set; aggregation operators; distance measures; pattern recognition; MCDM

1 Introduction

In any real-life problem-solving technique, the complexity characterizes the behavior of an object whose components interrelate in multiple ways and follow different logical rules, meaning there is no fixed rule to handle multiple challenges due to various uncertainties in real life circumstances. Many scholars from all over the world have apparently studied MCDM management techniques extensively. This effort resulted in a multitude of innovative solutions to complex real concerns. The frameworks for this objective are largely based on a summary of the issues at hand.

To deal with uncertainties the researchers have been proposed various mathematical techniques. Zadeh [1] initiated the idea of fuzzy set (FS) and membership degrees of objects/alternatives. Later, the intuitionistic fuzzy set (IFS) proposed by Atanassov [2] is the direct extension of FS by using membership degrees (MDs) and non-membership degrees (NMDs). Yager et al. [3,4] and Yager et al. [5] introduced Pythagorean fuzzy set and Pythagorean fuzzy membership grades. Zhang et al. [6,7] introduced an independent extension of fuzzy set named as bipolar fuzzy sets (BFSs) and Lee [8] presented some basics operations. A bipolar fuzzy information is used to express a property of an object as well as its counter property.

Alcantud et al. [9] initiated the notion of N-soft set approach to rough sets and introduced the concept of dual extended hesitant fuzzy sets [10]. Akram et al. [11,12] initiated MCDM based on Pythagorean fuzzy TOPSIS method and Pythagorean Dombi fuzzy AOs. Ashraf et al. [13] initiated spherical fuzzy Dombi AOs. Eraslan et al. [14] and Feng et al. [15] proposed new approaches for MCDM. Garg et al. [16–18] introduced some AOs on different sets also their applications to MCDM. Jose et al. [19] proposed AOs for MCDM. Karaaslan [20], Liu et al. [21], Liu et al. [22], Wang et al. [23], Yang et al. [24], Smarandache [25], and Liu et al. [26] initiated many different approaches including AOs on different extension of fuzzy set for MCDM. Naeem et al. [27,28], Peng et al. [29,30], Peng et al. [31] introduced some significant results for Pythagorean fuzzy sets.

Riaz et al. [32], initiated the concept of linear Diophantine fuzzy Set and its applications to MCDM. Riaz et al. [33] introduced some hybrid AOs, Einstein prioritized AOs [34], related to q-ROFSs. Riaz et al. [35] introduced cubic bipolar fuzzy set and related AOs. Cagman et al. [36], and Shabir et al. [37] independently introduced the notion of soft topological spaces.

Cuong [38] presented the idea of a picture fuzzy set (PFS) as a new paradigm distinguished with three functions that assign the positive membership degree (MD), the neutral MD and the negative membership degree (NMD) to each object/alternative. The basic restrictions on these degrees are that they lie in [0, 1] and their sum also lies in [0, 1]. Cuong [39] further introduced the concept of Pythagorean picture fuzzy sets and its basic notions. Garg [40], Jana et al. [41] and Wang et al. [42] proposed some AOs for picture fuzzy information aggregation. Pamucar [43] studied the notion of normalized weighted geometric Dombi Bonferoni mean operator with interval grey numbers: Application in multicriteria decision making. Pamucar et al. [44] proposed an application of the hybrid interval rough weighted Power-Heronian operator in multi-criteria decision making. Ramakrishnan et al. [45] introduced a cloud TOPSIS model for green supplier selection. Riaz et al. also introduced some AOs [46,47] related to green supplier selection. Si et al. [48] and Sinani [49] also presented different AOs in some extension of fuzzy set.

The first objective of this paper is to introduce bipolar picture fuzzy sets (BPFSs) as a new hybrid structure of bipolar fuzzy sets (BFSs) and picture fuzzy sets (PFSs). BPFSs are more efficient for dealing with the real-life situation when modeling needs to address the bipolarity (both positive and negative aspects) to each MD (belonging-ness), neutral membership (not decided), and non-membership degree (refusal). The second objective of BPFSs is propose bipolar picture fuzzy MCDM technique based on bipolar picture fuzzy AOs. The third objective of BPFSs is to define new distance measure and its application towards pattern recognition. Additionally, the proposed methodology can extend to solve various problems of artificial intelligence, computational intelligence and MCDM that involve bipolar picture fuzzy information.

The rest of the paper is as follows. The definitions of IFS, PFS and BFS are discussed in Section 2. Section 3 introduces the definition of BPFS. Section 4 indicates some bipolar picture AOs and new BPFS distance measures. Section 5 shows the generalizability of the suggested paradigm for pattern recognition. Section 6 introduces a new BPF-MCDM approach based on suggested AOs and a numerical example. Finally, Section 7 summarizes the findings of this research study.

2 Preliminaries

In this section, we give some basic definitions to IFSs, BFSs and PFSs.

Definition 2.1 [2] Let $\Psi = (\delta_1, \delta_2, \ldots \delta_n)$ be a crisp set, an IFS J in Ψ is defined by

$$J = \{\langle \delta, \mu_J(\delta), \nu_J(\delta)\rangle : \delta \in \Psi\} \tag{1}$$

where $0 \le \mu_J(\delta) \le 1$, $0 \le \nu_J(\delta) \le 1$ and $0 \le \mu_J(\delta) + \nu_J(\delta) \le 1$, $\forall \delta \in \Psi$. $\pi_J(\delta) = 1 - (\mu_J(\delta) + \nu_J(\delta))$ is called indeterminacy degree (ID) of J in Ψ. Also $0 \le \pi_J(\delta) \le 1$ $\forall \delta \in \Psi$.

Definition 2.2 [38] Let W be a crisp set, a PFS A in W is defined as follows:

$$A = \{\langle \delta, \mu_A(\delta), \lambda_A(\delta), \nu_A(\delta)\rangle \mid \delta \in W\} \tag{2}$$

where, $\mu_A(\delta) \in [0,1]$ is called positive MD of δ in A, $\lambda_A(\delta) \in [0,1]$ is called neutral MD of δ in A, $\nu_A(\delta) \in [0,1]$ is called negative MD of δ in A, and $\mu_A(\delta)$, $\lambda_A(\delta)$, $\nu_A(\delta)$ satisfy the condition $0 \le \mu_A(\delta) + \lambda_A(\delta) + \nu_A(\delta) \le 1$ $(\forall \delta \in W)$ and $1 - (\mu_A(\delta) + \lambda_A(\delta) + \nu_A(\delta))$ is called refusal MD of A.

A basic element $\langle \delta, \mu_A(\delta), \lambda_A(\delta), \nu_A(\delta)\rangle$ in a PFS A is denoted by $\tilde{\mathrm{A}} = \langle \mu_A, \lambda_A, \nu_A\rangle$, which is called picture fuzzy number (PFN).

Definition 2.3 [38] Some operational laws of picture fuzzy set as follows:

Let $P_1 = \{\langle \delta, \mu_{P_1}(\delta), \lambda_{P_1}(\delta), \nu_{P_1}(\delta)\rangle \mid \delta \in W\}$ and $P_2 = \{\langle \delta, \mu_{P_2}(\delta), \lambda_{P_2}(\delta), \nu_{P_2}(\delta)\rangle \mid \delta \in W\}$ be any two PFS. Then

1. $P_1 \subseteq P_2$ iff,

$$\mu_{P_1}(\delta) \le \mu_{P_2}(\delta), \lambda_{P_1}(\delta) \le \lambda_{P_2}(\delta), \nu_{P_1}(\delta) \ge \nu_{P_2}(\delta).$$

2. $P_1 = P_2$ iff,

$$\mu_{P_1}(\delta) = \mu_{P_2}(\delta), \lambda_{P_1}(\delta) = \lambda_{P_2}(\delta), \nu_{P_1}(\delta) = \nu_{P_2}(\delta).$$

3. The complement of P_1 is defined by

$$P_1^c = \{(\nu_{P_1}(\delta), \lambda_{P_1}(\delta), \mu_{P_1}(\delta)) \mid \delta \in X\}.$$

4. The union is defined by

$$P_1 \cup P_2 = \{(\delta, \max(\mu_{P_1}(\delta), \mu_{P_2}(\delta)), \min(\lambda_{P_1}(\delta), \lambda_{P_2}(\delta)), \min(\nu_{P_1}(\delta), \nu_{P_2}(\delta))) : \delta \in W\}.$$

5. The intersection is defined by

$$P_1 \cap P_2 = \{(\delta, \min(\mu_{P_1}(\delta), \mu_{P_2}(\delta)), \max(\lambda_{P_1}(\delta), \lambda_{P_2}(\delta)), \max(\nu_{P_1}(\delta), \nu_{P_2}(\delta))) : \delta \in W\}.$$

Definition 2.4 [40] The score function of a PFN $\delta = \langle \mu_A, \lambda_A, \nu_A\rangle$ is defined as

$$\Re(P) = \mu_A - \lambda_A - \nu_A$$

However, in certain circumstances, the previous score function may not rank any two PFNs. For example, $P_1 = (0.7, 0.2, 0.1)$ and $P_2 = (0.6, 0.1, 0.1)$ then they have same score function values, i.e., $R(P_1) = R(P_2)$. For this we use accuracy function given as

$$\Im(P) = \mu_A + \lambda_A + \nu_A$$

Let $P_1 = \langle \mu_1, \lambda_1, \nu_1 \rangle$ and $P_2 = \langle \mu_2, \lambda_2, \nu_2 \rangle$ are any two PFNs, and $R(P_1)$, $R(P_2)$ are the score function of P_1 and P_2, and $\Im(P_1)$, $\Im(P_2)$ are the accuracy function of P_1 and P_2, respectively, then

(1) If $R(P_1) > R(P_2)$, then $P_1 > P_2$.
(2) If $R(P_1) = R(P_2)$, then,

If $\Im(P_1) > \Im(P_2)$ then $P_1 > P_2$,

If $\Im(P_1) = \Im(P_2)$, then $P_1 = P_2$.

Definition 2.5 [6] Let X be a set, a BFS B in$X$ is defined as follows:

$$B = \{\langle \delta, \mu_B^+(\delta), \mu_B^-(\delta) \rangle \mid \delta \in X\} \tag{3}$$

where $\mu_B^+(\delta)\colon X \to [0,1]$ and $\mu_B^-(\delta)\colon X \to [-1,0]$. The positive MD $\mu_B^+(\delta)$ demonstrates the satisfaction degree of an element δ to the property corresponding to a BFS B, negative MD $\mu_B^-(\delta)$ denotes the satisfaction degree of an element δ to some implicit counter-property of B.

Definition 2.6 [6,7] Some operational laws of bipolar fuzzy set as follows:

Let $B = \{\langle \delta, \mu_B^+(\delta), \mu_B^-(\delta) \rangle \mid \delta \in W\}$, $B_1 = \{\langle \delta, \mu_{B_1}^+(\delta), \mu_{B_1}^-(\delta) \rangle \mid \delta \in W\}$ and $B_2 = \{\langle \delta, \mu_{B_2}^+(\delta), \mu_{B_2}^-(\delta) \rangle \mid \delta \in W\}$ be three bipolar fuzzy sets. Then

1. $B_1 \subseteq B_2$ iff,

$$\mu_{B_1}^+(\delta) \le \mu_{B_2}^+(\delta) \text{ and } \mu_{B_1}^-(\delta) \ge \mu_{B_2}^-(\delta)$$

2. $B_1 = B_2$ iff,

$$\mu_{B_1}^+(\delta) = \mu_{B_2}^+(\delta) \text{ and } \mu_{B_1}^-(\delta) = \mu_{B_2}^-(\delta).$$

3. The complement of B_1 is denoted by B_1^c,

$$B_1^c = \{(1 - \mu_{B_1}^+(\delta), -1 - \mu_{B_1}^+(\delta) \mid \delta \in W\}.$$

4. The union is defined by

$$B_1 \cup B_2 = \{(\delta, \max(\mu_{B_1}^+(\delta), \mu_{B_2}^+(\delta)), \min(\mu_{B_1}^+(\delta), \mu_{B_2}^+(\delta)) \text{ for all } \delta \in W\}.$$

5. The intersection is defined by

$$B_1 \cap B_2 = \{(\delta, \min(\mu_{B_1}^+(\delta), \mu_{B_2}^+(\delta)), \max(\mu_{B_1}^+(\delta), \mu_{B_2}^+(\delta)) \text{ for all } \delta \in W\}.$$

6. α-cut (B_α) of B,

$$B_\alpha = B_\alpha^+ \cup B_\alpha^-$$

$$B_\alpha^+ = \{\delta \mid \mu_B^+(\delta) \ge \alpha\}$$

$$B_\alpha^- = \{\delta \mid \mu_B^-(\delta) \le -\alpha\}$$

Here, B_α^+ is called positive α-cut and B_α^- is called negative α-cut.

7. Support (shortly, $\mathrm{Supp}(B)$) of B,

$\mathrm{Supp}(B) = \mathrm{Supp}(B)^+ \cup \mathrm{Supp}(B)^-$

$\mathrm{Supp}(B)^+ = \{\delta \mid \mu_B^+(\delta) > 0\}$

$\mathrm{Supp}(B)^- = \{\delta \mid \mu_B^-(\delta) < 0\}$

Here, $\mathrm{Supp}(B)^+$ is called positive α-cut and $\mathrm{Supp}(B)^-$ is called negative α-cut.

3 Bipolar Picture Fuzzy Set

The BFS assign positive and negative grades to the alternatives and PFS is characterized by three functions expressing the MD, the neutral MD and the NMD. Fuzzy set assign a membership grade to each alternatives δ in the unit closed interval $[0,1]$. In BFS the positive MD $\mu_\lambda^+(\delta)$ represent the satisfaction degree of an alternative δ to the property corresponding to a BFS λ, and negative MD $\mu_\lambda^-(\delta)$ represent the satisfaction degree of an element δ to some implicit counter-property of λ. In PFS there are three types of grades, $\mu_A(\delta) \in [0,1]$ is called positive MD of δ in A, $\lambda_A(\delta) \in [0,1]$ is called neutral MD of δ in A, $\nu_A(\delta) \in [0,1]$ is called negative MD of δ in A, and where μ_A, λ_A, ν_A satisfy the condition $0 \leq \mu_A(\delta) + \lambda_A(\delta) + \nu_A(\delta) \leq 1$, $\forall \delta \in X$.

We present the idea of BPFS as a new hybrid version of BFS and PFS. In this model of BPFS, we assign positive and negative grades for each MD (belonging-ness), neutral membership (not decided), and non-membership degree (refusal). We present specific examples to relate the proposed model with the real life applications. We define some operational laws of BPFS along with its score and accuracy functions.

Definition 3.1 A BPFS Ω on universe W is an abject of the form

$$\Omega = \{\langle \delta, \mu_\Omega^+(\delta), \lambda_\Omega^+(\delta), \nu_\Omega^+(\delta), \mu_\Omega^-(\delta), \lambda_\Omega^-(\delta), \nu_\Omega^-(\delta)\rangle : \delta \in W\} \tag{4}$$

where $\mu^+, \lambda^+, \nu^+ : W \to [0,1]$ and $\mu^-, \lambda^-, \nu^- : W \to [-1,0]$ with conditions

$$0 \leq \mu_\Omega^+ + \lambda_\Omega^+ + \nu_\Omega^+ \leq 1$$

$$-1 \leq \mu_\Omega^- + \lambda_\Omega^- + \nu_\Omega^- \leq 0$$

$$0 \leq \mu_\Omega^+ + \lambda_\Omega^+ + \nu_\Omega^+ - \mu_\Omega^- - \lambda_\Omega^- - \nu_\Omega^- \leq 2.$$

The positive MDs $\mu_\Omega^+(\delta)$, $\lambda_\Omega^+(\delta)$, $\nu_\Omega^+(\delta)$ demonstrate the truth MD, indeterminate MD and false MD of an element δ corresponding to a BPFS Ω and the negative MDs $\mu_\Omega^-(\delta)$, $\lambda_\Omega^-(\delta)$, $\nu_\Omega^-(\delta)$ demonstrate the truth MD, indeterminate MD and false MD of an element δ to some implicit counter-property corresponding to a BPFS Ω. Absolute BPFS assign $(1,0,0,-1,0,0)$ to each alternative, denoted by $\mathfrak{U}$ and null BPFS assign $(0,0,1,0,0,-1)$ to each alternative, denoted by $\mathfrak{N}$. Moreover $\rho_\Omega^+ = 1 - (\mu_\Omega^+ + \lambda_\Omega^+ + \nu_\Omega^+)$ is called the positive degree of refusal membership of δ in Ω and $\rho_\Omega^- = -1 - (\mu_\Omega^- + \lambda_\Omega^- + \nu_\Omega^-)$ is called the negative degree of refusal membership of δ in Ω.

Now we discuss some applications of proposed model to relate it with real life problems.

Business:

In the field of finance and business, we use two terms profit and loss. We can relate the decision-making applications based on business with BPFS. If a person invests some money, then he wants to earn max profit in some interval of time. Bipolar picture fuzzy number (BPFN) can be described as

$\langle \mu^+, \lambda^+, \nu^+, \mu^-, \lambda^-, \nu^- \rangle$.

The physical meaning of this structure in business terms is that, what is the satisfactions grade that he earns profit (μ^+), what is the dissatisfactions grade that he earns profit (ν^+), what is the abstinence grade that he earns profit (μ^+), what is the satisfactions grade that he gets loss (μ^-), what is the dissatisfactions grade that he gets loss (ν^-) and what is the abstinence grade that he gets loss (λ^-). We can see how BPFN is useful for the decision-making problems of real life problems. The detail of component of BPFN for finance and business is given in Tab. 1.

Table 1: Tabular representation of BPFN under business related problems

	Ω
μ^+	Satisfactions grade that he earns profit
λ^+	Abstinence grade that he earns profit
ν^+	Dissatisfactions grade that he earns profit
μ^-	Satisfactions grade that he gets loss
λ^-	Abstinence grade that he gets loss
ν^-	Dissatisfactions grade that he gets loss

Medication:

In the field of medical, we mostly focus on the effects and side effects of medicines related to every disease. If a patient get some medication according to his type of disease, then we can relate our model to the effects and side effects of that medicine in medical diagnosis, treatment and recovery terms. For the BPFN can be written as

$\langle \mu^+, \lambda^+, \nu^+, \mu^-, \lambda^-, \nu^- \rangle$.

μ^+ represents the positive effects of recommended medicine to the disease of the patient, ν^+ represents the dissatisfaction effects of recommended medicine, λ^+ represents the abstinence effects of recommended medicine, μ^- represents the negative or bad effects of recommended medicine, ν^- represents the dissatisfaction grade of bad effects of recommended medicine and λ^- represents the abstinence grades of side effects of recommended medicine. The detail of component of BPFN for medication is given in Tab. 2.

Table 2: Tabular representation of BPFS under medical related problems

	Ω
μ^+	Positive effects of recommended medicine
λ^+	Abstinence effects of recommended medicine
ν^+	Dissatisfaction effects of recommended medicine
μ^-	Negative or bad effects of recommended medicine
λ^-	Abstinence grades of side effects of recommended medicine
ν^-	Dissatisfaction grade of bad effects of recommended medicine

The proposed model is superior than these two models, in fact it is hybrid structure of BFS and PFS that assign six grades to the alternative.

Comparison Analysis:

In this part, we discuss about the terms and characteristics of proposed model and compare it with the existing techniques. There are various objectives to construct this hybrid structure and some of them are listed below:

1. The first objective to construct this hybrid model is to fill the research gap which exists in previous methodologies. The bipolar fuzzy set and picture fuzzy set can be used together in decision analysis. We can deal with the satisfaction, abstinence and dissatisfaction grades of the alternatives with its counter properties.
2. The second objective is that we can cover the evaluation space in a different manner. If we compare our model with the existing theories then we find that it is strong, valid and superior to others. The comparison analysis of BPFS with the existing models is given in Tab. 3.
3. The third objective is to represent the relationship of BPFS to the MCDM problems. We study some real life problems and convert the input data into BPF numeric values and deal it with the proposed aggregation operators. This novel structure is superior, flexible and easy to handle and can deal with the MCDM problems in the field of medical, business, artificial intelligence and engineering etc. The graphical representation of PFS and BPFS is given in the Figs. 1 and 2, respectively.
4. In bipolar neutrosophic set (see [50]) the conditions are as follows:

$$0 \leq T^+ + I^+ + F^+ \leq 3$$

$$0 \leq -T^- - I^- - F^- \leq 3$$

$$0 \leq T^+ + I^+ + F^+ - T^- - I^- - F^- \leq 6$$

However, in the proposed model BPFS the conditions are as follows:

$$0 \leq \mu_\Omega^+ + \lambda_\Omega^+ + \nu_\Omega^+ \leq 1$$

$$-1 \leq \mu_\Omega^- + \lambda_\Omega^- + \nu_\Omega^- \leq 0$$

$$0 \leq \mu_\Omega^+ + \lambda_\Omega^+ + \nu_\Omega^+ - \mu_\Omega^- - \lambda_\Omega^- - \nu_\Omega^- \leq 2$$

Table 3: Comparison of BPFS with the existing set theoretic models

Set theoretic models	Satisfaction grade (MD)	Abstinence grade (Neutral MD)	Dissatisfaction grade (NMD)	Bipolarity
Fuzzy set [1]	✓	×	×	×
IFS [2]	✓	×	✓	×
BFS [6]	✓	×	×	✓
PFS [38]	✓	✓	✓	×
Proposed BPFS	✓	✓	✓	✓

Definition 3.2 Let $\Omega_1 = \langle \delta, \mu_1^+(\delta), \lambda_1^+(\delta), \nu_1^+(\delta), \mu_1^-(\delta), \lambda_1^-(\delta), \nu_1^-(\delta) \rangle$ and $\Omega_2 = \langle \delta, \mu_2^+(\delta), \lambda_2^+(\delta), \nu_2^+(\delta), \mu_2^-(\delta), \lambda_2^-(\delta), \nu_2^-(\delta) \rangle$ be two BPFSs. Then $\Omega_1 \subseteq \Omega_2$ iff

$$\mu_1^+(\delta) \leq \mu_2^+(\delta), \lambda_1^+(\delta) \leq \lambda_2^+(\delta), \nu_1^+(\delta) \geq \nu_2^+(\delta)$$

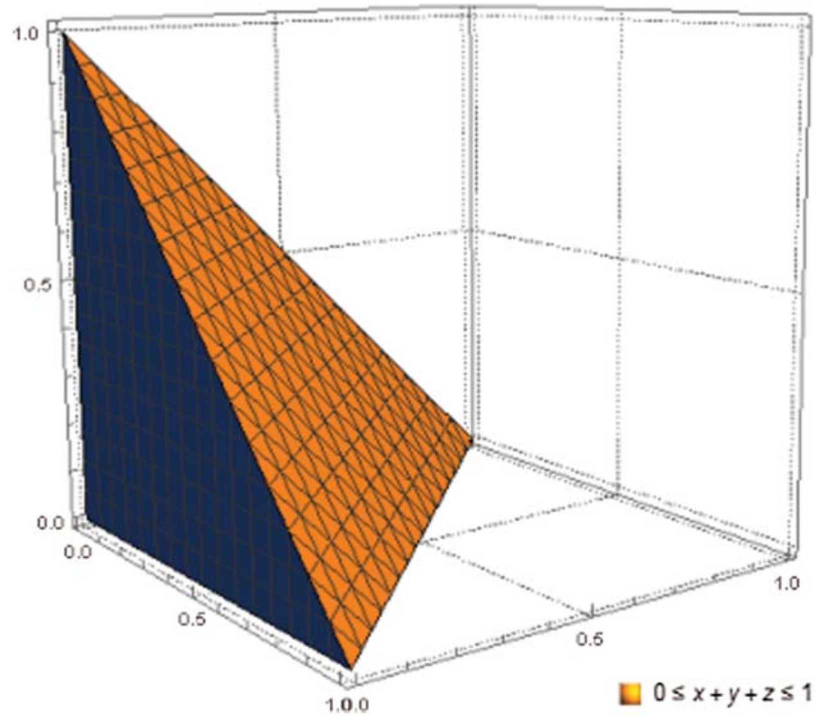

Figure 1: Graphical representation for satisfaction, abstinence and dissatisfaction grades of picture fuzzy set. $0 \leq x + y + z \leq 1$

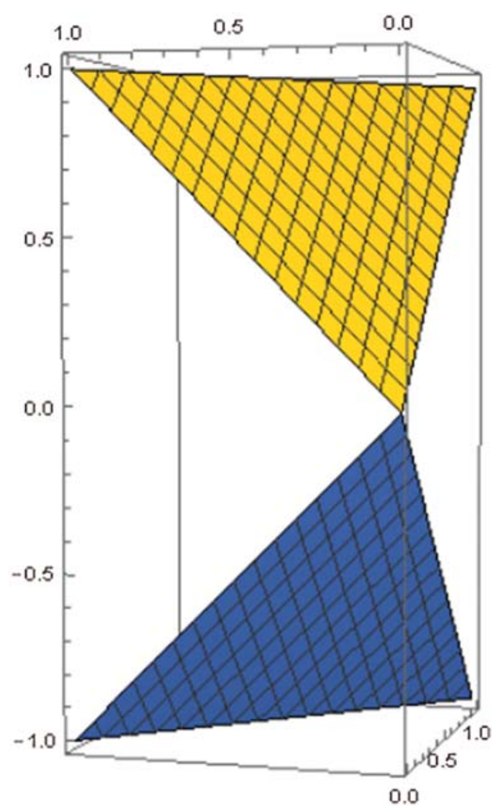

Figure 2: Graphical representation for grades of bipolar picture fuzzy set

and

$\mu_1^-(\delta) \geq \mu_2^-(\delta),\ \lambda_1^-(\delta) \geq \lambda_2^-(\delta),\ \nu_1^-(\delta) \leq \nu_2^-(\delta)$

Definition 3.3 Let $\Omega_1 = \langle \delta, \mu_1^+(\delta), \lambda_1^+(\delta), \nu_1^+(\delta), \mu_1^-(\delta), \lambda_1^-(\delta), \nu_1^-(\delta) \rangle$ *and* $\Omega_2 = \langle \delta, \mu_2^+(\delta), \lambda_2^+(\delta), \nu_2^+(\delta), \mu_2^-(\delta), \lambda_2^-(\delta), \nu_2^-(\delta) \rangle$ be two BPFSs. Then $\Omega_1 = \Omega_2$ iff

$\mu_1^+(\delta) = \mu_2^+(\delta),\ \lambda_1^+(\delta) = \lambda_2^+(\delta),\ \nu_1^+(\delta) = \nu_2^+(\delta)$

and

$\mu_1^-(\delta) = \mu_2^-(\delta),\ \lambda_1^-(\delta) = \lambda_2^-(\delta),\ \nu_1^-(\delta) = \nu_2^-(\delta)$.

Definition 3.4 Let $\Omega_1 = \langle \delta, \mu_1^+(\delta), \lambda_1^+(\delta), \nu_1^+(\delta), \mu_1^-(\delta), \lambda_1^-(\delta), \nu_1^-(\delta) \rangle$ and $\Omega_2 = \langle \delta, \mu_2^+(\delta), \lambda_2^+(\delta), \nu_2^+(\delta), \mu_2^-(\delta), \lambda_2^-(\delta), \nu_2^-(\delta) \rangle$ be two BPFSs.

The union of these two *BPFSs* is defined as

$$(\Omega_1 \cup \Omega_2)(\delta) = (\max(\mu_1^+(\delta), \mu_2^+(\delta)), \min(\lambda_1^+(\delta), \lambda_2^+(\delta)), \min(\nu_1^+(\delta), \nu_2^+(\delta)),$$
$$\min(\mu_1^-(\delta), \mu_2^-(\delta)), ma\delta(\lambda_1^-(\delta), \lambda_2^-(\delta)), ma\delta(\nu_1^-(\delta), \nu_2^-(\delta))).$$

Example 3.5 Let $X = \{\delta_1, \delta_2, \delta_3\}$. Let us consider are two BPFSs Ω_1, Ω_2 in X given by

$$\Omega_1 = \langle \delta_1, 0.5, 0.2, 0.2, -0.1, -0.2, -0.4 \rangle, \langle \delta_2, 0.3, 0.4, 0.3, -0.35, -0.3, -0.3 \rangle,$$
$$\langle \delta_3, 0.3, 0.4, 0.2, -0.5, -0.1, -0.2 \rangle$$
$$\Omega_2 = \langle \delta_1, 0.4, 0.3, 0.2, -0.2, -0.4, -0.2 \rangle, \langle \delta_2, 0.5, 0.1, 0.3, -0.3, -0.5, -0.2 \rangle,$$
$$\langle \delta_3, 0.3, 0.4, 0.1, -0.3, -0.2, -0.4 \rangle$$

Then their union is

$$\Omega_1 \cup \Omega_2 = \langle \delta_1, 0.5, 0.2, 0.2, -0.2, -0.2, -0.2 \rangle, \langle \delta_2, 0.5, 0.1, 0.3, -0.3, -0.3, -0.2 \rangle,$$
$$\langle \delta_3, 0.3, 0.4, 0.1, -0.5, -0.1, -0.2 \rangle$$

Definition 3.6 *Let* $\Omega_1 = \langle \delta, \mu_1^+(\delta), \lambda_1^+(\delta), \nu_1^+(\delta), \mu_1^-(\delta), \lambda_1^-(\delta), \nu_1^-(\delta) \rangle$ and $\Omega_2 = \langle \delta, \mu_2^+(\delta), \lambda_2^+(\delta), \nu_2^+(\delta), \mu_2^-(\delta), \lambda_2^-(\delta), \nu_2^-(\delta) \rangle$ be two BPFSs.

The intersection of these two BPFSs is defined as

$$(\Omega_1 \cap \Omega_2)(\delta) = (\min(\mu_1^+(\delta), \mu_2^+(\delta)), ma\delta(\lambda_1^+(\delta), \lambda_2^+(\delta)), ma\delta(\nu_1^+(\delta), \nu_2^+(\delta)),$$
$$\max(\mu_1^-(\delta), \mu_2^-(\delta)), \min(\lambda_1^-(\delta), \lambda_2^-(\delta)), \min(\nu_1^-(\delta), \nu_2^-(\delta))).$$

Example 3.7 Let $W = \{\delta_1, \delta_2, \delta_3\}$. Let us consider are two bipolar picture fuzzy set Ω_1, Ω_2 in W given by

$$\Omega_1 = \langle \delta_1, 0.5, 0.2, 0.2, -0.1, -0.2, -0.4 \rangle, \langle \delta_2, 0.3, 0.4, 0.3, -0.35, -0.3, -0.3 \rangle,$$
$$\langle \delta_3, 0.3, 0.4, 0.2, -0.5, -0.1, -0.2 \rangle$$
$$\Omega_2 = \langle \delta_1, 0.4, 0.3, 0.2, -0.2, -0.4, -0.2 \rangle, \langle \delta_2, 0.5, 0.1, 0.3, -0.3, -0.5, -0.2 \rangle,$$
$$\langle \delta_3, 0.3, 0.4, 0.1, -0.3, -0.2, -0.4 \rangle.$$

Then their intersection is

$$\Omega_1 \cap \Omega_2 = \langle \delta_1, 0.4, 0.3, 0.2, -0.1, -0.4, -0.4 \rangle, \langle \delta_2, 0.3, 0.4, 0.3, -0.3, -0.5, -0.3 \rangle,$$
$$\langle \delta_3, 0.3, 0.4, 0.2, -0.3, -0.2, -0.4 \rangle$$

Definition 3.8 Let $\Omega = \langle \delta, \mu^+(\delta), \lambda^+(\delta), \nu^+(\delta), \mu^-(\delta), \lambda^-(\delta), \nu^-(\delta) \rangle$ be a BPF set in W. Then the compliment of Ω is denoted by Ω^c and defined as, for all $\delta \in W$,

$$\Omega = \langle \delta, \nu^+(\delta), \lambda^+(\delta), \mu^+(\delta), \nu^-(\delta), \lambda^-(\delta), \mu^-(\delta) \rangle$$

Example 3.9 Let $W = \{\delta_1, \delta_2, \delta_3\}$. Consider a BPFS Ω in W given by

$$\Omega = \langle \delta_1, 0.3, 0.2, 0.4, -0.1, -0.3, -0.4 \rangle, \langle \delta_2, 0.3, 0.4, 0.3, -0.2, -0.2, -0.4 \rangle,$$
$$\langle \delta_3, 0.3, 0.4, 0.1, -0.2, -0.1, -0.4 \rangle$$

Then its complement is

$$\Omega^c = \langle \delta_1, 0.4, 0.2, 0.3, -0.4, -0.3, -0.1 \rangle, \langle \delta_2, 0.3, 0.4, 0.3, -0.4, -0.2, -0.2 \rangle,$$
$$\langle \delta_3, 0.1, 0.4, 0.3, -0.4, -0.1, -0.2 \rangle$$

Now we see that BFS and PFS are special cases of BPFS.

Proposition 3.10 BFS and PFS are special cases of BPFS, i.e., Bipolar fuzzy numbers (BFNs) and picture fuzzy numbers (PFNs) are special cases of the bipolar picture fuzzy numbers (BPFNs).

Proof. For any $\delta \in X$, consider a BPFN given by, $\langle \mu^+(\delta), \lambda^+(\delta), \nu^+(\delta), \mu^-(\delta), \lambda^-(\delta), \nu^-(\delta) \rangle$. Then by setting the components $\lambda^+(\delta)$, $\nu^+(\delta)$, $\lambda^-(\delta)$, $\nu^-(\delta)$ equals to zero, we obtain a BFN, $\langle \mu^+(\delta), \mu^-(\delta) \rangle$.

Similarly, by setting the components $\mu^-(\delta)$, $\lambda^-(\delta)$, $\nu^-(\delta)$ equal to zero, we obtain we obtain a PFN, $\langle \mu^+(\delta), \lambda^+(\delta), \nu^+(\delta) \rangle$ which can be written as, $\langle \mu(\delta), \lambda(\delta), \nu(\delta) \rangle$. This complete the proof.

Theorem 3.11 Let Ω_1, Ω_2 and Ω_3 be the BPFSs in a universe X, then we have

1. $\Omega_1 \cup \Omega_1 = \Omega_1$ and $\Omega_1 \cap \Omega_1 = \Omega_1$
2. $\Omega_1 \cup \Omega_2 = \Omega_2 \cup \Omega_1$ and $\Omega_1 \cap \Omega_2 = \Omega_2 \cap \Omega_1$
3. $(\Omega_1^c)^c = \Omega_1$
4. $(\Omega_1 \cup \Omega_2) \cup \Omega_3 = \Omega_1 \cup (\Omega_2 \cup \Omega_3)$
5. $(\Omega_1 \cap \Omega_2) \cap \Omega_3 = \Omega_1 \cap (\Omega_2 \cap \Omega_3)$
6. $\Omega_1 \cup (\Omega_1 \cap \Omega_2) = \Omega_1$
7. $\Omega_1 \cap (\Omega_1 \cup \Omega_2) = \Omega_1$
8. $\Omega_1 \cup (\Omega_2 \cap \Omega_3) = (\Omega_1 \cup \Omega_2) \cap (\Omega_1 \cup \Omega_3)$
9. $\Omega_1 \cap (\Omega_2 \cup \Omega_3) = (\Omega_1 \cap \Omega_2) \cup (\Omega_1 \cap \Omega_3)$
10. $\Omega_1 \cup \Omega_1^c \neq \mathfrak{U}$ and $\Omega_1 \cap \Omega_1^c \neq \mathfrak{N}$

Proof. The proof is obvious.

Theorem 3.12 Let O and P be the BPFSs in a universe X, then we have

1. $(O \cup P)^c \neq O^c \cap P^c$
2. $(O \cap P)^c \neq O^c \cup P^c$

We will denote the set of all *BPFSs* in X by $\mathfrak{X}$.

Definition 3.13 Let $\phi_1 = \langle \mu_1^+, \lambda_1^+, \nu_1^+, \mu_1^-, \lambda_1^-, \nu_1^- \rangle$ and $\phi_2 = \langle \mu_2^+, \lambda_2^+, \nu_2^+, \mu_2^-, \lambda_2^-, \nu_2^- \rangle$ be two BPFNs then

1. $\phi_1 \vee \phi_2 = (\max(\mu_1^+, \mu_2^+), \min(\lambda_1^+, \lambda_2^+), \min(\nu_1^+, \nu_2^+), \min(\mu_1^-, \mu_2^-), \max(\lambda_1^-, \lambda_2^-), ma\delta(\nu_1^-, \nu_2^-))$
2. $\phi_1 \wedge \phi_2 = (\min(\mu_1^+, \mu_2^+), \max(\lambda_1^+, \lambda_2^+), \max(\nu_1^+, \nu_2^+), \max(\mu_1^-, \mu_2^-), \min(\lambda_1^-, \lambda_2^-), \min(\nu_1^-, \nu_2^-))$
3. $\phi_1^c = \langle \nu_1^+, \lambda_1^+, \mu_1^+, \nu_1^-, \lambda_1^-, \mu_1^- \rangle$
4. $\phi_1^\lambda = ((\mu_1^+ + \lambda_1^+)^\lambda - (\lambda_1^+)^\lambda, (\lambda_1^+)^\lambda, 1 - (1 - \nu_1^+)^\lambda, ((-\mu_1^-) + (-\lambda_1^-))^\lambda - (-\lambda_1^-)^\lambda, (-\lambda_1^-)^\lambda, 1 - (1 - (-\nu_1^-))^\lambda)$
5. $\phi_1 \otimes \phi_2 = ((\mu_1^+ + \lambda_1^+)(\mu_2^+ + \lambda_2^+) - \lambda_1^+\lambda_2^+, \lambda_1^+\lambda_2^+, 1 - (1 - \nu_1^+)(1 - \nu_2^+), ((-\mu_1^-) + (-\lambda_1^-))((-\mu_2^-) + (-\lambda_2^-)) - (-\lambda_1^-)(-\lambda_2^-), (-\lambda_1^-)(-\lambda_2^-), 1 - (1 - (-\nu_1^-))(1 - (-\nu_2^-))$

4 Some Bipolar Picture Fuzzy Geometric Operators

In this section, firstly, we introduce score function, accuracy function, and certainty function for BPFNs. Secondly, we introduce BPFWG operator, BPFOWG operator, and BPFHG operator.

Definition 4.1 Let $\mathcal{T}_1 = \langle \mu_1^+, \lambda_1^+, \nu_1^+, \mu_1^-, \lambda_1^-, \nu_1^- \rangle$ be (*BPFN*). Then the score function $\Phi(\mathcal{T}_1)$, accuracy function $\Upsilon(\mathcal{T}_1)$ and certainty fruition $\Pi(\mathcal{T}_1)$ of *BPFN* are defined as follows:

1. $\Phi(\mathcal{T}_1) = (\mu_1^+ - \lambda_1^+ - \nu_1^+ + \mu_1^- - \lambda_1^- - \nu_1^-)/2$
2. $\Upsilon(\mathcal{T}_1) = (\mu_1^+ + \nu_1^+ - \mu_1^- - \nu_1^-)/2$
3. $\Pi(\mathcal{T}_1) = (\mu_1^+ - \nu_1^-)$

The range of score function $\Phi(\mathcal{T})$ is $[-1, 1]$, range of accuracy function $\Upsilon(\mathcal{T})$ is $[0, 1]$ and range of certainty fruition $\Pi(\mathcal{T})$ of *BPFN* is $[0, 1]$.

Definition 4.2 Let $T_1 = \langle \mu_1^+, \lambda_1^+, \nu_1^+, \mu_1^-, \lambda_1^-, \nu_1^- \rangle$ and $T_2 = \langle \mu_2^+, \lambda_2^+, \nu_2^+, \mu_2^-, \lambda_2^-, \nu_2^- \rangle$ be two (BPFN). The comparison method can be defined as follows:

1. if $\Phi(T_1) > \Phi(T_2)$, then T_1 greater than T_2 and denoted by $T_1 > T_2$.
2. if $\Phi(T_1) = \Phi(T_2)$ and $\Upsilon(T_1) > \Upsilon(T_2)$, then T_1 greater than T_2 and denoted by $T_1 > T_2$.
3. if $\Phi(T_1) = \Phi(T_2)$, $\Upsilon(T_1) = \Upsilon(T_2)$ and $\Pi(T_1) > \Pi(T_2)$ then T_1 greater than T_2 and denoted by $T_1 > T_2$.
4. if $\Phi(T_1) = \Phi(T_2)$, $\Upsilon(T_1) = \Upsilon(T_2)$ and $\Pi(T_1) = \Pi(T_2)$ then T_1 equal to T_2 and denoted by $T_1 = T_2$.

Definition 4.3 *Let* $T_j = \langle \mu_j^+, \lambda_j^+, \nu_j^+, \mu_j^-, \lambda_j^-, \nu_j^- \rangle$ $(j = 1, 2, \ldots, n)$ be an assemblage of BPFNs. A mapping *BPFWG*: $\mathfrak{X}_n \to \mathfrak{X}$ is called a bipolar picture fuzzy weighted geometric (BPFWG) operator.

$$BPFWG\,(T_1, T_2, \ldots, T_n) = \sum_{j=1}^{n} T_j^{P_j} = T_1^{P_1} \otimes T_2^{P_2}, \ldots \otimes, T_n^{P_n}$$

where P_j is the weight vector (WV) of T_j, $P_j \in [0, 1]$ and $\sum_{j=1}^n P_j = 1$.

Theorem 4.4 Let $T_j = \langle \mu_j^+, \lambda_j^+, \nu_j^+, \mu_j^-, \lambda_j^-, \nu_j^- \rangle$ $(j = 1, 2, \ldots, n)$ be an assemblage of BPFNs. We also find BPFWG by

$BPFWG(T_1, T_2, \ldots, T_n)$

$$= \left(\prod_{j=1}^{n} \left(\mu_j^+ + \lambda_j^+ \right)^{P_j} - \prod_{j=1}^{n} \left(\lambda_j^+ \right)^{P_j}, \prod_{j=1}^{n} \left(\lambda_j^+ \right)^{P_j}, 1 - \prod_{j=1}^{n} \left(1 - \nu_j^+ \right)^{P_j}, \right.$$

$$\left. \prod_{j=1}^{n} \left(\left(-\mu_j^- \right) - \left(-\lambda_j^- \right) \right)^{P_j} - \prod_{j=1}^{n} \left(-\lambda_j^- \right)^{P_j}, \prod_{j=1}^{n} \left(-\lambda_j^- \right)^{P_j}, 1 - \prod_{j=1}^{n} \left(1 - \left(-\nu_j^- \right) \right)^{P_j} \right) \tag{5}$$

Proof. Using mathematical induction to prove this theorem.

For $n = 2$

$$T_1^{P_1} = \left(\left(\mu_1^+ + \lambda_1^+ \right)^{P_1} - \left(\lambda_1^+ \right)^{P_1}, \left(\lambda_1^+ \right)^{P_1}, 1 - \left(1 - \nu_1^+ \right)^{P_1}, \left(\left(-\mu_1^- \right) + \left(-\lambda_1^- \right) \right)^{P_1} - \left(-\lambda_1^- \right)^{P_1}, \right.$$
$$\left. \left(-\lambda_1^- \right)^{P_1}, 1 - \left(1 - \left(-\nu_1^- \right) \right)^{P_1} \right)$$

$$T_2^{P_2} = \left(\left(\mu_2^+ + \lambda_2^+ \right)^{P_2} - \left(\lambda_2^+ \right)^{P_2}, \left(\lambda_2^+ \right)^{P_2}, 1 - \left(1 - \nu_2^+ \right)^{P_2}, \left(\left(-\mu_2^- \right) + \left(-\lambda_2^- \right) \right)^{P_2} - \left(-\lambda_2^- \right)^{P_2}, \right.$$
$$\left. \left(-\lambda_2^- \right)^{P_2}, 1 - \left(1 - \left(-\nu_2^- \right) \right)^{P_2} \right)$$

Then, it follows that

$$T_1^{P_1} \otimes T_2^{P_2} = \left(\left(\mu_1^+ + \lambda_1^+ \right)^{P_1} \left(\mu_2^+ + \lambda_2^+ \right)^{P_2} - \left(\lambda_1^+ \right)^{P_1} \left(\lambda_2^+ \right)^{P_2}, \left(\lambda_1^+ \right)^{P_1} \left(\lambda_2^+ \right)^{P_2}, 1 - \left(1 - \nu_1^+ \right)^{P_1} 1 - \left(1 - \nu_2^+ \right)^{P_2}, \right.$$
$$\left(\left(-\mu_1^- \right) + \left(-\lambda_1^- \right) \right)^{P_1} \left(\left(-\mu_2^- \right) + \left(-\lambda_2^- \right) \right)^{P_2} - \left(-\lambda_1^- \right)^{P_1} \left(-\lambda_2^- \right)^{P_2}, \left(-\lambda_1^- \right)^{P_1} \left(-\lambda_2^- \right)^{P_2},$$
$$\left. 1 - \left(1 - \left(-\nu_1^- \right) \right)^{P_1} 1 - \left(1 - \left(-\nu_2^- \right) \right)^{P_2} \right)$$

$$= \left(\prod_{j=1}^{2} \left(\mu_j^+ + \lambda_j^+ \right)^{P_j} - \prod_{j=1}^{2} \left(\lambda_j^+ \right)^{P_j}, \prod_{j=1}^{2} \left(\lambda_j^+ \right)^{P_j}, 1 - \prod_{j=1}^{2} \left(1 - \nu_j^+ \right)^{P_j}, \right.$$
$$\left. \prod_{j=1}^{2} \left(\left(-\mu_j^- \right) - \left(-\lambda_j^- \right) \right)^{P_j} - \prod_{j=1}^{2} \left(-\lambda_j^- \right)^{P_j}, \prod_{j=1}^{2} \left(-\lambda_j^- \right)^{P_j}, 1 - \prod_{j=1}^{2} \left(1 \left(-\nu_j^- \right) \right)^{P_j} \right)$$

This shows that it is true for $n = 2$, now let that it holds for $n = k$, i.e.,

$BPFWG\,(T_1, T_2, \ldots, T_k)$

$$= \left(\prod_{j=1}^{k} \left(\mu_j^+ + \lambda_j^+\right)^{P_j} - \prod_{j=1}^{k} \left(\lambda_j^+\right)^{P_j}, \prod_{j=1}^{k} \left(\lambda_j^+\right)^{P_j}, 1 - \prod_{j=1}^{k} \left(1 - \nu_j^+\right)^{P_j}, \right.$$

$$\left. \prod_{j=1}^{k} \left(\left(-\mu_j^-\right) - \left(-\lambda_j^-\right)\right)^{P_j} - \prod_{j=1}^{k} \left(-\lambda_j^-\right)^{P_j}, \prod_{j=1}^{k} \left(-\lambda_j^-\right)^{P_j}, 1 - \prod_{j=1}^{k} \left(1 - \left(-\nu_j^-\right)\right)^{P_j} \right)$$

Now $n = k + 1$, by operational laws of BPFNs we have

$BPFWG\left(T_1, T_2, \ldots, T_{k+1}\right)$

$$= BPFWG\,(T_1, T_2, \ldots, T_k) \otimes T_{k+1}^{P_{k+1}}$$

$$= \left(\prod_{j=1}^{k} \left(\mu_j^+ + \lambda_j^+\right)^{P_j} - \prod_{j=1}^{k} \left(\lambda_j^+\right)^{P_j}, \prod_{j=1}^{k} \left(\lambda_j^+\right)^{P_j}, 1 - \prod_{j=1}^{k} \left(1 - \nu_j^+\right)^{P_j}, \right.$$

$$\left. \prod_{j=1}^{k} \left(\left(-\mu_j^-\right) - \left(-\lambda_j^-\right)\right)^{P_j} - \prod_{j=1}^{k} \left(-\lambda_j^-\right)^{P_j}, \prod_{j=1}^{k} \left(-\lambda_j^-\right)^{P_j}, 1 - \prod_{j=1}^{k} \left(1 - \left(-\nu_j^-\right)\right)^{P_j} \right)$$

$$\otimes ((\mu_{k+1}^+ + \lambda_{k+1}^+)^{P_{k+1}} - (\lambda_{k+1}^+)^{P_{k+1}}, (\lambda_{k+1}^+)^{P_{k+1}}, 1 - (1 - \nu_{k+1}^+)^{P_{k+1}},$$

$$((-\mu_{k+1}^-) + (-\lambda_{k+1}^-))^{P_{k+1}} - (-\lambda_{k+1}^-)^{P_{k+1}}, (-\lambda_{k+1}^-)^{P_{k+1}}, 1 - (1 - (-\nu_{k+1}^-))^{P_{k+1}})$$

$$= \left(\prod_{j=1}^{k+1} \left(\mu_j^+ + \lambda_j^+\right)^{P_j} - \prod_{j=1}^{k+1} \left(\lambda_j^+\right)^{P_j}, \prod_{j=1}^{k+1} \left(\lambda_j^+\right)^{P_j}, 1 - \prod_{j=1}^{k+1} \left(1 - \nu_j^+\right)^{P_j}, \right.$$

$$\left. \prod_{j=1}^{k+1} \left(\left(-\mu_j^-\right) - \left(-\lambda_j^-\right)\right)^{P_j} - \prod_{j=1}^{k+1} \left(-\lambda_j^-\right)^{P_j}, \prod_{j=1}^{k+1} \left(-\lambda_j^-\right)^{P_j}, 1 - \prod_{j=1}^{k+1} \left(1 - \left(-\nu_j^-\right)\right)^{P_j} \right)$$

This shows that for $n = k + 1$, holds. Thus, by the principle of mathematical induction Theorem 4.4 holds for all n.

$BPFWG\,(T_1, T_2, \ldots, T_n)$

$$= \left(\prod_{j=1}^{n} \left(\mu_j^+ + \lambda_j^+\right)^{P_j} - \prod_{j=1}^{n} \left(\lambda_j^+\right)^{P_j}, \prod_{j=1}^{n} \left(\lambda_j^+\right)^{P_j}, 1 - \prod_{j=1}^{n} \left(1 - \nu_j^+\right)^{P_j}, \right.$$

$$\left. \prod_{j=1}^{n} \left(\left(-\mu_j^-\right) - \left(-\lambda_j^-\right)\right)^{P_j} - \prod_{j=1}^{n} \left(-\lambda_j^-\right)^{P_j}, \prod_{j=1}^{n} \left(-\lambda_j^-\right)^{P_j}, 1 - \prod_{j=1}^{n} \left(1 - \left(-\nu_j^-\right)\right)^{P_j} \right)$$

Below we define some of *BPFWA*'s appealing properties.

Theorem 4.5 (Idempotency) Let $T_j = \langle \mu_{\mathbf{j}}^{+}, \lambda_{\mathbf{j}}^{+}, \nu_{\mathbf{j}}^{+}, \mu_{\mathbf{j}}^{-}, \lambda_{\mathbf{j}}^{-}, \nu_{\mathbf{j}}^{-} \rangle$ be an assemblage of *BPFNs*. If, $T_j = T = \langle \mu^{+}, \lambda^{+}, \nu^{+}, \mu^{-}, \lambda^{-}, \nu^{-} \rangle$ for all j, then
$BPFWA(T_1, T_2, \ldots, T_n) = T$

Proof. Since, $T_1 = T_2 = \ldots = T_n = T$. By Theorem 4.4, we get
$BPFWG(T_1, T_2, \ldots, T_n)$

$$= \left(\prod_{j=1}^{n} \left(\mu_j^{+} + \lambda_j^{+}\right)^{P_j} - \prod_{j=1}^{n} \left(\lambda_j^{+}\right)^{P_j}, \prod_{j=1}^{n} \left(\lambda_j^{+}\right)^{P_j}, 1 - \prod_{j=1}^{n} \left(1 - \nu_j^{+}\right)^{P_j}, \right.$$

$$\left. \prod_{j=1}^{n} \left(\left(-\mu_j^{-}\right) - \left(-\lambda_j^{-}\right)\right)^{P_j} - \prod_{j=1}^{n} \left(-\lambda_j^{-}\right)^{P_j}, \prod_{j=1}^{n} \left(-\lambda_j^{-}\right)^{P_j}, 1 - \prod_{j=1}^{n} \left(1 - \left(-\nu_j^{-}\right)\right)^{P_j} \right)$$

$$= \left(\prod_{j=1}^{n} \left(\mu^{+} + \lambda^{+}\right)^{P_j} - \prod_{j=1}^{n} \left(\lambda^{+}\right)^{P_j}, \prod_{j=1}^{n} \left(\lambda^{+}\right)^{P_j}, 1 - \prod_{j=1}^{n} \left(1 - \nu^{+}\right)^{P_j}, \right.$$

$$\prod_{j=1}^{n} \left(\left(-\mu^{-}\right) - \left(-\lambda^{-}\right)\right)^{P_j} - \prod_{j=1}^{n} \left(-\lambda^{-}\right)^{P_j}, \prod_{j=1}^{n} \left(-\lambda^{-}\right)^{P_j}, 1 - \prod_{j=1}^{n} \left(1 - \left(-\nu^{-}\right)\right)^{P_j}$$

$$= \left(\left(\mu^{+} + \lambda^{+}\right)^{\sum_{j=1}^{n} P_j} - \left(\lambda^{+}\right)^{\sum_{j=1}^{n} P_j}, \left(\lambda^{+}\right)^{\sum_{j=1}^{n} P_j}, 1 - \left(1 - \nu^{+}\right)^{\sum_{j=1}^{n} P_j}, \right.$$

$$\left. ((-\mu^{-}) - (-\lambda^{-}))^{\sum_{j=1}^{n} P_j} - (-\lambda^{-})^{\sum_{j=1}^{n} P_j}, (-\lambda^{-})^{\sum_{j=1}^{n} P_j}, 1 - (1 - (-\nu^{-}))^{\sum_{j=1}^{n} P_j} \right)$$

We know, $\sum_{j=1}^{n} P_j = 1$,
$BPFWA(T_1, T_2, \ldots, T_n) = T$

Theorem 4.6 (Monotonicity) Let $T_j = \langle \mu_{\mathbf{j}}^{+}, \lambda_{\mathbf{j}}^{+}, \nu_{\mathbf{j}}^{+}, \mu_{\mathbf{j}}^{-}, \lambda_{\mathbf{j}}^{-}, \nu_{\mathbf{j}}^{-} \rangle$ and $T_j^{*} = \langle (\mu_{\mathbf{j}}^{+})^{*}, (\lambda_{\mathbf{j}}^{+})^{*}, (\nu_{\mathbf{j}}^{+})^{*}, (\mu_{\mathbf{j}}^{-})^{*}, (\lambda_{\mathbf{j}}^{-})^{*}, (\nu_{\mathbf{j}}^{-})^{*} \rangle$ be two families of BPFNs. If $T_j \leq T_j^{*}$ for all $(j = 1, 2, \ldots, n)$ then
$BPFWA(T_1, T_2, \ldots, T_n) \leq BPFWA(T_1^{*}, T_2^{*}, \ldots, T_n^{*})$

Proof. Here, we omit the proof.

Example 4.7 Let T_1, T_2, T_3, T_4 be the *BPFNs* as follows:
$T_1 = (0.3, 0.2, 0.1, -0.2, -0.1, -0.4)$
$T_2 = (0.3, 0.4, 0.1, -0.1, -0.2, -0.3)$
$T_3 = (0.2, 0.4, 0.3, -0.1, -0.3, -0.4)$
$T_4 = (0.2, 0.1, 0.2, -0.2, -0.5, -0.1)$

and $P = (0.3, 0.2, 0.1, 0.4)$ then

$BPFWG(T_1, T_2, T_3, T_4)$

$$= \left(\prod_{j=1}^{4} \left(\mu_j^+ + \lambda_j^+\right)^{P_j} - \prod_{j=1}^{4} \left(\lambda_j^+\right)^{P_j}, \prod_{j=1}^{4} \left(\lambda_j^+\right)^{P_j}, 1 - \prod_{j=1}^{4} \left(1 - \nu_j^+\right)^{P_j}, \right.$$

$$\left. \prod_{j=1}^{4} \left(\left(-\mu_j^-\right) - \left(-\lambda_j^-\right)\right)^{P_j} - \prod_{j=1}^{4} \left(-\lambda_j^-\right)^{P_j}, \prod_{j=1}^{4} \left(-\lambda_j^-\right)^{P_j}, 1 - \prod_{j=1}^{4} \left(1 - \left(-\nu_j^-\right)\right)^{P_j} \right)$$

$$= (0.257384, 0.186607, 0.162727, -0.189252, -0.225869, -0.272259)$$

When we need to weight the ordered positions of the bipolar picture fuzzy arguments instead of weighting the arguments themselves, $BPFWG$ can be generalized to $BPFOWG$.

Definition 4.8 Let $T_j = \langle \mu_j^+, \lambda_j^+, \nu_j^+, \mu_j^-, \lambda_j^-, \nu_j^- \rangle$ be a assemblage of BPFNs. A mapping $BPFOWG$: $\mathfrak{X}_n \to \mathfrak{X}$ is called a bipolar picture fuzzy ordered weighted geometric (BPFOWG) operator.

$$BPFOWG(T_1, T_2, \ldots, T_n) = \sum_{j=1}^{n} T_{\sigma(j)}^{P_j} = T_{\sigma(1)}^{P_1} \otimes T_{\sigma(2)}^{P_2}, \ldots \otimes, T_{\sigma(n)}^{P_n}$$

where P_j is the WV of $T_j (j = 1, 2, \ldots, n)$, $P_j \in [0, 1]$ and $\sum_{j=1}^{n} P_j = 1$.

According to the operational laws of the BPFNs, we can obtain the following theorems. Since their proofs are similar to those mentioned above, we are omitting them here.

Theorem 4.9 Let $T_j = \langle \mu_j^+, \lambda_j^+, \nu_j^+, \mu_j^-, \lambda_j^-, \nu_j^- \rangle$ $(j = 1, 2, \ldots, n)$ be a assemblage of BPFNs. We also find BPFOWG by

$BPFOWG(T_1, T_2, \ldots, T_n)$

$$= \left(\prod_{j=1}^{n} \left(\mu_{\sigma(j)}^+ + \lambda_{\sigma(j)}^+\right)^{P_j} - \prod_{j=1}^{n} \left(\lambda_{\sigma(j)}^+\right)^{P_j}, \prod_{j=1}^{n} \left(\lambda_{\sigma(j)}^+\right)^{P_j}, 1 - \prod_{j=1}^{n} \left(1 - \nu_{\sigma(j)}^+\right)^{P_j}, \right.$$

$$\left. \prod_{j=1}^{n} \left(\left(-\mu_{\sigma(j)}^-\right) - \left(-\lambda_{\sigma(j)}^-\right)\right)^{P_j} - \prod_{j=1}^{n} \left(-\lambda_{\sigma(j)}^-\right)^{P_j}, \prod_{j=1}^{n} \left(-\lambda_{\sigma(j)}^-\right)^{P_j}, 1 - \prod_{j=1}^{n} \left(1 - \left(-\nu_{\sigma(j)}^-\right)\right)^{P_j} \right) \quad (6)$$

Theorem 4.10 (Idempotency) Let $T_j = \langle \mu_{\mathbf{j}}^+, \lambda_{\mathbf{j}}^+, \nu_{\mathbf{j}}^+, \mu_{\mathbf{j}}^-, \lambda_{\mathbf{j}}^-, \nu_{\mathbf{j}}^- \rangle$ be a assemblage of *BPFNs*. If, $T_j = T = \langle \mu^+, \lambda^+, \nu^+, \mu^-, \lambda^-, \nu^- \rangle$ for all j, then

$BPFOWA(T_1, T_2, \ldots, T_n) = T$

Theorem 4.11 (Monotonicity) Let $T_j = \langle \mu_{\mathbf{j}}^+, \lambda_{\mathbf{j}}^+, \nu_{\mathbf{j}}^+, \mu_{\mathbf{j}}^-, \lambda_{\mathbf{j}}^-, \nu_{\mathbf{j}}^- \rangle$ and $T_j^* = \langle (\mu_{\mathbf{j}}^+)^*, (\lambda_{\mathbf{j}}^+)^*, (\nu_{\mathbf{j}}^+)^*, (\mu_{\mathbf{j}}^-)^*, (\lambda_{\mathbf{j}}^-)^*, (\nu_{\mathbf{j}}^-)^* \rangle$ be two families of BPFNs. If $T_j \le T_j^*$ for all $(j = 1, 2, \ldots, n)$ then

$BPFOWA(T_1, T_2, \ldots, T_n) \le BPFWA(T_1^*, T_2^*, \ldots, T_n^*)$

Theorem 4.12 (Commutativity) Let $T_j = \langle \mu_j^+, \lambda_j^+, \nu_j^+, \mu_j^-, \lambda_j^-, \nu_j^- \rangle$ be a assemblage of $BPFNs$.

$BPFOWG(T_{\sigma(1)}, T_{\sigma(2)}, \ldots, T_{\sigma(n)}) = BPFOWG(T_1, T_2, \ldots, T_n)$

where $(\sigma(1), \sigma(2), \ldots, \sigma(n))$ is any permutation of $(1, 2, \ldots, n)$.

When both the ordered positions of the bipolar picture fuzzy arguments and the arguments themselves need to be weighted, BPFWG can be generalized to the following bipolar picture fuzzy hybrid geometric operator.

Definition 4.13 $ABPFHG$ operator is a mapping $BPFHG\colon \mathfrak{X}_n \to \mathfrak{X}$ such that P_j is the WV of $T_j (j = 1, 2, \ldots, n)$, $P_j \in [0, 1]$ and $\sum_{j=1}^{n} P_j = 1$.

$$BPFHG(T_1, T_2, \ldots, T_n) = \sum_{j=1}^{n} \ddot{T}_{\sigma(j)}^{P_j}$$

$\ddot{T}_{\sigma(j)}$ is the j_{th} largest of the weighted BPFNs. Here $\ddot{T}_j = nw_j T_j$, $(1, 2, \ldots, n)$, n is the number of BPFNs and $w = (w_1, w_2 \ldots w_n)$ is the standard WV.

We can drive the following theorem based on the operations of the PFNs which is similar to Theorem 4.4.

Theorem 4.14 $Let T_j = \langle \mu_j^+, \lambda_j^+, \nu_j^+, \mu_j^-, \lambda_j^-, \nu_j^- \rangle$ $(j = 1, 2, \ldots, n)$ be a assemblage of $BPFNs$. We also find $BPFHG$ by

$$BPFHG(T_1, T_2, \ldots, T_n)$$

$$= \left(\prod_{j=1}^{n} (\ddot{\mu}_{\sigma(j)}^+ + \ddot{\lambda}_{\sigma(j)}^+)^{P_j} - \prod_{j=1}^{n} (\ddot{\lambda}_{\sigma(j)}^+)^{P_j}, \prod_{j=1}^{n} (\ddot{\lambda}_{\sigma(j)}^+)^{P_j}, 1 - \prod_{j=1}^{n} (1 - \ddot{\nu}_{\sigma(j)}^+)^{P_j}, \right.$$
$$\left. \prod_{j=1}^{n} ((-\ddot{\mu}_{\sigma(j)}^-) - (-\ddot{\lambda}_{\sigma(j)}^-))^{P_j} - \prod_{j=1}^{n} (-\ddot{\lambda}_{\sigma(j)}^-)^{P_j}, \prod_{j=1}^{n} (-\ddot{\lambda}_{\sigma(j)}^-)^{P_j}, 1 - \prod_{j=1}^{n} (1 - (-\ddot{\nu}_{\sigma(j)}^-))^{P_j} \right). \tag{7}$$

The weighting vector associated with the operator of BPFWG, the operator of BPFOWG and the operator of BPFHG can be assessed as identical to that of the other operators. For example, a normal distribution-based approach can be used to evaluate weights. The distinctive feature of the approach is that it can reduce the effect of bias claims on the outcome of the judgment by assigning low weights to the wrong ones.

4.1 Distance Measure of Bipolar Picture Fuzzy Sets

In this section of the paper, we define the distance measures of bipolar picture fuzzy sets.

Definition 4.15 A function $\tau\colon BPFS(X) \times BPFS(X) \to [0, +\infty)$ is a distance measure between BPFS-sets if it satisfies follow conditions:

1) $\tau(O, P) = 0$ iff $O = P$.
2) $(O, P) = \tau(P, O)$ $\forall\, O, P \in BPFS(X)$.
3) $(O, \mathcal{Q}) \leq \tau(O, P) + \tau(P, \mathcal{Q})$ $\forall\, O, P, \mathcal{Q} \in BPFS(X)$.

Theorem 4.16 *Given* $X = \{\delta_1, \delta_2, \ldots, \delta_n\}$ is a universe of discourse. *For* $O, P \in BPFS(X)$. We have some distance measure between BPFSs.

$$\tau_H(O,P) = \frac{1}{6n}\sum_{j=1}^{n}\langle|\mu_O^+(\delta_j) - \mu_P^+(\delta_j)| + |\lambda_O^+(\delta_j) - \lambda_P^+(\delta_j)| + |\nu_O^+(\delta_j) - \nu_P^+(\delta_j)|$$

$$+ |(-\mu_O^-(\delta_j)) - (-\mu_P^-(\delta_j))| + |(-\lambda_O^-(\delta_j)) - (-\lambda_P^-(\delta_j))| + |(-\nu_O^-(\delta_j)) - (-\nu_P^-(\delta_j))|\rangle$$

$$\tau_E(O,P) = \frac{1}{n}\left[\sum_{j=1}^{n}\langle|\mu_O^+(\delta_j) - \mu_P^+(\delta_j)| + |\lambda_O^+(\delta_j) - \lambda_P^+(\delta_j)| + |\nu_O^+(\delta_j) - \nu_P^+(\delta_j)|\right.$$

$$\left.+ |(-\mu_O^-(\delta_j)) - (-\mu_P^-(\delta_j))| + |(-\lambda_O^-(\delta_j)) - (-\lambda_P^-(\delta_j))| + |(-\nu_O^-(\delta_j)) - (-\nu_P^-(\delta_j))|\rangle\right]^{\frac{1}{2}}$$

$$\tau_H^m(O,P) = \frac{1}{6n}\sum_{j=1}^{n}\max\langle|\mu_O^+(\delta_j) - \mu_P^+(\delta_j)| + |\lambda_O^+(\delta_j) - \lambda_P^+(\delta_j)| + |\nu_O^+(\delta_j) - \nu_P^+(\delta_j)|$$

$$+ |(-\mu_O^-(\delta_j)) - (-\mu_P^-(\delta_j))| + |(-\lambda_O^-(\delta_j)) - (-\lambda_P^-(\delta_j))| + |(-\nu_O^-(\delta_j)) - (-\nu_P^-(\delta_j))|\rangle$$

$$\tau_E^m(O,P) = \frac{1}{n}\left[\sum_{j=1}^{n}\max\langle|\mu_O^+(\delta_j) - \mu_P^+(\delta_j)| + |\lambda_O^+(\delta_j) - \lambda_P^+(\delta_j)| + |\nu_O^+(\delta_j) - \nu_P^+(\delta_j)|\right.$$

$$\left.+ |(-\mu_O^-(\delta_j)) - (-\mu_P^-(\delta_j))| + |(-\lambda_O^-(\delta_j)) - (-\lambda_P^-(\delta_j))| + |(-\nu_O^-(\delta_j)) - (-\nu_P^-(\delta_j))|\rangle\right]^{\frac{1}{2}}$$

We can actually confirm that the functions in Theorem 4.16 satisfy distance measuring properties between bipolar picture fuzzy sets. In it, $\tau_E(O,P)$ is typically used to calculate object distance in geometry, and $\tau_H(O,P)$ is used in the theory of information.

Example 4.17 Assume there are three patterns denoted by BPFSs on $X = \{\delta_1, \delta_2, \delta_3\}$ as follows:

$\Omega_1 = \langle\delta_1, 0.5, 0.2, 0.2, -0.1, -0.2, -0.4\rangle, \langle\delta_2, 0.3, 0.4, 0.3, -0.35, -0.3, -0.3\rangle,$
$\langle\delta_3, 0.3, 0.4, 0.2, -0.5, -0.1, -0.2\rangle$

$\Omega_2 = \langle\delta_1, 0.4, 0.3, 0.2, -0.2, -0.4, -0.2\rangle, \langle\delta_2, 0.5, 0.1, 0.3, -0.3, -0.5, -0.2\rangle,$
$\langle\delta_3, 0.3, 0.4, 0.1, -0.3, -0.2, -0.4\rangle$

$\Omega_3 = \langle\delta_1, 0.2, 0.4, 0.1, -0.4, -0.1, -0.3\rangle, \langle\delta_2, 0.1, 0.1, 0.3, -0.3, -0.2, -0.3\rangle,$
$\langle\delta_3, 0.3, 0.1, 0.2, -0.3, -0.2, -0.2\rangle$

are three bipolar picture fuzzy set in X. Using Theorem 4.16, we get

$\tau_H(\Omega_1, \Omega_2) = 0.1194$

$\tau_H(\Omega_1, \Omega_3) = 0.1306$

$\tau_H(\Omega_2, \Omega_3) = 0.1445$

and

$$\tau_E(\Omega_1, \Omega_2) = 0.4888$$
$$\tau_E(\Omega_1, \Omega_3) = 0.5109$$
$$\tau_E(\Omega_2, \Omega_3) = 0.5375. \quad (8)$$

5 MCDM Based on BPFS to Pattern Recognition

In this section, we discuss some distance measures of BPFSs and their application to the pattern recognition. Pattern recognition is a science and technology discipline which aims to classify objects into a number of categories. This method is widely used in the identification of data analysis, shapes, pattern classification, traffic analysis & regulation, natural language processing, rock identification, biological stimuli, odor identification, understanding of the DNA sample, credit fraud detection, biometrics including fingerprints, palm vein technology & face recognition, medical diagnosis, weather forecasting, intelligence, informatics, voice to text transition, terrorism identification, radar tracking, and automatic military target recognition, etc. In Fig. 3 step by step method is shown of facial recognition which is example of pattern recognition.

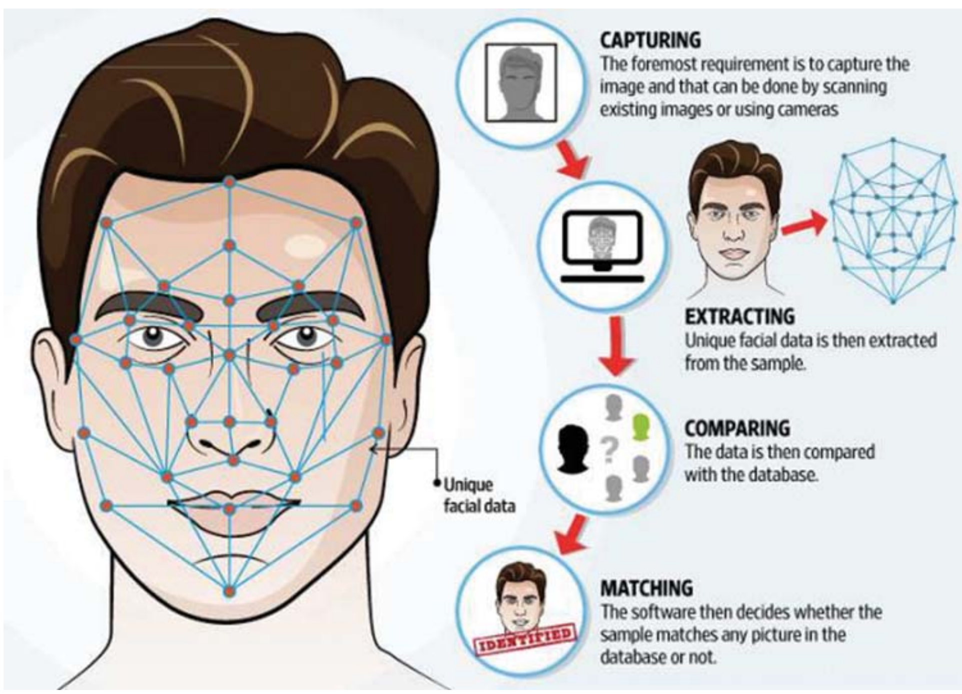

Figure 3: Method of facial recognition

5.1 Numerical Example for Using New Measures in Pattern Recognition

Example 5.1 Suppose that there are three patterns denoted by BPFSs on $X = \{\delta_1, \delta_2, \delta_3\}$ as follows:

$$\Omega_1 = \langle \delta_1, 0.35, 0.25, 0.20, -0.15, -0.20, -0.40 \rangle, \langle \delta_2, 0.10, 0.40, 0.35, -0.15, -0.35, -0.35 \rangle,$$
$$\langle \delta_3, 0.30, 0.45, 0.15, -0.35, -0.35, -0.20 \rangle$$

$$\Omega_2 = \langle\delta_1, 0.40, 0.35, 0.25, -0.20, -0.40, -0.25\rangle, \langle\delta_2, 0.35, 0.10, 0.35, -0.30, -0.50, -0.25\rangle,$$
$$\langle\delta_3, 0.30, 0.40, 0.15, -0.35, -0.25, -0.10\rangle$$
$$\Omega_3 = \langle\delta_1, 0.20, 0.40, 0.10, -0.40, -0.10, -0.30\rangle, \langle\delta_2, 0.10, 0.10, 0.35, -0.35, -0.25, -0.35\rangle,$$
$$\langle\delta_3, 0.35, 0.10, 0.25, -0.30, -0.25, -0.25\rangle$$

Now, there is a sample,

$$B = \langle\delta_1, 0.25, 0.45, 0.10, -0.45, -0.15, -0.35\rangle, \langle\delta_2, 0.15, 0.10, 0.30, -0.35, -0.20, -0.30\rangle,$$
$$\langle\delta_3, 0.35, 0.15, 0.25, -0.30, -0.25, -0.20\rangle$$

The question is, what pattern belongs to B? By applying the distance measure τ_E. Using Theorem 4.16, we get

$$\tau_E(\Omega_1, B) = 0.1111$$
$$\tau_E(\Omega_2, B) = 0.1167$$
$$\tau_E(\Omega_3, B) = 0.0306$$

We see that B belongs to pattern Ω_3 if we use the distance measure τ_E.

6 MCDM Based on Some Bipolar Picture Fuzzy Geometric Operators

MCDM method using the aggregation operators defined for BPFNs is presented in this section.

Suppose that $T = \{T_1, T_2, \ldots, T_p\}$ is the set of alternatives and $C = \{C_1, C_2, \ldots, C_q\}$ is the set of criterion. Let P be the WV, s.t $P_j \in [0,1]$ and $\sum_{j=1}^{n} P_j = 1$, $(j = 1, 2, \ldots, n)$ and P_j show the weight of C_j. Alternatives on an attribute are reviewed by the decision-maker (DM) and the assessment measurements has to be in the BPFN. Assume that $\lambda = (\alpha_{ij})_{p\times q}$ is the decision matrix provided by DM.(α_{ij}) represent a *BPFN* for alternative T_i associated with the criterion C_j. Here we have some conditions such that

1. $\mu^+_{ij}, \lambda^+_{ij}, \nu^+_{ij}, \mu^-_{ij}, \lambda^-_{ij}$ and $\nu^-_{ij} \in [0,1]$
2. $0 \le \mu^+_{ij} + \lambda^+_{ij} + \nu^+_{ij} \le 1$ and $-1 \le \mu^-_{ij} + \lambda^-_{ij} + \nu^-_{ij} \le 0$.

Indeed an algorithm is being developed to discuss MCDM.

Algorithm

Step 1.

The DM has given its personal opinion in the form of BPFNs. $\alpha_{ij} = \langle\mu^+_{ij}, \lambda^+_{ij}, \nu^+_{ij}, \mu^-_{ij}, \lambda^-_{ij}, \nu^-_{ij}\rangle$ towards the alternative T_i and hence construct a BPF decision matrix $\lambda = (\alpha_{ij})_{p\times q}$ as

$$\begin{bmatrix} (\mu^+_{11}, \lambda^+_{11}, \nu^+_{11}, \mu^-_{11}, \lambda^-_{11}, \nu^-_{11}) & \cdots & \left(\mu^+_{1q}, \lambda^+_{1q}, \nu^+_{1q}, \mu^-_{1q}, \lambda^-_{1q}, \nu^-_{1q}\right) \\ \vdots & \ddots & \vdots \\ \left(\mu^+_{p1}, \lambda^+_{p1}, \nu^+_{p1}, \mu^-_{p1}, \lambda^-_{p1}, \nu^-_{p1}\right) & \cdots & \left(\mu^+_{pq}, \lambda^+_{pq}, \nu^+_{pq}, \mu^-_{pq}, \lambda^-_{pq}, \nu^-_{pq}\right) \end{bmatrix}$$

Step 2.

Normalize the decision matrix. If there are different types of criteria or attributes like cost and benefit. By normalize the decision matrix we deal all criteria or attributes in the same way. Otherwise, different criterion or attributes should be aggregate in different ways.

$$r_{ij} = \begin{cases} \alpha_{ij}^{c}; & j \in \tau_c \\ \alpha_{ij}; & j \in \tau_b. \end{cases} \tag{9}$$

where α_{ij}^{c} show the compliment of α_{ij}.

Step 3.

Based on decision matrix acquired from Step 2, the aggregated value of the alternative T_i under various parameter C_j is obtained using either BPFWA or BPFOWA or BPFHA operators and hence get the collective value r_i for each alternative $T_i (i = 1, 2, \ldots m)$.

Step 4.

Calculate the score functions for all r_i for BPFNs.

Step 5.

Rank all r_i as per the score values to choose the most desirable option.

The flow chart of proposed algorithm is expressed in Fig. 4.

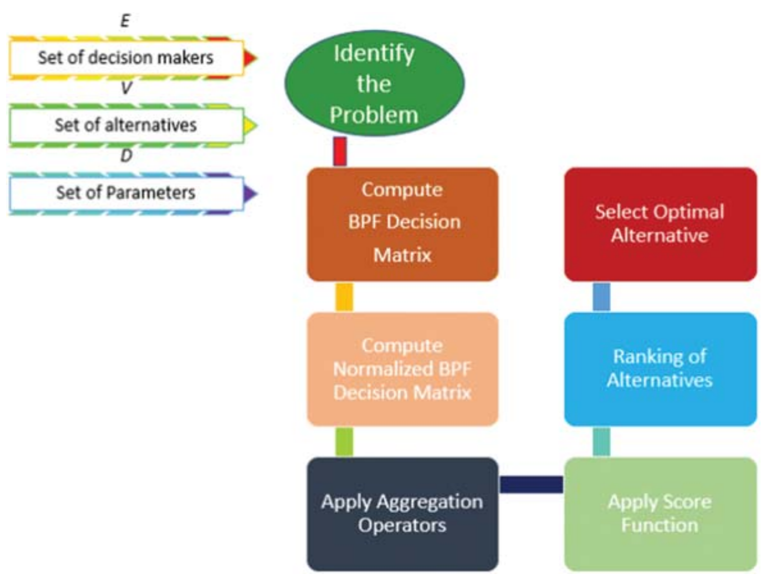

Figure 4: Flow chart of proposed algorithm

6.1 Case Study

We are considering quantitative examples in the selection of mushroom farming alternatives in Pakistan to show the effectiveness of the proposed processes. Filled with taste and an incredible nutritional composition to boot, oyster mushrooms will be a worthy complement to a balanced diet. Here several categories of oyster mushrooms that differ a little in flavor and nutritional benefits. In this paper, we will focus majorly on king oyster mushrooms (KOM) and their nutritional benefits. In all types of mushroom, preparation procedures that we describe can be included.

Pleurotus Eryngii (PE) (Fig. 5) is a real term for king oyster mushrooms. They are also known as French horn, royal trumpet, king brown, king trumpet and steppe boletus. The King oyster mushroom is, as its name suggests, the largest of all oyster mushrooms. This is evidently rising in the Middle East and North Africa. It is also extensively grown in Asia, in a variety of countries, as well as in Italy, Australia and the USA. Looking at the health benefits of oyster mushrooms, there are several positive features of this study. Very good sources of riboflavin, iron, niacin, phosphorus, potassium, copper, Protein, vitamin B6, pantothenic acid,folate, magnesium, zinc and manganese from the right source. Mostly limited in cholesterol and saturated fat. Only about 35 per 100 g king oyster mushroom calories. King oyster mushrooms have a good protein source and are the ideal complement to vegetarian or vegan diets. They are not a full source of protein and must ensure that a number of various sources of protein are included in your healthy diet. Alone last year, Americans grew more than two million pounds of exotic mushrooms. Oyster mushrooms, a type of exotic mushrooms, are among the best and fastest growing exotic mushrooms. They could even grow in about six weeks, and they are looking to sell around 6 Dollar a pound wholesale and 12 Dollar a pound retail. They looked incredibly easy to produce, they're growing quickly, and they can make you decent money—all the justifications whether you like to oyster mushrooms to grow for financial gain.

Figure 5: King oyster mushroom

KOM is an enormous, important food naturalized to Asia and the rest of Europe. Although hard to find in the wild, it is widely cultivated and famous for its buttery taste and eggplant-like flavor, particularly in some Asian and African cuisines. predominant Chinese medication has for centuries recognized the importance of KOM and other medicative mushrooms.

Here are among the most possibly the best-researched advantages of KOM.

1. Immune System Support

B-glucans in KOM enable them are some of the healthiest meals on the earth to support the immune function toward short-and long-term diseases [51]. Unlike other food products that either activate or inhibit the immune system, the mushrooms balanced the immune cells. Plus, KOM are filled with other antioxidants to help avoid harm caused by free radicals and oxidative stress so that the immune cells can protect itself against aging [52].

2. Reducing of High Blood Pressure

Your body requires nutrients like vitamin D to stabilize your heart rate and blood pressure. Do you think that the majority of people living in colder climates are deficient with vitamin D? One research found that edible mushrooms, such as oysters, reduced blood pressure in rats with chronic or uncertain high blood pressure [53].

3. Regulating Cholesterol Levels

Although mushrooms like KOM have a tasty taste and texture and no cholesterol, they are a fine replacement for meat in several steamed dishes. One study also initiate that in people with diabetes, the intake of oyster mushrooms decreased glucose and cholesterol levels [54].

4. Strong Bones

KOM provide a number of essential ingredients for building better bones. Vitamin D and magnesium in particular. While most persons concentrate on calcium, your body also requires vitamin D and magnesium to absorb and preserve calcium in your bones.

5. Anti-Inflammatory Properties

B-glucans and nutrients in KOM make it a perfect food to reduce inflammation. Some work indicates that, besides B-glucans, some of the anti-inflammatory effects of oysters come from a special and somewhat obscure amino acid called ergothioneine. According to study, ergothioneine reduces "systemic" inflammation around the body, frequently leading to diseases such as dementia and diabetes.

6. Anti-Cancer Properties

B-glucans in mushrooms, such as KOM, serve as powerful antioxidants that can offer some protection from cancer. One research showed that oyster mushrooms have the potential to be involved in some forms of cancer cells.

Various substrates, such as sawdust (SD) and rice straw (RS), have been used to grow KOM. Sun-dried SD, wheat bran and rice husk were combined together. Water was applied to change the water absorption and $CaCO_3$ was blended at a rate of 1% of the mixture. The substrate mixture was packed with airtight plastic polymer bottles. The bottles were sterilized, and after cool back to normal temperature, the sterilized bottles were tested separately. We were using $CaCO_3$, straw, sawdust, corn cob and rice bran to grow King oyster mushrooms. They are combined based on specific ratios, Nguyen et al. [55] take into account each combining formula as an alternative given in Tab. 4.

Table 4: Combining formulas for alternatives

Alternatives	Combining formula
T_1	1% $CaCO_3$, 40% straw, 29% sawdust, 30% corn cob and 0% rice bran
T_2	1% $CaCO_3$, 40% straw, 27% sawdust, 27% corn cob and 5% rice bran
T_3	1% $CaCO_3$, 40% straw, 24% sawdust, 25% corn cob and 10% rice bran
T_4	1% $CaCO_3$, 40% straw, 17% sawdust, 17% corn cob and 25% rice bran

6.2 Numerical Example

There are four types of alternatives $T_i (i = 1, 2, 3, 4)$, given in Tab. 4. Analysis the effects of rapidly increasing materials on the productivity growth of king oyster mushrooms. We consider C_1 = infection rate, C_2 = Biological productivity, C_3 = diameter of mushroom cap and C_4 = diameter of mushroom stalks as attributes. In this example we use BPFNs as input data for ranking the given alternatives under the given attributes. Also, the WV P is $(0.3, 0.2, 0.1, 0.4)$ and standard WV w is $(0.2, 0.3, 0.3, 0.2)$.

Using BPFWG operator

Step 1.

Construct the decision matrix given by the decision maker in Tab. 5 consist on bipolar picture fuzzy information.

Table 5: BPF decision matrix taking by decision maker

	C_1	C_2	C_3	C_4
T_1	(0.5, 0.3, 0.2, −0.1, −0.6, −0.2)	(0.5, 0.3, 0.2, −0.1, −0.6, −0.2)	(0.3, 0.3, 0.1, −0.1, −0.5, −0.2)	(0.4, 0.3,0.1, −0.1, −0.4, −0.2)
T_2	(0.2, 0.4, 0.1, −0.1, −0.3, −0.2)	(0.2, 0.4,0.3, −0.1, −0.2, −0.1)	(0.2, 0.2,0.4, −0.1, −0.2, −0.4)	(0.2, 0, 0.2, −0.1, −0.6, −0.2)
T_3	(0.3, 0.4, 0.1, −0.2, −0.3, −0.4)	(0.2, 0.1, 0.4, −0.4, −0.2, −0.1)	(0.1, 0.3, 0.4, −0.2, −0.2,−0.4)	(0.2, 0.4, 0.3, −0.1, −0.2, −0.3)
T_4	(0.1, 0.2, 0.3, −0.4, −0.1, −0.2)	(0.3, 0.4, 0.1, −0.1, −0.2, −0.3)	(0.2, 0.4, 0.3, −0.1, −0.3, −0.4)	(0.2, 0.1, 0.2, −0.2, −0.5, −0.1)

Step 2.

Normalize the decision matrix, because the attribute C_1 = price, given in Tab. 6.

Table 6: Normalized BPF decision matrix

	C_1	C_2	C_3	C_4
T_1	(0.2, 0.3, 0.5, −0.2, −0.6, −0.1)	(0.5, 0.3,0.2, −0.1, −0.6, −0.2)	(0.3, 0.3, 0.1, −0.1, −0.5, −0.2)	(0.4, 0.3,0.1, −0.1, −0.4, −0.2)
T_2	(0.1, 0.4,0.2, −0.2, −0.3, −0.1)	(0.2, 0.4,0.3, −0.1, −0.2, −0.1)	(0.2, 0.2,0.4, −0.1, −0.2, −0.4)	(0.2, 0, 0.2, −0.1, −0.6, −0.2)
T_3	(0.1, 0.4, 0.3, −0.4, −0.3, −0.2)	(0.2, 0.1, 0.4, −0.4, −0.2, −0.1)	(0.1, 0.3, 0.4, −0.2, −0.2,−0.4)	(0.2, 0.4, 0.3, −0.1, −0.2, −0.3)
T_4	(0.3, 0.2, 0.1, −0.2, −0.1, −0.4)	(0.3, 0.4, 0.1, −0.1, −0.2, −0.3)	(0.2, 0.4, 0.3, −0.1, −0.3, −0.4)	(0.2, 0.1, 0.2, −0.2, −0.5, −0.1)

Step 3.

Evaluate $r_i = BPFWG(r_{i1}, r_{i2}, \ldots, r_{ip})$.

$r_1 = (0.339975, 0.3, 0.263063, -0.126166, -0.500953, -0.171227)$

$r_2 = (0.351511, 0, 0.243171, -0.140243, -0.350514, -0.175535)$

$r_3 = (0.180331, 0.294547, 0.331635, -0.231444, -0.225869, -0.245551)$

$r_4 = (0.257384, 0.186607, 0.162727, -0.189252, -0.225869, -0.272259)$

Step 4.

Calculate the score functions for all r_i.

$\Phi(r_1) = 0.161463$

$\Phi(r_2) = 0.247073$

$\Phi(r_3) = -0.102937$

$\Phi(r_4) = 0.108463$

Step 5.

Rank all the $r_i (i = 1, 2, \ldots, p)$ according to the score values,

$r_2 \succ r_1 \succ r_4 \succ r_3$

r_2 corresponds to T_2, so T_2 is the best alternative.

Using BPFOWG operator

Step 1.

Construct the decision matrix given by decision maker consist on bipolar picture fuzzy information, given in Tab. 7.

Table 7: BPF decision matrix taking by decision maker

	C_1	C_2	C_3	C_4
T_1	(0.5, 0.3, 0.2, −0.1, −0.6, −0.2)	(0.5, 0.3, 0.2, −0.1, −0.6, −0.2)	(0.3, 0.3, 0.1, −0.1, −0.5, −0.2)	(0.4, 0.3,0.1, −0.1, −0.4, 0.2)
T_2	(0.2, 0.4, 0.1, −0.1, −0.3, −0.2)	(0.2, 0.4,0.3, −0.1, −0.2, −0.1)	(0.2, 0.2,0.4, −0.1, −0.2, −0.4)	(0.2, 0, 0.2, −0.1, −0.6, −0.2)
T_3	(0.3, 0.4, 0.1, −0.2, −0.3, −0.4)	(0.2, 0.1, 0.4, −0.4, −0.2, −0.1)	(0.1, 0.3, 0.4, −0.2, −0.2,−0.4)	(0.2, 0.4, 0.3, −0.1, −0.2, −0.3)
T_4	(0.1, 0.2, 0.3, −0.4, −0.1, −0.2)	(0.3, 0.4, 0.1, −0.1, −0.2, −0.3)	(0.2, 0.4, 0.3, −0.1, −0.3, −0.4)	(0.2, 0.1, 0.2, −0.2, −0.5, −0.1)

Step 2.

Normalize the decision matrix, because the attribute $C_1 =$ price, given in Tab. 8.

Table 8: Normalized BPF decision matrix

	C_1	C_2	C_3	C_4
T_1	(0.2, 0.3, 0.5, −0.2, −0.6, −0.1)	(0.5, 0.3,0.2, −0.1, −0.6, −0.2)	(0.3, 0.3, 0.1, −0.1, −0.5, −0.2)	(0.4, 0.3,0.1, −0.1, −0.4, −0.2)
T_2	(0.1, 0.4,0.2, −0.2, −0.3, −0.1)	(0.2, 0.4,0.3, −0.1, −0.2, −0.1)	(0.2, 0.2,0.4, −0.1, −0.2, −0.4)	(0.2, 0, 0.2, −0.1, −0.6, −0.2)
T_3	(0.1, 0.4, 0.3, −0.4, −0.3, −0.2)	(0.2, 0.1, 0.4, −0.4, −0.2, −0.1)	(0.1, 0.3, 0.4, −0.2, −0.2,−0.4)	(0.2, 0.4, 0.3, −0.1, −0.2, −0.3)
T_4	(0.3, 0.2, 0.1, −0.2, −0.1,−0.4)	(0.3, 0.4, 0.1, −0.1, −0.2, −0.3)	(0.2, 0.4, 0.3, −0.1, −0.3, −0.4)	(0.2, 0.1, 0.2, −0.2, −0.5, −0.1)

Step 3.

Evaluate $r_i = BPFOWG(r_{i1}, r_{i2}, \ldots, r_{ip})$.

$r_1 = (0.327119, 0.3, 0.313267, -0.124239, -0.543269, -0.161407)$

$r_2 = (0.396485, 0, 0.275882, -0.142954, -0.293016, -0.221633)$

$r_3 = (0.19018, 0.223225, 0.351926, -0.299446, -0.225869, -0.206661)$

$r_4 = (0.257384, 0.186607, 0.162727, -0.189252, -0.225869, -0.272259)$

Step 4.

Calculate the score functions for all r_i.

$\Phi(r_1) = 0.147144$

$\Phi(r_2) = 0.246149$

$\Phi(r_3) = -0.125943$

$\Phi(r_4) = 0.108463$

Step 5.

Rank all the $r_i (i = 1, 2, \ldots, p)$ according to the score values,

$$r_2 \succ r_1 \succ r_4 \succ r_3$$

r_2 corresponds to T_2, so T_2 is the best alternative.

Using BPFHG operator

Step 1.

Construct the decision matrix given by decision maker consist on bipolar picture fuzzy information given in Tab. 9.

Table 9: BPF decision matrix taking by decision maker

	C_1	C_2	C_3	C_4
T_1	(0.5, 0.3, 0.2, −0.1, −0.6, −0.2)	(0.5, 0.3, 0.2, −0.1, −0.6, −0.2)	(0.3, 0.3, 0.1, −0.1, −0.5, −0.2)	(0.4, 0.3,0.1, −0.1, −0.4, −0.2)
T_2	(0.2, 0.4, 0.1, −0.1, −0.3, −0.2)	(0.2, 0.4,0.3, −0.1, −0.2, −0.1)	(0.2, 0.2,0.4, −0.1, −0.2, −0.4)	(0.2, 0, 0.2, −0.1, −0.6, −0.2)
T_3	(0.3, 0.4, 0.1, −0.2, −0.3, −0.4)	(0.2, 0.1, 0.4, −0.4, −0.2, −0.1)	(0.1, 0.3, 0.4, −0.2, −0.2,−0.4)	(0.2, 0.4, 0.3, −0.1, −0.2, −0.3)
T_4	(0.1, 0.2, 0.3, −0.4, −0.1, −0.2)	(0.3, 0.4, 0.1, −0.1, −0.2, −0.3)	(0.2, 0.4, 0.3, −0.1, −0.3, −0.4)	(0.2, 0.1, 0.2, −0.2, −0.5, −0.1)

Step 2.

Normalize the decision matrix, because the attribute C_1 = price, given in Tab. 10.

Table 10: Normalized BPF decision matrix

	C_1	C_2	C_3	C_4
T_1	(0.2, 0.3, 0.5, −0.2, −0.6, −0.1)	(0.5, 0.3,0.2, −0.1, −0.6, −0.2)	(0.3, 0.3, 0.1, −0.1, −0.5, −0.2)	(0.4, 0.3,0.1, −0.1, −0.4, −0.2)
T_2	(0.1, 0.4,0.2, −0.2, −0.3, −0.1)	(0.2, 0.4,0.3, −0.1, −0.2, −0.1)	(0.2, 0.2,0.4, −0.1, −0.2, −0.4)	(0.2, 0, 0.2, −0.1, −0.6, −0.2)
T_3	(0.1, 0.4, 0.3, −0.4, −0.3, −0.2)	(0.2, 0.1, 0.4, −0.4, −0.2, −0.1)	(0.1, 0.3, 0.4, −0.2, −0.2,−0.4)	(0.2, 0.4, 0.3, −0.1, −0.2, −0.3)
T_4	(0.3, 0.2, 0.1, −0.2, −0.1, −0.4)	(0.3, 0.4, 0.1, −0.1, −0.2, −0.3)	(0.2, 0.4, 0.3, −0.1, −0.3, −0.4)	(0.2, 0.1, 0.2, −0.2, −0.5, −0.1)

Step 3.

Evaluate $r_i = BPFHG(r_{i1}, r_{i2}, \ldots, r_{ip})$.

Before evaluating r_i we use standard WV to find the $\ddot{T}_j$ given in Tab. 11, where $\ddot{T}_j = nw_j T_j$.

Table 11: Values of $\ddot{T}_j$

	C_1	C_2	C_3	C_4
$\ddot{T}_1$	(0.19, 0.38, 0.43, −0.17, −0.66, −0.08)	(0.45, 0.38,0.16, −0.09, −0.66, −0.16)	(0.28, 0.38, 0.08, −0.09, −0.57, −0.16)	(0.37, 0.38,0.08, −0.09, −0.48, −0.17)
$\ddot{T}_2$	(0.10, 0.33,0.23, −0.20, −0.24, −0.12)	(0.21, 0.33,0.35, −0.09, −0.15, −0.12)	(0.19, 0.15,0.46, −0.09, −0.15, −0.49)	(0.15, 0, 0.23, −0.11, −0.54, −0.23)
$\ddot{T}_3$	(0.10, 0.33, 0.35, −0.17, −0.06, −0.46)	(0.17, 0.06, 0.46, −0.40, −0.15, −0.12)	(0.10, 0.24, 0.46, −0.19, −0.15,−0.46)	(0.20, 0.33, 0.35, −0.09, −0.14, −0.35)
$\ddot{T}_4$	(0.30, 0.28, 0.08, −0.22, −0.16, −0.34)	(0.27, 0.48, 0.08, −0.11, −0.28, −0.25)	(0.18, 0.48, 0.25, −0.10, −0.38, −0.34)	(0.22, 0.16, 0.16, −0.18, −0.57, −0.08)

Now,

$$r_1 = (0.303975, 0.38, 0.260786, -0.119664, -0.610262, -0.130955)$$

$$r_2 = (0.30602, 0, 0.294798, -0.146554, -0.265846, -0.241942)$$

$$r_3 = (0.179348, 0.161636, 0.407547, -0.226579, -0.112387, -0.318708)$$

$$r_4 = (0.242059, 0.309948, 0.175018, -0.162835, -0.350119, -0.261469)$$

Step 4.

Calculate the score functions for all r_i.

$\Phi(r_1) = 0.284742$

$\Phi(r_2) = 0.372456$

$\Phi(r_3) = -0.185319$

$\Phi(r_4) = 0.205846$

Step 5.

Rank all the $r_i (i = 1, 2, \ldots, p)$ according to the score values,

$r_2 \succ r_1 \succ r_4 \succ r_3$

r_2 corresponds to T_2, so T_2 is the best alternative.

The proposed AOs BPFWG operator, BPFOWG operator and BPFHG operator are compared as shown in Tab. 12 below, which lists the final comparative study ranked among the top four alternatives. The best selection made by any of the proposed operators and current operators, as shown in Tab. 12, validates the consistency and authenticity of the proposed methods.

Table 12: Comparison analysis of the proposed operators and existing operators in the given numerical example

Method	Ranking of alternatives	The optimal alternative
PFWA (Garg [40])	$T_2 \succ T_1 \succ T_4 \succ T_3$	T_2
PFOWA (Garg [40])	$T_2 \succ T_1 \succ T_3 \succ T_4$	T_2
PFHA (Garg [40])	$T_2 \succ T_1 \succ T_3 \succ T_4$	T_2
PFWG (Wang et al. [42])	$T_2 \succ T_3 \succ T_4 \succ T_2$	T_2
PFOWG (Wang et al. [42])	$T_2 \succ T_3 \succ T_4 \succ T_2$	T_2
PFHG (Wang et al. [42])	$T_2 \succ T_4 \succ T_1 \succ T_3$	T_2
PFDWA (Jana et al. [41])	$T_2 \succ T_3 \succ T_4 \succ T_1$	T_2
PFDOWA (Jana et al. [41])	$T_2 \succ T_3 \succ T_4 \succ T_1$	T_2
PFDHWA (Jana et al. [41])	$T_2 \succ T_3 \succ T_1 \succ T_4$	T_2
PFDWG (Jana et al. [41])	$T_2 \succ T_1 \succ T_4 \succ T_3$	T_2
PFDOWG (Jana et al. [41])	$T_2 \succ T_1 \succ T_3 \succ T_4$	T_2
PFDHWG (Jana et al. [41])	$T_2 \succ T_1 \succ T_4 \succ T_3$	T_2
BPFWG (Proposed)	$T_2 \succ T_1 \succ T_4 \succ T_3$	T_2
BPFOWG (Proposed)	$T_2 \succ T_1 \succ T_4 \succ T_3$	T_2
BPFHG (Proposed)	$T_2 \succ T_1 \succ T_4 \succ T_3$	T_2

7 Conclusion

MCDM has been studied to solve complex real-world problems that involve uncertainty, imprecision and ambiguity due to vague and incomplete information. The MCDM techniques practically rely on fuzzy sets and fuzzy models that are considered to address vagueness and uncertainties. The existing fuzzy set theoretic models fail to deal with real life situations when modeling need to assign bipolarity (positive and negative aspects) to each of the degrees of MD (belonging-ness), neutral MD (not-decided), and NMD (refusal). In order to handle such MCDM problems, in this study, we introduced a new extension of fuzzy sets named as BPFS. A BPFS is the hybrid structure of BFS and PFS. The notion of a bipolar picture fuzzy number (BPFN) is superior than existing bipolar fuzzy number and picture fuzzy number. We introduced some algebraic operations and key properties of BPFSs as well as some new distance measures of BPFSs. We presented score function, accuracy function and certainty function for bipolar picture fuzzy information aggregation. Information aggregation plays an important role in the MCDM, and therefore in this study, some new aggregation operators (AOs) named as "bipolar picture fuzzy weighted geometric operator, bipolar picture fuzzy ordered weighted geometric operator, and bipolar picture fuzzy hybrid geometric operator" are developed. Additionally, on the basis of these AOs, a new MCDM approach has been developed for the ranking of objects using BPFNs. The presented scientific method is illustrated by a numerical model to demonstrate its effectiveness and sustainability.

In further research, we can extend proposed aggregation operators to some other MCDM techniques including; TOPSIS, VIKOR, AHP, ELECTRE family and PROMETHEE family. Long term work will pay special attention to Heronian mean, Einstein, Bonferroni mean, Dombi AOs and so on. We keep hoping that our research results will be beneficial for researchers working in the fields of information fusion, pattern recognition, image recognition, machine learning, decision support systems, soft computing and medicine.

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