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A New Method to Achieve Equivalent Plastic Strain Explicit Form of J2 Plastic Isotropic Kinematic Hardening Model and Numerical Verification

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Abstract: Based on the classic J2 Plastic Flow Theory, the explicit equation of the equivalent plastic strain was derived elaborately utilizing the tensor analysis method. And a modified method commenced with the plastic flow rule was employed to obtain the same explicit equation. Finally, a subroutine written with FORTRAN was imported into the ABAQUS-version6.11 to validate the robustness of the numerical integrated scheme above The derivation procedure for the explicit equation of the equivalent plastic strain based on J2 Plastic Flow Theory can be utilized as an alternate method to solve complicated constitute integrated problems. And the developed algorithm can save as much as 20% time on the same calculation scale, compared to the traditional iterative method.

Keywords: plasticity, J2 Plastic Theory, tensor analysis, numerical verification.

1 Introduction

Plastic behavior is one of the most important characters of the structure, especially for metal material which involves ductile behavior. The basic theory of plasticity can be found in the standard textbooks and papers, such as Sokolnikoff (1956), Gurtine (1972), Hill (1950,1958), Koiter (1953,1960) and Maier (1970,1979). There are various kinds of plastic constitute in solid mechanics, which can be employed to simulate the elastoplastic response of various kinds of materials, such as concrete [Lee (1998); Lubliner (1998)], steel [Hill (1950,1958); Dugdale (1960)], glass [Anand (1996,2003)], polymer [Boyce (1988); Donald (1997)] and so on. The most famous and classical plasticity model is the J2 Plastic Flow Theory. The

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code of this theory is the Huber-Von Mises yield condition. Meanwhile this model is widely used in almost every area in industry engineering because of its simple mathematical form. However, since then, the J2 Plastic Flow Theory has developed very slowly as boundary conditions are very complicated when solving plastic response. Analytic solutions, when adopting to this J2 theory, can only be obtained on structures with regular shapes. With the progress of computer science and numerical technology, the nonlinear solid mechanics problem can be solved using new numerical methods like Finite Element Method (FEM), Boundary Element Method (BEM), Smooth Particle Hydrodynamic (SPH), Discrete Element Method (DEM), Extended Finite Element Method (X-FEM), Free Mesh Method [Yagawa (1996,2000)]. Meanwhile, many numerical forms for plastic models have been developed, including effective numerical integrated forms for J2 Plastic Flow Theory in a state of plane stain, a state of a plane stress, or a three-dimensional condition. However, the lack of details in derivation processes, which were neglected in most papers, jeopardizes their popularity.

2 Objective and Scope

The aim of this paper is to develop an alternate and easy method to obtain the explicit equations of the equivalent plastic strain. The J2 Plastic Flow Theory was utilized to derive the explicit equations and the robust of the numerical integrated scheme above was validated by a simulation in ABAQUS-version 6.11 with a subroutine written in FORTRAN.

3 Methodology

Firstly, based on finite deformation theory, the concrete numerical process of classic three-dimensional J2 Plastic Flow Theory with Kinematic hardening for concrete was deduced in detail. Secondly, based on the plastic rule, the same explicit form of equivalent plastic-strain was also concluded. Furthermore, a user defined material subroutine was compiled on the software Finite Element platform (ABAQUS), using previous integrated numerical algorithm.

4 Elastic Theory

Two material parameters, i.e. Young's modulus E and Poisson's ratioν are used to calculate the isotropic elastic response. Nevertheless, these material parameters could not reflect the relationship among shear, volume deformation and stress. Therefore, another two material parameters were introduced as lame constants, as shown in eq.1.

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \qquad \mu = \frac{E}{2(1+\nu)} \tag{1}$$

λ and μ are lame constants. The stress tensor $\bar{\sigma}$ can been written with component form as eq.2

$$\sigma_{ij} = \begin{vmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{vmatrix} \tag{2}$$

$\bar{\sigma}$ can been decomposed as hydrostatic stress tensor $\sigma_0\bar{\delta}$ ($\sigma_0 = \frac{\sigma_{11}+\sigma_{22}+\sigma_{33}}{3}$) and deviatoric stress tensor $\bar{S}$, which can be written with component forms as eq.3

$$\sigma_{ij} = \begin{vmatrix} \sigma_0 & 0 & 0 \\ 0 & \sigma_0 & 0 \\ 0 & 0 & \sigma_0 \end{vmatrix} + \begin{vmatrix} \sigma_{11}-\sigma_0 & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22}-\sigma_0 & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33}-\sigma_0 \end{vmatrix} \tag{3}$$

and

$$\sigma_0\delta_{ij} = \begin{vmatrix} \sigma_0 & 0 & 0 \\ 0 & \sigma_0 & 0 \\ 0 & 0 & \sigma_0 \end{vmatrix} \tag{4}$$

$$S_{ij} = \begin{vmatrix} \sigma_{11}-\sigma_0 & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22}-\sigma_0 & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33}-\sigma_0 \end{vmatrix} \tag{5}$$

In eq.4, $\bar{\delta}$ is the Kronecker delta and second-order symmetric unit tensor. The stress tensor $\bar{\varepsilon}$ can be written with component form as eq. 6,

$$\varepsilon_{ij} = \begin{vmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{vmatrix} \tag{6}$$

$\bar{\varepsilon}$ can been decomposed as hydrostatic strain tensor $\bar{\varepsilon}_0$ ($\varepsilon_0 = \frac{\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33}}{3}$) and deviatoric strain tensor $\bar{e}$, which could be written with component forms as eq.7.

$$\varepsilon_{ij} = \begin{vmatrix} \varepsilon_0 & 0 & 0 \\ 0 & \varepsilon_0 & 0 \\ 0 & 0 & \varepsilon_0 \end{vmatrix} + \begin{vmatrix} \varepsilon_{11}-\varepsilon_0 & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22}-\varepsilon_0 & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33}-\varepsilon_0 \end{vmatrix} \tag{7}$$

and

$$\varepsilon_0\delta_{ij} = \begin{vmatrix} \varepsilon_0 & 0 & 0 \\ 0 & \varepsilon_0 & 0 \\ 0 & 0 & \varepsilon_0 \end{vmatrix} \tag{8}$$

$$e_{ij} = \begin{vmatrix} \varepsilon_{11}-\varepsilon_0 & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22}-\varepsilon_0 & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33}-\varepsilon_0 \end{vmatrix} \tag{9}$$

$\varepsilon_0\delta_{ij}$ and e_{ij} are component forms of hydrostatic strain tensor $\bar{\varepsilon}_0$ and deviatoric strain tensor $\bar{e}$ respectively.

The elastic constitutive relationship between stress tensor rate and strain tensor rate could been given in term of hooker law

$$\dot{\bar{\sigma}} = \lambda \times \dot{\bar{\varepsilon}}_0 : I + 2\mu \times \dot{\bar{e}} \tag{10}$$

$\bar{I}$ is called as identity tensor and a four-order symmetric unit tensor.

5 Constitutive Equations for Plastic Deformation

In this section we begin by summarizing the constitutive model for J2 plastic deformation of Isotropic Kinematic hardening conditions occurring in metal materials. This (isothermal) model is based on the finite deformation decomposition. The J2 plastic model also involves two internal variables: the equivalent plastic strain $\bar{\varepsilon}^{pl} \geq 0$ which defines isotropic hardening of the Von Mises yield surface, and back stress tensor $\bar{\alpha} \geq 0$ which defines the center of Von Mises yield surface in deviatoric stress space.

Then, in terms of the variables

$\bar{\alpha}$−Back stress tensor, is the center of the yield surface in deviatoric stress space

$\bar{\xi}$− The tensor is the stress measured from the center of the yield surface and is written as eq.11

$\bar{\varepsilon}^{el}$− elastic strain tensor

$\bar{\varepsilon}^{pl}$− plastic strain tensor

$\bar{\varepsilon}_0^{el}$− hydrostatic elastic strain tensor

$\bar{\varepsilon}_0^{pl}$− hydrostatic elastic strain tensor (always zero during plastic deformation)

$\bar{e}^{el}$− deviatoric elastic strain tensor

$\bar{e}^{pl}$− deviatoric plastic strain tensor

$\dot{\gamma}$− a plastic-scalar multiplier

$H-$ The slope of uniaxial yield stress versus plastic strain curve.

$f(\bar{\sigma})-$ Huber-Von Mises yield condition, which can be defined as Equ.12.

$R-$ The radius of the yield surface can be written as Equ.13.

$\bar{Q}-$ The normal to the Von Mises surface, which can be defined as Equ.14.

σ_Y- The equivalent yield stress, when the stress tensor is on the yield surface.

σ_s- The virtual equivalent yield stress during the calculation process.

and the symbol definitions:

The superscript - represents a tensor;

The superscript ˆ represents an equivalent scalar;

The superscript · represents a rate form;

The subscript *new* represents the status at the end of preceding numerical integration;

The subscript *old* represents the status at present numerical integration.

$$\bar{\xi} = \bar{S} - \bar{\alpha} \tag{11}$$

$$f(\bar{\sigma}) = \sqrt{\frac{3}{2} \times \xi : \xi} - \sigma_Y \tag{12}$$

$$R = \sqrt{\frac{2}{3}}\sigma_Y \tag{13}$$

$$\bar{Q} = \frac{\frac{3}{2} \times \ \xi}{\sigma_Y} \tag{14}$$

It can be obtained from eq.12 that:

$$f(\bar{\sigma}) = 0 \Rightarrow \sqrt{\frac{3}{2} \times \xi : \xi} = \sigma_Y \tag{15}$$

When considering Levy-Saint Venant flow rule, it follows that the plastic-strain rate is defined as:

$$\dot{\varepsilon}^{pl} = \dot{\gamma}\frac{\partial f(\bar{\sigma})}{\partial \bar{\sigma}} = \frac{1}{2} \times \frac{\frac{3}{2} \times 2\bar{\xi}}{\sqrt{\frac{3}{2} \times \bar{\xi} : \bar{\xi}}} : \frac{\partial(\bar{S} - \bar{\alpha})}{\partial \bar{\sigma}} \tag{16}$$

eq.17 can be gotten from the definition of $\bar{S}$, and can be written as:

$$\frac{\partial(\bar{S} - \bar{\alpha})}{\partial \bar{\sigma}} = \bar{I} \tag{17}$$

By substituting (12), (13) and (15) in (16), it obtains

$$\dot{\bar{\varepsilon}}^{pl} = \dot{\gamma} \times \frac{\partial f(\bar{\sigma})}{\partial \bar{\sigma}} = \dot{\gamma} \times \frac{\frac{3}{2} \times \xi}{\sqrt{\frac{3}{2} \times \xi : \xi}} = \dot{\gamma} \times \frac{\frac{3}{2} \times \xi}{\sigma_Y} \tag{18}$$

so the relationship between $\dot{\bar{\varepsilon}}^{pl}$ and $\bar{Q}$ can be obtained

$$\dot{\bar{\varepsilon}}^{pl} = \dot{\gamma} \times \bar{Q} \tag{19}$$

meanwhile

$$\bar{Q} : \bar{Q} = \frac{\frac{3}{2} \times \xi}{\sigma_Y} : \frac{\frac{3}{2} \times \xi}{\sigma_Y} = \frac{3}{2} \times \frac{\frac{3}{2} \times \xi : \xi}{\sigma_Y \times \sigma_Y} = \frac{3}{2} \times \frac{\sigma_Y \times \sigma_Y}{\sigma_Y \times \sigma_Y} = \frac{3}{2} \tag{20}$$

Based on finite deformation assumption,

$$\begin{aligned} &\bar{\varepsilon} = \bar{\varepsilon}^{el} + \bar{\varepsilon}^{pl} \\ &\bar{\varepsilon}_0 = \bar{\varepsilon}_0^{el} + \bar{\varepsilon}_0^{pl} \\ &\bar{e} = \bar{e}^{el} + \bar{e}^{pl} \end{aligned} \tag{21}$$

and the equivalent plastic-strain rate, $\dot{\hat{e}}^{pl}$, can be defined as

$$\dot{\hat{e}}^{pl} = \sqrt{\frac{2}{3} \times \dot{\bar{e}}^{pl} : \dot{\bar{e}}^{pl}} \tag{22}$$

By substituting eq. (18), (19) and (20) in (22), a very important relation can be deduced as eq.23.

$$\dot{\hat{e}}^{pl} = \sqrt{\frac{2}{3} \times \dot{e}^{pl} : \dot{e}^{pl}} = \sqrt{\frac{2}{3} \times \dot{\gamma} \times \bar{Q} : \bar{Q} \times \dot{\gamma}} = \sqrt{\frac{2}{3} \times \frac{3}{2} \times \dot{\gamma} \times \dot{\gamma}} = \dot{\gamma} \tag{23}$$

The evolution law for $\bar{\alpha}$ (back stress tensor) is given as

$$\dot{\bar{\alpha}} = \frac{2}{3} \times H \times \bar{Q} \tag{24}$$

When considering the stress tensor relation between proceeding status and present status, eq. 25 can be concluded.

$$\bar{\sigma}_{new} = \bar{\sigma}_{old} + \lambda \times tace(\Delta \bar{\varepsilon}) : I + 2 \times \mu \times \Delta \bar{\varepsilon} \tag{25}$$

From the definition of the hydrostatic elastic strain tensor, it can be found that $\bar{\varepsilon}_0^{pl}$ has no effort on plastic deformation, so eq. 25 can be written as eq. 26.

$$\bar{S}_{old} = \bar{S}_{new} - 2 \times \mu \times \Delta \bar{\varepsilon}^{pl} = \bar{S}_{new} - 2 \times \mu \times \Delta \lambda \times \bar{Q} \tag{26}$$

When considering the back stress tensor and equivalent plastic strain relation between previous status and present status, eq. 27 and eq. 28 can be concluded.

$$\bar{\alpha}_{new} = \bar{\alpha}_{old} + \frac{2}{3} \times H \times \Delta\gamma \times \bar{Q} \tag{27}$$

$$\hat{e}^{pl}_{new} = \hat{e}^{pl}_{old} + \Delta\gamma \tag{28}$$

Furthermore,

$$\begin{aligned} &\bar{Q} = \frac{\frac{3}{2} \times \xi}{\sqrt{\frac{3}{2} \times \xi : \xi}} = \frac{\frac{3}{2} \times \xi}{\sigma_Y} \Rightarrow \bar{Q} \times \sigma_Y = \frac{3}{2} \times \xi \\ &\Rightarrow \bar{Q} \times \sigma_Y = \frac{3}{2} \times (\bar{S}_{new} - \bar{\alpha}_{new}) \Rightarrow \frac{2}{3} \times \bar{Q} \times \sigma_Y = \bar{S}_{new} - \bar{\alpha}_{new} \end{aligned} \tag{29}$$

$$\bar{\alpha}_{new} + \frac{2}{3} \times \bar{Q} \times \sigma_Y = \bar{S}_{new} \tag{30}$$

$$\bar{\alpha}_{old} + \frac{2}{3} \times H \times \Delta\gamma \times \bar{Q} + \frac{2}{3} \times \bar{Q} \times \sigma_Y = \bar{S}_{new} - 2 \times \mu \times \Delta\gamma \times \bar{Q} \tag{31}$$

$$\begin{aligned} &(\bar{\alpha}_{old} + \frac{2}{3} \times H \times \Delta\gamma \times \bar{Q} + \frac{2}{3} \times \bar{Q} \times \sigma_Y) : \bar{Q} = (\bar{S}_{new} - 2 \times \mu \times \Delta\gamma \times \bar{Q}) : \bar{Q} \\ &\Rightarrow \bar{\alpha}_{old} : \bar{Q} + \frac{2}{3} \times H \times \Delta\gamma \times \bar{Q} : \bar{Q} + \frac{2}{3} \times \bar{Q} : \bar{Q} \times \sigma_Y = \bar{S}_{new} : \bar{Q} - 2 \times \mu \times \Delta\gamma \times \bar{Q} : \bar{Q} \\ &\Rightarrow \bar{\alpha}_{old} : \bar{Q} + \frac{2}{3} \times H \times \Delta\gamma \times \frac{3}{2} + \frac{2}{3} \times \frac{3}{2} \times \sigma_Y = \bar{S}_{new} : \bar{Q} - 2 \times \mu \times \Delta\gamma \times \frac{3}{2} \\ &\Rightarrow \frac{2}{3} \times H \times \Delta\gamma \times \frac{3}{2} + \frac{2}{3} \times \frac{3}{2} \times \sigma_Y + 2\mu \times \Delta\gamma \times \frac{3}{2} = \bar{S}_{new} : \bar{Q} - \bar{\alpha}_{old} : \bar{Q} \\ &\Rightarrow H \times \Delta\gamma + \sigma_Y + 3\mu \times \Delta\gamma = \bar{S}_{new} : \bar{Q} - \bar{\alpha}_{old} : \bar{Q} \\ &\Rightarrow H \times \Delta\gamma + 3 \times \mu \times \Delta\gamma = \bar{S}_{new} : \bar{Q} - \bar{\alpha}_{old} : \bar{Q} - \sigma_Y \\ &\Rightarrow (H + 3 \times \mu) \times \Delta\gamma = \bar{S}_{new} : \bar{Q} - \bar{\alpha}_{old} : \bar{Q} - \sigma_Y \\ &\Rightarrow (H + 3 \times \mu) \times \Delta\gamma = (\bar{S}_{new} - \bar{\alpha}_{old}) : \bar{Q} - \sigma_Y \\ &\Rightarrow (H + 3 \times \mu) \times \Delta\gamma = \frac{3}{2} \times (\bar{S}_{new} - \bar{\alpha}_{old}) : \frac{(\bar{S}_{new} - \bar{\alpha}_{old})}{\sigma_s} - \sigma_Y \\ &\Rightarrow (H + 3 \times \mu) \times \Delta\gamma = \frac{3}{2} \times \frac{(\bar{S}_{new} - \bar{\alpha}_{old}) : (\bar{S}_{new} - \bar{\alpha}_{old})}{\sigma_s} - \sigma_Y \\ &\Rightarrow (H + 3 \times \mu) \times \Delta\gamma = \frac{\sigma_s \times \sigma_s}{\sigma_s} - \sigma_Y \\ &\Rightarrow (H + 3 \times \mu) \times \Delta\gamma = \sigma_s - \sigma_Y \end{aligned} \tag{32}$$

Finally, the explicit equivalent plastic strain increment form can be concluded as eq.33.

$$\begin{aligned} &(H+3\mu)\times\Delta\hat{e}^{pl}=\sigma_s-\sigma_Y \\ &\Delta\hat{e}^{pl}=\frac{\sigma_s-\sigma_Y}{H+3\mu} \end{aligned} \tag{33}$$

6 Explicit Equivalent Plastic Strain Increment Equation

This equation for plastic-strain update could be concluded in another view point listed as below.

$$\sigma_s-\sigma_Y=\sqrt{\frac{3}{2}\times(\bar{S}_{new}-\bar{\alpha}_{new}):(\bar{S}_{new}-\bar{\alpha}_{new})}-\sigma_Y \tag{34}$$

Meanwhile,

$$\sigma_s-\sigma_Y=\Delta\sigma_s \tag{35}$$

Based on plasticity theory, the equivalent plastic strain $\hat{e}^{pl}$ is the only parameter of the increment of equivalent yield stress, so the differential form of $\Delta\sigma_s$ could be written as:

$$\Delta\sigma_s=\frac{\partial\sigma_s}{\partial\hat{e}^{pl}}\times\Delta\hat{e}^{pl}=\frac{\partial\sigma_s}{\partial\bar{e}^{pl}}:\frac{\partial\bar{e}^{pl}}{\partial\hat{e}^{pl}}\times\Delta\hat{e}^{pl} \tag{36}$$

combine eq. 36, 34 and 22, then can get:

$$\Delta\sigma_s=\frac{\frac{3}{2}\times(\bar{S}_{old}-\bar{\alpha}_{old})}{\sqrt{\frac{3}{2}\times(\bar{S}_{old}-\bar{\alpha}_{old}):(\bar{S}_{old}-\bar{\alpha}_{old})}}:\frac{\partial(\bar{S}_{old}-\bar{\alpha}_{old})}{\partial\bar{e}^{pl}}:\frac{\partial\bar{e}^{pl}}{\partial\hat{e}^{pl}}\times\Delta\hat{e}^{pl} \tag{37}$$

The new status deviatoric stress tensor can be given as

$$\begin{aligned} &\bar{S}_{new}=\bar{S}_{old}+2\times\mu\times(\Delta\bar{e}-\Delta\bar{e}^{pl}) \\ &\Rightarrow\bar{S}_{old}=\bar{S}_{new}-2\times\mu\times(\Delta\bar{e}-\Delta\bar{e}^{pl}) \end{aligned} \tag{38}$$

$$\begin{aligned} &\bar{\alpha}_{new}=\bar{\alpha}_{old}+\frac{2}{3}\times H\times\Delta\bar{e}^{pl} \\ &\Rightarrow\bar{\alpha}_{old}=\bar{\alpha}_{new}-\frac{2}{3}\times H\times\Delta\bar{e}^{pl} \end{aligned} \tag{39}$$

$$\Delta\sigma_s=\frac{\frac{3}{2}\times(\bar{S}_{old}-\bar{\alpha}_{old})}{\sqrt{\frac{3}{2}\times(\bar{S}_{old}-\bar{\alpha}_{old}):(\bar{S}_{old}-\bar{\alpha}_{old})}}:(2\times\mu+\frac{2}{3}\times H)\times I:\frac{\partial\bar{e}^{pl}}{\partial\hat{e}^{pl}}\times\Delta\hat{e}^{pl} \tag{40}$$

based on the flow rule,

$$\dot{\bar{e}}^{pl} = \dot{\gamma} \times \frac{\partial f(\xi)}{\partial(\bar{\sigma})} = \dot{\gamma} \times \frac{\frac{3}{2} \times (\bar{S} - \bar{\alpha})}{\sigma_Y} \tag{41}$$

$$\dot{\bar{e}}^{pl} = \dot{\hat{e}}^{pl} \times \frac{\frac{3}{2} \times (\bar{S} - \bar{\alpha})}{\sigma_Y} \tag{42}$$

$$\frac{\partial \bar{e}^{pl}}{\partial \hat{e}^{pl}} = \frac{3}{2} \times \frac{(\bar{S} - \bar{\alpha})}{\sigma_Y} \tag{43}$$

$$\begin{aligned}
\Delta\sigma_s &= \frac{\frac{3}{2} \times (\bar{S}_{old} - \bar{\alpha}_{old})}{\sqrt{\frac{3}{2} \times (\bar{S}_{old} - \bar{\alpha}_{old}) : (\bar{S}_{old} - \bar{\alpha}_{old})}} : I : \frac{(\bar{S}_{old} - \bar{\alpha}_{old})}{\sigma_Y} \\
&\times \frac{3}{2} \times (2 \times \mu + \frac{2}{3} \times H) \times \Delta\hat{\varepsilon}^{pl} \\
\Rightarrow \Delta\sigma_s &= \frac{\frac{3}{2} \times (\bar{S}_{old} - \bar{\alpha}_{old}) : (\bar{S}_{old} - \bar{\alpha}_{old})}{\sqrt{\frac{3}{2} \times (\bar{S}_{old} - \bar{\alpha}_{old}) : (\bar{S}_{old} - \bar{\alpha}_{old})} \times \sigma_Y} \times (3 \times \mu + H) \times \Delta\hat{\varepsilon}^{pl} \\
\Rightarrow \Delta\sigma_s &= \frac{\sigma_Y \times \sigma_Y}{\sigma_Y \times \sigma_Y} \times (3 \times \mu + H) \times \Delta\hat{\varepsilon}^{pl} = (3 \times \mu + H) \times \Delta\hat{\varepsilon}^{pl}
\end{aligned} \tag{44}$$

Finally, we could obtain the same form with Equ.33 using above method.

7 Numerical Validation

In order to validate the developed model, a standard example in ABAQUS was simulated with the developed model and the results were compared with the ones in the standard example.

A user defined material subroutine was inserted into the finite-element computer program ABAQUS/Standard and the J2 plastic constitutive model was implemented to simulate the elastoplastic performance of a slender beam under repeated loads. The list of parameters using in numerical test are: $E = 100Gpa$; $\nu = 0.25$; $\sigma_Y = 2Mpa$; $H = 1 \times 10^9$.

To numerically model the uniaxial tension cycling test, a long beam numerical specimen with cross size 0.2m×0.2m, and the length 2m has been meshed with 40000 C3D8R (An 8-node linear brick, reduced integration, hourglass control), shown as Fig.1. One cross is fixed the freedom of three-dimensional displacement, and the other cross has been enforced cycling regular triangle displacement loading with magnitude5×10^{-5}m. Thirteen minutes was consumed on a server with 4 CPUs and 8GB RAM memory.

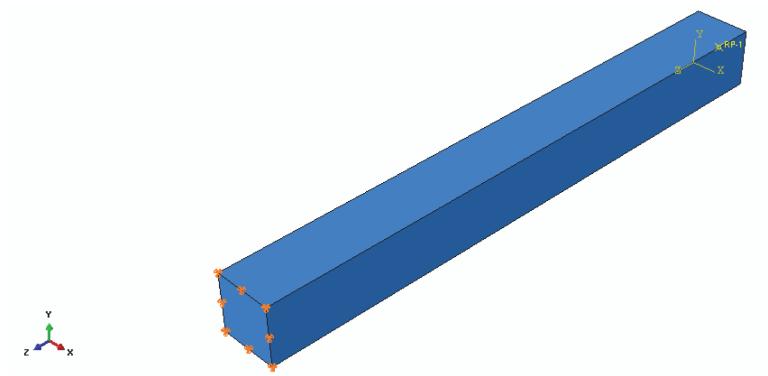

Figure 1: Geometry of the numerical specimen

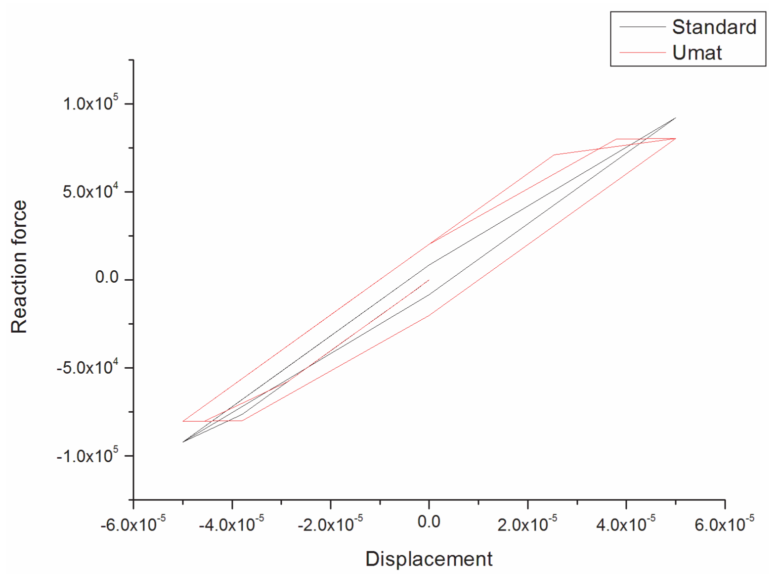

Figure 2: Load-displacement curves in the subroutine and in the standard example in ABAQUS

The convergence speed highly depends upon the numerical integration methods The most widely used method is the generalized midpoint rule, as mentioned (1986). For J2 plastic flow theory, Rice and Tracey (1973) has proposed the return map which is two-order accurate. Furthermore, other return schemes have been introduced to simulated different material by Krieg (1977); Schreyer, Kulak, and Kramer (1979); and Yoder and Whirley (1984); Loret and Prevost (1986); and

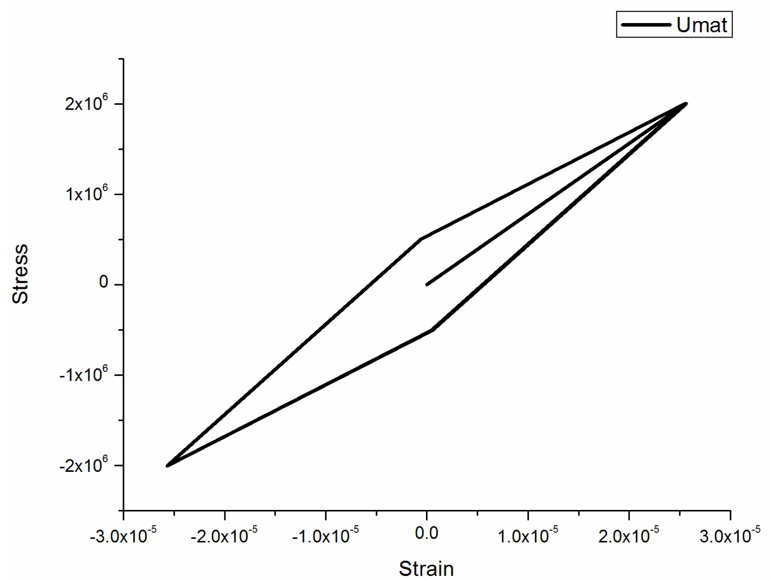

Figure 3: the relationship between stress and strain in middle of specimen

Lubliner (1972,1973,1984). In this paper, the return map numerical integrated scheme has been employed.

Fig.2 has shown the relation between displacement and reaction force in the reference point of the loading cross. Fig.3 has shown the relationship between stress and strain in middle of specimen. From Fig.2 and Fig.3, we can observe the Bauschinger effect: this effect is characterized by a reduced yield stress upon load reversal after plastic deformation has occurred during the initial loading. This phenomenon decreases with continued cycling.

8 Conclusions

This paper has manifested the whole concrete detail and conduction process of constitutive model for J2 plastic deformation of Isotropic Kinematic hardening. Meanwhile based on Levy-Saint Venant flow rule, a new method has been proposed to obtain the explicit form of equivalent plastic strain, and corresponding numerical verification has also been accomplished using Finite element method. Furthermore some conclusions can be summarized as follows:

1. Using tensor analysis method, the concrete detail of the deduction process is given, which could take a reference for self-development numerical calcula-

tion software to follow and accomplish program.

2. Based on the Levy-Saint Venant flow rule, explicit form for equivalent plastic strain can been gotten, which has offered a new method to solve the constitute integration problem, such as coupled plastic damage model and coupled visco-elastic damage model.

3. The subroutine of J2 Plastic Flow Theory simulated the similar results with the standard example in ABAQUS while the calculation time of it was about 20% less than the iterative methods.

4. A new user defined material subroutine has been developed using J2 plastic deformation of Isotropic Kinematic hardening theory, which has been deduced in this paper, and this program has also manifest accurate stress and strain response during the cycling loading conditions. At the same time, the *Bauschinger effect* has also been manifested in numerical test using this paper's program.

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