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The Direct Coupling Method of Natural Boundary Element and Finite Element on Elastic Plane Problem in Unbounded Domain

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Abstract: The advantage of the coupling method of natural boundary element method(NBEM) and finite element method (FEM) is introduced firstly. Then the principle of the direct coupling method of NBEM and FEM, and its implementation, are discussed. The comparison of results between the direct coupling method and FEM proves that the direct coupling method is simple, feasible and valid in practice.

Keywords: FEM; NBEM; coupling method; elastic plane problem in unbounded domain

1 Introduction

The FEM method is one of the most widely used computing method in Computational Mechanics. It is convenient to solve problems of nonlinear equation or asymmetric medium with high precision[1]. But the FEM method is not directly designed to account for problems in unbounded domain. Usually these problems were solved in bounded domain approximately, which would have bigger error. In order to overcome the shortcoming, some scholar developed a kind of coupling method of boundary element method(BEM), which can be used to solve problems in unbounded domain, and FEM method. Liu chunfeng and Zai Ruicai studied the coupling method of boundary element and finite element to calculate wave force[2]; He Yinnian solved Navier-Stokes equation with the coupling method[3]; G. N. Gatica and other scholars studied linear exterior boundary value problems with a domain decomposition method based on BEM and FEM[4] . But it is very complex to construct the rigidness matrix. And the results from the coupling method is not very ideal because some good properties can not maintain during boundary reducing. Yu Dehao and other scholars developed a new type boundary element method, natural boundary element method(NBEM), which has not only the advantage of being used

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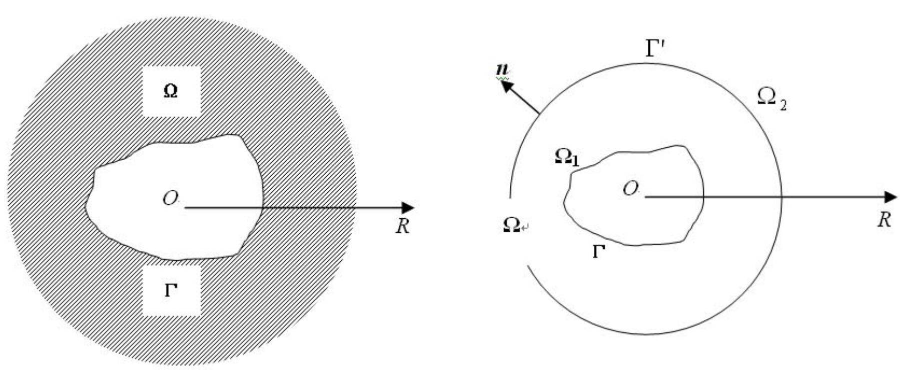
to solve problems in exterior domain, but also some other advantage such as direct derivation, unique form of equation, small calculation quantity and energy function being able to keep invariable before and after the boundary reduction. Especially NBEM and FEM base on the same invariational principle, which can lead to a direct and natural coupling. The coupling method was applied to study a torsion problem of the square cross-section bar with cracks by Zhao Huiming and et al[6]. Du Qikui studied the parabolic equation based on natural boundary reduction[7]. The NBEM is also called Dirichlet-to-Neumann (DtN) mapping method outside China. G. N. Gatica studied variational formulations of transmission problems via FEM, BEM and DtN mappings[8]. G. K. Gächter and M. J. Grote uesd DtN mapping method for three-dimensional elastic waves[9]. Miroslav Premrov and Igor Spacapan solved exterior problems of wave propagation based on an iterative variation of local DtN operators[10]. This paper focuses on solving elastic plane problem in unbounded domain by the direct coupling method of natural boundary element and finite element.

2 The Principle of Coupling Method of NBE and FE

Consider the following boundary problem in the domain noted in Fig. 1(a):

$$\begin{cases} \mu\Delta\vec{u}+(\lambda+\mu)\text{grad div}\vec{u}=0 & \text{in} \quad \Omega \\ \sum\limits_{j=1}^{2}\sigma_{ij}n_j=\overrightarrow{g} & \text{on} \quad \Gamma \end{cases} \tag{1}$$



(a) Unbounded domain with hole (b) Decomposed domain

Figure 1: The domain where a problem to be solved

Let:

$$
\begin{aligned}
&D(\vec{u},\vec{v}) = \iint\limits_{\Omega} \sum_{i,j=1}^{2} \sigma_{ij}(\vec{u})\varepsilon_{ij}(\vec{v})\,dp \\
&F(v) = \int_\Gamma \vec{g}\cdot\vec{v}ds
\end{aligned}
$$

Then the boundary problem (1) is equivalent to the following variational problem:

$$
\begin{cases}
\text{Find} \quad \vec{u}\in W_0^1(\Omega)^2 \quad \text{such that} \\
D(\vec{u},\vec{v}) = F(\vec{v}), \quad \forall\vec{v}\in W_0^1(\Omega)^2
\end{cases}
\tag{2}
$$

where

$$
W_0^1(\Omega)^2 = \left\{ \frac{u}{\sqrt{1+r^2}\ln(2+r^2)} \in L^2(\Omega),\ \frac{\partial u}{\partial x_i} \in L^2(\Omega),\ i=1,2,\ r=\sqrt{x_1^2+x_2^2} \right\}
$$

For elastic plane problem in unbounded domain, zero-strain state contains only rigid translational displacement. Let

$$
\Re = \{(C_1,\ C_2)\,|_{C_1,\,C_2\in R}\}
$$

And then variational problem (2) exits unique solution in quotient space $W_0^1(\Omega)^2/\Re$. Draw a circle Γ' with radius $R$to divide domain Ω into two parts, Ω_1 and Ω_2. Subdomain Ω_2 shown in Fig. 1(b) is an exterior circular domain. At the same time the acting domain of bilinear form $D(\vec{u},\vec{v})$ is decomposed into Ω_1 and Ω_2. So new bilinear forms $D_i(\vec{u},\vec{v})$, $i=1,2$ are obtained, and

$$
D(\vec{u},\vec{v}) = D_1(\vec{u},\vec{v}) + D_2(\vec{u},\vec{v}) \tag{3}
$$

The FEM method can be directly used in domain Ω_1 to construct its rigidness matrix while the NBEM method is applied in exterior domain Ω_2. Let K is a natural integral operator of elastic plane problem in an exterior domain outside a circle with radius R. So we can get:

$$
D_2(\vec{u},\vec{v}) = \int_{\Gamma'} \vec{v}_0\cdot K\vec{u}_0 ds
$$

Then the variational problem (2) is equivalent to the following variational problem:

$$
\begin{cases}
\text{Find} \quad \vec{u}\in W_0^1(\Omega)^2, \quad \text{such that} \\
D(\vec{u},\vec{v}) = F(\vec{v}), \quad \forall\vec{v}\in W_0^1(\Omega)^2
\end{cases}
\tag{4}
$$

3 Implementing the Direct Coupling Method of NBEM and FEM

The rigidness from FEM can be obtained by discretizingD_1in domain Ω_1, which will not be described in detail. Our focus is in implementing natural boundary reduction in domain Ω_2 for D_2.

Divide the artificial boundary Γ' into N equal parts. And the piecewise linear basis function can be expressed:

$$L_i(\theta) = \begin{cases} N(\theta-\theta_{i-1})/2\pi, & \theta_{i-1} \le \theta \le \theta_i, \\ N(\theta_{i+1}-\theta)/2\pi, & \theta_i \le \theta \le \theta_{i+1} \\ 0, & \text{other} \end{cases} \tag{5}$$

Let

$$u^h_{r0}(\theta) = \sum_{j=1}^{N} U_j L_j(\theta), \quad u^h_{\theta 0}(\theta) = \sum_{j=1}^{N} V_j L_j(\theta), \tag{6}$$

where $U_j$and V_j(j=1,. . . .., N) are undetermined coefficients. Then the rigidness matrix of NBEM in an exterior circle domain:

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \tag{7}$$

where

$$Q_{lm} = \left[q^{(lm)}_{ij} \right]_{i,j=1,\ldots,N,} \qquad l,\ m = 1,\ 2$$

$$q^{(11)}_{ij} = \hat{D}(L_j, \quad 0; \quad L_i, \quad 0), \quad q^{(12)}_{ij} = \hat{D}(0, \quad L_j; \quad L_i, \quad 0),$$

$$q^{(21)}_{ij} = \hat{D}(L_j, \quad 0; \quad 0, \quad L_i), \quad q^{(22)}_{ij} = \hat{D}(0, \quad L_j; \quad L_i \quad 0),$$

i, j=1, 2, . . . , N¡£

Using the method of series of integral kernel and following formula

$$-\frac{1}{4\sin^2\frac{\theta}{2}} = \sum_{n=1}^{\infty} n\cos n\theta, \quad \frac{1}{2}\mathrm{ctg}\frac{\theta}{2} = \sum_{n=1}^{\infty} \sin n\theta,$$

the matrix Q can be calculated as follows:

$$Q_{11} = Q_{22} = \frac{2ab}{a+b}((a_0, a_1, ..., a_{N-1})) + \frac{2\pi b^2}{3N(a+b)}((4,1,0,...,0,1)) + \frac{4\pi ab}{N^2(a+b)}((1,\ ...,\ 1))$$

$$Q_{12} = -Q_{21} = \frac{2ab}{a+b}((0,\ d_1\ ...,\ d_{N-1})) + \frac{b^2}{a+b}((0,\ 1,\ 0,\ ...,\ -1)),$$

where

$$a_k = \frac{4N^2}{\pi^3}\sum_{j=1}^{\infty}\frac{1}{j^3}\sin^4\frac{j\pi}{N}\cos\frac{jk}{N}2\pi,$$

$$d_k = \frac{4N^2}{\pi^3}\sum_{j=1}^{\infty}\frac{1}{j^3}\sin^4\frac{j\pi}{N}\sin\frac{jk}{N}2\pi,$$

k=0, 1, 2, …, N-1

Obviously Q_{11} and Q_{22} are symmetric circulant matrices, Q_{12} and Q_{21} are anti-symmetric circulant matrices, Q is a semi-positive defined symmetric matrix.

The so-called direct coupling of NBEM and FEM is that the domain in which NBEM is applied is regard as a special element of FEM while coupling. So the total rigidness matrix can be constructed by direct addition of the rigidness matrix from FEM and the one from NBEM. Finally the linear algebraic equations can be solved to obtain the solutions.

4 Example Analysis

Example 1 There is an elastic plane problem in unbounded domain with a square hole, shown in Fig. 2. Let elastic modulus E=40Gpa, Poisson's ratio μ=0.3. And there are uniformly distributed load q=100kN/m acting on edges of the square hole.

In order to guarantee the uniqueness of the solution and the symmetry of constraint, some condition is generally added. Here we assume the displacements at point (-2,0) and point (2,0) are zero. So we can only take one fourth of the model to study because of the symmetry of its structure, constraints and loads.

Some different values of R are taken to solve numerically the problem by the coupling method of NBEM and FEM. We first get the displacement of domain Ω_1, and compare these results with that obtained by FEM method. The programs of the

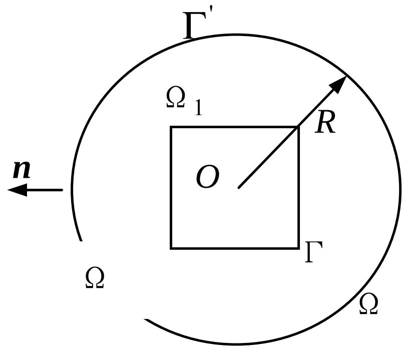


Figure 2: Elastic plane problem in unbounded domain with a square hole

coupling method and FEM method are coded by software MATHACAD. And their meshing style are the same, mesh density are similar. The below tables show the difference of these results at point (2,2) obtained by NBEM and FEM respectively.

Table 1: The displacement along x axis at point (2, 2)

Radius/m / Methods	5	15	30	50	100
Coupling Method	1.4273E-08	1.3354E-08	1.3234E-08	1.3175E-08	1.3126E-08
FEM	1.1101E-08	1.3452E-08	1.3268E-08	1.3187E-08	1.3126E-08

Table 2: The displacement along y axis at point (2, 2)

Radius/m / Methods	5	15	30	50	100
Coupling Method	4.08361E-08	4.5311E-08	4.4490E-08	4.4185E-08	4.3964E-08
FEM	2.2245E-08	4.1284E-08	4.3462E-08	4.3815E-08	4.3965E-08

From Tab. 1 and 2, it can be seen that the results from the coupling method of NBEM and FEM can easily approximate convergence value with small R. The FEM method can also approximate the convergence value, but it will cost more computational complexity with bigger R. The results from these two method approximate to be the same when R=100m. It shows that the results with sufficient precision can be obtained with small solving domain, which is what the significant of the coupling method is.

Example 2 With the same model with example, there is a concentrated force F =100kN acting at the middle of top edge.

The load is not symmetrical load. Can we carry out ideal results by the coupling method? Let's see its results. The results at point (2,2) are chosen to be compared in below table.

Table 3: The displacement along x axis at point (2, 2)

Radius/m Methods	5	15	50	100	200
Coupling Method	-6.0862E-07	-5.4047E-07	-5.1977E-07	-5.1559E-07	-5.1352E-07
FEM	-2.2684E-07	-4.1022E-07	-4.3859E-07	-4.4824E-07	-4.5540E-07

Table 4: The displacement along y axis at point (2, 2)

Radius/m Methods	5	15	50	100	200
Coupling Method	2.2367E-06	1.9027E-06	1.8022E-06	1.7821E-06	1.7723E-06
FEM	3.6947E-07	1.0670E-06	1.3012E-06	1.3659E-06	1.4131E-06

The results in Tab. 3 and 4 show that the tendency of convergence of the coupling method is the same with FEM. But these two methods do not converge to same value. What is the reason? We want to known which result is better. So the same problem is studied with different solving domain and different meshing style by software ANSYS.

(1) Let R, the radius of solving domain, be 50m. And use the item 'Smartsize' in *Meshtool* with mesh density up to grade 1. Under the circumstances the value of displacement at point (2,2) along x axis is -4.9989$\times 10^{-7}$m, which is more approximate to -5.1352$\times 10^{-7}$ m, the value from the coupling method. The value, 1.6987$\times 10^{-6}$m, of displacement at point (2,2) along y axis is also approximate to 1.7723$\times$10-8 m, the value from the coupling method¡£

(2) Let R be 50m with using the item 'Smartsize' in *Meshtool* with mesh density up to grade 1. And Also refine the elements near the square hole with refining grade up to 3. Under the circumstances the value of displacement at point (2,2) along x axis is -5.0169$\times 10^{-7}$m The difference between the value and that from the coupling method becomes smaller.

From the above examples it can be seen that the stability of the coupling method is better than FEM method, and that the precision of the coupling method is also higher. The FEM method would need more computational complexity to reach the similar value.

5 Conclusion

Based on the above research, some conclusions can be drew:

(1) The procedure for the coupling of NBEM and FEM is simple and direct. The sub-domain in which the NBEM method is applied can be regard as a special element of FEM method because FEM method is based on the same variational principle. So the total rigidness matrix can be easily constructed.

(2) The results from the coupling method can approximate value with ideal precision when R is very small. It will save lots of finite elements. So the computational efficient of the coupling method is higher than FEM method.

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