 SL, vol.8, no.2, pp.99-107, 2012

An Exact Boundary Integral Equation Formulation for BEM Thermoelastic Analysis of Transversely Anisotropic Solids

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Abstract: In BEM analysis of generally anisotropic solids, the additional volume integral associated with thermal effects that appears in the direct formulation of the boundary integral equation (BIE) has hitherto been successfully transformed in an analytically exact fashion into surface ones only for two-dimensions (2D), and not for the three-dimensional (3D) case. This is due to the mathematical complexity of the Green's function and its derivatives for the 3D solid. The presence of the domain integral destroys the distinctive feature of the boundary element method (BEM) as a truly boundary solution numerical analysis tool. As a precursor to treating this problem in 3D general anisotropy, the exact volume-to-surface integral transformation associated with thermal effects is successfully carried out in this study for the special case of 3D transverse isotropy and implemented in a BEM formulation. It follows a similar approach previously employed by the authors for the same task in 2D generally anisotropic thermoelastic BEM analysis. However, a numerical scheme needs to be introduced to evaluate some terms in the new surface integrals of the BIE. Two examples are presented to demonstrate the veracity of the analytical and numerical formulations implemented.

Keywords: Boundary element method, transverse isotropy, thermoelasticity, Green's function.

1 Introduction

A distinctive feature of the boundary element method (BEM) is that only the surface of the solution domain needs to be modeled in engineering analysis as the numerical solution involves solving a boundary integral equation (BIE). It is wellknown, however, that for BEM thermoelastic stress analysis, the steady state thermal effects give rise to an additional volume integral term in the integral equation. This

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destroys the notion of the BEM as a truly boundary solution method if it is not further transformed into surface ones. It has been successfully achieved in an analytically exact manner for 2D and 3D isotropic thermoelasticity (see, e.g., [Rizzo and Shippy (1977); Danson (1983)]) and for 2D general anisotropy, [Shiah and Tan (1999)]. The great appeal of the exact transformation method (ETM) is that it restores the analysis to a purely boundary or surface one without introducing additional simplifications and numerical approximations, unlike some other schemes; a review of these schemes may be found in, e.g., [Rashed and Brebbia (2003); Kögl and Gaul (2003)]. The extension of the ETM to BEM thermoelastic analysis of 3D generally anisotropic bodies has remained a challenge because of the mathematical complexity of the Green's function, or fundamental solution, and its derivatives. The volume-to-surface integral transformation (VIT) involves these quantities. Indeed, their efficient numerical evaluation in BEM formulations has been a focus of several investigations over the past three decades, see, e.g., [Wilson and Cruse (1978); Sales and Gray (1998); Wang and Denda (2007); Tan, Shiah, and Lin (2009); Shiah, Tan, Sun, and Chen (2010)]. It is only recently that fully explicit algebraic forms of these solutions have been successfully implemented for BEM stress analysis of generally anisotropic 3D solids, by the authors [Tan, Shiah, and Lin (2009); Shiah, Tan, Sun, and Chen (2010)]. As a requirement for the VIT process using the approach which has been successfully carried out so far is that the temperature terms can be expressed as a potential function which satisfies the Poisson's equation in potential theory.

As a precursor to examining the feasibility of the exact volume-to-surface integral transformation for BEM thermoelastic analysis of 3D generally anisotropic solids, this paper reports on its success and implementation for the special case of transverse isotropy. The primary focus is on the attainability of valid numerical results in the BEM implementation. For this special case of transverse isotropy, the Green's function and its derivatives are available in various explicit forms, e.g. [Willis (1965); Pan and Chou (1976); Tavara, Ortiz, Mantic, and Paris (2008)]; the one by Pan and Chou (1976) was chosen because it is relatively more concise even though it has some minor limitations. The key steps involved in the VIT follow closely those carried out in 2D for generally anisotropic solids by the authors [Shiah and Tan (1999)]. The veracity of the formulations and the BEM implementation is demonstrated by two examples.

2 The Bem for Transversely Isotropic Thermoelasticity

The analytical basis of the BEM for linear elasticity is the boundary integral equation (BIE) which is an integral constraint equation relating displacements, u_i, to the tractions, t_i, on the surface S of the domain V. In the presence of a temperature field

$\Theta(q)$, it may be written as follows,

$$C_{ij}(P)u_i(P) = \int_S t_i(Q)U_{ij}(P,Q)\,dS - \int_S u_i(Q)T_{ij}(P,Q)\,dS + \int_S \gamma_{ik}\Theta(Q)U_{ij}(P,Q)n_k\,dS - \int_V \gamma_{ik}\Theta_{,k}(q)U_{ij}(P,q)\,dV \qquad (1)$$

where the leading coefficient $C_{ij}(P)$ depends upon the local geometry of S at the source point P; $U_{ij}(P,Q)$ and $T_{ij}(P,Q)$ represent the fundamental solutions of displacements and tractions, respectively, in the x_i-direction at the field point Q due to a unit load in the x_j-direction at P in a homogeneous infinite body; and γ_{ik} is the thermoelasticity tensor or thermal moduli. Consider a unit point force applied in the direction normal to the plane of isotropy of an infinite transversely isotropic solid. The displacements derived by Pan and Chou (1976) are expressed as follows:

$$U_{13} = \sum_{i=1}^{2}\left[v_i A_i \frac{x_1}{R_i R_i^*} - v_i^2(A_i+B_i)\frac{x_1x_3}{R_i^3}\right], \qquad (2a)$$

$$U_{23} = \sum_{i=1}^{2}\left[v_i A_i \frac{x_2}{R_i R_i^*} - v_i^2(A_i+B_i)\frac{x_2x_3}{R_i^3}\right], \qquad (2b)$$

$$U_{33} = \sum_{i=1}^{2}\left[-\left(\frac{c_{11}B_i + c_{44}v_i^2 A_i}{c_{13}+c_{44}}\right)\frac{1}{R_i} - \frac{(A_i+B_i)v_i^2}{c_{13}+c_{44}}\left(\frac{c_{44}\rho^2 + c_{11}x_3^2}{R_i^3}\right)\right], \qquad (2c)$$

where the constants v_i, A_i, B_i, ρ, R_i and R^* are given in their paper. Although the fundamental solution $T_{ij}(P,Q)$ can be analytically derived, they can just as conveniently computed from the first order derivatives of the above displacement fundamental solution and invoking Hooke's law.

For a transversely isotropic material, all nonzero components of the thermal moduli are given by

$$\gamma_{11} = \gamma_{22} = \gamma_0 = (c_{11}+c_{12})\alpha_0 + c_{13}\alpha_0^{'}, \gamma_{33} = \gamma_0^{'} = 2c_{13}\alpha_0 + c_{33}\alpha_0^{'} \qquad (3)$$

where α_0 is the coefficient of thermal expansion on the isotropic $(x_1-x_2)-$plane, $\alpha_0^{'}$ represents that in the third direction, and c_{ij} are the elastic stiffness coefficients. Under steady state conditions, the thermal field can be solved independently and is first obtained before the solution of the elastostatic problem. The corresponding heat conduction equation is

$$K_0\left(\frac{\partial^2\Theta}{\partial x_1^2} + \frac{\partial^2\Theta}{\partial x_2^2}\right) + K^{'}\frac{\partial^2\Theta}{\partial x_3^2} = 0, \qquad (4)$$

where the invariants K_0 and $K_0^{'}$ are defined in terms of the thermal conductivity coefficients, K_{ij}, by $K_0 = K_{11} = K_{22}$, $K_0^{'} = K_{33}$. Clearly, the volume integral on the right-hand side of eq. (1) must be transformed to surface ones to restore the BEM formulation of the BIE as a truly boundary solution process. As a first step to this end, eq. (3) can be transformed into the standard Laplace equation by domain mapping $\hat{\mathbf{x}}^{\mathbf{T}} = \mathbf{F}\mathbf{x}^{\mathbf{T}}$ where $\mathbf{F}$ is a diagonal matrix with the elements $F_{11} = F_{22} = 1$ and $F_{33} = \sqrt{K_0/K_0^{'}}$. By writing the volume integral in the mapped domain, and using Green's theorem and second identity, it can be transformed into surface ones Shiah and Tan (2011), as follows:

$$\begin{aligned} V_j &= -\int_{\hat{V}} \Gamma_{\underline{ik}} \Theta_{,\underline{k}} U_{\underline{ij}}\, d\hat{V} \\ &= \int_{\hat{S}} \Gamma_{\underline{ik}} \left[\left(\Theta W_{\underline{ijk,t}} - W_{\underline{ijk}} \Theta_{,\underline{t}} \right) n_{\underline{t}} - \Theta U_{\underline{ij}} n_{\underline{k}} \right] d\hat{S} \end{aligned} \tag{5}$$

where the underscore in the indices denote the mapped domain; $W_{\underline{ijk}}$ is a new function introduced that satisfies $W_{\underline{ijk,mm}} = U_{\underline{ij,k}}$; and $\Gamma_{\underline{ik}}$ is the thermal moduli in the mapped domain. Details of this derivation and the numerical scheme to evaluate $W_{\underline{ijk}}$ and $W_{\underline{ijk,t}}$ are given in [Shiah and Tan (2011)]. Thus the BIE becomes:

$$\begin{aligned} C_{ij}(P)u_i(P) = &\int_S t_i(Q)U_{ij}(P,Q)\, dS \\ &- \int_S u_i(Q)T_{ij}(P,Q)\, dS + \int_S \gamma_{ik}\Theta(Q)U_{ij}(P,q)n_k\, dS \\ &+ \int_{\hat{S}} \Gamma_{\underline{ik}} \left[\left(\Theta W_{\underline{ijk,t}} - W_{\underline{ijk}} \Theta_{,\underline{t}} \right) n_{\underline{t}} - \Theta U_{\underline{ij}} n_{\underline{k}} \right] d\hat{S} \end{aligned} \tag{6}$$

The numerical solution of eq. (6) can be obtained by usual nodal collocation in 3D BEM formulations. In the present work, the quadratic isoparametric element formulation is employed [Tan, Shiah, and Lin (2009); Shiah, Tan, Sun, and Chen (2010)].

3 Numerical Results

Due to space limitations, only two examples are presented here. The first example serves to demonstrate the mathematical soundness of the volume-to-surface integral transformation given in eq. (5). The second example is a simple problem of a rectangular column with a thermal gradient, the BEM solutions for which are compare with those obtained using the FEM commercial code ANSYS. For both these problems, the following transverse isotropic material properties are used:

$$C_{11} = C_{22} = 4.65GPa, C_{33} = 5.63GPa, C_{44} = C_{55} = 2.33GPa,$$
$$C_{13} = C_{23} = 1.17GPa, C_{12} = 1.24GPa \quad (7a)$$
$$\alpha_{11} = \alpha_{22} = 6.5 \times 10^{-6}(/^{circ}C), \alpha_{33} = 5.0 \times 10^{-7}(/^{\circ}C) \quad (7b)$$
$$K_0 = 355(Wm/^{\circ}C), K_0^{'} = 89(Wm/^{\circ}C). \quad (7c)$$

The physical problem used for the verification of the VIT in eq. (5) is a simple cube of unit side lengths in the $\hat{\mathbf{x}}$-coordinate system, as shown in Figure 1. Each of the faces of the cube is modeled with one 8-node quadratic isoparametric element. The thermal boundary conditions are also as indicated in the figure; the temperature field is thus given by $\Theta = 100\hat{x}_3$. The numerical values of the volume integral and the corresponding transformed surface integrals as represented in eq. (5) in the the collocated BIE for each of the nodal points are listed in Table 1. The values from direct volume integration, denoted by $(V_j)_v$, are obtained using the mathematics software MathCAD, while the transformed surface integrals are evaluated using 8-point Gauss quadrature, denoted by $(V_j)_s$. The percentage deviations between the two sets of results can be seen to be very small indeed, being all significantly less than 0.1%.

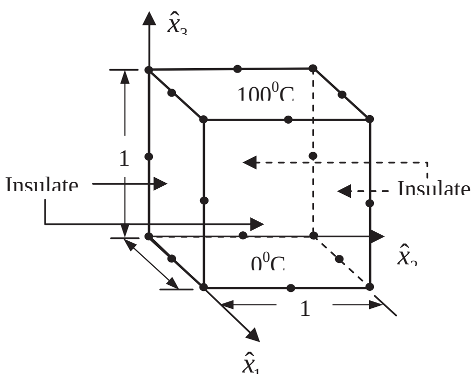

Figure 1: A unit cube with a temperature gradient – *Example I*

Figure 2 shows the second example treated with a relatively refined boundary element mesh of the full problem. The top and bottom surfaces which are fixed, have a temperature difference $\Delta\Theta = 100^{\circ}C$ while all the lateral surfaces are thermally insulated. The contour plots of the von Mises equivalent stress at the bottom $(x_3 = 0)$ and top $(x_3 = H)$ faces obtained from the BEM analysis are shown in Figures 3(a)

Table 1: Comparison between volume integrations and surface integrations for *Example I*

(x_1,x_2,x_3)	$(V_j)_v$			$(V_j)_s$			|%Diff.|		
	$j=1$	$j=2$	$j=3$	$j=1$	$j=2$	$j=3$	$j=1$	$j=2$	$j=3$
(0.0,0.0,0.0)	-0.619858	-0.619858	-6.092510	-0.620052	-0.620052	-6.092664	0.0313	0.0313	0.0025
(0.0,0.0,0.5)	-0.000000	0.000000	-6.945561	-0.001507	-0.001542	-6.946569	N/A	N/A	0.0145
(0.0,0.0,1.0)	0.619858	0.619858	-6.092510	0.620231	0.620231	-6.092351	0.0602	0.0602	0.0026
(1.0,0.0,0.0)	0.619858	-0.619858	-6.092510	0.620031	-0.620065	-6.092649	0.0279	0.0334	0.0023
(1.0,0.0,0.5)	0.000000	0.000000	-6.945561	0.001525	-0.001588	-6.946564	N/A	N/A	0.0144
(1.0,0.0,1.0)	-0.619858	0.619858	-6.092510	-0.620248	0.620190	-6.092361	0.0629	0.0536	0.0024
(1.0,1.0,0.0)	0.619858	0.619858	-6.092510	0.620044	0.620044	-6.092633	0.0300	0.0300	0.0020
(1.0,1.0,0.5)	0.000000	0.000000	-6.945561	0.001571	0.001571	-6.946558	N/A	N/A	0.0144
(1.0,1.0,1.0)	-0.619858	-0.619858	-6.092510	-0.620208	-0.620207	-6.092373	0.0565	0.0563	0.0022
(0.0,1.0,0.0)	-0.619858	0.619858	-6.092510	-0.620065	0.620030	-6.092649	0.0334	0.0277	0.0023
(0.0,1.0,0.5)	0.000000	0.000000	-6.945561	-0.001622	0.001525	-6.946564	N/A	N/A	0.0144
(0.0,1.0,1.0)	0.619858	-0.619858	-6.092510	0.620191	-0.620248	-6.092361	0.0537	0.0629	0.0024
(0.5,0.0,0.0)	0.000000	-0.889828	-7.477243	-0.000025	-0.889520	-7.478550	N/A	0.0346	0.0175
(0.5,0.0,1.0)	0.000000	0.889828	-7.477243	-0.000032	0.892791	-7.482662	N/A	0.3330	0.0725
(1.0,0.5,0.0)	0.889828	0.000000	-7.477243	0.889498	-0.000025	-7.478525	0.0371	N/A	0.0171
(1.0,0.5,1.0)	-0.889828	0.000000	-7.477243	-0.892805	-0.000033	-7.482683	0.3346	N/A	0.0728
(0.5,1.0,0.0)	0.000000	0.889828	-7.477243	-0.000025	0.889497	-7.478525	N/A	0.0372	0.0171
(0.5,1.0,1.0)	0.000000	-0.889828	-7.477243	-0.000032	-0.892805	-7.482683	N/A	0.3346	0.0728
(0.0,0.5,0.0)	-0.889828	0.000000	-7.477243	-0.889519	-0.000025	-7.478550	0.0347	N/A	0.0175
(0.0,0.5,1.0)	0.889828	0.000000	-7.477243	0.892790	-0.000033	-7.482661	0.3329	N/A	0.0725

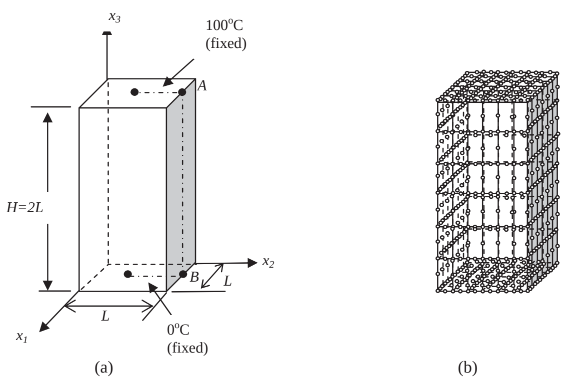

Figure 2: *Example II* - A rectangular column with a thermal load: (a) the physical problem, lateral surfaces thermally insulated, (b) the BEM mesh (216 boundary elements).

and 3(b), respectively; they have been normalized by $c_{11}\alpha_{11}\Delta\Theta$. As expected, the results are symmetric about the x_1- and x_2-axes. For verification of the BEM results, the problem was analyzed using 2000 SOLID90 and SOLID186 elements in ANSYS FEM for one-quarter model of the domain, taking advantage of symmetry. Due to space limitation, only the results for the displacements, as normalized by $\alpha_{11}\Delta\Theta L$, along the line AB (on the surface of the column at the $x_2 = L$ plane at $x_1 = 0.5L$) are presented here, in Figure 4. It can be seen that agreement between the BEM and FEM results is excellent indeed. The discrepancies between the computed results of the stresses obtained from both numerical methods are also relatively small as well; they will be shown in a forthcoming paper [Shiah and Tan (2011)].

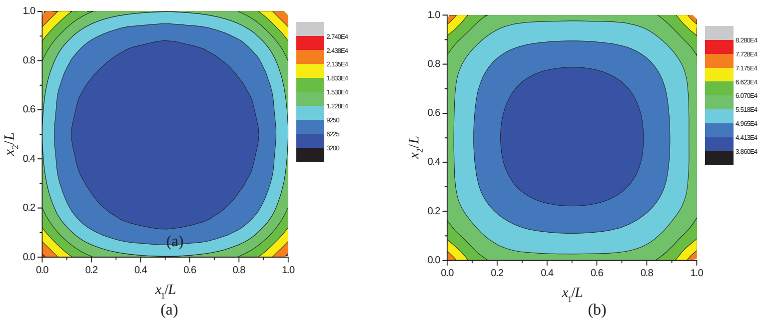

Figure 3: Contour plots of normalized von Mises equivalent stresses of the transversely isotropic rectangular column as obtained from the BEM analysis: (a) at the bottom face, $x_3 = 0$; and (b) at the top face, $x_3 = H$. (*Example II*)

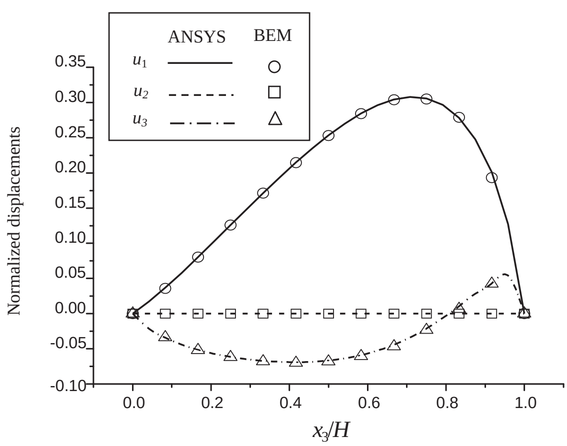

Figure 4: Variations of the normalized displacements along the line *AB* on the transversely isotropic rectangular column under thermal load – *Example I*

4 Conclusions

Thermal effects manifest themselves as an additional volume integral in the BIE in conventional BEM linear elastic stress analysis. In this study, the volume integral has been successfully transformed in an analytically exact manner into surface integrals for the special case of 3D transverse isotropy. Such a transformation restores the BEM for the thermoelastic analysis of the problem as a truly boundary solution analysis tool; it has not been successfully carried out previously for anisotropic solids because of the mathematical complexity of the associated spatial Green's function for such solids. The procedures followed similar steps as those successfully performed by the authors for the case of 2D general anisotropic thermoelasticity. It involved, first, mapping the temperature field into another domain to reduce the mathematical field problem to one governed by Laplace equation in the mapped domain. The volume integral associated with the thermal effects was then analytically transformed using Green's theorem and second identity in this mapped domain. Two examples have been presented to demonstrate the veracity of the mathematical formulation and numerical implementation. This work serves as a precursor for the similar task in the BEM thermoelastic analysis of 3D generally anisotropic solids in the near future.

Acknowledgement: The work described in this paper is supported by research grants received by the authors from the National Science & Engineering Research Council of Canada and the National Science Council of Taiwan, ROC (NSC 99-2221-E-035-MY3), respectively.

References

Danson, D. (1983): Linear isotropic elasticity with body forces. In Brebbia, C.(Ed): *Progress in Boundary Element Methods*. Pentech Press, London.

Kögl, M.; Gaul, L. (2003): A boundary element method for anisotropic coupled thermoelasticity. *Arch. Appl. Mech.*, vol. 73, pp. 377–398.

Pan, Y. C.; Chou, T. W. (1976): Point force solution for an infinite transversely isotropic solid. *ASME Journal of Applied Mechanics*, vol. 98, no. E, pp. 608¨C612.

Rashed, Y. F.; Brebbia, C. A. (2003): *Transformation of Domain Effects to the Boundary*. WIT Press, Southampton.

Rizzo, F. J.; Shippy, D. J. (1977): An advanced boundary integral equation method for three-dimensional thermoelasticity. *Int. J. Numer. Methods Engng*, vol. 11, pp. 1753–1768.

Sales, M. A.; Gray, L. J. (1998): Evaluation of the anisotropic green's function and its derivatives. *Comp. & Struct.*, vol. 69, pp. 247–254.

Shiah, Y. C.; Tan, C. L. (1999): Exact boundary integral transformation of the thermoelastic domain integral in bem for general 2d anisotropic elasticity. *Comput. Mech.*, vol. 23, pp. 87–96.

Shiah, Y. C.; Tan, C. L. (2011): A boundary element method for thermoelastic analysis of three dimensional transversely isotropic solid. *Int. J. Solids & Structures*. To be submitted.

Shiah, Y. C.; Tan, C. L.; Sun, W. X.; Chen, Y. H. (2010): On the displacement derivatives of the three-dimensional green's function for generally anisotropic bodies. In Zhang Ch., Aliabadi M.H., S. M.(Ed): *Advances in Boundary Element Techniques XI, Proc. BeTeq 2010 Conf., Berlin*, pp. 426–432.

Tan, C. L.; Shiah, Y. C.; Lin, C. W. (2009): Stress analysis of 3d generally anisotropic elastic solids using the boundary element method. *CMES Comp. Modeling Engng. Sc.*, vol. 41, pp. 195–214.

Tavara, L.; Ortiz, J. E.; Mantic, V.; Paris, R. (2008): Unique real-variable expressions of displacement and traction fundamental solutions covering all transversely isotropic materials for 3d bem. *Int. J. Numer. Methods Engng.*, vol. 74, pp. 776–798.

Wang, C. Y.; Denda, M. (2007): 3d bem for general anisotropic elasticity. *Int. J. Solids Struct.*, vol. 44, pp. 7073–7091.

Willis, J. R. (1965): The elastic interaction energy of dislocation loops in anisotropic media. *Q. J. Mech. & Appl. Math.*, vol. 18, pp. 419–433.

Wilson, R. B.; Cruse, T. A. (1978): Efficient implementation of anisotropic three dimensional boundary integral equation stress analysis. *Int. J. Numer. Methods Engng.*, vol. 12, pp. 1383–1397.