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Application of the Multi Scaling Characteristic Time Expansion Method for Estimating Nonlinear Restoring Forces

Yung-Wei Chen[1], **Jiang-Ren Chang**[2], **Fu-Hsuan Hsieh**[2] **and Che-Wei Chen**[2]

Abstract: Since numerical instability phenomena always arise in the solving process for parameters identification of structural vibration. To resolve such a problem, the multi scaling characteristic time expansion method (MSCTEM) in conjunction with the natural regularization method is adopted to overcome the higher order numerical oscillation problem when polynomial series expansion is necessary. Due to inclusion of the characteristic length (CL) in the scheme, the ill-posed problem of the constructed Vandwemonde matrix will be overcome and will also increase the term number of polynomial series. Thus, the ill condition and numerical instability of numerical calculations can be resolved. Besides, to overcome the numerical instability problem of a noise disturbance, in contrast to the conventional Tikhonov regularization method, the natural regularization method is again adopted to resolve the problem. It is shown that the MSCTEM with the natural regularization method can effectively resolve those above mentioned problems through three benchmark examples.

1 Introduction

Nonlinear dynamical system identification problems are usually encountered in engineering applications. For instance, to specify the parameters of dynamical systems is necessary in optimal processes, it is important to analyze and determine the parameters of the system using experimental testing and numerical methods. However, uses of these methods might arise some challenging problems in the structural mechanic field because a small measurement error can cause a large error in the parameter's identification results.

To overcome these inverse problems, some solutions proposed in the literatures

[1] Department of Marine Engineering, National Taiwan Ocean University, 2, Pei-Ning Road, Keelung 20224, Taiwan, ROC.
[2] Department of Systems Engineering & Naval Architecture, National Taiwan Ocean University, 2, Pei-Ning Road, Keelung 20224, Taiwan, ROC.

have included numerical techniques and experimental testing. Other various publications in [Gladwell (1986); Gladwell and Movahhedy (1993); Lancaster, Maroulas, He, and Yu (1987); Starek and Inman (1991); Starek, Inman, and Kress (1992); Starek and Inman (1995, 1997); Adhikari and Woodhouse (2001a,b); Feldman (2007)] recommended using the damping coefficient, stiffness, and external force for solving the inverse problems. Mode shape, frequency, displacement, and velocity at different times can also be used to successfully estimate these properties [Kerschen, Worden, Vakakis, and Golinval (2006)]. Huang (2001) has employed a conjugate gradient method (CGM) to solve the nonlinear inverse vibration problem for the estimation of the time-dependent stiffness coefficient. Recently Liu (2008a,c) has developed a Lie-group shooting method to study the inverse vibration problem for estimating the time-dependent damping and stiffness coefficients and, at the same time, derived a closed-form solution to estimate the parameters.

A complete review of the developments of some useful methods for the realm of nonlinear system identification can be found in [Liu (2008c)]. Masri and Caughey (1979) also proposed the idea of a force state mapping method which is a simple procedure that allows a direct identification of the restoring force for nonlinear mechanical systems. This idea was further extended in [Crawley and Aubert (1986); E. F. Crawley (1986); Duym, Schoukens, and Guillaume (1996)]. Recently, Namdeo and Manohar (2008) have modified the force state mapping technique with two alternative functional representation schemes: 1) reproducing kernel particle method and kriging technique and 2) estimating the nonlinear system parameters from measured time histories of response under known excitations.

The purpose of this paper is to develop a simple, multi-step regulation algorithm with easy numerical implementation and versatility. A simple power series can be considered as a fit for the time history of displacement response under known excitations. However, doing so will result an inaccurate matrix with a high-order function (the Vandermonde matrix), which has been described in [Gohberg and Olshevsky (1997)]. To resolve this problem, the characteristic length (CL) of computational time into power series can be used to maintain numerical stability. This concept was first proposed to deal with the Laplace equation using a physical quantity reported in [Liu (2008a,c, 2007a,b)]. Recently, CL has been successfully extended to deal with the Laplace equation and sloshing wave problem in [Chen, Liu, and Chang (2009); Chen, Liu, Chang, and Chang (2010); Chen, Yeih, Liu, and Chang (2012)]. It should be noted that the instability of the mathematical procedure and a small disturbance of measured data needs to be considered in the numerical procedure because they could cause an error of the parameter's identification. This paper will apply the natural regularization and multi-scaling CL technique, proposed in [Liu, Hong, and Atluri (2010); Liu and Atluri (2013)], to track ill-posed linear problems and to chose CLs in numerical procedures. One advantage of this regulation method is that it can determine whether a solution exists for a linear system.

Except for the current section, Section 2 of this paper will describe the mathematical formulation of the characteristic time expansion method and introduces the numerical procedure of the matrix CGM. Section 3 will demonstrate several numerical examples, including Duffing's oscillator, Duffing's oscillator with negative linear stiffness, Van der Pol's oscillator, and the seat model, to compare our method with the analytical solution. Finally, some concrete conclusions will be summarized in section 4.

2 Basic formulation

A second-order ordinary differential equation (ODE) for the equation of motion is expressed as:

$$\ddot{x} = H(x,\dot{x}) = P(t), \tag{1}$$

where x represents the displacement of response of a system and $P(t)$ and $H(x,\dot{x})$ are the external excitation and restoring force, respectively. In order to obtain H, a trivial rearrangement of (1) gives:

$$H(x,\dot{x}) = P(t) - \ddot{x}. \tag{2}$$

Here H can be obtained if the quantities, $P(t)$ and $\ddot{x}$, on the right-hand side are known. In general, it is easier to measure the displacement at some discrete sampling times than it is to directly measure velocities and accelerations. Therefore, if $x_1(t) = g(t)$ is denoted as the measured displacements can be expressed as follows:

$$x_1(t) = g(t), \tag{3}$$

$$\dot{x}_1 = x_2(t), \tag{4}$$

$$\dot{x}_2 = x_3(t). \tag{5}$$

This is however an index-three differential algebraic equations [Liu (2008b)], which is hard to solve.

2.1 The characteristic time expansion method

According to (1) above, the displacement can be expressed as a power series:

$$x(t_i) = \sum_{k=0}^{\infty} a_k \left(\frac{t_i}{T_k}\right)^k, 0 \le t_i \le t_f, t_f < T_k \tag{6}$$

where t_i denotes each discrete time, $x(t_i)$ denotes the displacement at each time, t_f denotes the final time, T_k denotes the CL, and a_k denote the unknown coefficients. Differentiation of (6) yields velocity and acceleration and is expressed as follows:

$$\dot{x}(t_i) = \sum_{k=0}^{\infty} \frac{k}{T_k} a_k \left(\frac{t_i}{T_k}\right)^{k-1}, \tag{7}$$

$$\ddot{x}(t_i) = \sum_{k=0}^{\infty} \frac{k(k-1)}{T_k^2} a_k \left(\frac{t_i}{T_k}\right)^{k-2}. \tag{8}$$

The power series in (6) can be used to describe the displacement of a system. Hence, (6) has admissible functions with finite terms, and one can be expressed as a linear equation system with $n = m+1$:

$$\begin{bmatrix} 1 & t_0/T_1 & (t_0/T_2)^2 & \cdots & (t_0/T_m)^m \\ 1 & t_1/T_1 & (t_1/T_2)^2 & \cdots & (t_1/T_m)^m \\ 1 & t_2/T_1 & (t_2/T_2)^2 & \cdots & (t_2/T_m)^m \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & t_m/T_1 & (t_m/T_2)^2 & \cdots & (t_m/T_m)^m \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix} = \begin{bmatrix} x(t_0) \\ x(t_1) \\ x(t_2) \\ \vdots \\ x(t_m) \end{bmatrix}. \tag{9}$$

We denote the above equation by:

$$\mathbf{R}\mathbf{c} = \mathbf{b}_1, \tag{10}$$

Where $\mathbf{c} = [a_0, a_1, a_2, \ldots, a_m]^T$ is the vector of unknown coefficients.
The norm of the first column of $\mathbf{R}$ is $\sqrt{n}$. According to the idea of "equilibrated matrix", we can choose the multiple-scale T_k, $k = 1, \ldots, m$ by

$$T_k = \left(\frac{1}{n}\sum_{i=0}^{m} t_i^{2k}\right)^{\frac{1}{2k}}, k = 1, \ldots, m \tag{11}$$

In order to satisfied $t_f < T_k$, at each time step, (11) can be expressed as follows:

$$T_{MR} = T_S + T_k, \tag{12}$$

where T_S is a positive integer chosen by the user.

2.2 The matrix CGM for ill-posed linear system

When a matrix is ill-posed and measured data contains noisy disturbances, it is difficult to determine the stability of the system using the conventional regularization techniques. Therefore, Liu, Hong, and Atluri (2010) proposed a natural regularization method, which proves that a solution exists when ill-posed matrix and noisy disturbances occur. This method can be described by the following matrix equation:

$$\mathbf{R}^T\mathbf{U}^T = \mathbf{I}_m, \text{ i.e., } (\mathbf{U}\mathbf{R})^T = \mathbf{I}_m \tag{13}$$

If $\mathbf{U}$ is the inversion of $\mathbf{R}$, then numerically, $\mathbf{U}$ is a left-inversion of $\mathbf{R}$. Then we have

$$(\mathbf{R}\mathbf{R}^T)\mathbf{U}^T = \mathbf{R} \tag{14}$$

Let

$$\mathbf{R}\mathbf{X}_0 = \mathbf{y}_0. \tag{15}$$

Given $\mathbf{X}_0$, say $\mathbf{X}_0 = \mathbf{I} = [\mathbf{1},\ldots,\mathbf{1}]^T$, $\mathbf{y}_0$ can be directly obtained because $\mathbf{R}$ is a given matrix. Hence, we have:

$$\mathbf{y}_0^T\mathbf{U}^T = \mathbf{X}_0^T, \text{ i.e., } \mathbf{X}_0 = \mathbf{U}\mathbf{y}_0 \tag{16}$$

When (13) and (16) are combined, they create an over-determined system to calculate $\mathbf{U}^T$. The over-determined system can be written as:

$$\mathbf{B}\mathbf{U}^T = \begin{bmatrix} \mathbf{I}_m \\ \mathbf{X}_0^T \end{bmatrix}, \tag{17}$$

where:

$$\mathbf{B} = \begin{bmatrix} \mathbf{R}^T \\ \mathbf{y}_0^T \end{bmatrix}, \tag{18}$$

is an $n \times m$ matrix with $n = m+1$, we can obtain an $m \times m$ matrix equation:

$$[\mathbf{R}\mathbf{R}^T + \mathbf{y}_0\mathbf{y}_0^T]\mathbf{U}^T = \mathbf{R} + \mathbf{y}_0\mathbf{X}_0^T, \tag{19}$$

Besides the primal system in (10), we need to solve the dual system with

$$\mathbf{R}^T\mathbf{y} = \mathbf{b}_1, \tag{20}$$

Applying the operators in (19) to $\mathbf{b}_1$, and utilizing the above equation, i.e., $\mathbf{y} = \mathbf{R}^T\mathbf{b}_1$, we can obtain

$$[\mathbf{R}\mathbf{R}^T + \mathbf{y}_0\mathbf{y}_0^T]\mathbf{y} = \mathbf{R}\mathbf{b}_1 + (\mathbf{X}_0 \cdot \mathbf{b}_1]\mathbf{y}_0 \tag{21}$$

where $\mathbf{y}_0 = \mathbf{R}\mathbf{X}_0$.

Replacing the $\mathbf{R}$ in (21) by $\mathbf{R}^T$, we have a similar equation for the primal system in (10):

$$[\mathbf{R}^T\mathbf{R} + \mathbf{y}_0\mathbf{y}_0^T]\mathbf{c} = \mathbf{R}^T\mathbf{b}_1 + (\mathbf{X}_0 \cdot \mathbf{b}_1)\mathbf{y}_0 \tag{22}$$

where $\mathbf{y}_0 = \mathbf{R}^T\mathbf{X}_0$.

Finally, when $\mathbf{c}$ of (22) is calculated by CGM, the restoring force and acceleration can be obtained from (2), (7) and (8).

3 Numerical examples

Example 1

In this case, we consider a Duffing oscillator and a second-order ODE to describe the forced vibration of a nonlinear structure by:

$$\ddot{x} + \gamma\dot{x} + \beta x + \alpha x^3 = P(t) \tag{23}$$

where the parameters are fixed as $\alpha = 1$, $\beta = -1$ and $\gamma = 0.3$. The restoring force can be expressed as follows:

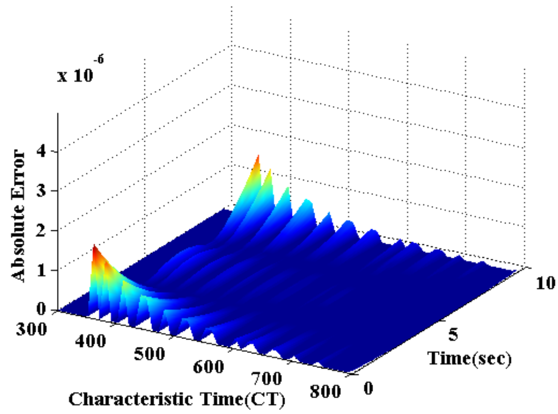
$$H(x) = x^3 - x \tag{24}$$

In order to identify the restoring force H as a function of x, a monotonic function of t is required.

In this instance, $x(t) = t^2 - 8$ is used to obtain the external force, x, and is given by:

$$P(t) = (t^2 - 8)^3 - t^2 + 0.6t + 10. \tag{25}$$

In order to test the stability of the MSCTEM and single scaling characteristic time expansion method (SSCTEM), the order of the polynomial and computational time are increased. The restoring force in the initial and final time changed very rapidly. To understand a CL effect, $m = 200$, $\mathbf{X}_0 = \mathbf{I}$, and $\varepsilon = 1 \times 10^{-16}$ have been fixed. The maximum estimation error of H, shown in Fig. 1 is smaller than 10^{-6}. From Fig. 1(b), we can observe that the MSCTEM can effectively overcome numerical oscillation in the initial and final time, and then the numerical errors decrease varied with T_s increasing.



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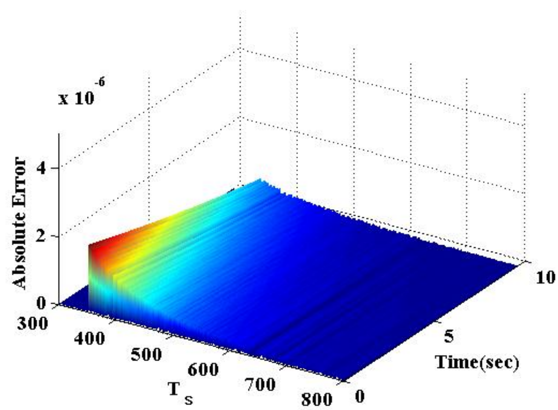

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Figure 1: For Example 1 showing the error of estimation with different CL.

It can be seen that including the CL into this case is efficiently to overcome an ill-posed matrix. Furthermore, when fixing $T_S = 800$, the exact solutions for velocity and acceleration can be determined. The numerical results are shown in Figs. 2-4. According to the numerical results, the maximum estimation errors are found to be smaller than 10^{-9}. Applying a CL and matrix regularization method can provide highly stability and accuracy. In order to further test the stability of the present method we also consider:

$$\hat{x}_i = x_i + \sigma R(i) \tag{26}$$

as an input into the estimation equations, where R_i is a random number in $[-1, 1]$, and σ is a noise level. Using different noise levels, with $\sigma = 1\%$ and 5%, the computed profile of restoring forces is shown in Fig. 5. Also, Fig. 5 shows that the maximum estimated errors are smaller than 10^{-1} with noisy disturbances. We can see that the present method has a high numerical accuracy and stability for noisy disturbances.

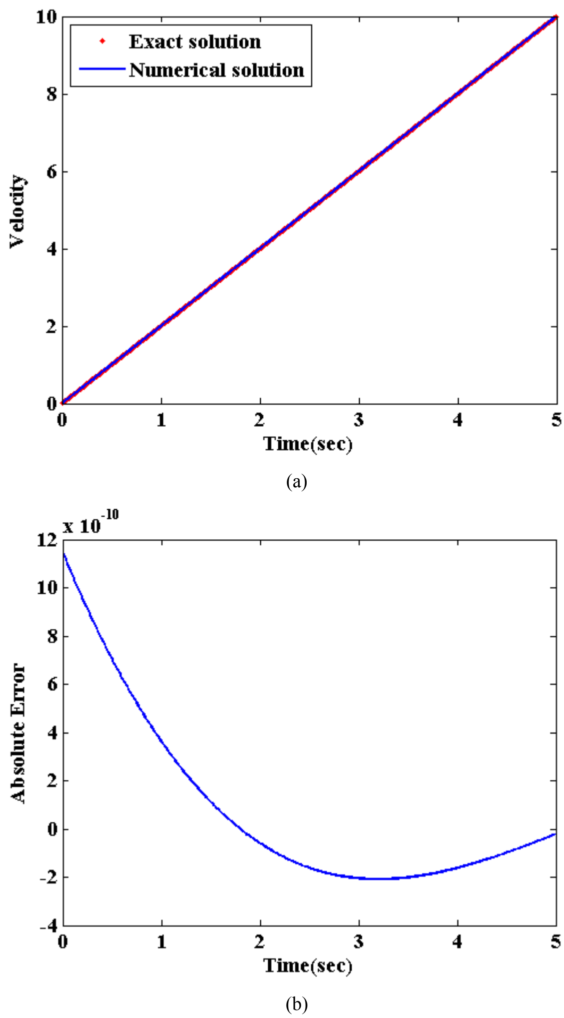
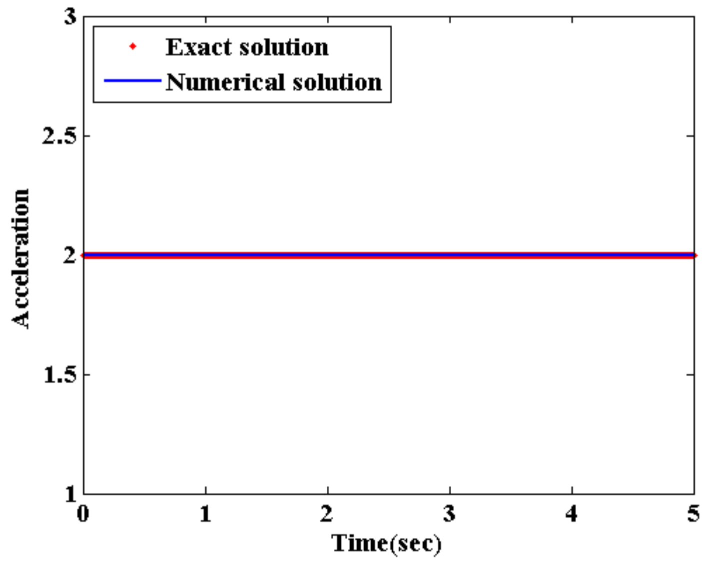


Figure 2: For Example 1: (a) comparing numerical solution and exact velocity, and (b) displaying the error of estimation.



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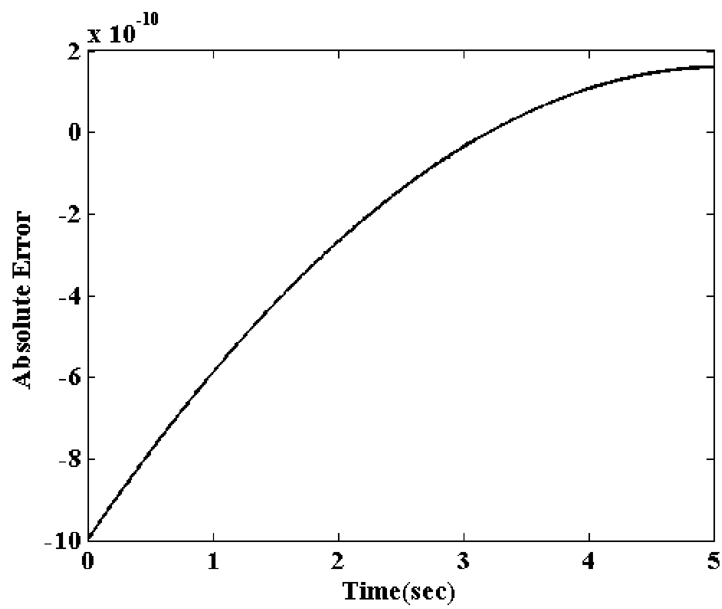

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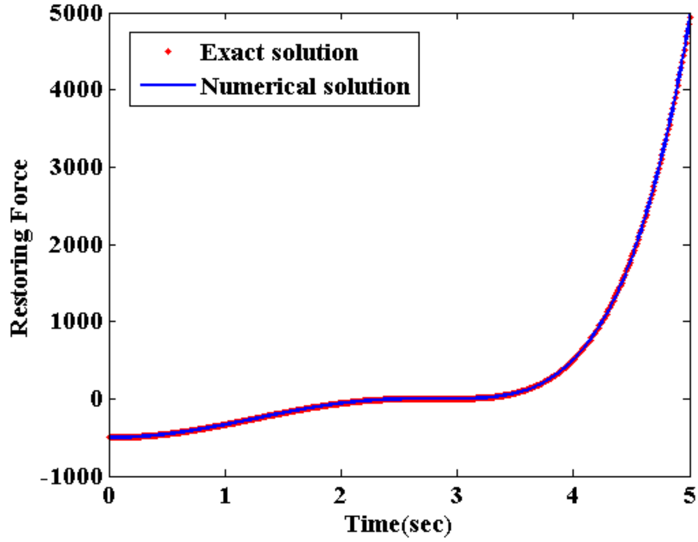
Figure 3: For Example 1: (a) comparing numerical solution and exact acceleration, and (b) displaying the error of estimation.



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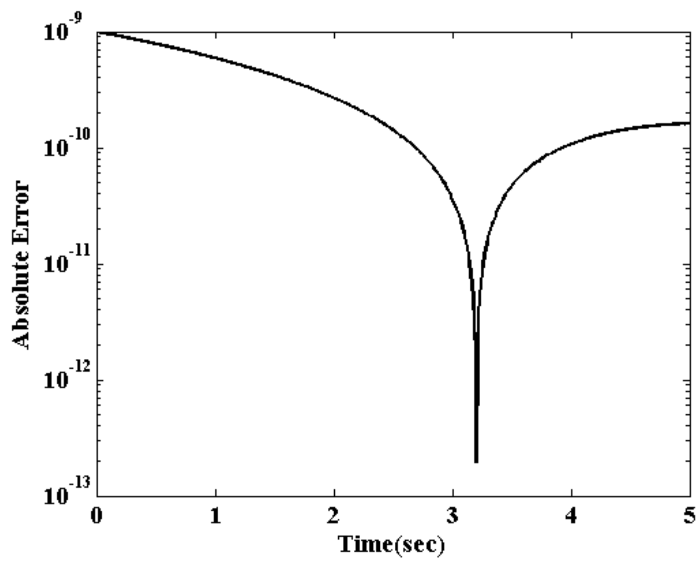
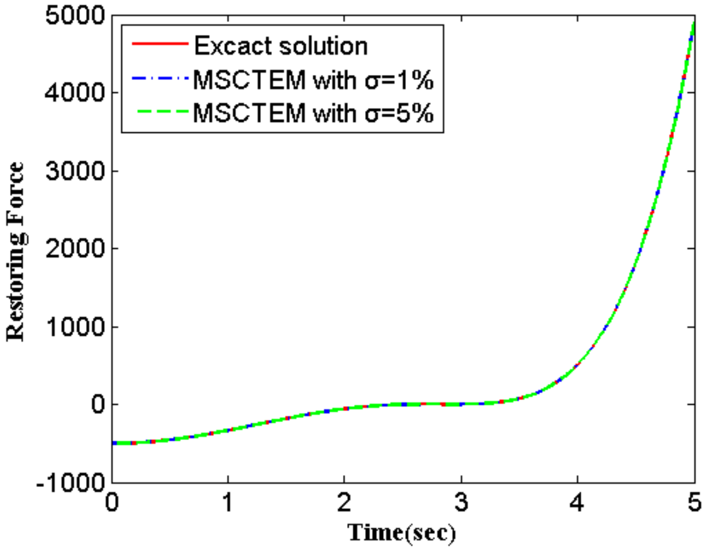


(b)

Figure 4: For Example 1: (a) comparing numerical solution and exact restoring force, and (b) displaying the error of estimation.



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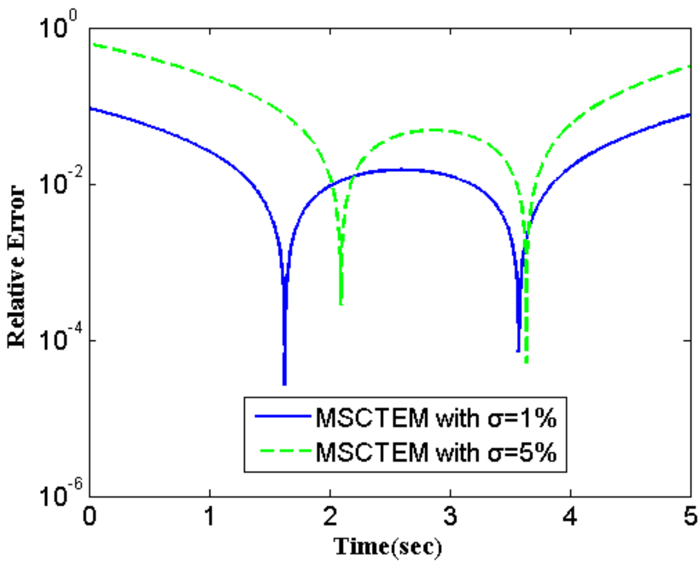


(b)

Figure 5: For Example 1: (a) comparing estimated and exact restoring forces under different noise level, and (b) displaying the error of estimation.

Example 2

In this case, $H(x,\dot{x})$ of the Van der Polo scillator is given by:

$$H(x,\dot{x}) = x + (x^2 - 1)\dot{x} \tag{27}$$

In this equation, x is given by $x(t) = t^3/3 - 8t$ and then the external force can be obtained as:

$$P(t) = \left(\frac{t^3}{3} - 8t\right) + \left[\left(\frac{t^3}{3} - 8t\right)^2 - 1\right](t^2 - 8) + 2t \tag{28}$$

In this calculation, by fixing $\varepsilon = 1 \times 10^{-16}$, $\mathbf{X}_0 = \mathbf{I}$, and $T_f = 4$, he numerical accuracy and stability of different parameters can be tested, which include: $m = 100$ and 500 and different T_S, respectively. The maximum numerical error of H shown in Fig. 6 and 7 are smaller than 10^{-3}. From numerical results, we can find that the maximum numerical error approximate stability when T_S is much larger than T_f.

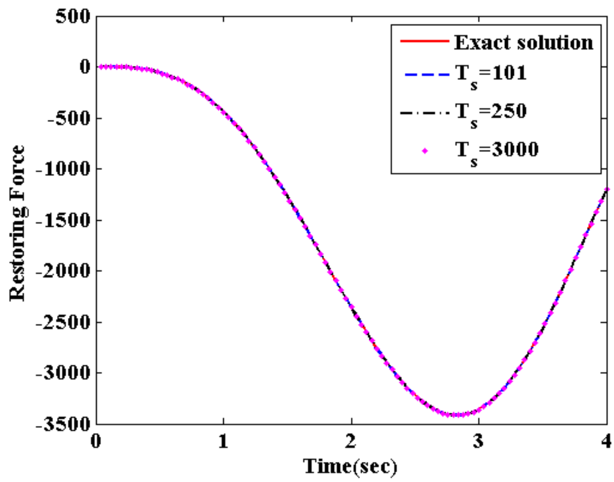
To test the numerical stability of increasing the computational time by 10 seconds, the parameters are fixed as: $T_S = 450$ and $m = 500$. The maximum estimation errors of H, shown in Fig. 8, is smaller than 10^{-7}. Then, when using for the noisy disturbances with 1 and 5%, the computed profile of H is plotted in Fig. 9. Fig. 9(a) compares the restoring force with exact one, and the maximum estimation error of H shown in Fig. 9(b) is smaller than 10^{-1}. From these results, it is shown that the restoring force can be recovered very well, even adding noisy disturbances into measuring data, which shows the stability and accuracy of the proposed scheme.

Example 3

A one-degree of freedom of first seat-person system model (i.e., Fig. 10), considered in [Wei and Griffin (1998)], is given by:

$$M\ddot{x} + c_1\dot{x} + c_2|\dot{x}|\dot{x} + \frac{k_1}{1 + k_2|x|}x = P(t). \tag{29}$$

The model parameters are given $k_1 = 48000(N/m)$, $k_2 = 24000(N/m)$, $c_1 = 300(N-s/m)$, $c_2 = 1500(N-s/m)$. $M = 8(kg)$, and $M_1 = 42(kg)$. Here the external force is given by $P(t) = 0.04\cos(t)$, and the parameters are given by $T_S = 2$, $t_f = 10$, $\mathbf{X}_0 = \mathbf{I}$, and $\varepsilon = 1 \times 10^{-16}$, respectively. The computed profile of H by $m = 51$ and 151 is shown in Fig. 11(a), and the maximum estimated absolute error of H, shown in Fig. 11(b), is smaller than 10^{-1}. From these figures, the numerical results of the proposed method for solving restoring force problem with discontinuous solution are extremely accurate and stable. In addition, using noise level with $\sigma = 1\%$, the computed profile of restoring forces is shown in Fig. 12. From the figure, the numerical errors by $m = 51$ and 151 are smaller than 10^{-1}. Therefore, we can conclude that the proposed scheme can stably and efficiently obtain accurate restoring force, even adding noise to the measuring data.



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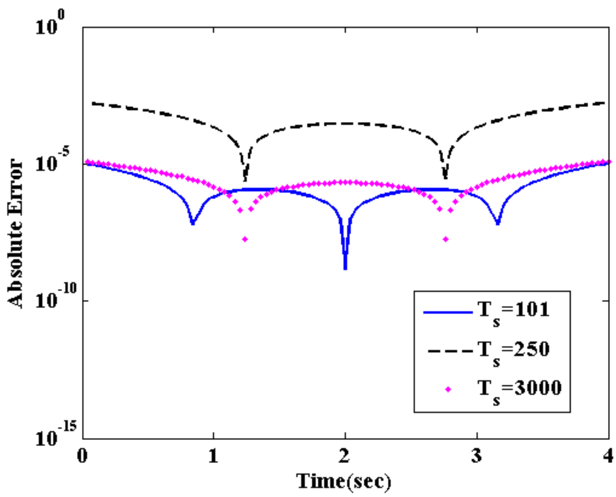
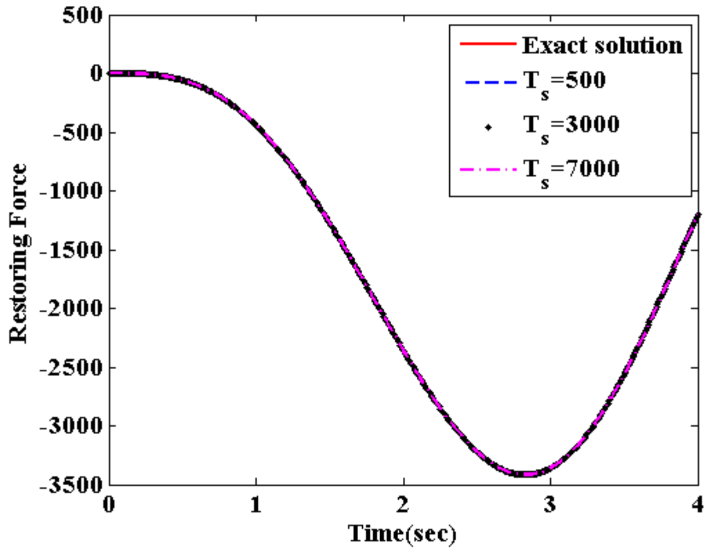


(b)

Figure 6: For Example 2 showing the error of estimation with different T_S and $m = 101$ under 4 seconds



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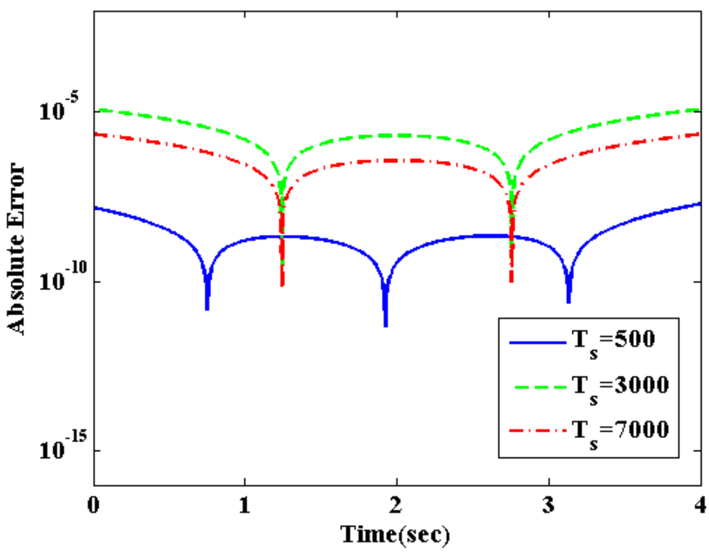

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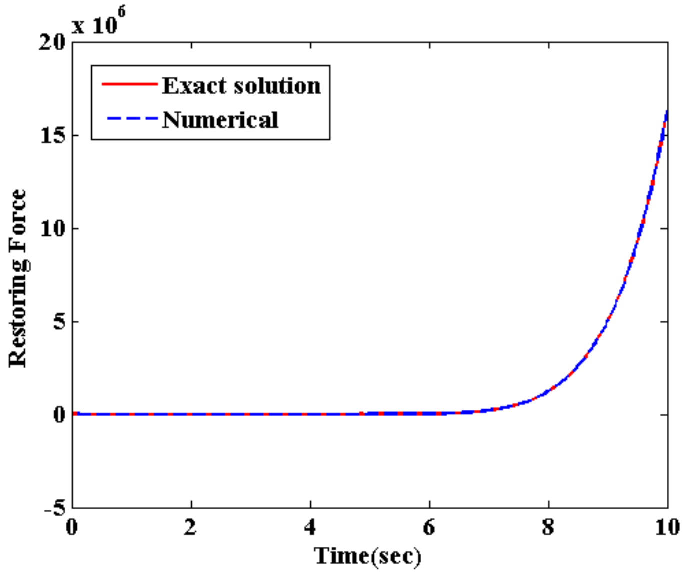
Figure 7: For Example 2 showing the error of estimation with different T_S and $m = 500$ under 4 seconds.



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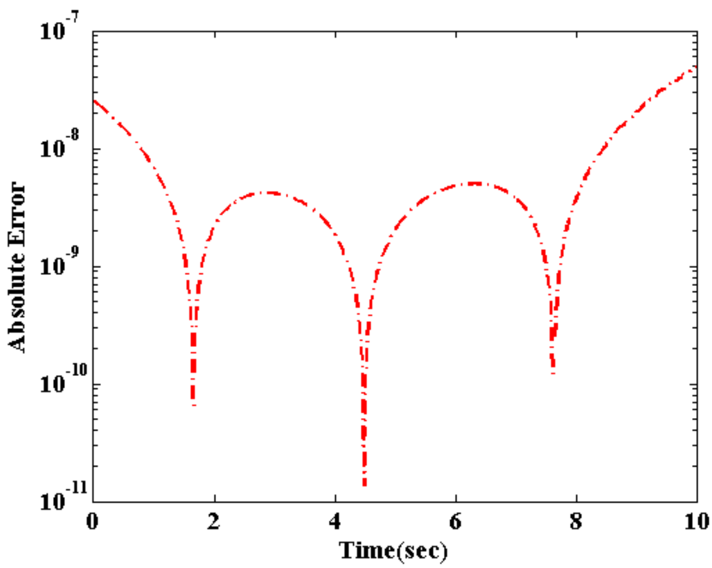

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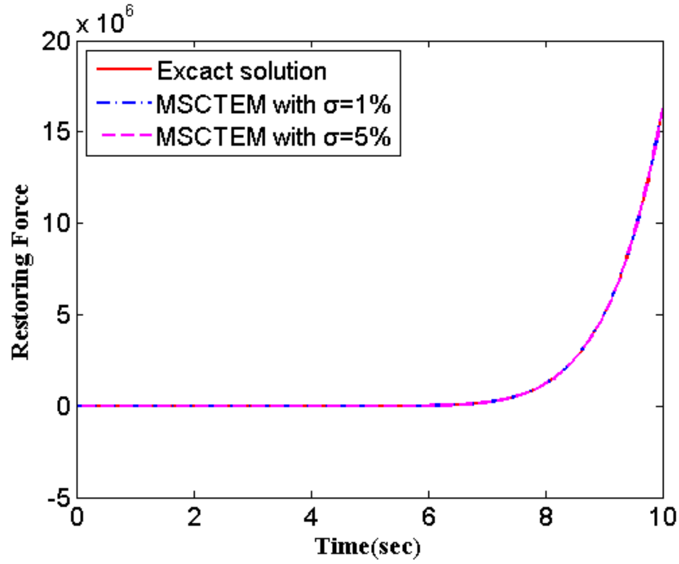
Figure 8: For Example 2 showing the error of estimation with $T_S = 450$ and $m = 500$ under 10 seconds.



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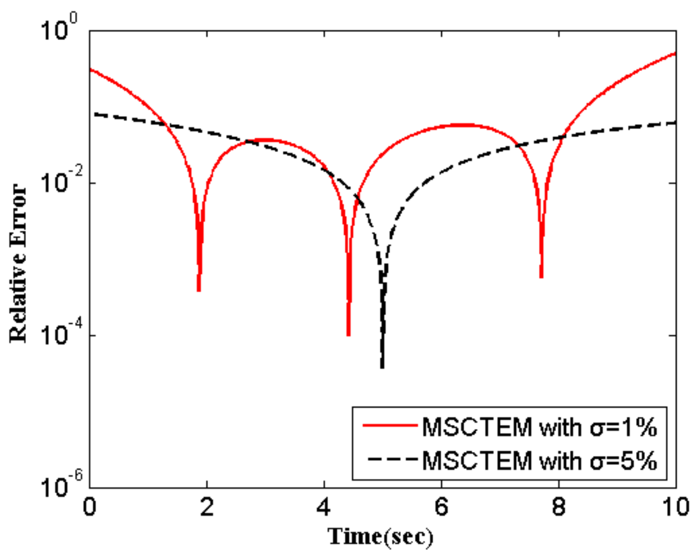

(b)

Figure 9: For Example 2: (a) comparing estimated and exact restoring forces, and (b) displaying the error of estimation.

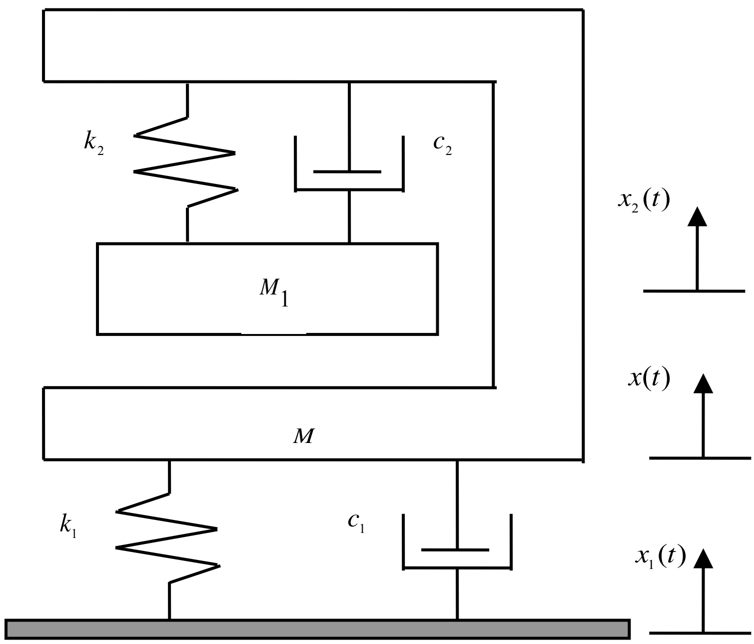

Figure 10: For Example 3: First seat-person system model.

4 Concluding remarks

In this articles, a combination of the MSCTEM and natural regularization technique have been proposed to stably and accurately analyze the restoring force identification problem. In nonlinear mechanical system analysis, the inverse vibration problem is difficult to solve under the measured data with noise. This paper has successfully combined the MSCTEM with a natural regularization technique to determine the unknown restoring force. Due to the inclusion of different CLs to retain high accuracy and stability, the MSCTEM can avoid the numerical instability caused by a high-order polynomial function. Furthermore, when the measured data is contaminated by a large noise, the errors can be controlled by utilizing a natural regularization technique and increasing the CL. In summary, the method presented is a more effective and convenient approach to solve inverse vibration problems.

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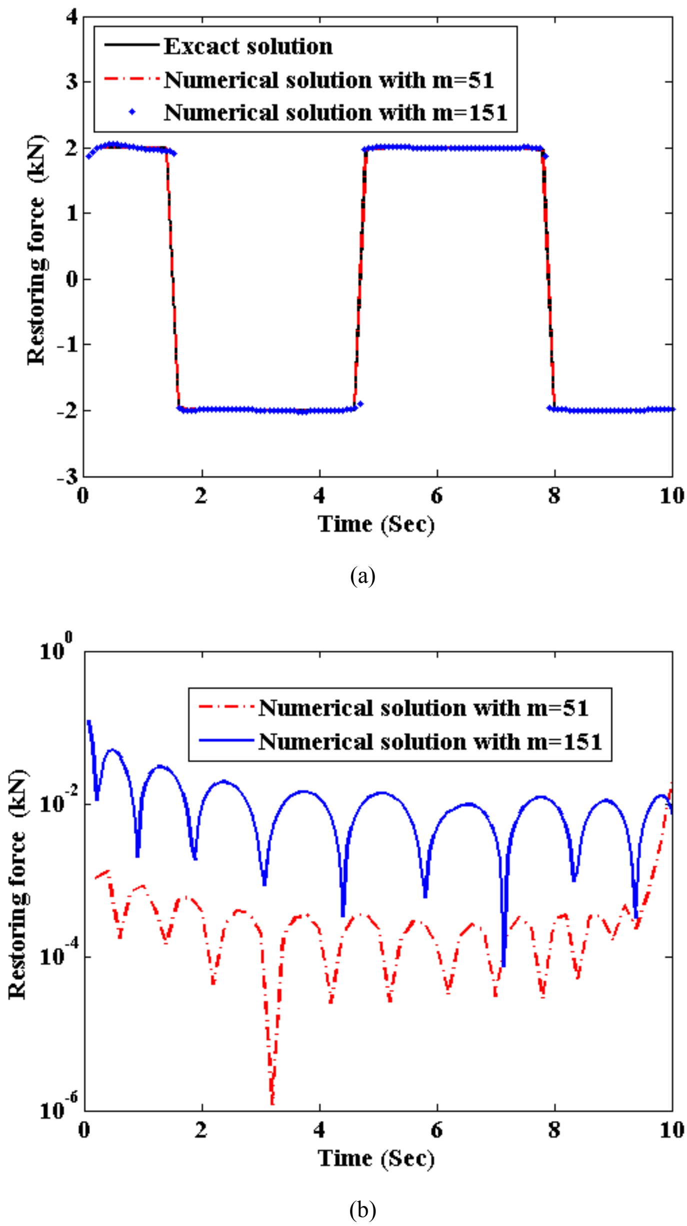

Figure 11: For Example 3: (a) comparing estimated and exact restoring forces; (b) displaying the error of estimation with $m = 51$ and 201.

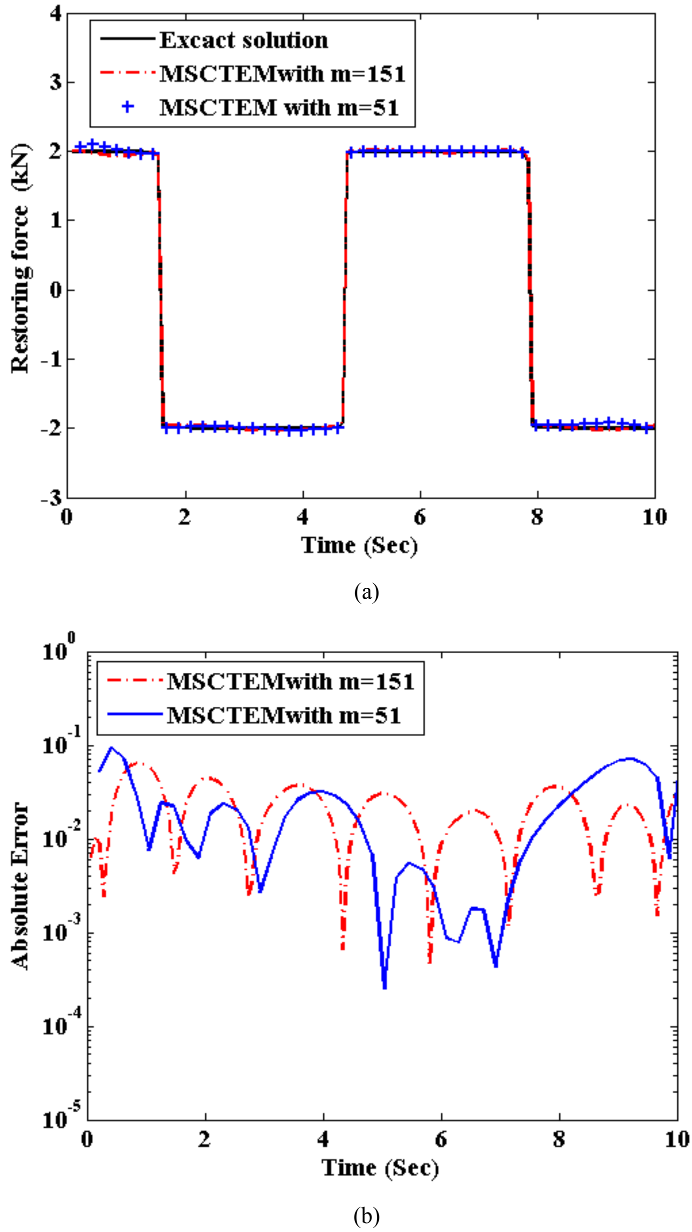

Figure 12: For Example 3: (a) comparing estimated and exact restoring forces under 1% noise level, and (b) displaying the error of estimation.

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