Thermal Bending of Circular Plates for Non-axisymmetrical Problems

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Abstract: Due to the complexity of thermal elastic problems, analytic solutions have be obtained only for some axisymmetrical problems and simply problems. Using the Green function, the boundary integral formula and natural boundary integral equation for the boundary value problems of biharmonic equation is obtained. Then based on bending solutions to circular plates subjected to the non-axisymmetrical load, by the Fourier series and convolution formulae, the bending solutions under non-axisymmetrical thermal conditions are gained. The formulas for the solutions have high convergence velocity and computational accuracy, and the calculating process is simpler. Examples show the discussed methods are effective.

Keywords: thermal bending problems, circular plate, boundary integral formula, natural boundary integral equation,

1 Introduction

Thermal elastic problems are important one of solid mechanics. Due to the complexity of thermal elastic problems, analytic solutions have been obtained only for some axisymmetrical problems and simply problems. For general loads and general boundary conditions, the numerical computation is the main method. For bending problems of solid circular plates, Fu Bao-lian adopted the reciprocal theorem and took the solution of the clamped circular plate as the basic solution to discuss some bending solutions under axis-symmetrical loads. Wang An-wen introduced the point source function to discuss the non-symmetrical bending problems under the concentrated force; Yu De-hao discussed bending problems of plates with the natural boundary element method. Using the above method, Li Shun-cai discussed the bending problems of solid circular plates] and bending deflections for annular infinite plates under the boundary loads. On the basis of the same method, expanding the boundary slope into Fourier series, and using several convolution formulae, the boundary integral formula and natural boundary integral equation for the boundary value problems of thermal bending of Circular Plates are obtained. The formula for the solutions has high convergence velocity and computational accuracy, and the calculating process is simple. Examples show the discussed methods are effective.

2 Boundary integral formula and natural boundary integral equation

The differential equation of elastic plate bending problems is:

$$\Delta^2 u = \frac{q(r,\theta)}{D} = f(r,\theta) \ \Omega \tag{1}$$

Where, Δ is the Laplacian operator, *u* is the deflection of the plate, *q* is the surface density of external loads,*D* is the bending rigidity of the plate, Ω is the plate in a circle domain. For convenient, suppose the circle is a unit circle.

Using the Green formula of the bending problems for thin plates, we get:

$$\iint_{\Omega} (u\Delta^2 v - v\Delta^2 u) dp = \int_{\Gamma} (u\frac{\partial}{\partial n}\Delta v - \frac{\partial u}{\partial n}\Delta v + \frac{\partial v}{\partial n}\Delta u - v\frac{\partial}{\partial n}\Delta u) ds + \iint_{\Omega} vfdp \quad (2)$$

Where dp = dxdy, Γ is the edge of the circular plate. Suppose u = u(p) satisfying the biharmonic equation, and let v = G(p, p'), which is the Green function of the biharmonic equation in Ω , and then the Poisson integral equation of the bending problem of the plate can be found

$$u(p) = \int_{\Gamma} \left[\frac{\partial}{\partial n'} \Delta' G(p, p') u_0(p') - \Delta' G(p, p') u_n(p') \right] \mathrm{d}s' + \iint_{\Omega} G(p, p') f(p') \mathrm{d}p',$$
$$p \in \Omega, \quad (3)$$

Where p = (x, y), p' = (x', y'), $u_n = \frac{\partial u}{\partial n}|_{\Gamma}$, dp' = dx'dy', Δ' is the Laplacian related to p'. The Green function outer the unit circular domain can be obtained from the basic solution of the biharmonic equation

$$G(p,p') = \frac{1}{16\pi} \left\{ \left[r^2 + r'^2 - 2rr'\cos(\theta - \theta') \right] \ln \frac{r^2 + r'^2 - 2rr'\cos(\theta - \theta')}{1 + r^2r'^2 - 2rr'\cos(\theta - \theta')} + (1 - r^2)(1 - r'^2) \right\}$$
(4)

Where, *P* and *P'* represent the polar coordinate (r, θ) and (r', θ') respectively. Thus

$$-\Delta' G|_{r'=1} = -\frac{(1-r^2)^2}{4\pi \left[1+r^2-2r\cos(\theta-\theta')\right]}$$

$$\left. \frac{\partial}{\partial n'} \Delta' G \right|_{r'=1} = \frac{(1-r^2)^2 \left[1 - r\cos(\theta - \theta') \right]}{2\pi \left[1 + r^2 - 2r\cos(\theta - \theta') \right]^2}$$

Hence, the Poisson integral formula of the bending circular plates

$$u(r,\theta) = \frac{(1-r^2)^2 [1-r\cos(\theta-\theta')]}{2\pi [1+r^2-2r\cos(\theta-\theta')]^2} * u_0(\theta) - \frac{(1-r^2)^2}{4\pi [1+r^2-2r\cos(\theta-\theta')]} * u_n(\theta) + \iint_{\Omega} G(r,\theta;r',\theta') f(r',\theta') r' dr' d\theta'$$

Where, * is the convolution with regard to θ , $u_0(\theta)$, $u_n(\theta)$ denotes the deflection and slope at the edge. For the supported edge, $u_0 = 0$, the above equation will be educed to

$$u(r,\theta) = \iint_{\Omega} G(r,\theta;r',\theta') f(r',\theta') r' dr' d\theta' - \frac{(1-r^2)^2}{4\pi [1+r^2-2r\cos(\theta)]} * u_n(\theta)$$
(5)

Suppose M is the differential boundary operator in the polar coordinate system, the bending moment Mu

$$Mu = \left[\mu\Delta u + (1-\mu)\frac{\partial^2}{\partial r^2}u\right]_{\Gamma} = -\frac{M_r}{D}$$
(6)

Where, μ is Poisson ratio. Let the boundary operator acts on Eq. (5), and use the limit formula of generalized function, the natural boundary integral equation of the bending problems [Li (2002)] can be obtained as

$$Mu = \iint_{\Omega} MG(r,\theta;r',\theta')f(r',\theta')r'dr'd\theta' - \pounds 1 + \mu \pounds u_n(\theta) - \frac{1}{2\pi \sin^2(\frac{\theta}{2})} * u_n(\theta)$$
(7)

3 Thermal elastic equation and boundary conditions

The thermal elastic equation is

$$\Delta^2 u = \frac{q^*(r,\theta)}{D} = f(r,\theta)$$

Where q^* is the surface distribution density of the equivalent load. Suppose *h* is the thickness of the plate, *E* is elastic modulus, α is the thermal expansion coefficient. *D* is the bending rigidity. In general, suppose the thermal linear distribution along the plate thickness:

$$q^* = -\frac{1}{1-\mu}\Delta M_T, f(r,\theta) = -\frac{1}{(1-\mu)D}\Delta M_T = -\frac{\alpha(1+\mu)}{h}\Delta T(r,\theta)$$

Where $T(r, \theta)$ is the thermal distribution function on the surface of the plate,

$$M_T = \alpha E \int_{-\frac{h}{2}}^{\frac{h}{2}} T(r, \theta) z dz = \frac{\alpha E h^2}{12} T(r, \theta)$$

The equivalent boundary condition of the clamped bending plate is $u|_{\Gamma} = 0$, $u_n|_{\Gamma} = 0$. The equivalent boundary condition of the simply bending plate is

$$u|_{\Gamma} = 0,$$

$$Mu = -\frac{M_T}{D(1-\mu)} = -\frac{\alpha T(1,\theta)(1+\mu)}{h} \pounds \text{ on } \Gamma$$

If in the plate there is no heat source, $\Delta T(r, \theta) = 0$, $q^*=0$, for the simply plate, $u_0(\theta) = 0$, equation (5) and (6), will be reduced to

$$u(r,\theta) = -\frac{(1-r^2)^2}{4\pi \left[1+r^2 - 2r\cos(\theta - \theta')\right]} * u_n(\theta)$$
(8)

$$-\frac{\alpha T(1,\theta)(1+\mu)}{h} = \frac{1+\mu}{R^2} u_n(\theta) - \frac{1}{2\pi R^2 \sin^2(\frac{\theta}{2})} * u_n(\theta)$$
(9)

For the clamped plate, if in the plate there is no heat source, $\Delta T(r, \theta) = 0$, $q^*=0$, there are no deflections. The following are the plates on the head sources.

Example 1 Suppose $T(r, \theta) = 1 + r$

$$f(r,\theta) = -\frac{\Delta M_T}{D(1-\mu)} = \frac{\alpha(1+\mu)}{hr} \pounds$$

For the clamped plate on the heat source, from eq. (5)

$$u(r,\theta) = \int_{0}^{2\pi} \int_{0}^{1} G(r,\theta;r',\theta') f(r',\theta') r' dr' d\theta'$$

$$u(r,\theta) = \int_{0}^{2\pi} \int_{0}^{1} G(r,\theta;r',\theta') f(r',\theta') r' dr' d\theta' = \frac{\alpha \pounds 1 + \mu}{h} \int_{0}^{2\pi} \int_{0}^{1} G(r,\theta;r',\theta') dr' d\theta'$$

The numerical computation is according to the axisymmetrical solution

$$u(r,\theta) = \frac{\alpha \pounds 1 + \mu}{2h} \left(\frac{1}{3}\left(\frac{2}{3}r^3 - r^2\right) + \frac{1}{9}\right)$$

For the simply plate on the heat source, firstly using the following equation to get u_n

$$Mu - \int_{0}^{2\pi} \int_{0}^{1} G(r,\theta;r',\theta') f(r',\theta')r'dr'd\theta' = \frac{1+\mu}{R^2} u_n(\theta) - \frac{1}{2\pi R^2 \sin^2(\frac{\theta}{2})} * u_n(\theta)$$

$$MG(r,\theta;r',\theta') = \frac{\pounds 1 - r^2)^2}{4\pi (1+r'^2 - 2r'\cos(\theta - \theta'))}$$
(10)

$$-\frac{\alpha T(1,\theta)(1+\mu)}{h} + \frac{\alpha \pounds 1 + \mu \pounds}{3h} = (1+\mu)u_n(\theta)$$
$$u_n(\theta) = -\frac{5\alpha}{3h}$$

Then using the convolution formula

$$\frac{1-r^2}{2\pi(1+r^2-2r\cos\theta)}*\cos k\theta=r^k\cos k\theta$$

We can get

$$u(r,\theta) = -\frac{(1-r^2)^2}{4\pi[1+r^2-2r\cos(\theta-\theta')]} * u_n(\theta) + \int_{0}^{2\pi} \int_{0}^{1} G(r,\theta;r',\theta') f(r',\theta') r' dr' d\theta'$$

= $\frac{5\alpha(1-r^2)}{6h} - \frac{\alpha\epsilon(1+\mu)}{2h} \left(\frac{1}{3}\left(\frac{2}{3}r^3 - r^2\right) + \frac{1}{9}\right)$

Example 2 Suppose the center of $T(r, \theta)$ is $(\frac{1}{2}, 0)$

$$T(r,\theta) = \sqrt{r^2 + \left(\frac{1}{2}\right)^2 - 2r \cdot \frac{1}{2}\cos\theta}$$

$$f(r,\theta) = -\frac{1}{(1-\mu)D} \Delta M_T = -\frac{\alpha(1+\mu)}{h} \Delta T(r,\theta) = \frac{\alpha(1+\mu)}{h} \times \frac{2}{\sqrt{4r^2 - 4\cos\theta + 1}}$$



120 60 0.8 0.6 150 30 0.4 180 0 0.61488 210 0.50364 330 0.33677 16991 240 300 270

90 1

Figure 1: Deflections of the clamped circular plate

Figure 2: Deflections of the clamped plate

For the clamped plate on the heat source, from eq. (5)

$$u(r,\theta) = \int_{0}^{2\pi} \int_{0}^{1} G(r,\theta;r',\theta') f(r',\theta') r' dr' d\theta'$$

For the simply plate on the heat source, firstly using eq. (11) to get u_n , suppose

$$u_n(\theta) = \sum_{m=0}^{\infty} b_m \cos m\theta + \sum_{m=1}^{\infty} b'_m \sin m\theta$$

The left of Eq. (11) is expanded to series

$$\frac{\alpha(1+\mu)}{h} \times T(1,\theta) + \int_{0}^{2\pi} \int_{0}^{1} \frac{2\ell 1 - r^2}{4\pi (1+r'^2 - 2r'\cos(\theta - \theta'))\sqrt{4r'^2 - 4\cos\theta' + 1}} dr' d\theta$$
$$= \sum_{m=0}^{\infty} a_m \cos m\theta + \sum_{m=1}^{\infty} a'_m \sin m\theta$$

Substituting the above equation into Eq. (11) which is an integral with a strongly singular Poisson kernel, and using the convolution formula, we can get

$$b_k = \frac{1}{1 + \mu + 2k} a_k b'_k = \frac{1}{1 + \mu + 2k} a'_k$$

Then,

$$\begin{split} u(r,\theta) &= -\frac{(1-r^2)^2}{4\pi [1+r^2-2r\cos(\theta)]} * \sum_{k=0}^{\infty} b_k \cos k\theta - \frac{(1-r^2)^2}{4\pi [1+r^2-2r\cos(\theta)]} * \sum_{k=0}^{\infty} b'_k \sin k\theta \\ &+ \int_0^{2\pi} \int_0^1 G(r,\theta;r',\theta') f(r',\theta') r' dr' d\theta' \\ &= \sum_{k=0}^{\infty} \frac{a_k (1-r^2)r^k}{2(1+\mu+2k)} \cos k\theta + \sum_{k=1}^{\infty} \frac{a'_k (1-r^2)r^k}{2(1+\mu+2k)} \sin k\theta + \int_0^{2\pi} \int_0^1 G(r,\theta;r',\theta') f(r',\theta') r' dr' d\theta' \end{split}$$

Suppose $\mu=0.3$, D=1, $\alpha \dot{A}/h=1$, the deflections of the circular plates from example 2 are following

4 Conclusions

Based on Green function method, the boundary integral formula and natural boundary integral equation with strongly singular kernel are educed for the biharmonic equation of the thermal bending problem of the plate supported at the boundary. The convolution formulae are utilized to get the solutions of deflection and slope directly for simple problems. As to complex problems, the Fourier series is be used to get the solutions which have nicer convergence velocity and computational accuracy, and the calculating process is simpler. For the other complicated load, it can be solved with the similar method or by the superposition with the solutions of above examples.

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