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ARTICLE

Some Identities of the Higher-Order Type 2 Bernoulli Numbers and Polynomials of the Second Kind

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ABSTRACT

We introduce the higher-order type 2 Bernoulli numbers and polynomials of the second kind. In this paper, we investigate some identities and properties for them in connection with central factorial numbers of the second kind and the higher-order type 2 Bernoulli polynomials. We give some relations between the higher-order type 2 Bernoulli numbers of the second kind and their conjugates.

KEYWORDS

Bernoulli polynomials of the second kind; higher-order type 2 Bernoulli polynomials of the second kind; higher-order conjugate type 2 Bernoulli polynomials of the second kind

1 Introduction

For $n \geq 0$, the central factorials $x^{[n]}$ are given by [1–3]

$$x^{[0]} = 1, \quad x^{[n]} = x\left(x + \frac{n}{2} - 1\right)\left(x + \frac{n}{2} - 2\right)\cdots\left(x - \frac{n}{2} + 1\right), \quad (n \geq 1),$$

and the central factorial numbers of the second kind $T(n, k)$ by

$$x^n = \sum_{k=0}^{n} T(n, k) x^{[k]}, \quad (n \geq 0), \quad (\text{see [4–6]}). \tag{1}$$

As is well known, the Bernoulli polynomials are defined by the generating function as

$$\frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \quad (\text{see [7,8]}).$$

When $x=0$, $B_n=B_n(0)$ are called the Bernoulli numbers. Whereas the cosecant polynomials are defined by

$$\frac{2t}{e^t-e^{-t}}e^{xt}=\frac{t}{\sinh t}e^{xt}=\sum_{n=0}^{\infty}D_n(x)\frac{t^n}{n!}.$$

When $x=0$, $D_n=D_n(0)$ are called the cosecant numbers which have been already studied in p.458 of [9]. Here we observe that $D_n(x)=2^nB_n\left(\frac{x+1}{2}\right)$, $(n\geq 0)$. Also, we note that $b_n(x)=\frac{1}{2}D_n(x)$ is called the type 2 Bernoulli polynomials in [10]. Let n be a positive integer and let k be a nonnegative integer. As is well known, Bernoulli polynomials appear in the following expressions of the sums of powers of consecutive integers. That is

$$\sum_{l=0}^{n-1}l^k=\frac{B_{k+1}(n)-B_{k+1}(0)}{k+1}. \tag{2}$$

On the other hand, in [11] it is noted that

$$\sum_{l=0}^{n-1}(2l+1)^k=\frac{1}{2(k+1)}(D_{k+1}(2n)-D_{k+1}). \tag{3}$$

Further, in [10] we considered a random variable cooked from random variables having Laplace distributions and showed its moment is closely connected with the type 2 Bernoulli numbers [10]. Yet another thing is that we obtained some symmetric identities involving type 2 Bernoulli polynomials and power sums of consecutive odd positive integers in (3) by means of Volkenborn p-adic integrals on $\mathbb{Z}_p$.

It is known that the Euler polynomials are given by

$$\frac{2}{e^t+1}e^{xt}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!}. \tag{4}$$

When $x=0$, $E_n=E_n(0)$ are called the Euler numbers.

Whereas the type 2 Euler polynomials are defined by

$$\operatorname{sech} t\, e^{xt}=\frac{2}{e^t+e^{-t}}e^{xt}=\sum_{n=0}^{\infty}E_n^*(x)\frac{t^n}{n!}. \tag{5}$$

When $x=0$, $E_n=E_n(0)$, $(n\geq 0)$, are called the type 2 Euler numbers. We observe that $E_n^*(x)=2^nE_n\left(\frac{x+1}{2}\right)$.

Here we would like to mention that in the literature both Euler and type 2 Euler polynomials are called Euler polynomials. Sometimes this is very confusing. Let n be a positive integer. Then, according to the definition (4), all the even Euler numbers $E_{2n}=0$. Whereas, according to the definition (5), all the odd Euler numbers $E^*_{2n+1} = 0$. To avoid a possible confusion, we call the polynomials in (5) the type 2 Euler polynomials, while reserving the term Euler polynomials for the ones in (4).

Let n be an odd positive integer. As is well known, Euler polynomials and numbers appear in the expressions of the alternating sums of powers of consecutive integers. That is

$$\sum_{l=0}^{n-1}(-1)^l l^k = \frac{E_k(n)+E_k}{2}.$$

On the other hand, it is shown in [10] that

$$\sum_{l=0}^{n-1}(-1)^l(2l+1)^k = \frac{E^*_k(2n)+E^*_k}{2}. \tag{6}$$

Again, in [10] we considered a random variable constructed from random variables having Laplace distributions and showed its moment is closely connected with the type 2 Euler numbers [10]. Still another thing is that we deduced certain symmetric identities involving type 2 Euler polynomials and alternating power sums of consecutive odd positive integers in (6) by using fermionic p-adic integrals on $\mathbb{Z}_p$.

As is well known, the Stirling numbers of the second kind are given by

$$\frac{1}{k!}(e^t-1)^k = \sum_{n=k}^{\infty} S_2(n,k)\frac{t^n}{n!}, \quad (k \geq 0), \quad \text{(see [2])},$$

and the Stirling numbers of the first kind by

$$\frac{1}{k!}(\log(1+t))^k = \sum_{n=k}^{\infty} S_1(n,k)\frac{t^n}{n!}, \quad \text{(see [2])}.$$

From (6), we can derive

$$\frac{1}{k!}(e^{\frac{t}{2}}-e^{-\frac{t}{2}})^k = \sum_{n=k}^{\infty} T(n,k)\frac{t^n}{n!}, \quad (k \geq 0), \tag{7}$$

the proof of which can be found in [2].

Thus, by (7), we get

$$T(n,k) = \frac{1}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\left(l-\frac{k}{2}\right)^n, \quad (k \geq 0). \tag{8}$$

It is well known that the Bernoulli polynomials of the second kind are defined by

$$\frac{t}{\log(1+t)}(1+t)^x = \sum_{n=0}^{\infty} b_n(x)\frac{t^n}{n!}, \quad \text{(see [2,9,11,12])}. \tag{9}$$

Sometimes $\frac{1}{n!}b_n(x)$ are called Bernoulli polynomials of the second kind, whereas $b_n(x)$ are called Cauchy polynomials (see [2,11]). However, we will stick to our definition for the Bernoulli polynomials of the second kind.

When $x=0$, $b_n = b_n(0)$ are variously called Bernoulli numbers of the second kind, Gregory coefficients, reciprocal logarithmic numbers, and Cauchy numbers of the first kind (see [9,13–15]). Here we remark that

$$b_n = B_n^{(n)}(1), \quad (n \geq 0), \quad (\text{see (11)}),$$

where $B_n^{(k)}(x)$ are the Bernoulli polynomials of order k given by

$$\left(\frac{t}{e^t-1}\right)^k e^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x)\frac{t^n}{n!}, \quad k \in \mathbb{Z}, \quad (\text{see [16,17]}).$$

In [9], Howard studied the polynomials $\alpha_n^{(z)}(\lambda)$ given by

$$\left(\frac{\lambda t}{1-(1-t)^{\lambda}}\right)^z = \sum_{n=0}^{\infty} \alpha_n^{(z)}(\lambda)\frac{t^n}{n!}. \tag{10}$$

For any real number $\lambda \neq 0, 1$, Korobov defined the degenerate Bernoulli polynomials of the second given by

$$\frac{\lambda t}{(1+t)^{\lambda}-1}(1+t)^x = \sum_{n=0}^{\infty} b_n(x;\lambda)\frac{t^n}{n!}.$$

Then we see that $\lim_{\lambda\to 0} b_n(x;\lambda) = b_n(x)$. In fact, Korobov introduced what he called 'special polynomials' $p_n(x)$ given by $b_n(x;p) = n!p_n(x)$, for any integer p with $p \geq 2$ (see [18]). Here we note that $b_n(x;\lambda)$ are also called the Korobov polynomials of the first kind and denoted by $K_n(x;\lambda)$ (see [12]).

When $x=0$, $b_n(\lambda) = b_n(0;\lambda)$ are called the degenerate Bernoulli numbers of the second kind. It is immediate to see that $b_n(\lambda) = (-1)^n \alpha_n^{(1)}(\lambda)$ (see (10)). Further, in [19] Howard considered the degenerate Bernoulli numbers of the second kind which is denoted by $\alpha_n(\lambda)$. Note also that $b_n(\lambda) = K_n(0;\lambda)$. In light of these considerations, $b_n(\lambda)$ may be variously called the degenerate Bernoulli numbers of the second, Howard numbers and Korobov numbers of the first kind (see [20]).

In the next section, we will introduce the higher-order type 2 Bernoulli numbers and polynomials of the second kind as variants of the usual higher-order Bernoulli numbers and polynomials of the second kind. We will study some properties and identities for them that are associated with central factorial numbers of the second kind and the higher-order type 2 Bernoulli polynomials. We will deduce some relations between the higher-order type 2 Bernoulli numbers of the second kind and their conjugates.

2 The Higher-Order Type 2 Bernoulli Numbers and Polynomials of the Second Kind

The Bernoulli polynomials of the second kind with order r are defined by the generating function

$$\left(\frac{t}{\log(1+t)}\right)^r (1+t)^x = \sum_{n=0}^{\infty} b_n^{(r)}(x)\frac{t^n}{n!}, \quad r \in \mathbb{Z}.$$

We note from [21–23] that

$$b_n^{(r)}(x) = B_n^{(n-r+1)}(x+1), \quad (n, r \geq 0). \tag{11}$$

From (9), we have

$$\int_0^1 (1+t)^{x+y} dy = \frac{t}{\log(1+t)}(1+t)^x = \sum_{n=0}^{\infty} b_n(x)\frac{t^n}{n!}, \tag{12}$$

and

$$\int_0^1 (1+t)^{x+y} dy = \sum_{n=0}^{\infty} \int_0^1 (x+y)_n dy \frac{t^n}{n!}, \tag{13}$$

where $(x)_0 = 1$, $(x)_n = x(x-1)\ldots(x-n+1)$, $(n \geq 1)$.

By (12) and (13), we get

$$\int_0^1 (x+y)_n dy = b_n(x), \quad (n \geq 0).$$

We observe that

$$2\int_0^1 (1+t)^{2y-1+x} dy = \frac{(1+t)-(1+t)^{-1}}{\log(1+t)}(1+t)^x. \tag{14}$$

Now, we define the type 2 Bernoulli polynomials of the second kind by

$$\frac{(1+t)-(1+t)^{-1}}{2\log(1+t)}(1+t)^x = \sum_{n=0}^{\infty} b_n^*(x)\frac{t^n}{n!}. \tag{15}$$

When $x = 0$, $b_n^* = b_n^*(0)$ is called the type 2 Bernoulli numbers of the second kind.

We observe that

$$\int_0^1 (1+t)^{2y-1+x} dy = \sum_{n=0}^{\infty} \int_0^1 (2y-1+x)_n dy \frac{t^n}{n!}. \tag{16}$$

Therefore, by (14)–(16), we obtain the following theorem:

Theorem 2.1. For $n \geq 0$, we have

$$\int_0^1 (2y-1+x)_n dy = b_n^*(x).$$

In particular,

$$b_n^* = \sum_{l=0}^{n} S_1(n, l) \frac{1}{2(l+1)} \left(1 + (-1)^l\right),$$

and

$$b_n^*(1) = \sum_{l=0}^{n} 2^l S_1(n, l) \frac{1}{l+1}.$$

We illustrate a few values of b_n^* in the following example.

Example 1: We observe first that $b_n^* = \sum_{1 \le l \le n,\, l\, even} S_1(n, l) \frac{1}{l+1}$.

$$b_1^* = 0,$$

$$b_2^* = S_1(2, 2)\frac{1}{3} = \frac{1}{3},$$

$$b_3^* = S_1(3, 2)\frac{1}{3} = (-3) \times \frac{1}{3} = -1,$$

$$b_4^* = S_1(4, 2)\frac{1}{3} + S_1(4, 4)\frac{1}{5} = 11 \times \frac{1}{3} + \frac{1}{5} = \frac{58}{15},$$

$$b_5^* = S_1(5, 2)\frac{1}{3} + S_1(5, 4)\frac{1}{5} = (-50) \times \frac{1}{3} + (-10) \times \frac{1}{5} = -\frac{56}{3},$$

$$b_6^* = S_1(6, 2)\frac{1}{3} + S_1(6, 4)\frac{1}{5} + S_1(6, 6)\frac{1}{7} = 274 \times \frac{1}{3} + 85 \times \frac{1}{5} + \frac{1}{7} = \frac{11390}{105}.$$

For $\alpha \in \mathbb{R}$, let us define the type 2 Bernoulli polynomials of the second kind with order α by

$$\left(\frac{(1+t) - (1+t)^{-1}}{2\log(1+t)}\right)^{\alpha} (1+t)^x = \sum_{n=0}^{\infty} b_n^{*(\alpha)}(x) \frac{t^n}{n!}. \tag{17}$$

When $x = 0$, $b_n^{*(\alpha)} = b_n^{*(\alpha)}(0)$ are called the type 2 Bernoulli numbers of the second kind with order α.

From (17) and with $\alpha = k \in \mathbb{N}$, we have

$$\sum_{n=0}^{\infty} b_n^{*(k)}(x) \frac{t^n}{n!} = \left(\frac{(1+t) - (1+t)^{-1}}{2\log(1+t)}\right)^{k} (1+t)^x. \tag{18}$$

By replacing t by $e^{\frac{t}{2}}-1$ in (18), we get

$$\frac{k!}{t^k}\frac{1}{k!}\left(e^{\frac{t}{2}}-e^{-\frac{t}{2}}\right)^k e^{\frac{t}{2}x}=\sum_{l=0}^{\infty}b_l^{*(k)}(x)\frac{1}{l!}\left(e^{\frac{t}{2}}-1\right)^l$$
$$=\sum_{n=0}^{\infty}\left(\frac{1}{2^n}\sum_{l=0}^{n}b_l^{*(k)}(x)S_2(n,\, l)\right)\frac{t^n}{n!}. \tag{19}$$

On the other hand, by making use of (7) we have

$$\frac{k!}{t^k}\frac{1}{k!}\left(e^{\frac{t}{2}}-e^{-\frac{t}{2}}\right)^k e^{\frac{t}{2}x}=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\frac{\binom{n}{l}}{\binom{l+k}{l}}T(l+k,\, k)2^{-n+l}x^{n-l}\right)\frac{t^n}{n!}. \tag{20}$$

Therefore, by (19) and (20), we obtain the following theorem:

Theorem 2.2. For $n\geq 0$ and $k\in\mathbb{N}$, we have

$$\sum_{l=0}^{n}b_l^{*(k)}(x)S_2(n,\, l)=\sum_{l=0}^{n}\frac{\binom{n}{l}}{\binom{l+k}{l}}T(l+k,\, k)2^l x^{n-l}.$$

In particular, we have

$$2^nT(n+k,\, k)=\binom{n+k}{n}\sum_{l=0}^{n}b_l^{*(k)}S_2(n,\, l),\quad b_n^{*(k)}=\sum_{l=0}^{n}S_1(n,\, l)\frac{2^lT(l+k,\, k)}{\binom{l+k}{l}}.$$

We illustrate a few values of $b_n^{*(2)}$ in the following example:

Example 2: Let $n\geq 2$ be any integer.

Then we have from (8) that $T(n,\,2)=\frac{1}{2!}\sum_{l=0}^{2}\binom{2}{l}(-1)^{2-l}(l-1)^n=\begin{cases}1, & \text{if } n \text{ even},\\ 0, & \text{if } n \text{ odd}.\end{cases}$

Thus, for $n\geq 1$, we have $b_n^{*(2)}=\sum_{1\leq l\leq n,\, l\, even}S_1(n,\, l)\frac{2^l}{\binom{l+2}{2}}.$

$$b_1^{*(2)}=0,$$

$$b_2^{*(2)}=S_1(2,\,2)\frac{2^2}{\binom{4}{2}}=\frac{4}{6}=\frac{2}{3},$$

$$b_3^{*(2)} = S_1(3,2)\frac{2^2}{\binom{4}{2}} = (-3) \times \frac{4}{6} = -2,$$

$$b_4^{*(2)} = S_1(4,2)\frac{2^2}{\binom{4}{2}} + S_1(4,4)\frac{2^4}{\binom{6}{2}} = 11 \times \frac{4}{6} + \frac{16}{15} = \frac{42}{5},$$

$$b_5^{*(2)} = S_1(5,2)\frac{2^2}{\binom{4}{2}} + S_1(5,4)\frac{2^4}{\binom{6}{2}} = (-50) \times \frac{4}{6} + (-10) \times \frac{16}{15} = -44,$$

$$b_6^{*(2)} = S_1(6,2)\frac{2^2}{\binom{4}{2}} + S_1(6,4)\frac{2^4}{\binom{6}{2}} + S_1(6,6)\frac{2^6}{\binom{8}{2}} = 274 \times \frac{4}{6} + 85 \times \frac{16}{15} + \frac{64}{28} = \frac{5788}{21}.$$

For $\alpha \in \mathbb{R}$, we recall that the cosecant polynomials of order α are defined by

$$\left(\frac{2t}{e^t - e^{-t}}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} D_n^{(\alpha)}(x)\frac{t^n}{n!}. \tag{21}$$

For $k \in \mathbb{N}$, let us take $\alpha = -k$ and replace t by $\log(1+t)$ in (21). Then we have

$$\begin{aligned}\left(\frac{(1+t)-(1+t)^{-1}}{2\log(1+t)}\right)^{k}(1+t)^x &= \sum_{l=0}^{\infty} D_l^{(-k)}(x)\frac{1}{l!}(\log(1+t))^l \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n} S_1(n,l)D_l^{(-k)}(x)\right)\frac{t^n}{n!}. \end{aligned} \tag{22}$$

Therefore, by (18) and (22), we obtain the following theorem:

Theorem 2.3. For $n \geq 0$, $k \in \mathbb{N}$, we have

$$b_n^{*(k)}(x) = \sum_{l=0}^{n} S_1(n,l)D_l^{(-k)}(x).$$

Replacing t by $2\log(1+t)$ in (7), we derive the following equation:

$$\begin{aligned}\frac{1}{k!}\left((1+t)-(1+t)^{-1}\right)^k &= \sum_{l=k}^{\infty} T(l,k)2^l\frac{1}{l!}(\log(1+t))^l \\ &= \sum_{l=k}^{\infty} T(l,k)2^l \sum_{n=l}^{\infty} S_1(n,l)\frac{t^n}{n!}\end{aligned}$$

$$=\sum_{n=k}^{\infty}\left(\sum_{l=k}^{n}T(l,k)2^{l}S_{1}(n,l)\right)\frac{t^{n}}{n!}. \tag{23}$$

On the other hand, we also have

$$\frac{1}{k!}\left((1+t)-(1+t)^{-1}\right)^{k}=\frac{1}{k!}\left(\frac{(1+t)-(1+t)^{-1}}{2\log(1+t)}\right)^{k}(2\log(1+t))^{k}$$
$$=2^{k}\sum_{l=0}^{\infty}b_{l}^{*(k)}\frac{t^{l}}{l!}\sum_{m=k}^{\infty}S_{1}(m,k)\frac{t^{m}}{m!}$$
$$=2^{k}\sum_{n=k}^{\infty}\left(\sum_{m=k}^{n}S_{1}(m,k)\binom{n}{m}b_{n-m}^{*(k)}\right)\frac{t^{n}}{n!}. \tag{24}$$

Therefore, by (23) and (24), we obtain the following theorem:

Theorem 2.4. For $n,\ k\geq 0$, we have

$$\sum_{l=k}^{n}T(l,k)2^{l}S_{1}(n,l)=2^{k}\sum_{l=k}^{n}S_{1}(l,k)\binom{n}{l}b_{n-l}^{*(k)}$$
$$=2^{k}\sum_{l=0}^{n-k}S_{1}(n-l,k)\binom{n}{l}b_{l}^{*(k)}.$$

We observe that

$$\int_{0}^{1}\cdots\int_{0}^{1}(1+t)^{2(x_{1}+\cdots+x_{k})-k+x}dx_{1}dx_{2}\cdots dx_{k}$$
$$=\left(\frac{(1+t)-(1+t)^{-1}}{2\log(1+t)}\right)^{k}(1+t)^{x}=\sum_{n=0}^{\infty}b_{n}^{*(k)}(x)\frac{t^{n}}{n!}. \tag{25}$$

Thus, by (25), we get

$$\frac{1}{n!}b_{n}^{*(k)}(x)=\int_{0}^{1}\cdots\int_{0}^{1}\binom{2(x_{1}+\cdots+x_{k})-k+x}{n}dx_{1}dx_{2}\cdots dx_{k}.$$

Now, for $\alpha\in\mathbb{R}$ we define the conjugate type 2 Bernoulli polynomials of the second kind with order α by

$$\left(\frac{(1+t)-(1+t)^{-1}}{2(1+t)\log(1+t)}\right)^{\alpha}(1+t)^{x}=\sum_{n=0}^{\infty}\widehat{b}_{n}^{*(\alpha)}(x)\frac{t^{n}}{n!}. \tag{26}$$

Then, by (26), we get

$$\int_0^1 \cdots \int_0^1 (1+t)^{-2(x_1+\cdots+x_k)+x} dx_1 dx_2 \cdots dx_k$$

$$= \left(\frac{(1+t)-(1+t)^{-1}}{2(1+t)\log(1+t)} \right)^k (1+t)^x = \sum_{n=0}^{\infty} \widehat{b}_n^{*(k)}(x) \frac{t^n}{n!}. \tag{27}$$

By (27), we get

$$\frac{1}{n!} \widehat{b}_n^{*(k)}(x) = \int_0^1 \cdots \int_0^1 \binom{-2(x_1+\cdots+x_k)+x}{n} dx_1 dx_2 \cdots dx_k. \tag{28}$$

When $x = 0$, $\widehat{b}_n^{*(\alpha)} = \widehat{b}_n^{*(\alpha)}(0)$ is called the conjugate type 2 Bernoulli numbers of the second kind with order α.

For $k \in \mathbb{N}$, by (28), we get

$$\frac{1}{n!} \widehat{b}_n^{*(k)}(k) = \int_0^1 \cdots \int_0^1 \binom{-2(x_1+\cdots+x_k)+k}{n} dx_1 dx_2 \cdots dx_k$$

$$= (-1)^n \int_0^1 \cdots \int_0^1 \binom{2(x_1+\cdots+x_k)-k+n-1}{n} dx_1 \cdots dx_k$$

$$= (-1)^n \sum_{m=0}^{n} \binom{n-1}{n-m} \int_0^1 \cdots \int_0^1 \binom{2(x_1+\cdots+x_k)-k}{m} dx_1 \cdots dx_k$$

$$= (-1)^n \sum_{m=1}^{n} \binom{n-1}{m-1} \frac{1}{m!} b_m^{*(k)}. \tag{29}$$

Therefore, by (29), we obtain the following theorem:

Theorem 2.5. For $n, k \in \mathbb{N}$, we have

$$(-1)^n \frac{1}{n!} \widehat{b}_n^{*(k)}(k) = \sum_{m=1}^{n} \binom{n-1}{m-1} \frac{1}{m!} b_m^{*(k)}.$$

Remark. Likewise, for $n, k \in \mathbb{N}$, we have

$$(-1)^n \frac{1}{n!} b_n^{*(k)}(k) = \sum_{m=1}^{n} \binom{n-1}{m-1} \frac{1}{m!} \widehat{b}_m^{*(k)}.$$

3 Conclusions

In Section 2, we introduced the higher-order type 2 Bernoulli numbers and polynomials of the second kind and the higher-order conjugate type 2 Bernoulli numbers of the second kind. In Theorems 2–4, we obtained some properties and identities for them that are associated with central factorial numbers of the second kind and higher-order cosecant polynomials and the Stirling numbers of the first kind. In Theorem 5, we derived the relation between the higher-order type 2 Bernoulli numbers of the second kind and their conjugates.

Many problems in science and engineering can be modeled by polynomial optimization which concerns optimizing a polynomial subject to polynomial equations and inequalities. Thanks to an adoption of tools from real algebraic geometry, semidefinite programming and the theory of moments, etc., there has been tremendous progress in this field. We hope that the polynomials newly introduced in the present paper or their possible multivariate versions will play some role in near future.

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